Econ 714 Homework 2

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There is a representative household with preferences:

$$U_t = \left[\left(\log c_t - \eta \frac{l_t^2}{2} \right)^{0.5} + 0.99(\mathbb{E}_t U_{t+1}^{-9})^{-\frac{0.5}{9}} \right]^{\frac{1}{0.5}}$$

The household consumes, saves, and works, with a budget constraint:

$$c_t + i_t = w_t l_t + r_t k_t$$

There is a production function:

$$c_t + i_t = e^{z_t} k_t^{\alpha_t} l_t^{1-\alpha_t}$$

with a law of motion for capital:

$$k_{t+1} = 0.9 k_t + i_t$$

where $i_t \geq 0$ and a technology level z_t that follows a Markov Chain that takes vales in:

$$z_t \in \{-0.0673, -0.0336, 0, 0.0336, 0.0673\}$$

with transition matrix:

$$\begin{pmatrix} 0.9727 & 0.0273 & 0 & 0 & 0 \\ 0.0041 & 0.9806 & 0.0153 & 0 & 0 \\ 0 & 0.0082 & 0.9836 & 0.0082 & 0 \\ 0 & 0 & 0.0153 & 0.9806 & 0.0041 \\ 0 & 0 & 0 & 0.0273 & 0.9727 \end{pmatrix}$$

and

$$\alpha_t \in \{0.25, 0.3, 0.35\}$$

with transition matrix:

$$\begin{pmatrix} 0.9 & 0.07 & 0.03 \\ 0.05 & 0.9 & 0.05 \\ 0.03 & 0.07 & 0.9 \end{pmatrix}$$

1 Social Planer (5 points)

Find the associated social planner's problem to this model and write it recursively.

Let ψ where $1-\psi = 0.5$ be the parameter for intertemporal elasticity of substitution, γ where $1-\gamma = -9$ be the parameter for risk aversion, $\beta = 0.99$ be the discount factor, and δ where $1-\delta = 0.9$ be the persistence of capital stock. In addition, let Γ and Π denote the transition matrices for the elasticity of output with respect to capital α and TFP z respectively. The social planner maximizes the total utility of the household subject to resource constraints and aggregate laws of motion:

$$\mathbb{V}(z,\alpha,k)^{1-\psi} = \max_{c,k',l}\left\{\Big(\log c - \eta\frac{l^2}{2}\Big)^{1-\psi} + \beta\Big[\sum_{\alpha',z'}\Gamma_{\alpha'|\alpha}\Pi_{z'|z}\Big(\mathbb{V}(z',\alpha',k')\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}}\right\} \tag{1}$$

$$\begin{aligned}
\text{s.t. } & c+i = e^z k^\alpha l^{1-\alpha}\\
& c,k,l \geq 0\\
& k' = (1-\delta)k + i
\end{aligned}$$

2 Steady State (5 points)

Compute the deterministic steady state of the model when $z_{ss}=0$ and $\alpha_{ss}=0.3$.

Since the model is a standard RBC model with no frictions or externalities, the First Welfare Theorem holds, which means that we can calculate the steady state of the equilibrium from the social planner's problem. Let λ be the Lagrange multiplier on the resource constraint. The FOCs are shown below:

$$\begin{aligned}
&[c](1-\psi)\Big(\log c - \eta\frac{l^2}{2}\Big)^{-\psi}\frac{1}{c} = \lambda\\
&[k']\beta(1-\psi)\Big[\mathbb{E}\Big(\mathbb{V}(z',\alpha',k')^{1-\gamma}\Big)\Big]^{\frac{\gamma-\psi}{1-\gamma}}\mathbb{E}\Big(\mathbb{V}(z',\alpha',k')^{-\gamma}\mathbb{V}_{k'}(z',\alpha',k')\Big) = \lambda\\
&[l](1-\psi)\Big(\log c - \eta\frac{l^2}{2}\Big)^{-\psi}(-\eta l) = -\lambda(1-\alpha)e^z k^\alpha l^{-\alpha}
\end{aligned}$$

$$\frac{(1-\alpha)e^z k^\alpha l^{-\alpha}}{\alpha e^z k^{\alpha-1}l^{1-\alpha}} = \frac{1-\alpha}{\alpha}\frac{k}{l}$$

and the envelope condition is:

$$(1-\psi)\mathbb{V}(z,\alpha,k)^{-\psi}\mathbb{V}_k(z,\alpha,k) = \lambda(\alpha e^z k^{\alpha-1}l^{1-\alpha} + 1 - \delta)$$

In steady state, the exogenous states are assumed to be at deterministic levels $z_{ss}=0$ and $\alpha_{ss}=0.3$, so the discounted continuation value becomes $\beta\mathbb{V}(z_{ss},\alpha_{ss},k')^{1-\psi}$. Substitute the law of motion for capital into the value function for the next period, then the FOCs are:

$$[c](1-\psi)\Big(\log c - \eta\frac{l^2}{2}\Big)^{-\psi}\frac{1}{c} = \lambda \tag{2}$$

$$[k']\beta(1-\psi)\mathbb{V}(z_{ss},\alpha_{ss},k')^{-\psi}\mathbb{V}_{k'}(z_{ss},\alpha_{ss},k') = \lambda \tag{3}$$

$$[l](1-\psi)\Big(\log c - \eta\frac{l^2}{2}\Big)^{-\psi}(-\eta l) = -\lambda(1-\alpha_{ss})e^{z_{ss}}k^{\alpha_{ss}}l^{-\alpha_{ss}} \tag{4}$$

and the envelope condition is:

$$(1-\psi)\mathbb{V}(z_{ss},\alpha_{ss},k)^{-\psi}\mathbb{V}_k(z_{ss},\alpha_{ss},k) = \lambda(\alpha_{ss}e^{z_{ss}}k^{\alpha_{ss}-1}l^{1-\alpha_{ss}} + 1 - \delta) \tag{5}$$

Substituting equation 3 into the envelope condition 5, and using the condition that $k'=k=k_{ss}$ in the steady state, we get

$$1 = \beta(\alpha_{ss}e^{z_{ss}}k^{\alpha_{ss}-1}l^{1-\alpha_{ss}} + 1 - \delta)$$

In addition, dividing equation 4 by equation 2 yields

$$\eta l c = (1-\alpha_{ss})e^{z_{ss}}k^{\alpha_{ss}}l^{-\alpha_{ss}} \tag{6}$$

Together with the resource constraint, we have 3 equations and 4 unknowns (3 steady state levels and η). Let c_{ss}, k_{ss} and l_{ss} be the steady state levels of consumption, capital stock and labor respectively, and collect the equations above:

$$\begin{cases} \frac{k_{ss}}{l_{ss}} = \left(\frac{1-\beta(1-\delta)}{\beta\alpha_{ss}e^{z_{ss}}}\right)^{\frac{1}{\alpha_{ss}-1}} \\ l_{ss}c_{ss} = \frac{(1-\alpha_{ss})e^{z_{ss}}}{\eta}\left(\frac{k_{ss}}{l_{ss}}\right)^{\alpha_{ss}} \\ \frac{c_{ss}}{l_{ss}} + \delta\frac{k_{ss}}{l_{ss}} = e^{z_{ss}}\left(\frac{k_{ss}}{l_{ss}}\right)^{\alpha_{ss}} \end{cases}$$

We can use the first equation to calculate the capital-labor ratio. Note however that the level of labor l_{ss} is not determined. To pin down the steady state levels, we normalize the steady state output $e^{z_{ss}}k_{ss}^{\alpha_{ss}}l_{ss}^{1-\alpha_{ss}}$ to 100. Then we get the following steady state levels and η:

$$\begin{cases} c_{ss} = & 72.7523 \\ k_{ss} = & 272.4771 \\ l_{ss} = & 65.0774 \\ \eta = & 2.2719e-04 \end{cases}$$

3 Value Function Iteration (VFI) with a Fixed Grid (10 points)

Fix a grid of 250 points of capital, centered around k_{ss} with a coverage of $\pm 30\%$ of k_{ss} and equally spaced. Iterate on the Value function implied by the Social Planner's Problem using linear interpolation until the change in the sup norm between two iterations is less than 10^{-6}. Compute the Policy function. Describe the responses of the economy to a technology shock and to a shock to α_t.

