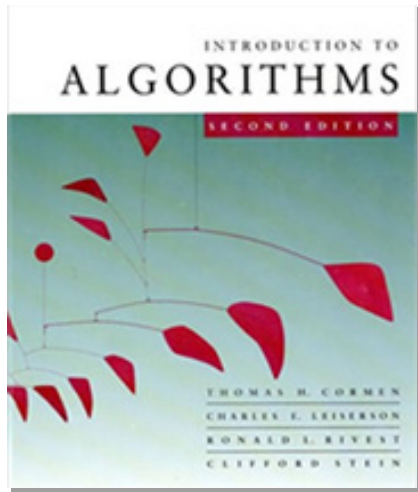


Introduction to Algorithms

6.046J/18.401J

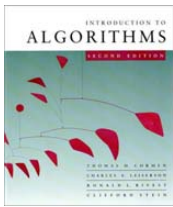


LECTURE 13

Amortized Analysis

- Dynamic tables
- Aggregate method
- Accounting method
- Potential method

Prof. Charles E. Leiserson



How large should a hash table be?

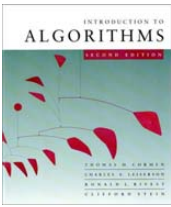
测试能否引入

Goal: Make the table as small as possible, but large enough so that it won't overflow (or otherwise become inefficient).

Problem: What if we don't know the proper size in advance?

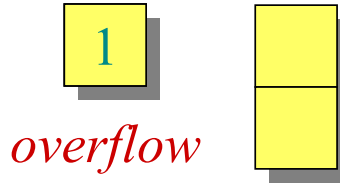
Solution: *Dynamic tables.*

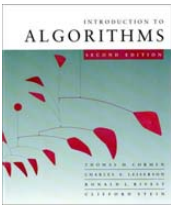
IDEA: Whenever the table overflows, “grow” it by allocating (via **malloc** or **new**) a new, larger table. Move all items from the old table into the new one, and free the storage for the old table.



