# Precautionary Mismatch*

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**Abstract**

How does wealth affect the extent to which the "right" workers are matched with the "right" jobs? Using the NLSY79 and O*NET, we document that wealth-poor workers are more mismatched with their jobs. We develop a model featuring worker and firm heterogeneity, search frictions, and incomplete markets. Workers and firms jointly face a trade-off between the speed and payoff of forming a match. A lack of wealth induces workers to trade off wages for finding a job faster due to precautionary motives, which in turn gives a wider range of firms the incentive to match. We refer to this phenomenon as "precautionary mismatch" and show that it leads to substantial within-type earnings and productivity gaps between the wealth-rich and the wealth-poor, especially among high-skilled workers. We estimate that total output would be 3% higher in the US if all employed workers were allocated to the right jobs. In a quantitative experiment, we find that wealth transfers from incumbent workers to young labor market entrants reduce within-type earnings and productivity inequality, improve sorting, and enhance labor productivity. Most of the productivity increase comes from reduced under-employment of high-skilled workers.

**Keywords:** Incomplete Markets, Search and Matching, Mismatch

**JEL Codes:** J64, E21, D31

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# 1 Introduction

An important question in the study of labor markets is the determinants of sorting between workers and firms. A classical insight of Becker's theory of assortative matching is that under perfect labor market competition, workers of different levels of talent optimally match with firms of different productivity, resulting in an efficient allocation of talents. However, we live in a world where market frictions are pervasive and workers do not always find their ideal jobs. The inefficiency in labor allocation can be exacerbated if other forms of market imperfections are present, such as incomplete financial markets.

There is growing empirical evidence that consumption insurance plays a role in workers' search decisions and labor market outcomes (e.g. Katz and Meyer (1990), Chetty (2008), and Herkenhoff et al. (2022), among many others). So far, research on the labor market implications of consumption insurance has been mostly focused on the trade-off on the side of workers[1]. In this paper, we take a step forward and consider a joint trade-off facing heterogeneous workers and heterogeneous firms. More precisely, we aim to study how wealth holdings, and ultimately the ability to self insure, affect the extent to which the "right" workers are assigned to the "right" jobs. Answering this question calls for a framework with very rich heterogeneity.

To this end, we propose a framework with three key elements. First, there is ex-ante heterogeneity in productivity among workers and firms, so that sorting can be studied. Second, the labor market is frictional and meetings are random, so that it takes time for workers and jobs to find "good" matches[2]. Third, workers are risk-averse and the financial market is incomplete, so that workers need to accumulate precautionary savings in order to self-insure against unemployment risk. Our framework organically nests three classes of canonical models in the macro and labor literature: an assignment model by Becker (1973), a Diamond-Mortensen-Pissarides search and matching model, and an incomplete markets model in the spirit of Bewley (1977), İmrohoroğlu (1989), Huggett (1993), and Aiyagari (1994).

The mechanism of our framework works as follows. Absent frictions, workers would efficiently sort into the "right" firms as in Becker's assignment model. However, due to search frictions, a trade-off exists between the speed of forming a match and the payoff from the match, resulting in an equilibrium where workers and jobs agree on a range of acceptable matches that deviate from the most efficient ones. In order to smooth consumption, wealth-poor workers are willing to find a job quickly at the cost of potentially lower payoff. On the firm side, wages offered to wealth-poor workers (who otherwise have the same productivity as others) are lower due

[1] A notable exception is Herkenhoff et al. (2022), which we will discuss in the literature review.

[2] In reality, production types are hard to precisely observe ex-ante. Both sides face randomness in the types of counterparts they meet in the labor market, and that's why multiple interviews are needed before successful matches are formed. We believe that random search is a suitable framework to characterize this randomness.

to their lower bargaining positions[3], and the matches are thus profitable even if the worker is mismatched. As a result, wealth-poor workers tend to deviate more from efficient matches. We refer to this phenomenon as "precautionary mismatch," as it reflects mismatches that occur due to workers' precautionary motives.

The mechanism at work in the model generates three key predictions. First, as a direct implication of the precautionary mismatch motive, wealth-poor workers tend to be more mismatched with their jobs. Second, wealth-poor workers have higher job finding rates since they are less picky about their job choices. Third, wealthier workers (of any given production type) receive higher wages due to smaller mismatch and higher bargaining positions. Using the NLSY79 and O*NET data, we find empirical evidence supporting all three predictions.