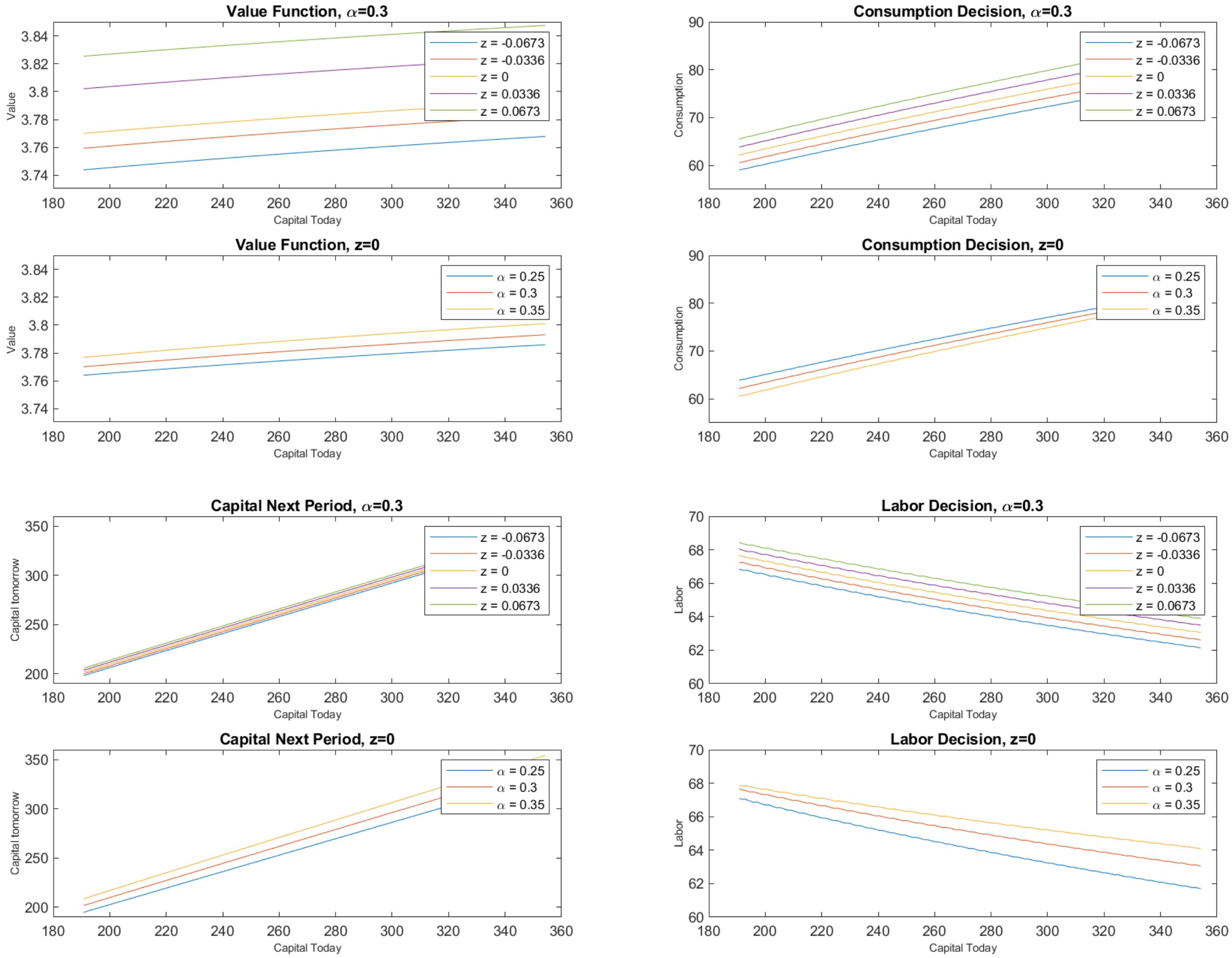
We start the value function iteration by guessing initial values for $V(z,\alpha,k)$ for each discrete state (z,α,k). The initial guesses are set to be the steady state utility. Then we use linear interpolation to derive continuous functions of V with respect to k for given each (z,α) combination. We solve for k' and l which maximizes the RHS of the value function $(1-\beta)\Big(\log c - \eta\frac{l^2}{2}\Big)^{1-\psi} + \beta\Big[\mathbb{E}\Big(V(z',\alpha',k')\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}}$ subject to $c + k' = e^z k^\alpha l^{1-\alpha} + (1-\delta)k$, where the flow utility is now normalized by a factor $1-\beta$ for stationarity of results. The optimal value for l can be jointly pinned down by the FOCs with respect to c and l, which yields $\eta lc = (1-\alpha)e^z k^\alpha l^{-\alpha}$. Therefore for any current state (z,α,k), the goal is to find:

$$\begin{aligned} \mathbb{V}(z,\alpha,k)^{1-\psi} = \max_{k'} &(1-\beta)\Big(\log c - \eta\frac{l^2}{2}\Big)^{1-\psi} + \beta\Big[\mathbb{E}\Big(\mathbb{V}(z',\alpha',k')\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}} \\ \text{s.t. } & c + k' = e^z k^\alpha l^{1-\alpha} + (1-\delta)k \\ & \eta lc = (1-\alpha)e^z k^\alpha l^{-\alpha} \end{aligned}$$

We fix a grid of 250 points of capital, centered around k_{ss} with a coverage of $\pm 30\%$ of k_{ss} and equally spaced. We iterate on the Value function implied by the Social Planner's Problem using linear interpolation until the change in the sup norm between to iterations is less than 10^{-6}. The whole algorithm takes 32458 seconds (9 hours) to run and 569 iterations to converge.

Figure 1 shows the value and policy functions for different values of z and α, with one of the shocks being fixed:

Figure 1: Value Functions and Policy Functions with One Shock Fixed (VFI)

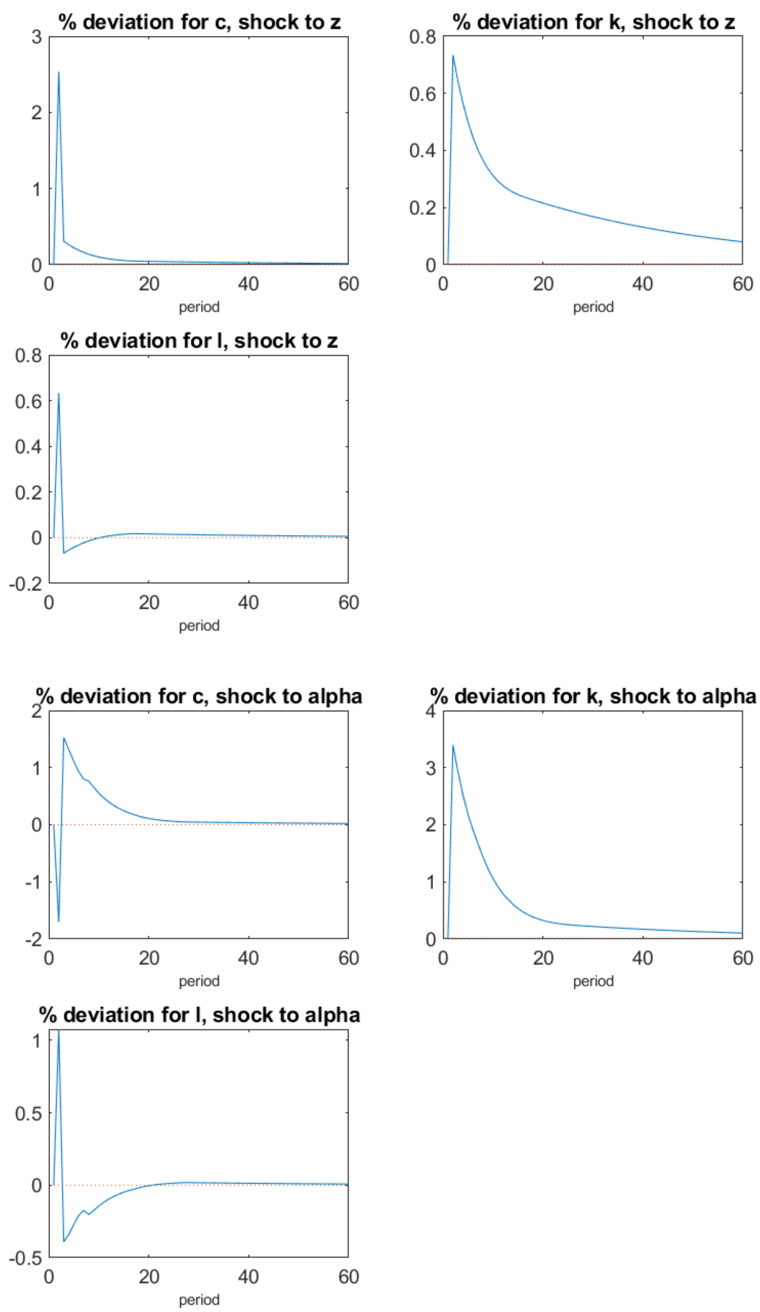


Evidently, the value function, consumption and capital for the next period all increase with current capital stock, and labor decreases with current capital. This is because higher wealth today enables households to consume more and, through consumption smoothing, enjoy higher wealth tomorrow and higher consumption going forward. On the other hand, since there is quadratic disutility from work and log utility from consumption, the wealth effect dominates and therefore wealthier households work less. Higher realized productivity leads to higher levels of the value function, higher consumption, higher capital for the next period and higher labor choice. This is because a higher marginal return to labor induces households to work more and therefore enables them to earn higher income, which in turn makes it possible for them consume more and save more for future.

In addition, higher α leads to higher levels of the value function, lower consumption, higher capital for the next period, and higher labor choice. This is because as α is higher, for given levels of capital and labor, capital return increases and wage decreases. The substitution effect induces households to increase their level of capital for the next period and work less. However, the lower wages also lead to a wealth effect, which increases the labor effort and reduces consumption. Therefore the ultimate effect is a decrease in consumption and an increase in labor and capital. Higher wealth in the future leads to higher continuations value, which increases value today.

We plot the Impulse Response Functions (IRFs) of consumption, capital next period, and labor to a one-period shock in z and α respectively in Figure 2 below:

Figure 2: Impulse Response Functions



The plots represent the % deviation of each variable from steady state, after a single positive shock to z and α. When a positive shock to productivity hits the economy, on impact output increases and therefore leads to higher returns of labor and capital. As a result, consumption shoots up, labor supply increases and investment increases. In the following periods, productivity returns to the steady state level, which immediately decreases returns of labor and capital. Therefore consumption and capital decreases, but are still high relative to steady state since capital stock is high due to investment in the initial period. On the other hand, labor undershoots the steady state due to an immediate decrease in wages. The variables gradually converge to their steady state levels as the effect of the shock dies out.

A positive shock to α increases the elasticity of output with respect to capital. This induces households to invest more and therefore increases capital stock. Due to the resource constraint, consumption has to decrease. In order to smooth consumption, households increase labor supply to offset the impact on consumption. In the following periods, α returns to its steady state level, which leads to lower elasticity of output with respect to capital. Consequently, investment in capital decreases, and capital increases immediately. This in turn leads to a change in labor supply in the opposite direction, and since capital level is still high, labor undershoots the steady state. Later on, as the effect of the shock dies out, the variables converge to the steady state levels.

The Matlab file that executes the VFI is called *"HW2_ Q3_ VFI.m"* and is located in the "Code" folder.

4 Value Function Iteration (VFI) with an Endogenous Grid (10 points)

Repeat previous exercise with an endogenous grid. Compare the solutions in 3) and 4) in terms of: 1) accuracy, 2) computing time, and 3) complexity of implementation. Present evidence to support your claims.