Example of a dynamic table

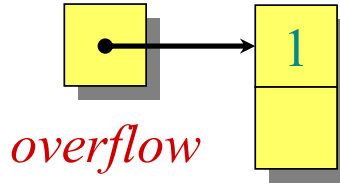
1. INSERT
2. INSERT

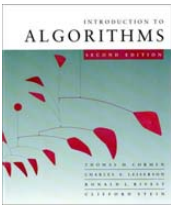




Example of a dynamic table

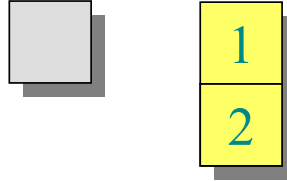
1. INSERT
2. INSERT

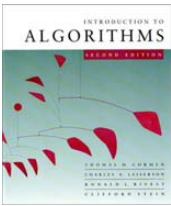




Example of a dynamic table

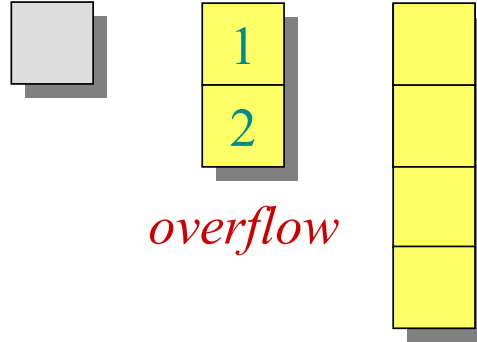
1. INSERT
2. INSERT

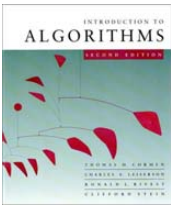




Example of a dynamic table

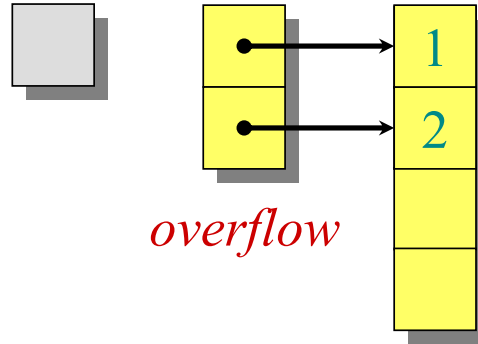
1. INSERT
2. INSERT
3. INSERT

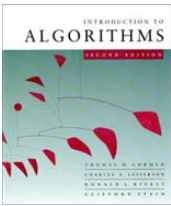




Example of a dynamic table

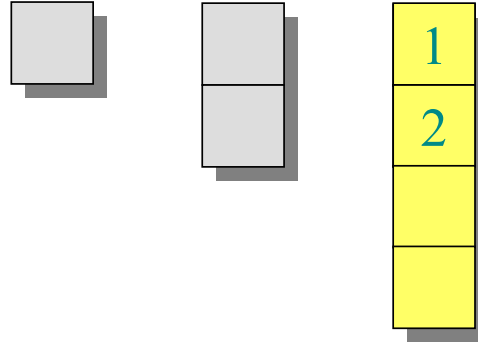
1. INSERT
2. INSERT
3. INSERT

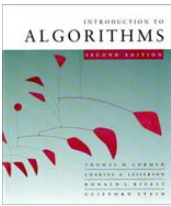




Example of a dynamic table

