Problem Set: Introduction to Measure Theory

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1 Measure Spaces

Exercise 1.

- $\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open }\}$ is not closed under complements. For example, $(0, +\infty)$ is an open set in $\mathbb{R}$ so it is included in $\mathcal{G}_1$, but its complement $(-\infty, 0]$ is not open. Thus $\mathcal{G}_1$ is not an algebra.

- If $\emptyset$ is not included in $\mathcal{G}_2$, then it is not an algebra by definition. If $\emptyset$ is included in $\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$, then it is an algebra as it is closed under complements (the complement would be a finite intersection of $(-\infty, a] \cup (b, \infty), (b, \infty)$ and $(-\infty, a]$, which is either empty or could be written as a finite union of the above forms) and closed under finite unions (a finite union of the sets that are finite unions of intervals of the form$(\text{a}, \text{b}], (-\infty, \text{b}], \text{ and } (\text{a}, \infty)$ is still a finite union of intervals of these forms). It is not a $\sigma-$algebra. For example, the countable union
$$\cup_{n=2}^{\infty} \left(0, \frac{n-1}{n}\right] = (0, 1)$$
is not in $\mathcal{G}_2$.

- $\mathcal{G}_3 = \{A : A \text{ is a countable union of } (\text{a}, \text{b}], (-\infty, \text{b}], \text{ and } (\text{a}, \infty)\}$ is a $\sigma-$algebra if we include $\emptyset$ in it. The proof follows similarly as above but now "finite" applies more generally to "countable" too.

Exercise 2.

- If $\mathcal{A}$ is a $\sigma-$algebra, then $\emptyset \in \mathcal{A}$. The complement of $\emptyset$, X must also be in $\mathcal{A}$. Thus $\{0, X\} \subset \mathcal{A}$.

- Note that if $S \subset X$, then $S^c = X \setminus S \subset X$. Also note that finite unions of subsets of X is still a subset of X. If $\mathcal{A}$ is a $\sigma-$algebra generated from some subsets of X, then $\mathcal{A} \subset \mathcal{P}(X)$ as $\mathcal{P}(X) = \{A : A \subset X\}$ contains all the subsets of X by definition.

Exercise 3.

- Since $\emptyset \in \mathcal{S}_\alpha, \forall \alpha$, we thus have $\emptyset \in \cap_\alpha \mathcal{S}_\alpha$.

- Pick any $X \in \cap_\alpha \mathcal{S}_\alpha$ and hence $X \in \mathcal{S}_\alpha, \forall \alpha$. Since a $\sigma-$algebra is closed under complements, we have $X^c \in \mathcal{S}_\alpha, \forall \alpha$. Therefore, $X^c \in \cap_\alpha \mathcal{S}_\alpha$.

- Pick countable sets $X_1, X_2, \ldots \in \cap_\alpha \mathcal{S}_\alpha$, and hence $X_i \in \mathcal{S}_\alpha, \forall \alpha, \forall i$. Since a $\sigma-$algebra is closed under countable unions, thus $\cup_i X_i \in \mathcal{S}_\alpha, \forall \alpha$. Therefore, $\cup_i X_i \in \cap_\alpha \mathcal{S}_\alpha$.

The above propositions mean that $\cap_\alpha \mathcal{S}_\alpha$ is a $\sigma-$algebra.

Exercise 4.

- Let $C = B \setminus A = B \cap A^c$. μ is a nonnegative measure and hence $\mu(C) \geq 0$. Note that $B = A \cup C$ and $A \cap C = \emptyset$. Therefore

$$\mu(B) = \mu(A) + \mu(C) \geq \mu(A).$$

- Let $B_n = \cup_{i=1}^n A_i$. Note that $B_1 \subset B_2 \subset B_3 \subset \cdots$. By Theorem 1.25 (i), we have $\lim_{n\to\infty} \mu(B_n) = \mu(\cup_{i=1}^\infty B_i)$.
  - Note that $\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty A_i$, hence $\mu(\cup_{i=1}^\infty B_i) = \mu(\cup_{i=1}^\infty A_i)$.
  - Note that $\lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(\cup_{i=1}^n A_i) \leq \lim_{n\to\infty} \sum_{i=1}^n \mu(A_i) = \sum_{i=1}^\infty \mu(A_i)$.
  - Therefore, $\mu(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu(A_i)$.

Exercise 5.

- $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.