Following Carroll's (2005) change of variables method, we define a new variable for market resources: $Y_t = c_t + k_{t+1} = e^{z_t} k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t$. We can rewrite the problem as

$$V(z_t, \alpha_t, Y_t)^{1-\psi} = \max_{k_{t+1}, l_t} \left\{ (1-\beta) \left[\log(Y_t - k_{t+1}) - \eta \frac{l_t^2}{2} \right]^{1-\psi} + \beta \left[\mathbb{E}_t \Big(V(z_{t+1}, \alpha_{t+1}, Y_{t+1}) \Big)^{1-\gamma} \right]^{\frac{1-\psi}{1-\gamma}} \right\}$$

where we replaced c_t using $c_t = Y_t - k_{t+1}$.

If we knew the policy function for l, then Y_{t+1} is just a function of z_{t+1}, α_{t+1} and k_{t+1} only. We can rewrite $\tilde{V}(z_t, \alpha_t, k_{t+1}) = \beta \left[\mathbb{E}_t \Big(V(z_{t+1}, \alpha_{t+1}, Y_{t+1}) \Big)^{1-\gamma} \right]^{\frac{1-\psi}{1-\gamma}}$, so that

$$V(z_t, \alpha_t, Y_t)^{1-\psi} = \max_{k_{t+1}} \left\{ (1-\beta) \left[\log(Y_t - k_{t+1}) - \eta \frac{l_t^2}{2} \right]^{1-\psi} + \tilde{V}(z_t, \alpha_t, k_{t+1}) \right\}$$

The FOC with respect to c yields

$$(1-\beta)(1-\psi) \left[\log c_t^* - \eta \frac{l_t^{*2}}{2} \right]^{-\psi} \frac{1}{c_t^*} = \tilde{V}_k(z_t, \alpha_t, k_{t+1})$$

where $c_t^* = Y_t^* - k_{t+1}^*$. If we knew $\tilde{V}_k(z_t, \alpha_t, k_{t+1})$ then we can solve for c_t^* numerically using the FOC above. Then we can get rid of the maximization in the problem above and write

$$V(z_t, \alpha_t, Y_t)^{1-\psi} = (1-\beta) \left[\log(c_t^*) - \eta \frac{l_t^{*2}}{2} \right]^{1-\psi} + \tilde{V}(z_t, \alpha_t, k_{t+1})$$

Define a grid on capital tomorrow $G_{k_{t+1}} = \{k_1, k_2, ..., k_{n_k}\}$, TFP $G_z = \{z_1, z_2, ..., z_{n_z}\}$ and elasticity of production with respect to capital $G_\alpha = \{\alpha_1, \alpha_2, ..., \alpha_{n_\alpha}\}$, and compute a grid of market resources for tomorrow: $G_{Y_{t+1}} = e^{z_{t+1}} k_{t+1}^{\alpha_{t+1}} l_{ss}^{1-\alpha_{t+1}} + (1-\delta)k_{t+1}$ for each $k \in G_{k+1}$. Following Barillas and Fernandez-Villaverde (2007), we use the following steps to perform value function iteration with an endogenous grid method (EGM):

1. EGM with labor fixed at l_{ss}

   (a) Start from $n=0$. Guess $\tilde{V}^0(z_t,\alpha_t,k_{t+1}) = \beta\Big[\mathbb{E}\Big(V^0(z_{t+1},\alpha_{t+1},Y_{t+1})\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}}$, where $\tilde{V}^0(z_t,\alpha_t,k_{t+1})$ should be increasing in capital.

   (b) For each value of $\tilde{V}^n(z_t,\alpha_t,k_{t+1})$ on the grid, compute the derivative $\tilde{V}_k^n(z_t,\alpha_t,k_{t+1})$ at the grid points for capital tomorrow. We use the average of the slopes of the linearly interpolated value function to compute the derivatives.

   (c) Compute the optimal c_t^* using the equation $(1-\beta)(1-\psi)\Big[\log c_t^* - \eta\frac{l_{ss}^2}{2}\Big]^{-\psi}\frac{1}{c_t^*} = \tilde{V}_k^n(z_t,\alpha_t,k_{t+1})$, for each point in the grid for z, α and k_{t+1}.

   (d) Find the endogenously determined market resources $Yend_t = c_t^*(z_t,\alpha_t,kend_t)+G_{k_{t+1}}$ and update the value function $V^n(z_t,\alpha_t,Yend_t) = \Big\{(1-\beta)\Big[\log c_t^*(z_t,\alpha_t,kend_t)-\eta\frac{l_{ss}^2}{2}\Big]^{1-\psi}+\tilde{V}^n(z_t,\alpha_t,k_{t+1})\Big\}^{\frac{1}{1-\psi}}$. Since we want the new value function to be defined on the grid $G_{Y_{t+1}}$, we interpolate $V^{n+1}(z_t,\alpha_t,Yend_t)$ on $G_{Y_{t+1}}$ to get $V^{n+1}(z_{t+1},\alpha_{t+1},Y_{t+1})$.

   (e) Update $\tilde{V}^{n+1}(z_t,\alpha_t,k_{t+1}) = \beta\Big[\mathbb{E}_t\Big(V^{n+1}(z_{t+1},\alpha_{t+1},Y_{t+1})\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}}$.

   (f) Stop if $\sup_{i,j,m}\Big|\tilde{V}^{n+1}(z_i,\alpha_j,k_m)-\tilde{V}^n(z_i,\alpha_j,k_m)\Big| \le 1e^{-6}$. Otherwise, $n \to n+1$ and go back to step (a).

   (g) After the value function $\tilde{V}$ has converged, retrieve $kend_t(z_i,\alpha_j)$ for all values of $z_i \in G_z$ and $\alpha_j \in G_\alpha$ from $Yend_t(z_i,\alpha_j,kend_t) = e^{z_i}kend_t^{\alpha_j}l_{ss}^{1-\alpha_j}+(1-\delta)kend_t$. Then interpolate the value function defined on the grid Y_{t+1}, i.e. $V(z_t,\alpha_t,Y_{t+1})$, on the grid $G_{k_{t+1}}$, to retrieve $V(z_t,\alpha_t,k_{t+1})$. We will use it as the first guess for step 2.

2. One VFI with grid search to recover policy functions for c_t, l_t and k_{t+1} with flexible l_t (one iteration only).

   (a) Use $V(z_{t+1},\alpha_{t+1},Y_{t+1})$ obtained in the previous EGM with fixed l_{ss} to compute $V(z_t,\alpha_t,k_t)$.

   (b) Use $V(z_t,\alpha_t,k_t)$ as a first guess for VFI to obtain policy functions for c_t, l_t and k_{t+1}.

3. Apply EGM using the policy functions l, k' from the previous step

   (a) Compute $kend_{z_i,\alpha_j,k_m}$ that satisfies $k'(z_i,\alpha_j,k_{z_i,\alpha_j,k_m}) \in G_{k_{t+1}}$.

   (b) Use $kend_{z_i,\alpha_j,k_m}$ and $l(z_i,\alpha_j,G_{k_{t+1}})$ from step 2 to obtain $l(z_i,\alpha_j,kend_{z_i,\alpha_j,k_m})$.

   (c) Set $n=0$ and compute values for $\tilde{V}^0(z_t,\alpha_t,k_{t+1}) = \beta\Big[\mathbb{E}\Big(V^0(z_{t+1},\alpha_{t+1},Y_{t+1})\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}}$.

   (d) For each value of $\tilde{V}^n(z_t,\alpha_t,k_{t+1})$ on the grid, compute the derivative $\tilde{V}_k^n(z_t,\alpha_t,k_{t+1})$ at points on $G_{k_{t+1}}$, and compute the optimal level of consumption c_t^* using $(1-\psi)\Big[\log c_t^* - \eta\frac{l_{ss}^2}{2}\Big]^{-\psi}\frac{1}{c_t^*} = \tilde{V}_k^n(z_t,\alpha_t,k_{t+1})$. Then solve for $kend_t$ by the resource constraint $c_t^*(z_t,\alpha_t,kend_t)+G_{k_{t+1}} = e^{z_t}kend_t^{\alpha_t}l(z_i,\alpha_j,kend_{z_i,\alpha_j,k_m})^{1-\alpha_t}+(1-\delta)kend_t$.

   (e) Update the value function

$$V^n(z_t,\alpha_t,kend_t)^{1-\psi} = (1-\beta)\Big[\log c_t^*(z_t,\alpha_t,kend_t)-\eta\frac{l(z_t,\alpha_t,kend_{z_i,\alpha_j,k_m})^2}{2}\Big]^{1-\psi}+\tilde{V}^n(z_t,\alpha_t,k_{t+1})$$

   and interpolate $V^{n+1}(z_t,\alpha_t,kend_t)$ on G_{k+1} using the value of $kend_t$ computed in step (d) to obtain $V^{n+1}(z_{t+1},\alpha_{t+1},k_{t+1})$.

   (f) Use $kend_t$ computed in (d) and policy function $l(z_i,\alpha_j,G_{k_{t+1}})$ from step 2 to interpolate $l(z_i,\alpha_j,kend_t)$.

(g) Compute $\tilde{V}^{n+1}(z_t, \alpha_t, k_{t+1}) = \beta\Big[\mathbb{E}\Big(V^{n+1}(z_{t+1}, \alpha_{t+1}, k_{t+1})\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}}$.

(h) If $\sup_{i,j,m}\Big|V^{n+1}(z_i, \alpha_j, k_m) - V^n(z_i, \alpha_j, k_m)\Big| < 1e^{-7}$, stop.