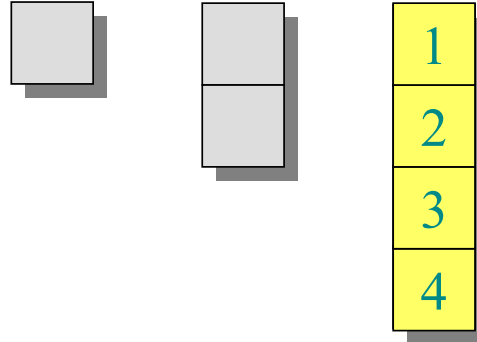
1. INSERT
2. INSERT
3. INSERT

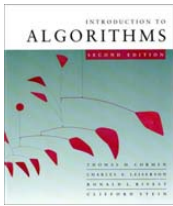




Example of a dynamic table

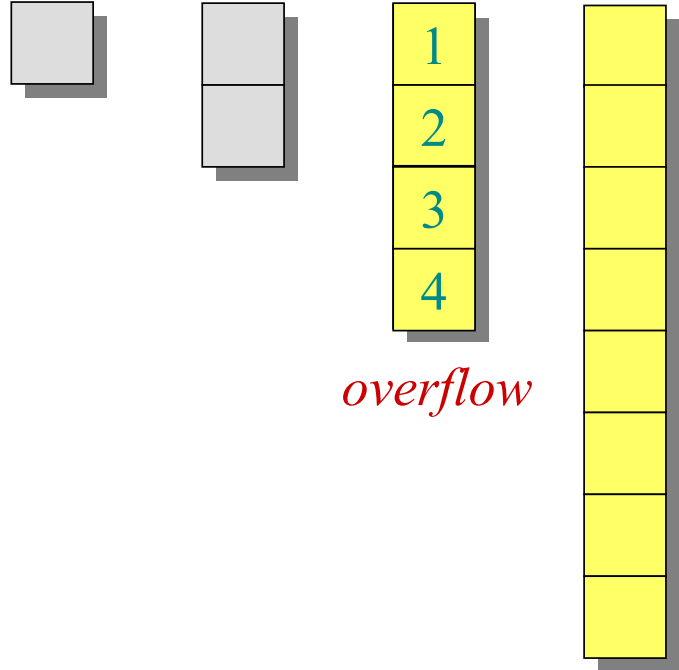
1. INSERT
2. INSERT
3. INSERT
4. INSERT

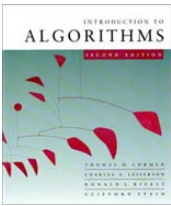




Example of a dynamic table

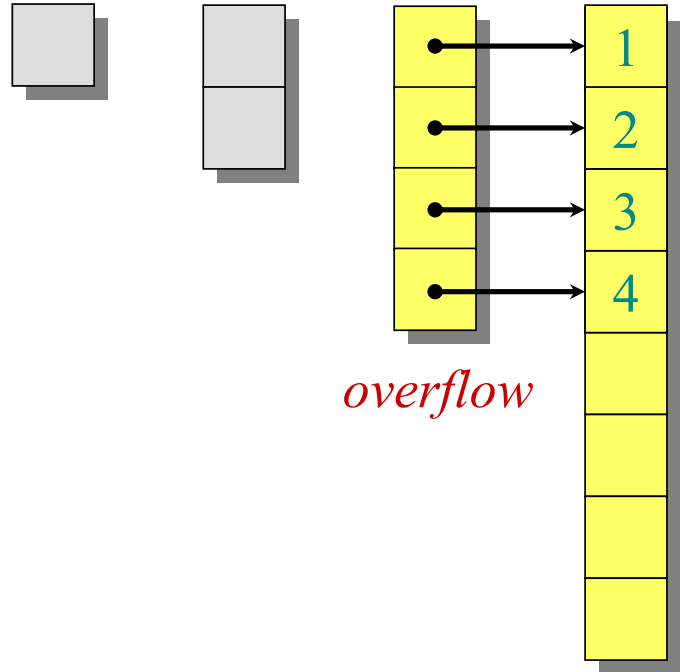
1. INSERT
2. INSERT
3. INSERT
4. INSERT
5. INSERT

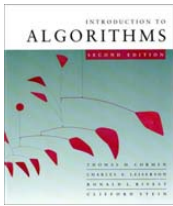




Example of a dynamic table

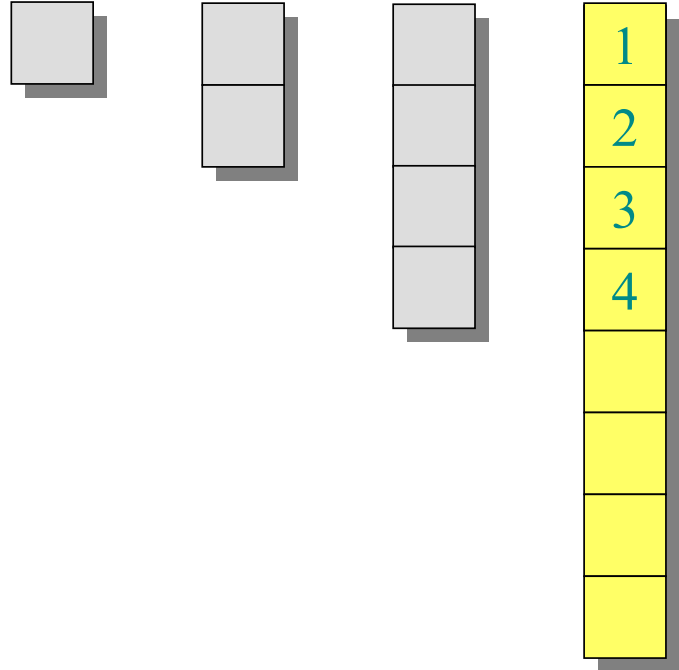
1. INSERT
2. INSERT
3. INSERT
4. INSERT
5. INSERT

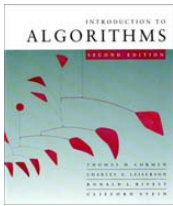




Example of a dynamic table

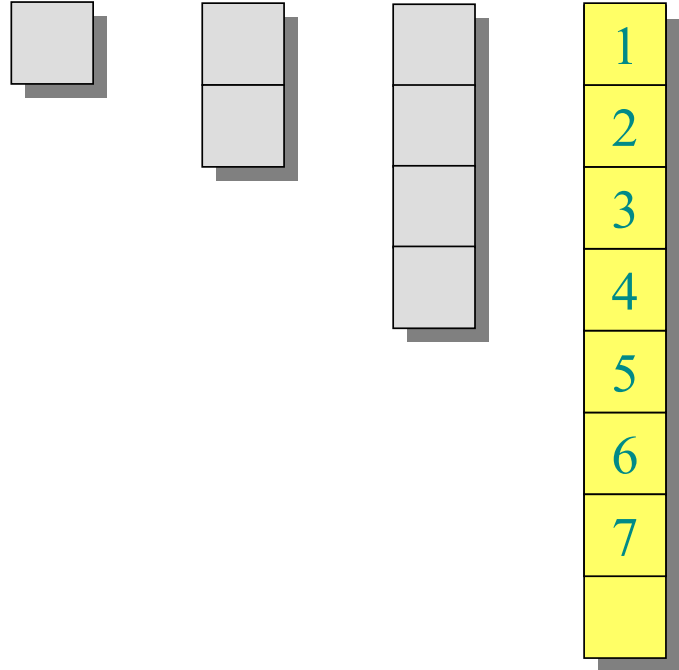
1. INSERT
2. INSERT
3. INSERT
4. INSERT
5. INSERT

