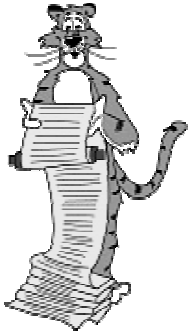


Lecture T2: Turing Machines



Overview

Attempt to understand essential nature of computation by studying properties of simple machine models.

Goal: simplest machine that is "as powerful" as conventional computers.

Surprising Fact 1.



Surprising Fact 2.



2

Adding Power to FSA

FSA advantages:

- Extremely simple and cheap to build.
- Well suited to certain important tasks.
 - pattern matching, filtering, dishwashers, remote controls, traffic lights, sequential circuits

FSA disadvantages:

- Not sufficiently "powerful" to solve numerous problems of interest.

How can we make FSAs more powerful?

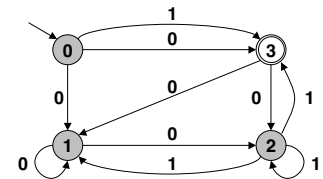
- NFSA = FSA + "nondeterminism."
(ability to guess the right answer!)

3

Nondeterministic Finite State Automata

Nondeterministic FSA (NFSA).

- Simple machine with N states.
- Start in state 0.
- Read a bit.
- Depending on current state and input bit
 - move to any of several new states
- Stop when last bit read.
- Accept if ANY choice of new states ends in state X, reject otherwise.

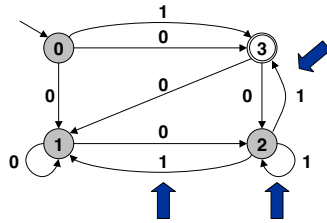


4

Nondeterministic Finite State Automata

Nondeterministic FSA (NFSA).

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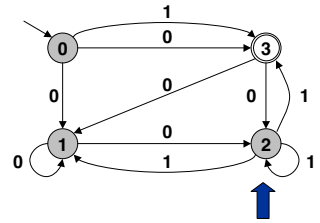
If in state 2, and next bit is 1:
 can move to state 1
 can move to state 2
 can move to state 3

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Nondeterministic Finite State Automata

Nondeterministic FSA (NFSA).

- Simple machine with N states.
- Start in state 0.
- Read a bit.
- Depending on current state and input bit
 - move to any of several new states
- Stop when last bit read.
- Accept if ANY choice of new states ends in state X, reject otherwise.



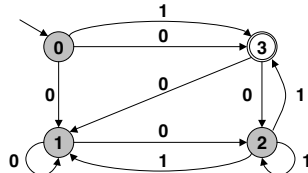
If in state 2, and next bit is 0:
 can't move to any state

6

Nondeterministic Finite State Automata

Nondeterministic FSA (NFSA).

- Simple machine with N states.
- Start in state 0.
- Read a bit.
- Depending on current state and input bit
 - move to any of several new states
- Stop when last bit read.
- Accept if ANY choice of new states ends in state X, reject otherwise.



Which strings are accepted? ▶

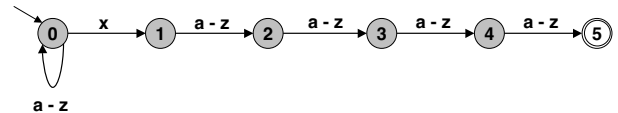
- ✓ 0010001
- ✗ 00
- ✗ 10000111001100
- ✓ 10000111001101

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NFSA Example 2

Build an NFSA to match all strings whose 5th to last character is 'x'.

- % egrep 'x....\$' /usr/dict/words
- asphyxiate
- carboxylic
- contextual
- inflexible



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A Systematic Method for NFSA

Harder to determine whether an NFSA accepts a string than an FSA.

- For FSA, only one possible path to follow.
- For NFSA, need to consider many paths.

Systematic method for NFSA. 

- Keep track of ALL possible states that the NFSA could be in for a given input.
- Accept if one of possible ending states is accept state.

Power of nondeterminism is very useful, but is it essential?

9

FSA - NFSA Equivalence

Theorem: FSA and NFSA are "equally powerful".

- Given any NFSA, can construct FSA that accepts same inputs.

Notation: $X \subseteq Y$.

- Y is at least as powerful as X.
- Machine class Y can be "programmed" to accept all the languages that X can (and maybe more).

Proof (Part 1): $FSA \subseteq NFSA$.