In the calibration exercise, we match the model-implied relationships between wealth holdings and job finding rates as well as wages with their empirical counterparts from NLSY79. Then, using the calibrated model, we estimate the effect of wealth holdings on labor market allocations. We find that precautionary mismatch leads to pronounced earnings and productivity inequality between wealth-poor and wealth-rich workers of the same production type. In particular, among the highest-skilled group, the earnings gap and productivity gap between workers in the 1st and the 99th wealth percentiles are 31.5% and 40.8% respectively. This suggests that there could be substantial gain in output if mismatched workers could be efficiently reallocated. Indeed, we estimate that total output from the labor market would be 3% higher if all workers were allocated to the "right" jobs.

Finally, we conduct a policy experiment where wealth is transferred from incumbent workers (who tend to be wealthy) and distributed equally to young labor market entrants. We consider a quite generous payout worth of 0.5 year of average earnings in the baseline economy to each entering worker, but as the entry rate of new workers is low (1/45 of the population per year), the amount of taxes per capita needed to balance the budget is trivial. We find that over the long run, both within-type earnings and productivity inequality would shrink by 20% with the policy, and aggregate labor productivity would increase by a modest 0.15%. Most of the productivity increase can be attributed to improved allocation of high-skilled workers: in the baseline economy, high-skilled young workers tend to be underemployed due to precautionary mismatch, and wealth enables them to be more thorough with job search and find higher productivity jobs.

This paper makes three major contributions. Our first contribution is theoretical: we provide a joint theory of wealth, wages and labor productivity. Specifically, we offer a novel perspective on the productivity effect of precautionary savings (or the lack thereof) through the lens of labor market (mis)allocation, which in turn shapes the wealth distribution as labor market

[3]We will discuss the assumption of observable wealth holdings in more detail in Section 2.

outcomes feed back into wealth accumulation. By incorporating two-sided labor market heterogeneity, our model provides a clear notion of the "right" workers for the "right" jobs and the cost of misallocation, which is absent in models without this element. In doing so we also contribute to the rapidly growing research agenda on the macroeconomic implications of micro-level heterogeneity and the rich interactions between distributions and the macroeconomy.

Our second contribution is empirical: we document the relationship between mismatch and worker wealth holdings. Following Lindenlaub and Postel-Vinay (2020) as well as Guvenen et al. (2020), we construct a measure of mismatch and find that it is negatively associated with wealth, even after controlling for a variety of confounding factors. In a companion paper (currently in progress), we aim to identify sorting based on unobserved heterogeneity and revisit this relationship.

Our third contribution is methodological: we develop an efficient algorithm to compute a model with rich heterogeneity, in which the equilibrium depends on an infinite-dimensional object (i.e., distributions), both in and out of the steady state. We extend the state-of-the-art continuous time method developed by Achdou et al. (2022) designed for incomplete markets models to a setting with two-sided heterogeneity and frictional sorting. Under the continuous-time formulation, the computation is recast as Mean Field Games so that the equilibrium conditions boil down to two coupled systems of partial differential equations. We derive a wage function that can be expressed by readily-computed equilibrium objects so that a guess-and-update procedure is avoided[4]. We elaborate on the relation to the existing literature in the following subsection.

## Related Literature

Our framework features two limiting economies: without heterogeneity in production types on both sides, the model is close to Krusell et al. (2010) in which wealth and wages correspond one-to-one; without workers' risk aversion, out model becomes Shimer and Smith (2000) in which workers possess different skills but not wealth. In this regard, out paper makes direct contributions to both strands of the literature.

Theoretically, our paper builds on earlier important works integrating labor search with incomplete markets, including Lentz and Tranæs (2005), Krusell et al. (2010), Lise (2013), Eeckhout and Sepahsalari (2018), Krusell, Luo and Rios-Rull (2019), Griffy (2021), Chaumont and Shi (2022), and Herkenhoff, Phillips and Cohen-Cole (2022), among many others. Compared with previous studies, we introduce two-sided production heterogeneity, which allows us

[4] Krusell et al. (2010) point out that computation of an equilibrium where assets enter Nash bargaining problem is difficulty in discrete time.

to study the effect of wealth on sorting and thus the allocative efficiency of labor. The addition of worker productivity types also generates heterogeneity in workers' value of working (vs. not working) and production impacts, enabling us to look at the heterogeneous effects of wealth in the population. On the other hand, we also contribute to the literature on frictional sorting, including Burdett and Coles (1997), Shimer and Smith (2000), Gautier and Teulings (2006), Eeckhout and Kircher (2011), Bagger and Lentz (2019), Hagedorn et al. (2017) and Bonhomme, Lamadon and Manresa (2019) by allowing for self-insurance to affect sorting.