- For any $\{A_i\}_{i=1}^\infty \subset \mathcal{S}$s.t.$A_i \cap A_j = \emptyset \forall i \neq j$, we have

$$\begin{aligned}\lambda(\cup_{i=1}^\infty A_i) &= \mu((\cup_{i=1}^\infty A_i) \cap B) = \mu(\cup_{i=1}^\infty (A_i \cap B)) \\ &= \sum_{i=1}^\infty \mu(A_i \cap B) = \sum_{i=1}^\infty \lambda(A_i).\end{aligned}$$

Exercise 6. Let $B_n = A_1 \cap A_n^c$. Since

$$\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty (A_1 \cap A_i^c) = A_1 \cap (\cup_{i=1}^\infty A_i^c) = A_1 \setminus (\cap_{i=1}^\infty A_i),$$

we have $\mu\left(\cup_{i=1}^{\infty} B_i\right) = \mu\left(A_1\right) - \mu\left(\cap_{i=1}^{\infty} A_i\right)$. In addition,

$$\lim_{n\to\infty} \mu\left(B_n\right) = \lim_{n\to\infty} \mu\left(A_1 \cap A_n^c\right) = \lim_{n\to\infty} \left(\mu\left(A_1\right) - \mu\left(A_n\right)\right) = \mu\left(A_1\right) - \lim_{n\to\infty} \mu\left(A_n\right).$$

Note that $B_1 \subset B_2 \subset B_3 \subset \cdots$. By Theorem 1.25 (i) we have

$$\lim_{n\to\infty} \mu\left(B_n\right) = \mu\left(\cup_{i=1}^{\infty} B_i\right).$$

Therefore,

$$\mu\left(A_1\right) - \lim_{n\to\infty} \mu\left(A_n\right) = \mu\left(A_1\right) - \mu\left(\cap_{i=1}^{\infty} A_i\right) \Longrightarrow \lim_{n\to\infty} \mu\left(A_n\right) = \mu\left(\cap_{i=1}^{\infty} A_i\right).$$

2 Construction of Lebesgue Measure

Exercise 7. Since μ^* is an outer measure, it is countably subadditive. Note that $(B \cap E) \cup (B \cap E^c) = B$, thus

$$\mu^*\left(B\right) \le \mu^*\left(B \cap E\right) + \mu^*\left(B \cap E^c\right).$$

If in addition $\mu^*\left(B\right) \ge \mu^*\left(B \cap E\right) + \mu^*\left(B \cap E^c\right)$, it must be that $\mu^*\left(B\right) = \mu^*\left(B \cap E\right) + \mu^*\left(B \cap E^c\right)$.

Exercise 8. Denote $\mathcal{O}$ the collection of open sets of $\mathbb{R}$. By definition, the Borel $\sigma-$algebra of $\mathbb{R}$ is the $\sigma-$algebra generated by $\mathcal{O}$, i.e., $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O})$. Let ν be a premeasure on $\mathbb{R}$ and μ^* the outer measure generated by ν, and $\mathcal{M}$ the $\sigma-$algebra from the Caratheodory construction. By Theorem 2.12, we have $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O}) \subset \mathcal{M}$.

3 Measurable Functions

Exercise 9. Consider a countable set $\{x_n\}_{n=1}^{\infty}$. Note that $\{x_n\} \subset \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right], \forall \varepsilon > 0$. Note that

$$\sum_{n=1}^{\infty} \left[\left(x_n + \frac{\varepsilon}{2^{n+1}}\right) - \left(x_n - \frac{\varepsilon}{2^{n+1}}\right)\right] = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

By definition of Lebesgue measure,

$$
\begin{aligned}
\lambda\left(\{x_n\}_{n=1}^{\infty}\right) &= \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n) : \{x_n\}_{n=1}^{\infty} \subset \bigcup_{i=1}^{\infty}(a_i, b_i]\right\} \\
&\leq \inf\left\{\sum_{n=1}^{\infty}\left[\left(x_n + \frac{\varepsilon}{2^{n+1}}\right) - \left(x_n - \frac{\varepsilon}{2^{n+1}}\right)\right] : \{x_n\}_{n=1}^{\infty} \subset \bigcup_{i=1}^{\infty}\left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right], \varepsilon > 0\right\} \\
&= \inf\{\varepsilon : \varepsilon > 0\} = 0.
\end{aligned}
$$

Therefore, $\lambda\left(\{x_n\}_{n=1}^{\infty}\right) = 0$.

Exercise 10. Since $\mathcal{M}$ is a $\sigma-$algebra, it is closed under complements and countable unions.

1. If $\forall a, \{x \in X : f(x) \geq a\} \in \mathcal{M}$, then its complement $\{x \in X : f(x) < a\} \in \mathcal{M}$ too. So (*) can be replaced by $\{x \in X : f(x) \geq a\}$.

2. If $\forall a, \{x \in X : f(x) > a\} \in \mathcal{M}$, then a countable union

$$\cup_{n=1}^{\infty}\{x \in X : f(x) > a + \frac{1}{n}\} = \{x \in X : f(x) \geq a\}$$

   is also in $\mathcal{M}$. By (1), (*) can by replaced by $\{x \in X : f(x) > a\}$.