4. Second VFI to recover policy function

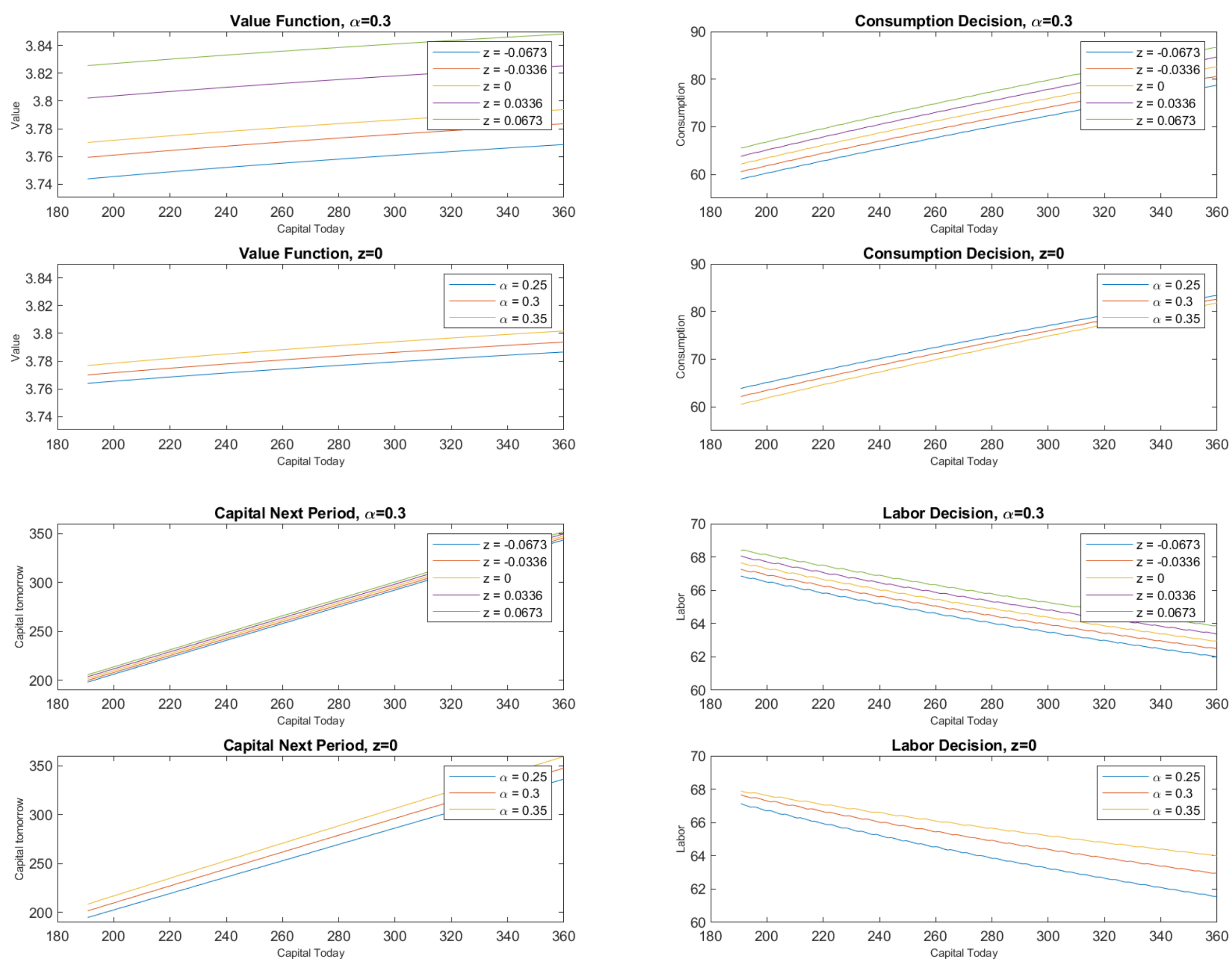
   (a) Use $V^{n+1}(z_{t+1}, \alpha_{t+1}, k_{t+1})$ to recover policy functions defined on $G_{k_{t+1}}$.

   (b) Stop if $\sup_{i,j,m}\Big|V^{n+1}(z_i, \alpha_j, k_m) - V^n(z_i, \alpha_j, k_m)\Big| \leq 1e^{-7}$.

Caveat: When the upper and lower bounds for capital grid are set to $\pm 30\%$ from k_{ss}, the EGM fails to converge, due to the fact that at the higher end of the capital distribution, the upper bound for capital policy becomes binding, which prevents the next period capital from getting to its optimal level. Changing the upper bound to 1.5 times k_{ss} fixes this problem. In addition, the policy functions using VFI with grid search in the last step produces weird impulse response functions, due to a lot of flat parts in the policy functions. To generate better results we use VFI with linear interpolation instead. The results for EGM below incorporates these changes.

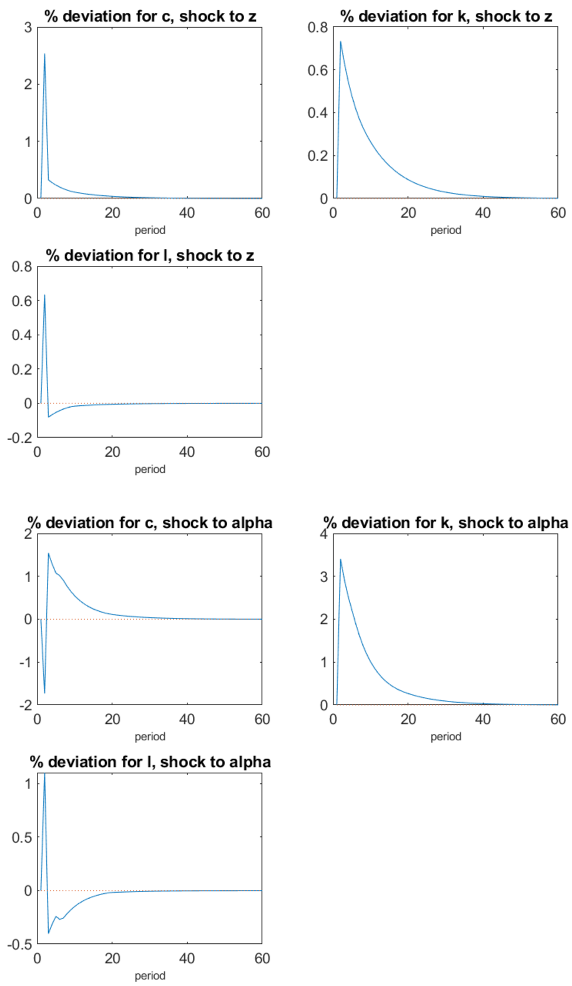
The value and policy functions from the Endogenous Grid Method are shown in Figure 3 below:

Figure 3: Value Functions and Policy Functions with One Shock Fixed (EGM)



The IRFs with orthogonal shocks to z and α respectively are shown in Figure 4 below:

Figure 4: Impulse Response Functions



The time and number of iterations for each step, as well as comparison to the regular VFI in part 3, are shown in Table 1. Although the EGM uses more iterations in total, it is 3 times faster than the regular VFI because the majority of iterations occur in the first step, where no max operator is needed.

Table 1: Running speed

	EGM (Part 4)				VFI (Part 3)
	1st EGM	1st VFI	2nd EGM	2nd VFI	Regular VFI
Iterations	1012	1	8	94	569
Seconds	3873	9	58	5744	32458

The accuracy of the algorithm can be represented by the Euler Equation Errors (EEE). To calculate the EEE, we first find the intertemporal Euler Equation (which is shown by equation 7 in part 7 below). Then we take the ratios between the right hand side and left hand side of the Euler Equation for each state and calculate their distances from 1. The log 10-based distance is the EEE. Table 2 shows the maximum Euler Equation Errors in the whole grid calculated from questions 3 and 4 respectively. The endogenous grid method would have been less accurate than the VFI with interpolation if we use grid search for the last step, which makes sense since it prohibits the endogenous variables from taking values different than grid point values. If instead we use linear interpolation for the VFI in step 4, we get roughly the same accuracy as the VFI in part 3. While the EGM is much faster and achieves similar accuracy as regular VFI with interpolation, the shortcoming is that it is difficult to implement.

Table 2: Euler Equation Errors

	EGM	EGM w/ Interpolation	Regular VFI
Max Euler Equation Error	-2.1404	-2.7061	-2.8451

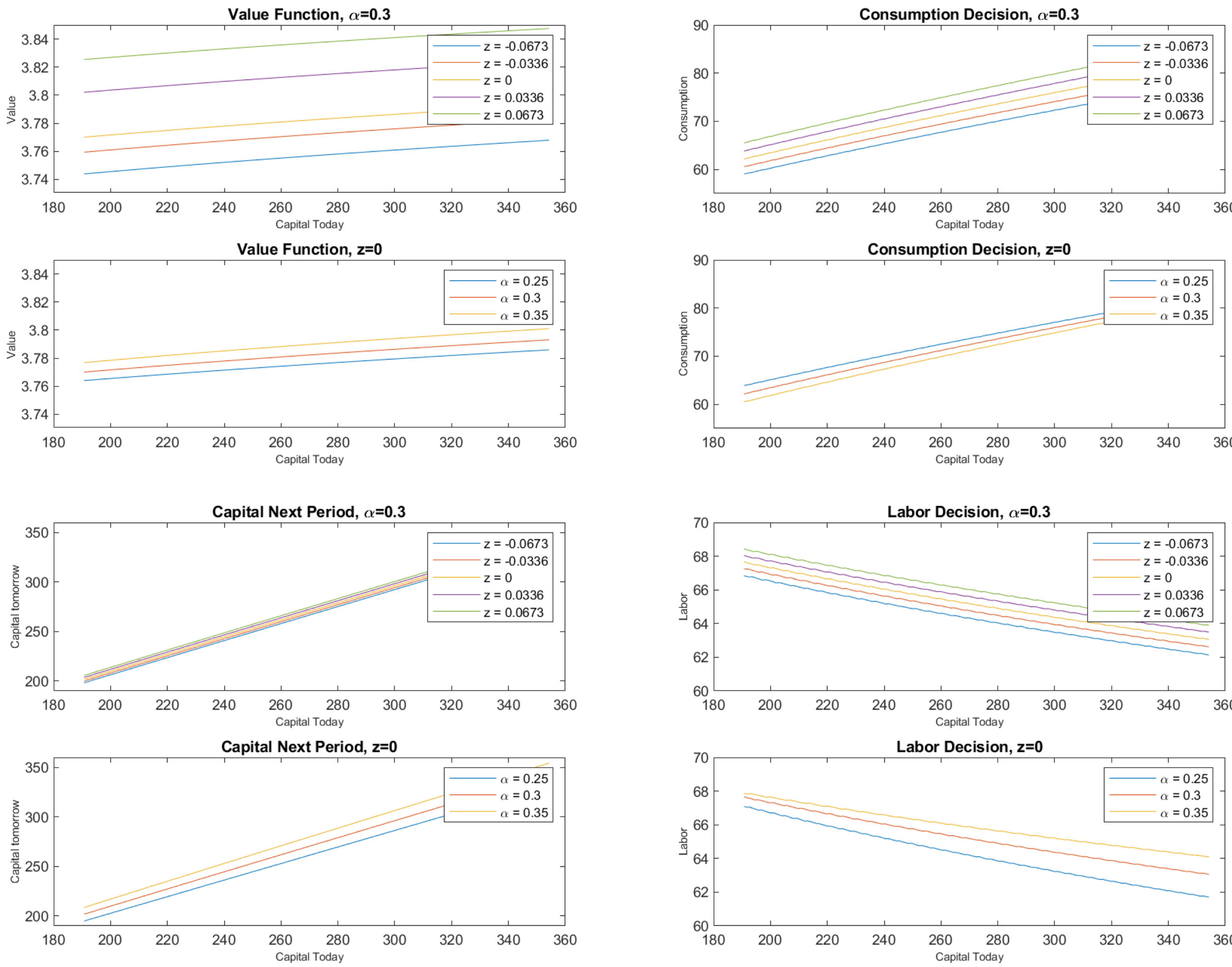
The Matlab file that executes the EGM algorithm is called "*HW2_Q4_EGM.m*" and is located in the "Code" folder.