Among the papers above, Herkenhoff et al. (2022) is the closest to ours, in that both papers consider frictional sorting with risk averse workers. While they study the effect of self insurance through relaxing credit access, we study the effect of wealth through precautionary savings. The two papers differ significantly in the way sorting is generated. In Herkenhoff et al. (2022), wages are assumed to be a piece-rate of match-specific production, and each worker direct their search to firms of a certain productivity type. A logical prediction from their framework is that better consumption insurance induces all workers, regardless of their productivity, to search for higher productivity firms, generating ambiguous effects of sorting. In our framework, there is positive assortative matching due to competition among workers and firms – a classical insight from Becker (1973). Higher wealth holdings induce workers to find jobs closer to their own levels of productivity, as these are the "right" jobs from their own perspective[5], and thus the effect of wealth on sorting is always positive.

Computationally, we develop a continuous-time technique based on Achdou et al. (2022), which uses a continuous-time approach to cast rather complex optimization problems and equilibrium conditions in incomplete-markets models into two coupled systems of partial differential equations that are much easier to compute. In our model, the equilibrium depends on an infinite-dimensional distribution, and wages are a three-dimensional function of worker state variables. Computation of this model would be almost impossible in discrete time. The continuous-time method not only enables us to compute the steady state efficiently, but also makes it possible to look at out-of-steady-state transitional dynamics.

Empirically, our paper is related to a large literature documenting relations between asset holdings and job search behavior (see, for example, Card et al. (2007), Rendon (2007), Lentz (2009), Chetty (2008), Herkenhoff et al. (2022), among many others). These papers show overwhelmingly that increasing the ability to smooth consumption, either through unemployment insurance, wealth or access to credit, leads to longer unemployment duration and higher accepted wages. These findings provide us with an important guidance to think about the implications of the observed search behavior in the context of labor market sorting. A natu-

[5]For low productivity workers, wages from high productivity firms can be lower than those offered by low productivity firms as the high productivity firms need to be compensated for the option value of waiting for better workers. We provide suggestive evidence of this prediction in the empirical section.

ral prediction from a longer unemployment duration is that the match quality of unemployed workers with new jobs also increases. To our knowledge, we are the first paper to document the effect of worker asset holdings on mismatch following an unemployment spell. Our approach to measure worker-firm mismatch follows recent papers including Lise and Postel-Vinay (2020) and Guvenen et al. (2020), which also use observable worker and job characteristics from NLSY79 and O*NET to estimate skills mismatch and effects on wages.

The rest of the paper is organized as follows. In Section 2, we describe the model and the algorithm to solve it. In Section 3, we discuss several key theoretical results regarding the connections between wealth, job search behavior and labor market outcomes. In Section 4, we describe the data sets we use for empirical analysis and the methods to estimate worker and firm types, and present empirical evidence on the relationship between liquid wealth, skill mismatch and wages. In Section 5 we present model calibrations and quantitative exercises. Section 6 concludes.

# 2 Model

## 2.1 Environment

Time is continuous and there is no aggregate uncertainty. We assume that there is a unit measure of workers and jobs.

*Demography.* We set up the demographic structure as in a perpetual youth model (Blanchard, 1985). Workers exit the economy stochastically at rate $\delta$. Exiting workers are replaced by newborns that enters the economy in unemployment with zero asset. Workers draw their skill level $x \in \mathbb{X}$ from a probability density function $d_w(x)$, and $x$ stays fixed throughout lifetime.

*Preference.* Workers maximize expected present value according to a common discount rate $\rho$, and jobs maximize the present value of expected profits discounted at rate $r$, equal to the risk-free interest rate of the economy. Workers are risk averse with flow utility $\mathfrak{u}(c)$ and firms are risk neutral. The utility function $\mathfrak{u}(\cdot)$ exhibits common properties $\mathfrak{u}' > 0, \mathfrak{u}'' < 0$.