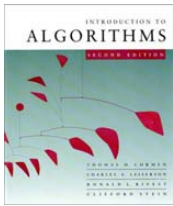




Example of a dynamic table

1. INSERT
2. INSERT
3. INSERT
4. INSERT
5. INSERT
6. INSERT
7. INSERT



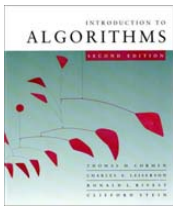


Worst-case analysis

Consider a sequence of n insertions. The worst-case time to execute one insertion is $\Theta(n)$. Therefore, the worst-case time for n insertions is $n \cdot \Theta(n) = \Theta(n^2)$.

WRONG! In fact, the worst-case cost for n insertions is only $\Theta(n) \ll \Theta(n^2)$.

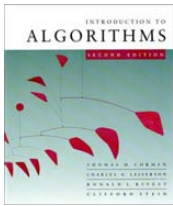
Let's see why.



Tighter analysis

Let $c_i =$ the cost of the i th insertion
$$= \begin{cases} i & \text{if } i - 1 \text{ is an exact power of } 2, \\ 1 & \text{otherwise.} \end{cases}$$

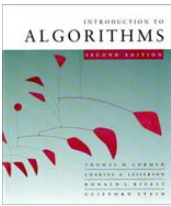
i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
c_i	1	2	3	1	5	1	1	1	9	1



Tighter analysis

Let $c_i =$ the cost of the i th insertion
 $= \begin{cases} i & \text{if } i - 1 \text{ is an exact power of } 2, \\ 1 & \text{otherwise.} \end{cases}$

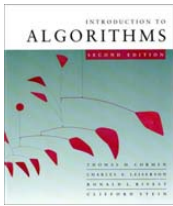
i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
c_i	1	1	1	1	1	1	1	1	1	1
		1	2		4				8	



Tighter analysis (continued)

$$\begin{aligned}\text{Cost of } n \text{ insertions} &= \sum_{i=1}^n c_i \\ &\leq n + \sum_{j=0}^{\lfloor \lg(n-1) \rfloor} 2^j \\ &\leq 3n \\ &= \Theta(n).\end{aligned}$$

Thus, the average cost of each dynamic-table operation is $\Theta(n)/n = \Theta(1)$.

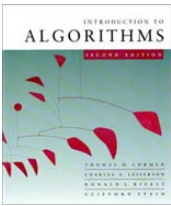


Amortized analysis

An *amortized analysis* is any strategy for analyzing a sequence of operations to show that the average cost per operation is small, even though a single operation within the sequence might be expensive.

Even though we're taking averages, however, probability is not involved!

- An amortized analysis guarantees the average performance of each operation in the *worst case*.



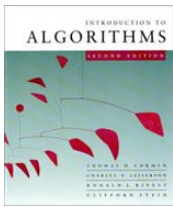
Types of amortized analyses

Three common amortization arguments:

- the *aggregate* method,
- the *accounting* method,
- the *potential* method.

We've just seen an aggregate analysis.

The aggregate method, though simple, lacks the precision of the other two methods. In particular, the accounting and potential methods allow a specific *amortized cost* to be allocated to each operation.



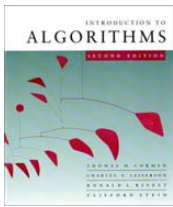
Accounting method

- Charge i th operation a fictitious *amortized cost* \hat{c}_i , where \$1 pays for 1 unit of work (*i.e.*, time).
- This fee is consumed to perform the operation.
- Any amount not immediately consumed is stored in the *bank* for use by subsequent operations.
- The bank balance must not go negative! We must ensure that

$$\sum_{i=1}^n c_i \leq \sum_{i=1}^n \hat{c}_i$$

for all n .

- Thus, the total amortized costs provide an upper bound on the total true costs.



