CS711008Z Algorithm Design and Analysis

Lecture 8. An example of cycling in simplex algorithm

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Cycling in simplex algorithm

- Cycling: If a sequence of pivots starting from some basic feasible solution ends up at the exact same basic feasible solution, then we refer to this as cycling. If the simplex method cycles, it can cycle forever.
- In 1953, Hoffman gave the first cycling example, which had 11 variables and 3 equations.
- In 1955, E. M. L. Beale gave a smaller example, which had 7 variables and 3 equations.

Standard form:

$$
\begin{array}{lrcrcrcrcc}
\min & -\frac{3}{4}x_1 & + & 150x_2 & - & \frac{1}{50}x_3 & + & 6x_4 & & \\
s.t. & \frac{1}{4}x_1 & - & 60x_2 & - & \frac{1}{25}x_3 & + & 9x_4 & \leq & 0 \\
 & \frac{1}{2}x_1 & - & 90x_2 & - & \frac{1}{50}x_3 & + & 3x_4 & \leq & 0 \\
 & & & & & x_3 & & & \leq & 1 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

Slack form:

$$
\begin{array}{lrrrrrrrrcc}
\min & -\frac{3}{4}x_1 & +150x_2 & -\frac{1}{50}x_3 & +6x_4 & & & & & & \\
s.t. & \frac{1}{4}x_1 & -60x_2 & -\frac{1}{25}x_3 & +9x_4 & +x_5 & & & & \leq & 0 \\
 & \frac{1}{2}x_1 & -90x_2 & -\frac{1}{50}x_3 & +3x_4 & & +x_6 & & & \leq & 0 \\
 & & & & x_3 & & & + & x_7 & \leq & 1 \\
 & x_1 & , x_2 & , x_3 & , x_4 & & & & \geq & 0 &
\end{array}
$$

Step 1.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}=-\frac{3}{4}$	$\overline{c_2}$=150	$\overline{c_3}=-\frac{1}{50}$	$\overline{c_4}$=6	$\overline{c_5}$=0	$\overline{c_6}$=0	$\overline{c_7}$=0
$\mathbf{x_{B1}} = b'_1$=0	$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0
$\mathbf{x_{B2}} = b'_2$=0	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0
$\mathbf{x_{B3}} = b'_3$=1	0	0	1	0	0	0	1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_5}, \mathbf{a_6}, \mathbf{a_7}\}$.
- Solution: $\mathbf{x} = [\mathbf{x_B}, \mathbf{x_N}] = [\mathbf{B^{-1}b}, \mathbf{0}] = [0, 0, 0, 0, 0, 0, 1]$.
- Pivoting (in red): choose $\mathbf{a_1}$ to enter basis since $\overline{c_1} = -\frac{3}{4} < 0$; choose $\mathbf{a_5}$ to exit since $\theta = \min_{\mathbf{a}_i \in \mathbf{B}, \lambda_i > 0} \frac{b'_i}{\lambda_i} = \frac{b'_1}{\lambda_1} = 0$.
- Here, the corresponding λ is stored in the 1-st column (Why? the basis $\mathbf{B}$ forms an identity matrix.)

Step 2.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}$=0	$\overline{c_2}$=-30	$\overline{c_3}=-\frac{7}{50}$	$\overline{c_4}$=33	$\overline{c_5}$=3	$\overline{c_6}$=0	$\overline{c_7}$=0
$\mathbf{x_{B1}} = b'_1$=0	1	-240	$-\frac{4}{25}$	36	4	0	0
$\mathbf{x_{B2}} = b'_2$=0	0	30	$\frac{3}{50}$	-15	-2	1	0
$\mathbf{x_{B3}} = b'_3$=1	0	0	1	0	0	0	1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_1}, \mathbf{a_6}, \mathbf{a_7}\}$.
- Solution: $\mathbf{x} = [\mathbf{x_B}, \mathbf{x_N}] = [\mathbf{B^{-1}b}, \mathbf{0}] = [0, 0, 0, 0, 0, 0, 1]$.
- Pivoting (in red): choose $\mathbf{a_2}$ to enter basis since $\overline{c_2} = -30 < 0$; choose $\mathbf{a_6}$ to exit since $\theta = \min_{\mathbf{a}_i \in \mathbf{B}, \lambda_i > 0} \frac{b'_i}{\lambda_i} = \frac{b'_2}{\lambda_2} = 0$.
- Here, the corresponding λ is stored in the 2-nd column (Why? the basis $\mathbf{B}$ forms an identity matrix.)