*Production.* Production happens when a worker successfully matches with a job. Jobs draw their skill requirements $y \in \mathbb{Y}$ from a density function $d_j(y)$. The production function of a matched pair is denoted $f(x, y) : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}_+$. We impose technical assumptions on $f$ to guarantee existence. Unemployed workers produce $b(x)$ (e.g., leisure, unemployment benefits, and home production). Assume $b(x) = b_0 * f(x, \underline{y})$ where $b_0 \in [0, 1]$.

*Search and Matching.* The labor market is frictional. Search and matching is random via a *meeting* function $M(u, v)$ that is constant returns to scale (CRS), where $u$ denotes unemployment and $v$ vacancies. We denote by $\theta = v/u$ the labor market tightness. Due to CRS, the meeting rate for an unemployed worker can be written as $p(\theta) := M(u,v)/u = M(1,\theta)$. Similarly, the meeting rate for a vacancy can be written as $q(\theta) := M(u,v)/v = M(\theta^{-1},1)$. Note that $q(\theta) = p(\theta)/\theta$. The difference between *meetings* and successful *matches* is worth noting. Once a worker and a job meet, they can decide whether to start production or not. Some meetings may not end up with a successful match if the agents prefer to continue searching. Jobs are destroyed exogenously with a Poisson rate $\sigma$. In addition, at rate $\tilde{\sigma}$ workers receive the opportunity to quit voluntarily if they find it optimal. We abstract from on-the-job search. Wage is determined by Nash bargaining with worker bargaining power denoted $\eta$.

*Incomplete Market.* There is not a complete set of Arrow securities. Instead, there is only one asset that agents can save at a risk-free rate $r$ to smooth consumption against fluctuations in labor income. Workers face a borrowing constraint $\underline{a}$. There is an annuity market such that the insurance company collects the wealth when people die, and pays people alive an annuity income flow $\delta a$. As a result, the effective return obtained by households is $r + \delta$.

## 2.2 Characterization

### 2.2.1 Distribution

Before characterizing the value functions, it proves useful to define several relevant measures. The population distributions over worker types and job types are given by $d_w(x)$ and $d_j(y)$, respectively. For the convenience of notations, we refer to matches as $m$, employed workers $e$, unemployed workers $u$, producing jobs $p$, and vacant jobs $v$, all using the first letter the words. For example, the density function of producing matches is denoted $d_m(a,x,y) : \mathbb{R} \times \mathbb{X} \times \mathbb{Y} \to \mathbb{R}_+$. We could define other densities in a similar fashion, with density of employed workers $d_e(a,x) = \int d_m(a,x,y)\,\mathrm{d}y$, density of unemployed workers $d_u(a,x)$, density of producing jobs $d_p(y) = \iint d_m(a,x,y)\,\mathrm{d}a\mathrm{d}x$, and density of vacant jobs $d_v(y) = d_j(y) - d_p(y)$. Notice that the aggregate unemployment and vacancy are given by $u = \iint d_u(a,x)\,\mathrm{d}a\mathrm{d}x$ and $v = \int d_v(y)\,\mathrm{d}y$, respectively. These add-up properties are summarized in Table 1.

### 2.2.2 Hamilton-Jacobi-Bellman Equations

**Worker Values** Employed workers make decisions on consumption-savings and whether to separate from the current job.[6] Let $U(a,x)$ denote the value of an unemployed worker of type $x$ with with wealth $a$, and $W(a,x,y)$ the value of an employed worker of type $x$ with asset $a$

[6]As we will discuss later, a worker will choose to separate if she accumulates enough assets and the job is close to the edge of the matching set, in which case the worker is better-off quitting and wait for a better match. Moreover, Nash bargaining ensures that workers' quitting decisions align with firms' separation decisions.

Table 1: Distribution Add-up Properties

| Description | Add-up Property |
|---|---|
| Workers | $d_w(x) = \int d_u(a,x)\,\mathrm{d}a + \iint d_m(a,x,y)\,\mathrm{d}y\mathrm{d}a$ |
| Total unemployment | $u = \iint d_u(a,x)\,\mathrm{d}x\mathrm{d}a$ |
| Firms | $d_j(y) = d_v(y) + \iint d_m(a,x,y)\,\mathrm{d}x\mathrm{d}a$ |
| Total vacancies | $v = \int d_v(y)\,\mathrm{d}y$ |

*Notes*: The table summarizes the aggregation properties relating densities $d_u(a,x)$, $d_v(y)$, $u$, $v$ and the match density $d_m(a,x,y)$.