5 Accelerator (5 points)

Recompute your solution to 3) using an accelerator, i.e., skipping the max operator in the Bellman equation 9 out of each 10 times. Compare accuracy and computing time between the simple grid scheme implemented in 3) and the results from the accelerator scheme. Present evidence to support your claims.

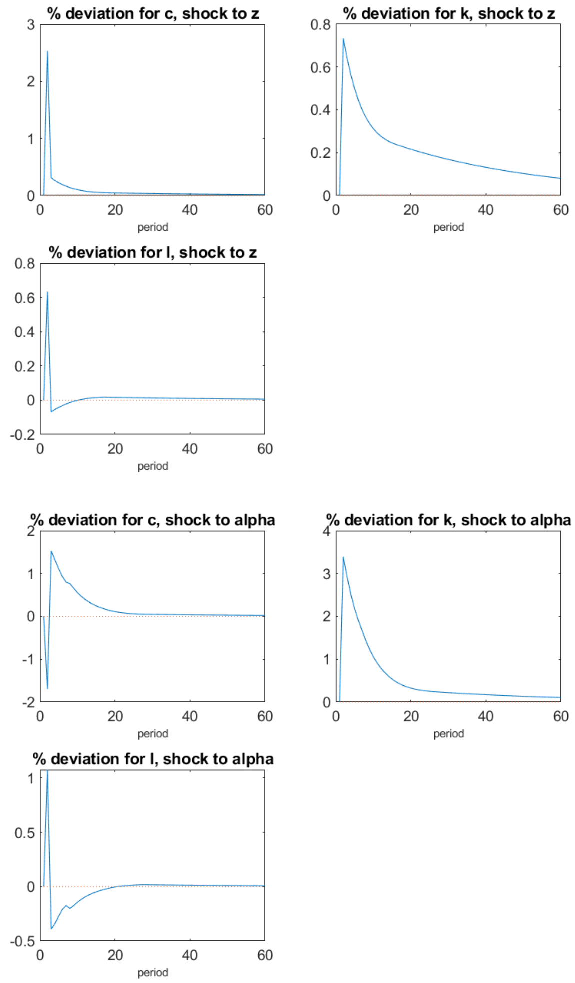
The value and policy functions from the VFI with accelerator are shown in Figure 5 below:

Figure 5: Value Functions and Policy Functions with One Shock Fixed (Accelerator)



The IRFs with orthogonal shocks to z and α respectively are shown in Figure 6 below:

Figure 6: Impulse Response Functions



We apply the max operator once in every 10 iterations. For the remaining 9 out of 10 iterations, we do not use the max operator and just apply the policy functions so far to calculate endogenous variables and calculate value functions. The accuracy compared with the regular VFI in part 3 is shown in Table 3 below. It turns out that the two methods have almost exactly the same accuracy. It is possibly due to the fact that both the accelerator and the VFI in part 3 use linear interpolation for value functions.

Table 3: Euler Equation Errors

	Accelerator	Regular VFI
Max Euler Equation Error	-2.8451	-2.8451

The comparison of computing time is shown in Table 4 below:

Table 4: Running speed

	Accelerator	Regular VFI
Iterations	554	569
Seconds	5467	32458

The Matlab file that executes the accelerator is called "*HW2_ Q5_ Accelerator.m*" and is located in the "Code" folder.

6 Multigrid (5 points)

Implement a multigrid scheme (Chow and Tsitsiklis, 1991) for a Value function iteration, with the grid centered around k_{ss} with a coverage of $\pm 30\%$ of k_{ss} and equally spaced (you can keep the grid of investment fixed). You will have 100 capital grid points in the first grid, 500 capital grid points in the second, and 5000 capital grid points in the third. Compare accuracy and computing time between the simple grid scheme implemented in 3) and the results from the multigrid scheme. Present evidence to support your claims.

Figure 7 shows the value and policy functions from the multigrid scheme:

Figure 7: Value Functions and Policy Functions with One Shock Fixed (Multigrid)

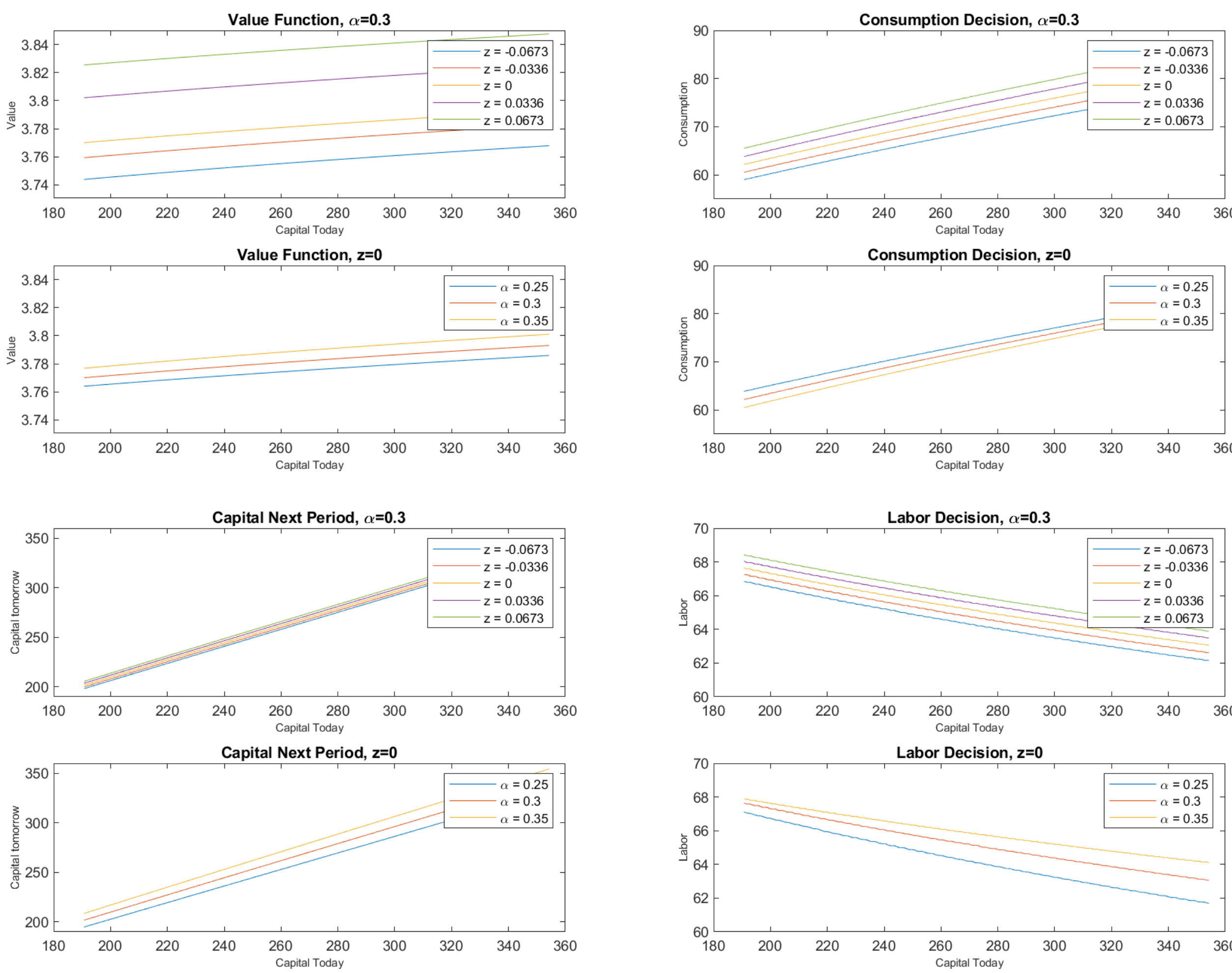


Figure 8 shows the IRFs with orthogonal shocks to z and α respectively. Compared with other methods, the path is more bumpy, which implies that the multigrid method might be less accurate.