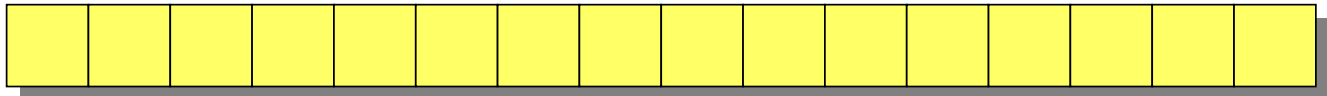
Accounting analysis of dynamic tables

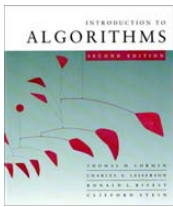
Charge an amortized cost of $\hat{c}_i = \$3$ for the i th insertion.

- \$1 pays for the immediate insertion.
- \$2 is stored for later table doubling.

When the table doubles, \$1 pays to move a recent item, and \$1 pays to move an old item.

Example:





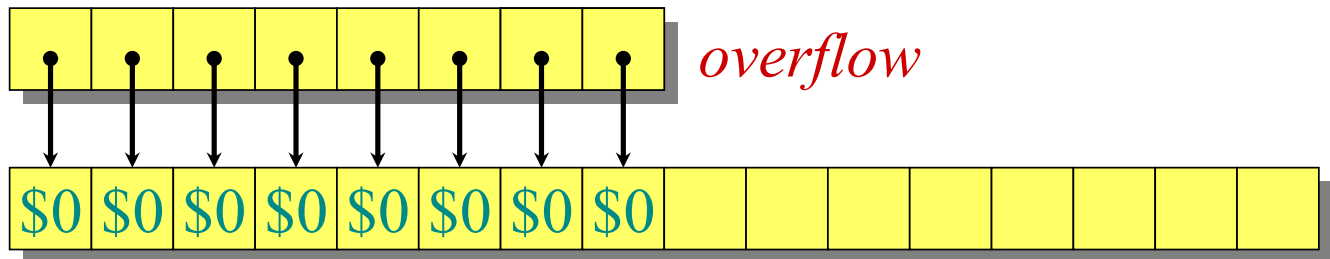
Accounting analysis of dynamic tables

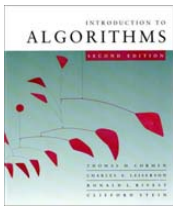
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- **\$1** pays for the immediate insertion.
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Example:





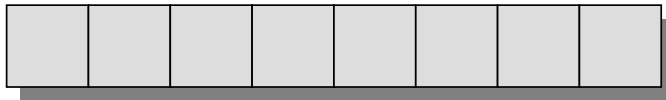
Accounting analysis of dynamic tables

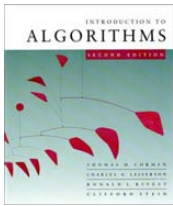
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Example:



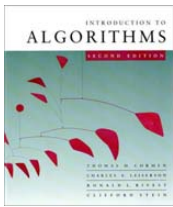


Accounting analysis (continued)

Key invariant: Bank balance never drops below 0. Thus, the sum of the amortized costs provides an upper bound on the sum of the true costs.

i	1	2	3	4	5	6	7	8	9	10
$size_i$	1	2	4	4	8	8	8	8	16	16
c_i	1	2	3	1	5	1	1	1	9	1
\hat{c}_i	2*	3	3	3	3	3	3	3	3	3
$bank_i$	1	2	2	4	2	4	6	8	2	4

*Okay, so I lied. The first operation costs only \$2, not \$3.

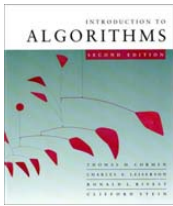


Potential method

IDEA: View the bank account as the potential energy (*à la* physics) of the dynamic set.

Framework:

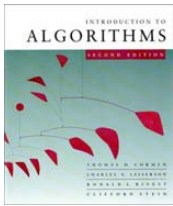
- Start with an initial data structure D_0 .
- Operation i transforms D_{i-1} to D_i .
- The cost of operation i is c_i .
- Define a **potential function** $\Phi : \{D_i\} \rightarrow \mathbb{R}$, such that $\Phi(D_0) = 0$ and $\Phi(D_i) \geq 0$ for all i .
- The **amortized cost** \hat{c}_i with respect to Φ is defined to be $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$.