- A FSA is a special type of NFSA.

10

FSA - NFSA Equivalence


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Notation: $X \subseteq Y$.

- Y is at least as powerful as X.
- Machine class Y can be "programmed" to accept all the languages that X can (and maybe more).

Proof (Part 2): $NFSA \subseteq FSA$.

- Given a nondeterministic FSA, we give recipe to construct a deterministic FSA that recognizes the same language.
- One state in FSA for every set of states in the NFSA.
- N-state NFSA $\Rightarrow 2^N$ state FSA. 

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RE – FSA Equivalence

Theorem: FSA and RE are "equally powerful".

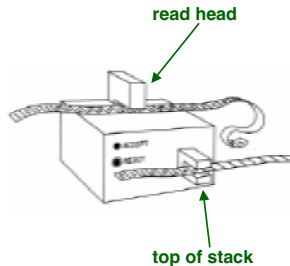
- We'll spare you the details. ☺
- Interested students: see supplemental lecture slides.

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Pushdown Automata

How can we make FSA's more powerful?

- Nondeterminism didn't help.
- Instead, add "memory" to the FSA.
- A pushdown stack (amount of memory is arbitrarily large).



Pushdown Automata (PDA).

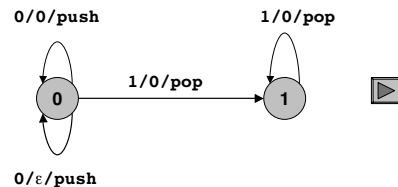
- Simple machine with N states.
- Start in state 0.
- Read a bit, check bit at top of stack.
- Depending on current state/input bit/stack bit:
 - move to new state
 - push the input onto stack, or pop topmost element from stack
- Stop when last bit is read.
- Accept if stack is EMPTY, reject otherwise. } different accept / reject mechanism

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Pushdown Automata

PDA for deciding whether input is of form $0^N 1^N$.

- N 0's followed by N 1's for some N .
- $\epsilon, 01, 0011, 000111, 00001111, \dots$
- Use notation $x/y/z$
- If input is x and top of stack is y , then do z .



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Pushdown Automata

How can we make FSA more powerful?

- PDA = FSA + stack.

Did it help?

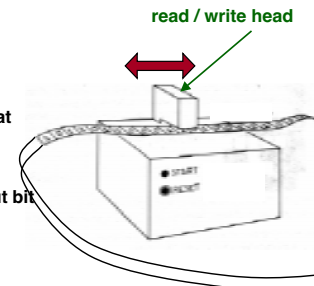
- More powerful, can recognize:
 - all bit strings with an equal number of 0's and 1's
 - all bit strings of the form $0^N 1^N$
 - all "balanced" strings in alphabet: $(, \{, [,], \},)$
- Still can't recognize language of all palindromes.
 - amanaplanacanalpanama
 - $11 \times 181 = 1991 = 181 \times 11$
 - murderforajarofredrum
- More powerful machines still needed.

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Turing Machine

Turing Machine.



- Simple machine with N states.
- Start in state 0.
- Input on an arbitrarily large TAPE that can be read from *and* written to.
- Read a bit from tape.
- Depending on current state and input bit:
 - write a bit to tape
 - move tape right or left
 - move to new state
- Stop if enter yes or no state.
- Accept if yes, reject if no or does not terminate. } new accept / reject mechanism



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Some Examples

Build Turing machines that accepts following languages:

- Equal number of 0's and 1's. 
#1100#, #0011#, #01101110000#
- Even length palindromes of 0's and 1's.
#0110#, #110011#, #10111000011101#
- Power of two 1's. 
#1#, #11#, #1111#, #11111111#

Notation.

- $x/y/z$: if TM head contains character x , then change it to y , and move head in direction z .
- # special character.

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C Program to Simulate Turing Machine

Three character alphabet (0 is 'blank').

Position on tape.

- head

Input: description of machine (9 integers per state s).

- $next[i][s] = t$: if currently in state s and input character read in is i , then transition to state t .
- $out[i][s] = w$: if currently in state s and input character read in is i , then write w to current tape position.
- $move[i][s] = \pm 1$: if currently in state s and input character is i , then move head one position to left or right.
- $tape[i]$ is i^{th} character on tape initially.

Details missing:

- Might run off end of tape.