3. If $\forall a, \{x \in X : f(x) \leq a\} \in \mathcal{M}$, then its complement $\{x \in X : f(x) > a\} \in \mathcal{M}$ too. By (2), (*) can by replaced by $\{x \in X : f(x) \leq a\}$.

Exercise 11.

- Let $F(f,g) = f + g$ which is continuous. By (4), $f + g$ is measurable.

- Let $F(f,g) = f \cdot g$ which is continuous. By (4), $f \cdot g$ is measurable.

- Let $f_n = f$ for n odd and $f_n = g$ for n even. Thus $\sup_{n\in\mathbb{N}} f_n(x) = \max(f,g)$. By (2), $\max(f,g)$ is measurable.

- Let $f_n = f$ for n odd and $f_n = g$ for n even. Thus $\inf_{n\in\mathbb{N}} f_n(x) = \min(f,g)$. By (2), $\min(f,g)$ is measurable.

- First, $g(x) = -1$ is measurable. Thus $-f = f \cdot g$ is measurable. Next note that $|f| = \max(f, -f)$, so $|f|$ is also measurable.

Exercise 12. We construct a partition as in the proof. If f is bounded, $\exists M$ s.t. $f(x) < M, \forall x$. For $n > M$, $\forall x \in X$, there exists some i s.t. $x \in E_i^n$. Thus $s_n(x) = \frac{i-1}{2^n}$ for this i and $|f(x) - s_n(x)| < \frac{1}{2^n}$. $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \epsilon$. Therefore, $\forall n > \max(N, M)$,we have

$$|f(x) - s_n(x)| < \frac{1}{2^n} < \varepsilon,$$

hence the convergence in is uniform.

4 Lebesgue Integration

Exercise 13. Since $f^+ = \max\{f(x), 0\} \in [0, M)$, and $\mu(E) < \infty$, then

$$0 \leq \int_E f^+ d\mu \leq M\mu(E) < \infty.$$

Similarly, $f^- = \max\{-f(x), 0\} \in [0, M)$, then

$$0 \leq \int_E f^- d\mu \leq M\mu(E) < \infty.$$

Both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite, so $f \in \mathscr{L}^1(\mu, E)$.

Exercise 14. Suppose there exists $X \subset E$ with $\mu(X) > 0$, and $f(x) = \infty, \forall x \in X$. Then

$$\int_E |f| \, d\mu \geq \int_A |f| \, d\mu \geq \int_A f d\mu = \infty,$$

contradictory to f being integrable.

Exercise 15. $f \leq g$ implies that $\{s : 0 \leq s \leq f, s \text{ simple, measurable }\} \subset \{s : 0 \leq s \leq g, s \text{ simple, measurable }\}$, and thus $\sup\left\{\int_E s d\mu : 0 \leq s \leq f, s \text{ simple, measurable }\right\} \leq \sup\left\{\int_E s d\mu : 0 \leq s \leq g, s \text{ simple, measurable }\right\}$. By definition this is $\int_E f d\mu \leq \int_E g d\mu$.

Exercise 16. Consider any $s(x) = \sum_{i=1}^N c_i \chi_E$ simple, measurable. $A \subset E$ implies that $A \cap E_i \subset E \cap E_i, \forall i$. Hence $\mu(A \cap E_i) \leq \mu(E \cap E_i)$. Thus

$$\int_A s d\mu = \sum_{i=1}^N c_i \mu(A \cap E_i) \leq \sum_{i=1}^N c_i \mu(E \cap E_i) = \int_E s d\mu.$$

Therefore,

$$\begin{aligned}
\int_A f^+ d\mu &= \sup\left\{\int_A s d\mu : 0 \leq s \leq f^+, \text{ s simple, measurable }\right\} \\
&\leq \sup\left\{\int_E s d\mu : 0 \leq s \leq f^+, \text{ s simple, measurable }\right\} \\
&= \int_E f^+ d\mu < \infty,
\end{aligned}$$

and similarly $\int_A f^- d\mu < \infty$. So $f \in \mathscr{L}^1(\mu, A)$.

Exercise 17. Let $X_1 = A \cap B$ and $X_2 = A - B$. Note that $X_1 \cap X_2 = \emptyset$ and $A = X_1 \cup X_2$. Since $B \subset A$, we have $A \cap B = B$. Thus

$$\int_A f d\mu = \int_B f d\mu + \int_{A-B} f d\mu.$$

Since $\mu(A - B) = 0$, by Proposition 4.6 we have $\int_{A-B} f d\mu = 0$. Therefore $\int_A f d\mu = \int_B f d\mu$.