Understanding potentials

$$\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}_{\text{potential difference } \Delta\Phi_i}$$

- If $\Delta\Phi_i > 0$, then $\hat{c}_i > c_i$. Operation i stores work in the data structure for later use.
- If $\Delta\Phi_i < 0$, then $\hat{c}_i < c_i$. The data structure delivers up stored work to help pay for operation i .

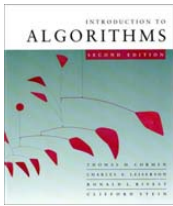


The amortized costs bound the true costs

The total amortized cost of n operations is

$$\sum_{i=1}^n \hat{c}_i = \sum_{i=1}^n (c_i + \Phi(D_i) - \Phi(D_{i-1}))$$

Summing both sides.

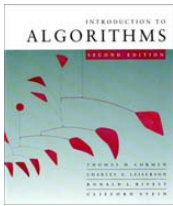


The amortized costs bound the true costs

The total amortized cost of n operations is

$$\begin{aligned}\sum_{i=1}^n \hat{c}_i &= \sum_{i=1}^n (c_i + \Phi(D_i) - \Phi(D_{i-1})) \\ &= \sum_{i=1}^n c_i + \Phi(D_n) - \Phi(D_0)\end{aligned}$$

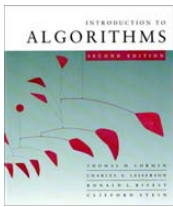
The series telescopes.



The amortized costs bound the true costs

The total amortized cost of n operations is

$$\begin{aligned}\sum_{i=1}^n \hat{c}_i &= \sum_{i=1}^n (c_i + \Phi(D_i) - \Phi(D_{i-1})) \\ &= \sum_{i=1}^n c_i + \Phi(D_n) - \Phi(D_0) \\ &\geq \sum_{i=1}^n c_i \quad \text{since } \Phi(D_n) \geq 0 \text{ and } \Phi(D_0) = 0.\end{aligned}$$



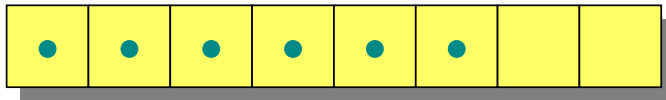
Potential analysis of table doubling

Define the potential of the table after the i th insertion by $\Phi(D_i) = 2i - 2^{\lceil \lg i \rceil}$. (Assume that $2^{\lceil \lg 0 \rceil} = 0$.)

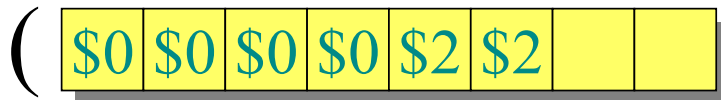
Note:

- $\Phi(D_0) = 0$,
- $\Phi(D_i) \geq 0$ for all i .

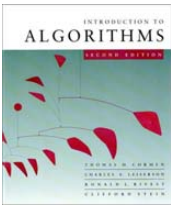
Example:



$$\Phi = 2 \cdot 6 - 2^3 = 4$$



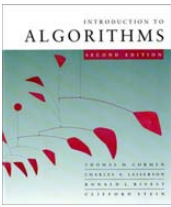
(accounting method)



Calculation of amortized costs

The amortized cost of the i th insertion is

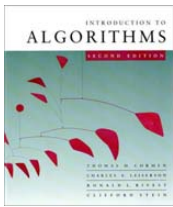
$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$



Calculation of amortized costs

The amortized cost of the i th insertion is

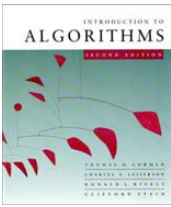
$$\begin{aligned}\hat{c}_i &= c_i + \Phi(D_i) - \Phi(D_{i-1}) \\ &= \left\{ \begin{array}{l} i \text{ if } i-1 \text{ is an exact power of } 2, \\ 1 \text{ otherwise;} \end{array} \right\} \\ &\quad + (2i - 2^{\lceil \lg i \rceil}) - (2(i-1) - 2^{\lceil \lg (i-1) \rceil})\end{aligned}$$