working at a firm of type $y$. The HJB equation for the value of being employed is [7]:

$$\begin{aligned}(\rho+\delta)W(a,x,y) = \max_c\, & u(c) + \sigma\left[U(a,x) - W(a,x,y)\right] + \tilde{\sigma}\left[U(a,x) - W(a,x,y)\right]^+ \\ & + \dot{a}W_a(a,x,y) \\ \text{s.t. }\; & \dot{a} = (r+\delta)a + \omega(a,x,y) - c \\ & a \geq \underline{a}\end{aligned} \tag{1}$$

where $[\bullet]^+ := \max\{\bullet, 0\}$. An employed worker receives flow interest and annuity $(r+\delta)a$ and wage $\omega(a,x,y)$ which is determined by Nash bargaining. At Poisson rate $\sigma$, the match is destroyed for exogenous reasons and becomes unemployed and at rate $\tilde{\sigma}$, she gets the opportunity to quit the job. The optimal consumption-saving decision is characterized by the first order condition

$$\mathfrak{u}'(c^e) = W_a(a,x,y)\,. \tag{2}$$

Unemployment workers make consumption-savings and job acceptance decisions. The Hamilton-Jacobi-Bellman (HJB) equation for the value of being unemployed is:

$$\begin{aligned}(\rho+\delta)U(a,x) = \max_c\, & \mathfrak{u}(c) + p(\theta)\int \frac{d_v(y)}{v}\left[W(a,x,y) - U(a,x)\right]^+ \mathrm{d}y + \dot{a}U_a(a,x) \\ \text{s.t. }\; & \dot{a} = (r+\delta)a + b(x) - c \\ & a \geq \underline{a}\end{aligned} \tag{3}$$

An unemployed worker makes home production $b(x)$ and receives flow interest and annuity $(r+\delta)a$. The unemployed worker meets a vacant job at rate $p(\theta)$, which is randomly sampled from the distribution of all vacancies. The first order condition for the consumption-saving

[7] In Appendix I.1 we provide derivations of value functions.

decision is given by

$$\mathfrak{u}'(c^u) = U_a(a, x). \tag{4}$$

**Firm Values** Let $V(y)$ denote the value of a vacant job of type $y$, and $J(a, x, y)$ the value of a producing job of type $y$, with an employee of type $x$ who has asset $a$. The HJB equation for the producing job is

$$\begin{aligned} rJ(a,x,y) = f(x,y) - \omega(a,x,y) + (\sigma+\delta)[V(y) - J(a,x,y)] + \tilde{\sigma}[V(y) - J(a,x,y)]^+ \\ + \dot{a}^e J_a(a,x,y), \end{aligned} \tag{5}$$

where $\dot{a}^e := (r+\delta)a + \omega(a,x,y) - c^e(a,x,y)$ is the optimal saving policy of the employee. The firm retains the remaining output net of wage paid to the worker. The match can be separated due to job destruction shock, worker exit shock or if the firm finds it optimal to separate and wait for a better match.

The value of a vacant job is

$$rV(y) = q(\theta) \iint \frac{d_u(a,x)}{u} [J(a,x,y) - V(y)]^+ \, dadx. \tag{6}$$

The vacancy meets an unemployed worker at rate $q(\theta)$ that is randomly drawn from the distribution of all unemployed workers.

The mass of jobs is determined by a free-entry condition. We assume that entrepreneurs need to pay a fixed entry cost of $\kappa$ before the skill requirement type $y$ is realized according to a cumulative density function $G(.)$.

$$\kappa = \int V(y) \, dG(y). \tag{7}$$

In equilibrium, the value of firms adjust so that the expected value of entry is zero.

### 2.2.3 Wage Determination

Wages are determined by Nash bargaining, denoted by $\omega(a, x, y)$. Appendix I.2 proves that the Nash solution can be characterized by

$$\eta \frac{J(a,x,y) - V(y)}{1 - J_a(a,x,y)} = (1-\eta) \frac{W(a,x,y) - U(a,x)}{W_a(a,x,y)}, \tag{8}$$

where $\eta \in (0, 1)$ represents the bargaining power of the worker. In addition, we can derive an expression for wages (Appendix I.2, equation (A2)) which can be easily computed once we obtain the value and policy functions of workers and firms.