Figure 8: Impulse Response Functions

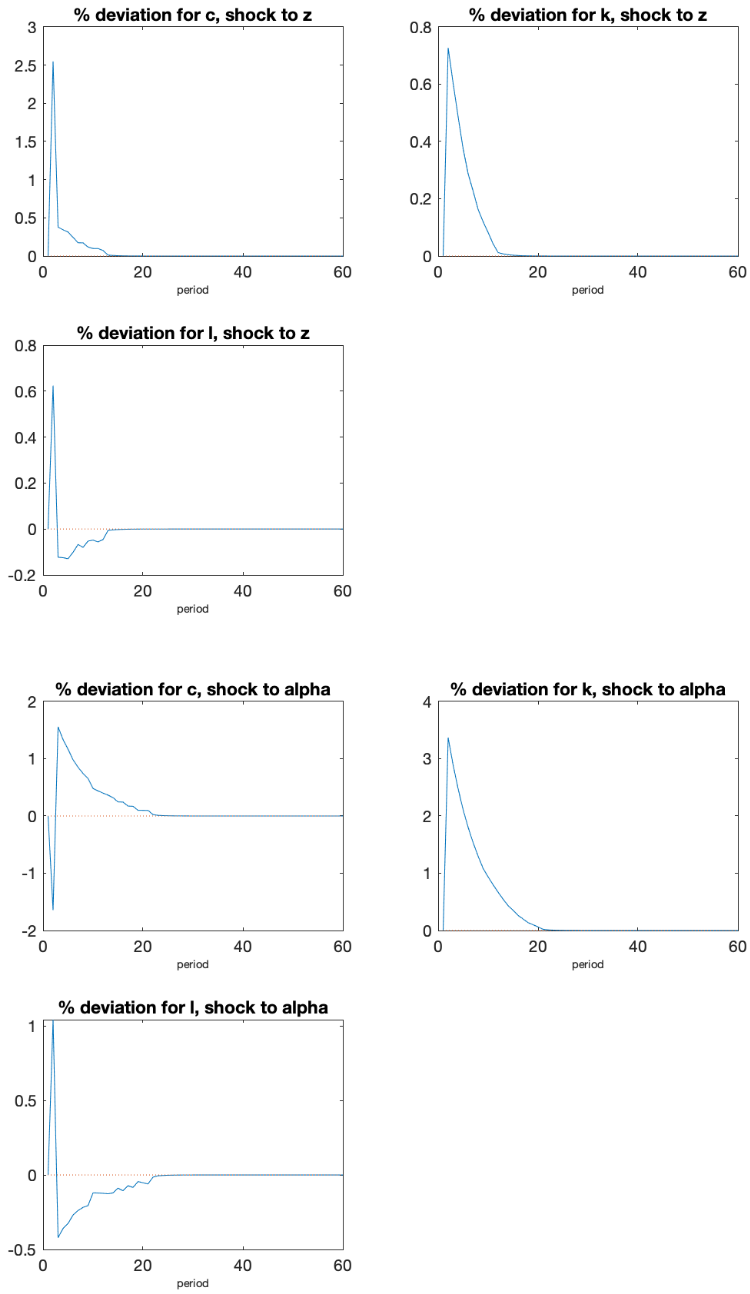


Table 5 below compares the Euler Equation Errors from the multigrid scheme and the regular VFI from part 3. Indeed, the multigrid method is no more accurate than the regular value function iteration, even though in the last step of multigrid we use 5000 grid points. My guess is that in the last two steps (500 and 5000 capital grid points), the initial guesses for value function are already good, which limits the improvement for policy functions. As a result, the Euler Equation Error, which heavily relies on the policy functions, is not in any sense lower compared with the VFI method in part 3, which uses 250 grid points.

Table 5: Euler Equation Errors

	Multigrid	Regular VFI
Max Euler Equation Error	-2.7975	-2.8174

Table 6 below compares the running time and number of iterations for the regular value function iteration and multigrid method. The number of iterations and total running time are the sums of those from the runs with 100, 500 and 5000 grid points respectively.

Table 6: Running speed

	Multigrid	Regular VFI
Iterations	575	569
Seconds	15143	32458

The Matlab code that executes the multigrid scheme is called "*HW2_ Q6_ Multigrid.m*" and is located in the "Code" folder.

7 Chebyshev (10 points)

Compute the solution to the previous model when you use 8 Chebyshev polynomials on capital. Compare the solution with the one from previous questions.

Let $u(c,l)$ denote the flow utility specified above, and u_c denote the partial derivatives of the flow utility with respect to c and l. From the envelope condition and FOC with respect to c we get

$$\mathbb{V}_k(z,\alpha,k) = \frac{u_c(c,l)(\alpha e^z k^{\alpha-1} l^{1-\alpha} + 1 - \delta)}{(1-\psi)\mathbb{V}(z,\alpha,k)^{-\psi}}$$

Plugging the above equation into the FOC with respect to k' we get the Euler equation:

$$u_c(c,l) = \beta(1-\psi)\Big[\mathbb{E}\Big(\mathbb{V}(z',\alpha',k')^{1-\gamma}\Big)\Big]^{\frac{\gamma-\psi}{1-\gamma}}\mathbb{E}\left(\mathbb{V}(z',\alpha',k')^{-\gamma}\frac{u_c(c',l')(\alpha' e^{z'} k'^{\alpha'-1} l'^{1-\alpha'} + 1 - \delta)}{(1-\psi)\mathbb{V}(z',\alpha',k')^{-\psi}}\right)$$

$$\Rightarrow u_c(c,l) = \beta\Big[\mathbb{E}\Big(\mathbb{V}(z',\alpha',k')^{1-\gamma}\Big)\Big]^{\frac{\gamma-\psi}{1-\gamma}}\mathbb{E}\Big(\mathbb{V}(z',\alpha',k')^{\psi-\gamma}u_c(c',l')(\alpha' e^{z'} k'^{\alpha'-1} l'^{1-\alpha'} + 1 - \delta)\Big) \tag{7}$$

where

$$u(c,l) = \left(\log c - \eta\frac{l^2}{2}\right)^{1-\psi}$$
$$u_c(c,l) = (1-\psi)\left(\log c - \eta\frac{l^2}{2}\right)^{-\psi}\frac{1}{c}$$

Let n be the number of collocation points for Chebyshev polynomials. We approximate the decision rule for labor and value function as

$$l(z,\alpha,k|\theta_i^l) = \sum_{i=1}^{n}\theta_i^l\psi_i^l(z,\alpha,k) \tag{8}$$

$$V(z,\alpha,k|\theta_i^V) = \sum_{i=1}^{n}\theta_i^V\psi_i^V(z,\alpha,k) \tag{9}$$

where $\{\psi_i^l(z,\alpha,k)\}_{i=1}^n$, $\{\psi_i^V(z,\alpha,k)\}_{i=1}^n$ are basis functions for labor and value functions, and $\{\theta_i^l\}_{i=1}^n$, $\{\theta_i^V\}_{i=1}^n$ are unknown coefficients.

Given the approximated policy function for l, we can use the equation 6 to derive optimal consumption:

$$c(z,\alpha,k) = \frac{1-\alpha}{\eta} e^z k^\alpha l(z,\alpha,k|\theta_i^l)^{-\alpha-1} \tag{10}$$

and from budget constraint, we get optimal capital for the next period:

$$k'(z,\alpha,k) = e^z k_t^\alpha l(z,\alpha,k|\theta_i^l)^{1-\alpha} - c(z,\alpha,k) + (1-\delta)k \tag{11}$$

To solve for the coefficients, we plug equations 8, 9, 10 and 11 into the Euler equation 7, then we have

$$\begin{aligned}(1-\psi)\left(\log c - \eta\frac{l^2}{2}\right)^{-\psi}\frac{1}{c} = \beta\Big[\sum_{\alpha',z'}\Gamma_{\alpha'|\alpha}\Pi_{z'|z}\Big(\sum_{i=1}^n \theta_i^V\psi_i^V(z',\alpha',k')\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}} \\ \Big[\sum_{\alpha',z'}\Gamma_{\alpha'|\alpha}\Pi_{z'|z}\Big(\sum_{i=1}^n \theta_i^V\psi_i^V(z',\alpha',k')\Big)^{\psi-\gamma}(1-\psi)\left(\log c' - \eta\frac{l'^2}{2}\right)^{-\psi}\frac{1}{c'}(\alpha' e^{z'}k'^{\alpha'-1}l'^{1-\alpha'}+1-\delta)\Big]\end{aligned}$$

where $c = c(z,\alpha,k)$, $l = l(z,\alpha,k|\theta_i^l)$, $c(z',\alpha') = c(z',\alpha',k')$, $l(z',\alpha') = l(z',\alpha',k')$ and $k' = k'(z,\alpha,k)$.