Calculation of amortized costs

The amortized cost of the i th insertion is

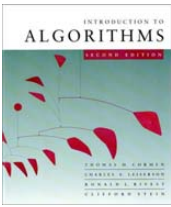
$$\begin{aligned}\hat{c}_i &= c_i + \Phi(D_i) - \Phi(D_{i-1}) \\ &= \left\{ \begin{array}{l} i \text{ if } i-1 \text{ is an exact power of } 2, \\ 1 \text{ otherwise;} \end{array} \right\} \\ &\quad + (2i - 2^{\lceil \lg i \rceil}) - (2(i-1) - 2^{\lceil \lg(i-1) \rceil}) \\ &= \left\{ \begin{array}{l} i \text{ if } i-1 \text{ is an exact power of } 2, \\ 1 \text{ otherwise;} \end{array} \right\} \\ &\quad + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg(i-1) \rceil}.\end{aligned}$$



Calculation

Case 1: $i - 1$ is an exact power of 2.

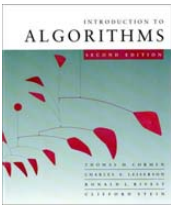
$$\hat{c}_i = i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil}$$



Calculation

Case 1: $i - 1$ is an exact power of 2.

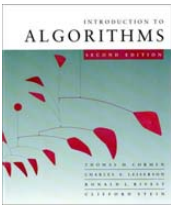
$$\begin{aligned}\hat{c}_i &= i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\ &= i + 2 - 2(i - 1) + (i - 1)\end{aligned}$$



Calculation

Case 1: $i - 1$ is an exact power of 2.

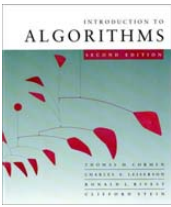
$$\begin{aligned}\hat{c}_i &= i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\ &= i + 2 - 2(i - 1) + (i - 1) \\ &= i + 2 - 2i + 2 + i - 1\end{aligned}$$



Calculation

Case 1: $i - 1$ is an exact power of 2.

$$\begin{aligned}\hat{c}_i &= i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\ &= i + 2 - 2(i - 1) + (i - 1) \\ &= i + 2 - 2i + 2 + i - 1 \\ &= 3\end{aligned}$$



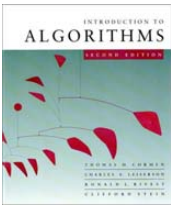
Calculation

Case 1: $i - 1$ is an exact power of 2.

$$\begin{aligned}\hat{c}_i &= i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\ &= i + 2 - 2(i - 1) + (i - 1) \\ &= i + 2 - 2i + 2 + i - 1 \\ &= 3\end{aligned}$$

Case 2: $i - 1$ is *not* an exact power of 2.

$$\hat{c}_i = 1 + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil}$$



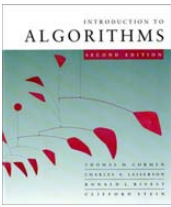
Calculation

Case 1: $i - 1$ is an exact power of 2.

$$\begin{aligned}\hat{c}_i &= i + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\ &= i + 2 - 2(i - 1) + (i - 1) \\ &= i + 2 - 2i + 2 + i - 1 \\ &= 3\end{aligned}$$

Case 2: $i - 1$ is *not* an exact power of 2.

$$\begin{aligned}\hat{c}_i &= 1 + 2 - 2^{\lceil \lg i \rceil} + 2^{\lceil \lg (i-1) \rceil} \\ &= 3 \quad (\text{since } 2^{\lceil \lg i \rceil} = 2^{\lceil \lg (i-1) \rceil})\end{aligned}$$



Calculation

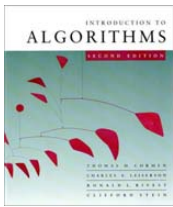
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Therefore, n insertions cost $\Theta(n)$ in the worst case.



Calculation

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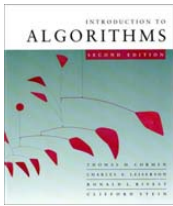
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Therefore, n insertions cost $\Theta(n)$ in the worst case.

Exercise: Fix the bug in this analysis to show that the amortized cost of the first insertion is only 2.



Conclusions

- Amortized costs can provide a clean abstraction of data-structure performance.
- Any of the analysis methods can be used when an amortized analysis is called for, but each method has some situations where it is arguably the simplest or most precise.
- Different schemes may work for assigning amortized costs in the accounting method, or potentials in the potential method, sometimes yielding radically different bounds.