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C Program to Simulate Turing Machine

```

turing.c

#define MAX_TAPE_SIZE    2000
#define STATES           100
#define ACCEPT_STATE     99
. . .
int next[3][STATES], out[3][STATES], move[3][STATES];
char tape[MAX_TAPE_SIZE];
int in, d, state = 0, head = MAX_TAPE_SIZE / 2;

. . . /* read in machine from file */

while (scanf("%ld", &d) != EOF) {
    tape[head++] = d;
}

while (state != ACCEPT_STATE) {
    in = tape[cursor];
    state = next[in][state];
    tape[head] = out[in][state];
    head += move[in][state];
}
    
```

← read in tape (consists of 0, 1, 2)

← simulate Turing machine until accept state reached

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Nondeterministic Turing Machine

TM with extra ability:

- Choose one of several possible transition states given current tape contents and state.
- No more powerful than deterministic TM.
- Faster than TM? (Stay tuned for NP-Completeness).

Exercise:

- Nondeterministic TM to recognize language of all bit strings of the form ww for some w .
– 110110
– 100011110001111
– 001100011100001111001100011100001111

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Abstract Machine Hierarchy

Each machine is strictly more powerful than the previous.

- Power = can recognize more languages.

Are there limits to machine power?

Corresponding hierarchy exists for languages.

- Essential connection between machines and languages.
(See Lecture T3.)

Machine	Nondeterminism adds power?
Finite state automata	No
Pushdown automata	Yes
Linear bounded automata	Unknown
Turing machine	No

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Summary

Abstract machines are foundation of all modern computers.

- Simple computational models are easier to understand.
- Leads to deeper understanding of computation.

Goal: simplest machine "as powerful" as conventional computers.

Abstract machines.

- FSA: simplest machine that is still interesting.
 - pattern matching, sequential circuits (Lecture T1)
 - can't recognize: equal number of 0's and 1's
- PDA: add read/write memory in the form of a stack.
 - compiler design (Lecture T3)
 - can't recognize: palindromes
- TM: add memory in the form of an arbitrarily large array.
 - general purpose computers (Lecture T4)
 - can't recognize: stay tuned

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Lecture T2: Extra Slides



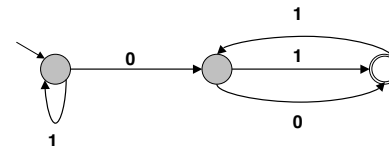
FSA, NFSA, and RE Are Equivalent

Theorem: FSA, NFSA, and RE are "equally powerful".

- $\text{NFSA} \subseteq \text{FSA}$

Proof sketch (part 2): $\text{FSA} \subseteq \text{RE}$

- Goal: given an FSA, find a RE that matches all strings accepted by the FSA and no other strings.
- Main idea: consider
 - paths from start state(s) to accept state(s): $00 \mid 01$
 - directed cycles: $(1^*)(00 \mid 01)(11 \mid 10)^*$



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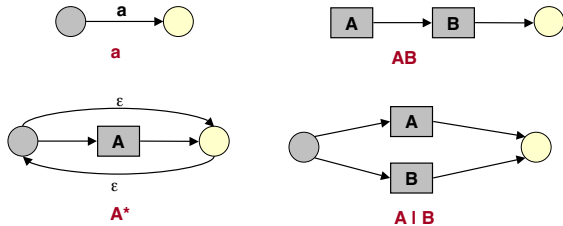
FSA, NFSA, and RE Are Equivalent

Theorem: FSA, NFSA, and RE are "equally powerful".

$$\text{NFSA} \subseteq \text{FSA} \subseteq \text{RE}$$

Proof sketch (part 3): $\text{RE} \subseteq \text{NFSA}$

- Goal: given a RE, construct a NFSA that accepts all strings matched by the RE, and rejects all others.
- Use the following rules to construct NFSA:

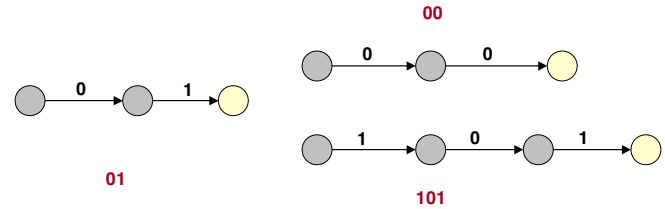


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FSA, NFSA, and RE Are Equivalent