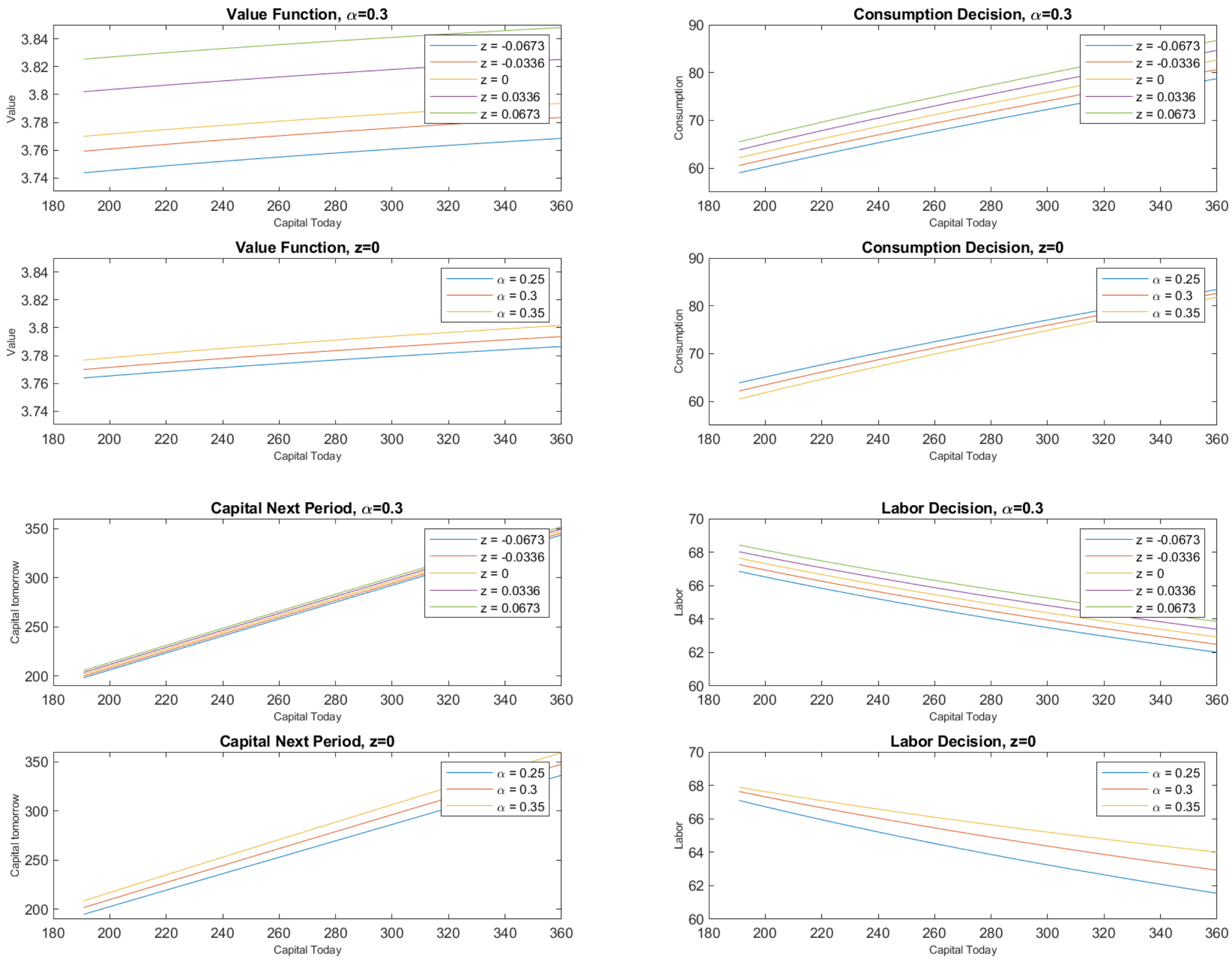
Similarly, plugging the equations 8, 9, 10 and 11 into the Bellman equation 1, then we have

$$\sum_{i=1}^n \theta_i^V\psi_i^V(z,\alpha,k) = \left\{(1-\beta)\Big(\log c - \eta\frac{l^2}{2}\Big)^{1-\psi} + \beta\Big[\sum_{\alpha',z'}\Gamma_{\alpha'|\alpha}\Pi_{z'|z}\Big(\sum_{i=1}^n \theta_i^V\psi_i^V(z',\alpha',k')\Big)^{1-\gamma}\Big]^{\frac{1-\psi}{1-\gamma}}\right\}^{\frac{1}{1-\psi}}$$

Caveat: for the same reason as was mentioned in part 4: EGM, the accuracy of this method deteriorates near the upper bound for the capital grid. Increasing the maximum capital to 1.5 times k_{ss} solves this problem. The results below incorporates this change.

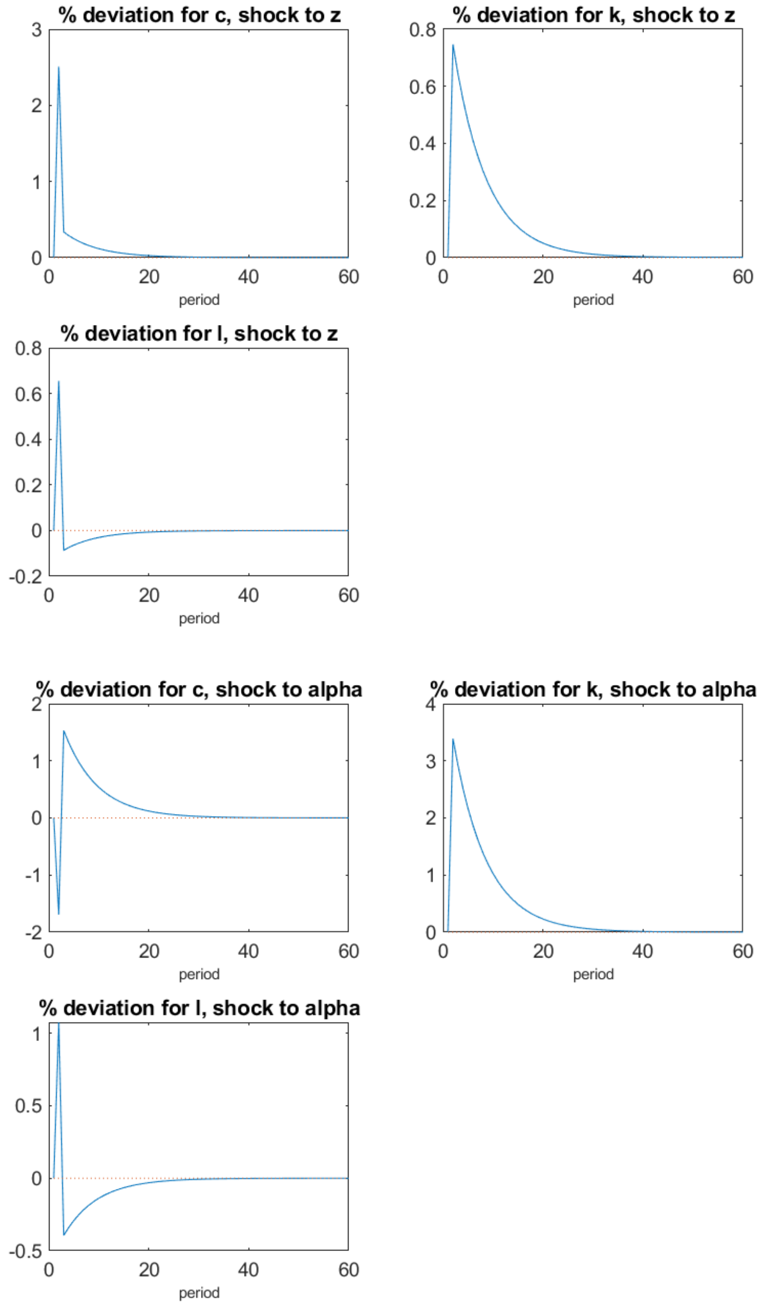
We use the solver in Matlab to solve numerically for the coefficients of the basis functions, which are set to be Chebyshev polynomials. Since calculating the coefficients for 8th order Chebyshev polynomials is numerically difficult, we start by calculating the coefficients of 3rd order polynomials, and increasing the order by 1 each time using the previous results as initial guesses for the existing polynomials. The resulting value and policy functions are plotted below:

Figure 9: Value Functions and Policy Functions with One Shock Fixed (Chebyshev)



The IRFs with orthogonal shocks to z and α are shown in Figure 10 below:

Figure 10: Impulse Response Functions



The computing time and number of iterations for the projection method, along with comparisons with earlier results, are shown in Table 7 below. Unsurprisingly, the projection method spends the least amount of time and fewest iterations, as we only needs to solve for 16 coefficients in total.

Table 7: Running time

	VFI	EGM (total)	Accelerator	Multigrid	Chebyshev
Iterations	569	1115	554	575	22
Seconds	32458	9684	5467	15143	47.84

The Euler Equation errors from the projection method, along with comparison with earlier results, are shown in Table 8 below. With the given range ($0.7k_{ss} - 1.3k_{ss}$) for the grid for capital, the projection method is actually slightly less accurate than the other methods. After expanding the maximum capital grid point to 1.5 times the steady state level, the projection method becomes much more accurate than the other methods.

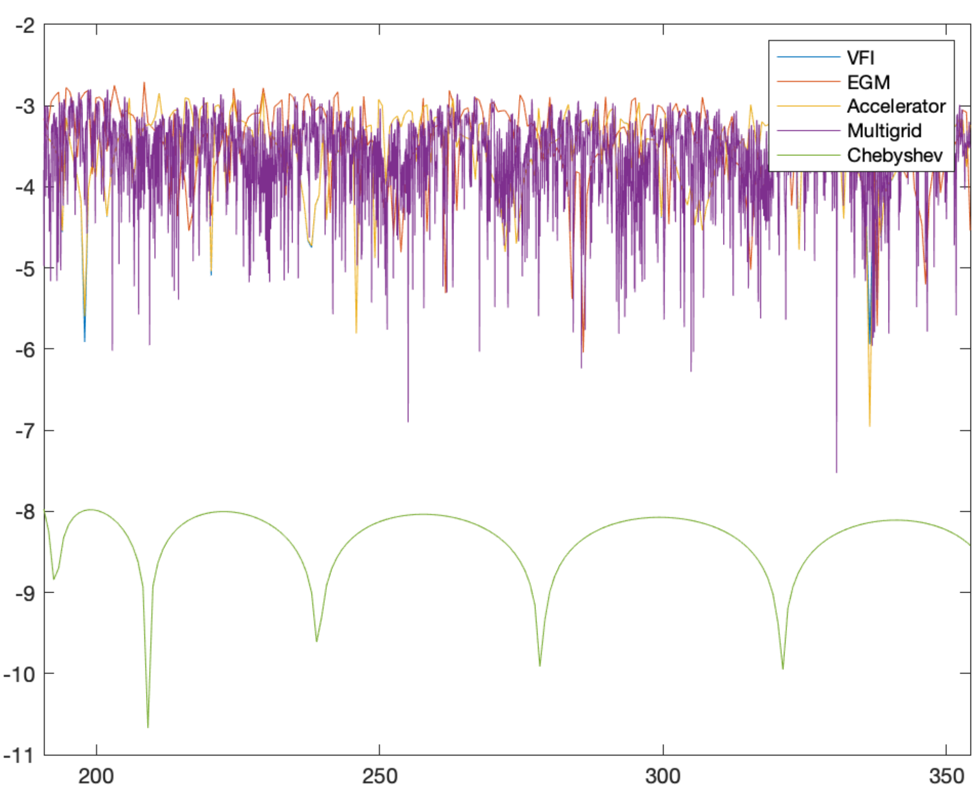
Table 8: Euler Equation Errors

	VFI	EGM	Accelerator	Multigrid	Chebyshev	Chebyshev (max capital = $1.5 \times k_{ss}$)
Max EEE	-2.8451	-2.7061	-2.8451	-2.7975	-2.0438	-7.9715

The file that executes the projection method using Chebyshev polynomials is called *"HW2_ Q7_ Chebyshev.m"* and is located in the "Code" folder.