To gain intuitions, it is useful to contrast our result with common Nash solutions in the case of linear utility. In an environment with linear utility, the match surplus is defined by the sum of worker's surplus and the job's surplus:

$$S\left(a, x, y\right) \coloneqq W(a, x, y) - U\left(a, x\right) + J\left(a, x, y\right) - V\left(y\right).$$

Then Nash bargaining works in a way that the worker and the job are splitting the match surplus according to $\eta$. However, it does not make sense to directly add up the worker surplus and the firm surplus if they are not measured in the same units, as is in the case when we have curvature in the utility function. In particular, once we introduce curvature in the flow utility function, worker values are measured in present discounted util, while firm values are measured in present discounted numeraire. It turns out that $W_a$ and $1 - J_a$ are the right adjustment terms so that we could add up the adjusted worker value and firm value. That is, consider the adjusted surplus

$$\hat{S}\left(a, x, y\right) \coloneqq \frac{1}{W_a(a, x, y)}\left[W(a, x, y) - U\left(a, x\right)\right] + \frac{1}{1 - J_a(a, x, y)}\left[J\left(a, x, y\right) - V\left(y\right)\right]. \tag{9}$$

The worker and the firm are splitting the adjusted surplus according to the bargaining power $\eta$.

It is obvious that $W_a$ properly measures the marginal value of a dollar to the worker. Now we illustrate the intuition why $1 - J_a$ is the right adjustment term for the firm. Think of the scenario of a marginal dollar transfer between the worker and the firm. If the worker transfers one additional dollar to the firm, there is a direct 1 dollar increase in firm's value and an indirect impact to the firm through asset decumulation of the worker, i.e., $-J_a$. Thus the total marginal value of additional dollar to the firm is properly captured by $1 - J_a$.

From equations (8) and (9) it is easy to see that worker-job matching and separation decisions are privately efficient, in the sense that workers' surplus from the match is positive whenever firms' surplus (and total surplus) is positive, and vice versa. Note that the separation decision is just the flip side of the acceptance decision: a match would end if the worker and job post separation would not have agreed to form the match.

In the formal proof in Appendix I.2, we write down the full Nash problem by defining values of deviating wages with tilde notations, e.g., $\tilde{W}\left(w, a, x, y\right)$. We show that

$$\frac{-\tilde{J}_w(w, a, x, y)}{\tilde{W}_w(w, a, x, y)} = \frac{1 - J_a(a, x, y)}{W_a(a, x, y)},$$

which implies that the adjusted surplus could alternatively be written as

$$\hat{S}\left(a,x,y\right) \coloneqq \frac{1}{\tilde{W}_w}\left[W(a,x,y)-U\left(a,x\right)\right]+\frac{1}{\left(-\tilde{J}_w\right)}\left[J\left(a,x,y\right)-V\left(y\right)\right].$$

This provides further intuition to the bargaining solution – the worker's surplus is adjusted by $\tilde{W}_w$ to the dollar value, and the firm's surplus is adjusted by $\left(-\tilde{J}_w\right)$ to the dollar value. Workers and firms split the adjusted surplus.

Finally, notice that as the curvature of the utility function goes to 0, i.e., as the utility function goes to linear, then $W_a = 1$ and $J_a = 0$. In this case, our adjusted surplus collapses to the standard definition of surplus.

**Discussion on the assumption of observable wages**

### 2.2.4 Steady State

We consider a stationary equilibrium. The steady state could be characterized by two sets of Kolmogorov Forward (KF) equations. Let $\Phi\left(a,x,y\right)$ denote the matching decision of workers/firms which equals 1 if the match is formed and 0 otherwise, and $1-\Phi\left(a,x,y\right)$ is thus workers'/firms' separation decision. The first one characterizes the inflow-outflow balancing equation for employed workers $d_m\left(a,x,y\right)$, i.e.,

$$\begin{aligned}0=&-\frac{\partial}{\partial a}\left[\dot{a}^e\left(a,x,y\right)d_m\left(a,x,y\right)\right]-\{\sigma+\delta+\tilde{\sigma}\left[1-\Phi\left(a,x,y\right)\right]\}d_m\left(a,x,y\right)\\&+d_u(a,x)p\left(\theta\right)\frac{d_v\left(y\right)}{v}\Phi\left(a,x,y\right),\end{aligned}\tag{10}$$

for all $a, x, y$. Outflow from the employed state $(a, x, y)$ occurs due to asset accumulation and separation (both exogenous and endogenous), and inflow occurs due to job acceptance. The second one characterizes the inflow-outflow balancing equation for unemployed workers $d_u(a, x)$, i.e.,