Step 3.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}$=0	$\overline{c_2}$=0	$\overline{c_3}=-\frac{2}{25}$	$\overline{c_4}$=18	$\overline{c_5}$=1	$\overline{c_6}$=1	$\overline{c_7}$=0
$\mathbf{x_{B1}} = b'_1$=0	1	0	$\frac{8}{25}$	-84	-12	8	0
$\mathbf{x_{B2}} = b'_2$=0	0	1	$\frac{1}{500}$	-$\frac{1}{2}$	$-\frac{1}{15}$	$\frac{1}{30}$	0
$\mathbf{x_{B3}} = b'_3$=1	0	0	1	0	0	0	1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_7}\}$.
- Solution: $\mathbf{x} = [\mathbf{x_B}, \mathbf{x_N}] = [\mathbf{B^{-1}b}, \mathbf{0}] = [0, 0, 0, 0, 0, 0, 1]$.
- Pivoting (in red): choose $\mathbf{a_3}$ to enter basis since $\overline{c_3} = -\frac{2}{25} < 0$; choose $\mathbf{a_1}$ to exit since $\theta = \min_{\mathbf{a}_i \in \mathbf{B}, \lambda_i > 0} \frac{b'_i}{\lambda_i} = \frac{b'_1}{\lambda_1} = 0$.
- Here, the corresponding λ is stored in the 3-rd column (Why? the basis $\mathbf{B}$ forms an identity matrix.)

Step 4.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}=\frac{1}{4}$	$\overline{c_2}$=0	$\overline{c_3}$=0	$\overline{c_4}$=-3	$\overline{c_5}$=-2	$\overline{c_6}$=3	$\overline{c_7}$=0
$\mathbf{x_{B1}} = b'_1$=0	$\frac{25}{8}$	0	1	$-\frac{525}{2}$	$-\frac{75}{2}$	25	0
$\mathbf{x_{B2}} = b'_2$=0	$-\frac{1}{60}$	1	0	$\frac{1}{40}$	$\frac{1}{120}$	$-\frac{1}{60}$	0
$\mathbf{x_{B3}} = b'_3$=1	$-\frac{25}{8}$	0	0	$\frac{525}{2}$	$\frac{75}{2}$	-25	1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_2}, \mathbf{a_3}, \mathbf{a_7}\}$.
- Solution: $\mathbf{x} = [\mathbf{x_B}, \mathbf{x_N}] = [\mathbf{B^{-1}b}, \mathbf{0}] = [0, 0, 0, 0, 0, 0, 1]$.
- Pivoting (in red): choose $\mathbf{a_4}$ to enter basis since $\overline{c_4} = -3 < 0$; choose $\mathbf{a_2}$ to exit since $\theta = \min_{\mathbf{a}_i \in \mathbf{B}, \lambda_i > 0} \frac{b'_i}{\lambda_i} = \frac{b'_2}{\lambda_2} = 0$.
- Here, the corresponding λ is stored in the 4-th column (Why? the basis $\mathbf{B}$ forms an identity matrix.)

Step 5.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}$=$-\frac{1}{2}$	$\overline{c_2}$=120	$\overline{c_3}$=0	$\overline{c_4}$=0	$\overline{c_5}$=-1	$\overline{c_6}$=1	$\overline{c_7}$=0
$\mathbf{x_{B1}} = b'_1$=0	$-\frac{125}{2}$	10500	1	0	50	-150	0
$\mathbf{x_{B2}} = b'_2$=0	$-\frac{1}{4}$	40	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0
$\mathbf{x_{B3}} = b'_3$=1	$\frac{125}{2}$	10500	0	0	50	150	1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_3}, \mathbf{a_4}, \mathbf{a_7}\}$.
- Solution: $\mathbf{x} = [\mathbf{x_B}, \mathbf{x_N}] = [\mathbf{B^{-1}b}, \mathbf{0}] = [0, 0, 0, 0, 0, 0, 1]$.
- Pivoting (in red): choose $\mathbf{a_5}$ to enter basis since $\overline{c_5} = -1 < 0$; choose $\mathbf{a_3}$ to exit since $\theta = \min_{\mathbf{a}_i \in \mathbf{B}, \lambda_i > 0} \frac{b'_i}{\lambda_i} = \frac{b'_1}{\lambda_1} = 0$.
- Here, the corresponding λ is stored in the 5-th column (Why? the basis $\mathbf{B}$ forms an identity matrix.)