Example.

$$\text{RE: } 01(00 | 101)^*$$

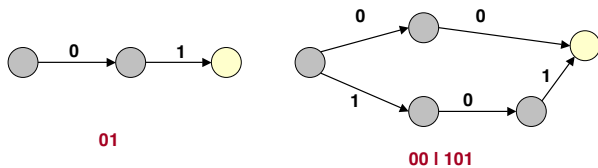


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FSA, NFSA, and RE Are Equivalent

Example.

$$\text{RE: } 01(00 | 101)^*$$

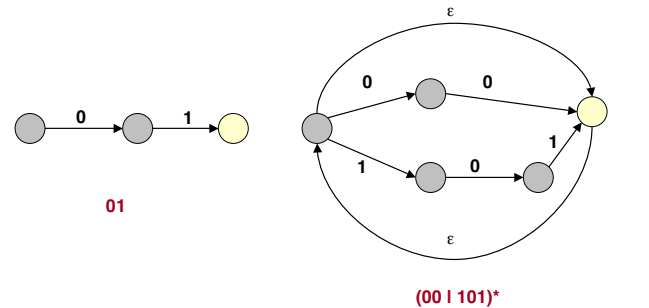


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FSA, NFSA, and RE Are Equivalent

Example.

$$\text{RE: } 01(00 | 101)^*$$

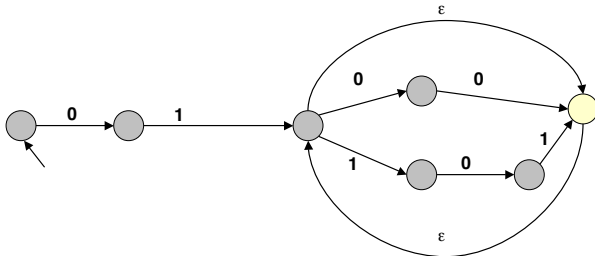


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FSA, NFSA, and RE Are Equivalent

Example.

- RE: $01(00 \mid 101)^*$



$01(00 + 101)^*$

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FSA, NFSA, and RE Are Equivalent

Theorem: FSA, NFSA, and RE are "equally powerful".

- $\text{NFSA} \subseteq \text{FSA} \subseteq \text{RE} \subseteq \text{NFSA}$

Equivalence of languages and machine models is essential in the theory of computation.

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Nondeterminism Does Help PDA's

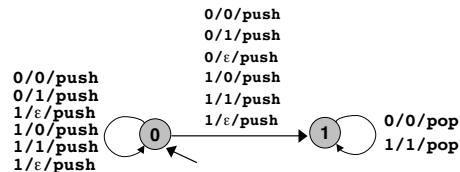
Nondeterministic pushdown automata (NPDA).

- Same as PDA, except depending on current state/input bit/stack bit
 - move to **ANY OF SEVERAL** new states
 - push the input onto stack, or pop top-most element from stack

NPDA to recognize all (even length) palindromes.



- Bit string is the same forwards and backwards.



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Nondeterminism Does Help PDA's

Nondeterministic pushdown automata (NPDA).

- Same as PDA, except depending on current state/input bit/stack bit
 - move to **ANY OF SEVERAL** new states
 - push the input onto stack, or pop top-most element from stack

NPDA to recognize all (even length) palindromes.

- Bit string is the same forwards and backwards.

Nondeterministic PDA more powerful than deterministic PDA.

- $\text{PDA} \subseteq \text{NPDA}$ trivially.
- PDA cannot recognize language of all (even length) palindromes, but NPDA can.
- Therefore $\text{PDA} \subset \text{NPDA}$.

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Pushdown Automata

How can we make FSA more powerful?

- NPDA = FSA + stack + nondeterminism.

Did it help?

- Can recognize language of all palindromes.
- Can't recognize some languages:
 - equal number of 0's 1's and 2's
 - $0^N 1^N 2^N$
 - bit strings with a power of two 1's
- Need still more powerful machines.

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Linear Bounded Automata

Turing machine.

- No limit on length of tape.

Linear bounded automata (LBA).

- A single tape TM that can only write on the portion of the tape containing the input.
- Note: allowed to increase alphabet size if desired.

LBA is strictly less powerful than TM.

- There are languages that can be recognized by TM but not a LBA.
- We won't dwell on LBA in this course.

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