To more intuitively compare the accuracy of all the methods from parts 3-7, we plot the maximum log 10 based EEE for each method on the grid for capital:

Figure 11: Euler Equation Errors for all Methods



8 Perturbation (10 points)

Compute the solution to the previous model using a third-order approximation and plot the IRFs initialized at the mean of the ergodic distribution. You need to substitute the process for productivity by:

$$\boldsymbol{z_t = 0.95 z_{t-1} + 0.005 \epsilon_t^1}$$

where $\boldsymbol{\epsilon_t^1}$ is a normalized gaussian innovation and the process for $\boldsymbol{\alpha}$ by:

$$\boldsymbol{\alpha_t = 0.03 + 0.9 \alpha_{t-1} + 0.01 \epsilon_t^2}$$

where $\boldsymbol{\epsilon_t^2}$ is a normalized gaussian innovation.

The Euler equation in part 7 can be rewritten as

$$\mathbb{E}[m'(\alpha' e^{z'} k'^{\alpha'-1} l'^{1-\alpha'} + 1 - \delta)] = 1$$

$$\begin{aligned} m' &= \beta \frac{u_{c'}(c', l')}{u_c(c, l)} \frac{\mathbb{V}'^{\psi-\gamma}}{\mathbb{E}(\mathbb{V}'^{1-\gamma})^{\frac{\psi-\gamma}{1-\gamma}}} \\ &= \beta \frac{\left(\log c' - \eta \frac{l'^2}{2}\right)^{-\psi} \frac{1}{c'}}{\left(\log c - \eta \frac{l^2}{2}\right)^{-\psi} \frac{1}{c}} \frac{\mathbb{V}'^{\psi-\gamma}}{\mathbb{E}(\mathbb{V}'^{1-\gamma})^{\frac{\psi-\gamma}{1-\gamma}}} \end{aligned}$$

where m' is the stochastic discount factor, $c' = g_c(z', \alpha', k')$, $k' = g_k(z, \alpha, k)$, $l' = g_l(z', \alpha', k')$ are the optimal policy responses, and $\mathbb{V}' = \mathbb{V}(z', \alpha', k')$.

Introduce a new control variable ξ defined by

$$\xi^{1-\gamma} = \mathbb{E}\Big(\mathbb{V}'^{1-\gamma}\Big)$$

The full model is given by

$$\begin{aligned} 1 &= \mathbb{E}[m'(\alpha' e^{z'} k'^{\alpha'-1} l'^{1-\alpha'} + 1 - \delta)] && (12) \\ \eta l c &= (1-\alpha) e^z k^\alpha l^{-\alpha} && (13) \\ k' &= e^z k^\alpha l^{1-\alpha} + (1-\delta) k - c && (14) \\ z' &= \rho_z z + \sigma_z \epsilon_z && (15) \\ \alpha' &= \mu_\alpha + \rho_\alpha \alpha + \sigma_\alpha \epsilon_\alpha && (16) \\ m' &= \beta \frac{\left(\log c - \eta \frac{l^2}{2}\right)^{\psi} c}{\left(\log c' - \eta \frac{l'^2}{2}\right)^{\psi} c'} \frac{\mathbb{V}'^{\psi-\gamma}}{\xi^{\psi-\gamma}} && (17) \\ \mathbb{V}'^{1-\psi} &= (1-\beta)\Big(\log c' - \eta \frac{l'^2}{2}\Big)^{1-\psi} + \beta \xi'^{1-\psi} && (18) \\ 1 &= \xi^{\gamma-1} \mathbb{E}\Big(\mathbb{V}'^{1-\gamma}\Big) && (19) \end{aligned}$$

The model, which is specified by equations 12-19 above, can be computed using Dynare. The Impulse Response Functions of value function, consumption, capital for the next period and labor with respect to shocks in z and α are shown below:

Figure 12: IRF with respect to shocks in z

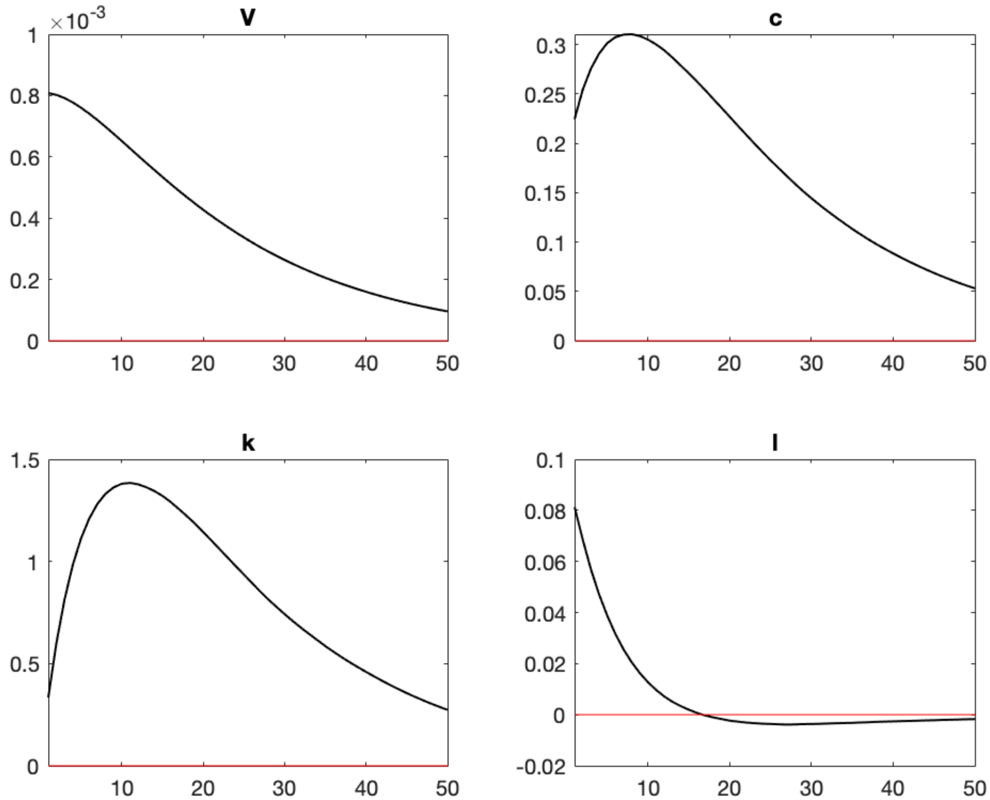
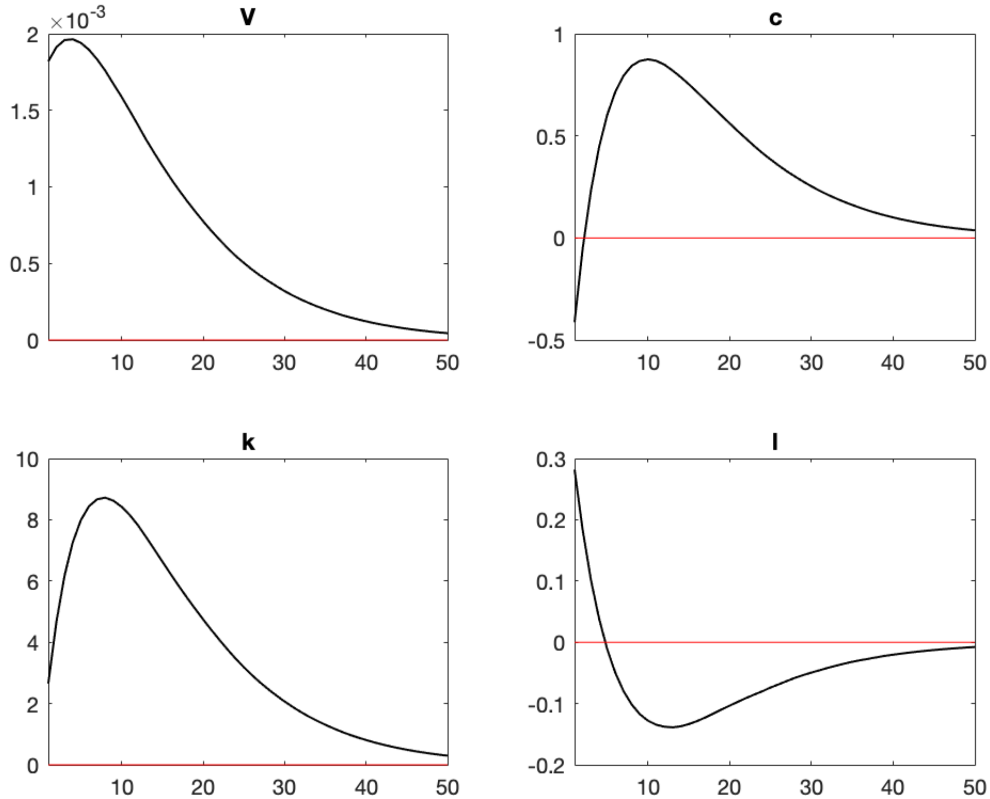


Figure 13: IRF with respect to shocks in α



The Dynare code that executes the perturbation method is called "*HW2_Q8_Perturbation.mod*" and is located in the "Code/Dynare_code" folder.