Step 6.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}=-\frac{7}{4}$	$\overline{c_2}$=330	$\overline{c_3}=\frac{1}{50}$	$\overline{c_4}$=0	$\overline{c_5}$=0	$\overline{c_6}$=-2	$\overline{c_7}$=0
$\mathbf{x_{B1}} = b_1'$=0	$-\frac{5}{4}$	210	$\frac{1}{50}$	0	1	-3	0
$\mathbf{x_{B2}} = b_2'$=0	$\frac{1}{6}$	-30	$-\frac{1}{150}$	1	0	$\frac{1}{3}$	0
$\mathbf{x_{B3}} = b_3'$=1	0	0	1	0	0	0	1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_4}, \mathbf{a_5}, \mathbf{a_7}\}$.
- Solution: $\mathbf{x} = [\mathbf{x_B}, \mathbf{x_N}] = [\mathbf{B^{-1}b}, \mathbf{0}] = [0, 0, 0, 0, 0, 0, 1]$.
- Pivoting (in red): choose $\mathbf{a_6}$ to enter basis since $\overline{c_6} = -2 < 0$; choose $\mathbf{a_4}$ to exit since $\theta = \min_{\mathbf{a}_i \in \mathbf{B}, \lambda_i > 0} \frac{b_i'}{\lambda_i} = \frac{b_2'}{\lambda_2} = 0$.
- Here, the corresponding λ is stored in the 6-th column (Why? the basis $\mathbf{B}$ forms an identity matrix.)

Step 7.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}=-\frac{3}{4}$	$\overline{c_2}=150$	$\overline{c_3}=-\frac{1}{50}$	$\overline{c_4}=6$	$\overline{c_5}=0$	$\overline{c_6}=0$	$\overline{c_7}=0$
$\mathbf{x_{B1}} = b_1'=0$	$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0
$\mathbf{x_{B2}} = b_2'=0$	$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0
$\mathbf{x_{B3}} = b_3'=1$	0	0	1	0	0	0	1

- Same as step 1. A cycle!

Bland rule to avoid cycling

- Cycling: If a sequence of pivots starting from some basic feasible solution ends up at the exact same basic feasible solution, then we refer to this as cycling. If the simplex method cycles, it can cycle forever.
- Bland indexing rule:
  1. choose $\mathbf{a}_j$ to enter: $j = \min\{j : \bar{c}_j \leq 0\}$.
  2. choose $\mathbf{a}_i$ to exit: choose the smallest l to break ties.

Let's see how Bland rule works for this example.

Step 5'.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-z= 0	$\overline{c_1}=-\frac{1}{2}$	$\overline{c_2}$=120	$\overline{c_3}$=0	$\overline{c_4}$=0	$\overline{c_5}$=-1	$\overline{c_6}$=1	$\overline{c_7}$=0
$\mathbf{x_{B1}} = b'_1$=0	$-\frac{125}{2}$	10500	1	0	50	-150	0
$\mathbf{x_{B2}} = b'_2$=0	$-\frac{1}{4}$	40	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0
$\mathbf{x_{B3}} = b'_3$=1	$\frac{125}{2}$	10500	0	0	50	150	1

- Basis (in blue): $\mathbf{B} = \{\mathbf{a_3}, \mathbf{a_4}, \mathbf{a_7}\}$.
- Solution: $\mathbf{x} = [\mathbf{x_B}, \mathbf{x_N}] = [\mathbf{B^{-1}b}, \mathbf{0}] = [0, 0, 0, 0, 0, 0, 1]$.
- Pivoting (in red): choose $\mathbf{a_1}$ to enter basis since $\overline{c_1} = -\frac{1}{2} < 0$; choose $\mathbf{a_7}$ to exit since $\theta = \min_{\mathbf{a}_i \in \mathbf{B}, \lambda_i > 0} \frac{b'_i}{\lambda_i} = \frac{b'_3}{\lambda_3} = \frac{2}{125}$.
- Here, the corresponding λ is stored in the 1-st column (Why? the basis $\mathbf{B}$ forms an identity matrix.)
- Note: $\theta \neq 0$. Escaped from the cycle!