$$\begin{aligned}0=&-\frac{\partial}{\partial a}\left[\dot{a}^u\left(a,x\right)d_u\left(a,x\right)\right]-\int p\left(\theta\right)\frac{d_v\left(y\right)}{v}\Phi\left(a,x,y\right)d_u(a,x)\mathrm{d}y-\delta d_u\left(a,x\right)\\&+\sigma\int d_m\left(a,x,y\right)\mathrm{d}y+\tilde{\sigma}\int\left[1-\Phi\left(a,x,y\right]\right)d_m\left(a,x,y\right)\mathrm{d}y+\delta d_x\cdot\mathbb{1}\{a=0\},\end{aligned}\tag{11}$$

for all $a, x$. Similarly, outflow from the unemployed state occurs due to asset decumulation, job finding and exit shock, and inflow occurs due to separation of employed workers. For zero-asset state, inflow also happens due to new-born workers.

In addition, there is an add-up condition that density integrates to 1:

$$1=\int_{\underline{a}}^{\infty} d_m(a,x,y)\,dadxdy+\int_{\underline{a}}^{\infty} d_u(a,x)\,dadx \tag{12}$$

as well as

$$d_x=\int_{\underline{a}}^{\infty} d_m(a,x,y)\,dady+\int_{\underline{a}}^{\infty} d_u(a,x)\,da \tag{13}$$

$$d_y=\int_{\underline{a}}^{\infty} d_m(a,x,y)\,dadx+d_v(y) \tag{14}$$

## 2.3 Equilibrium

For the sake of tractability and in order to focus on sorting between workers and firms, we assume an open asset market so that the economy takes the interest rate as given[8]. We also assume that the marginal distributions of workers and firms over production types are predetermined.

### 2.3.1 Formal Equilibrium Definition

Given interest rate $r$ and marginal densities of worker and firm types $d_x$, $d_y$, a stationary equilibrium consists of a set of value functions $\{W(a,x,y), U(a,x), J(a,x,y), V(y)\}$ for employed workers, unemployed workers, producing jobs, and vacant jobs, respectively; a set of policy functions including consumption policy $\{c^e(a,x,y), c^u(a,x)\}$ and matching acceptance decision conditional on meeting $\Phi(a,x,y)$; a wage policy $\omega(a,x,y)$; and an invariant distribution of employed workers $d_m(a,x,y)$ and unemployed workers $d_u(a,x)$, and market tightness $\theta$ such that [9]:

1. The value functions and policy functions solve worker and firm's optimization problem (1, 3, 5, 6);

2. Wage setting and matching acceptance decision satisfy Nash bargaining (8);

3. The stationary distributions satisfy the Kolmogorov Forward equations and add-up conditions (10 - 14);

4. Market tightness adjusts so that free entry condition in equation (7) gives zero economic profits to firms prior to entry.

[8]In the calibration exercise, we treat workers' savings as liquid assets, which is just a small fraction of total wealth. Therefore the assumption of a fixed interest rate is not completely divorced from reality.

[9]Without loss of generality we assume measure 1 of workers and firms in steady state so that $\theta = 1$

### 2.3.2 Model Outputs

This model provides a joint characterization of employment, wages, and wealth distributions.

First, it characterizes standard labor market variables of interest. Since the baseline model assumes an exogeneous separation rate, it is thus the job losing rate in the economy $\pi_{eu} = \sigma$. The job finding rate (not job meeting) in the economy is

$$\pi_{ue} = p(\theta) \int \frac{d_v(y)}{v} \Phi(a, x, y) \, \mathrm{d}y.$$

The steady state unemployment rate is given by

$$u = \frac{\sigma}{\sigma + \pi_{ue}},$$

which is known as the Beveridge curve.

Second, the model allows for a joint characterization of wage and wealth distributions. Specifically, the joint distribution of wealth and wage (among employed workers) is characterized by

$$h(a, w) = \frac{1}{e} \iint d_m(a, x, y) \mathbb{1}\{\omega(a, x, y) = w\} \, \mathrm{d}x\mathrm{d}y.$$

Third, the model allows us to study the determinants of aggregate output and productivity. The total output and measure of employed is

$$\begin{aligned} y &= \iiint f(x, y) \, d_m(a, x, y) \, \mathrm{d}a\mathrm{d}x\mathrm{d}y, \\ e &= \iiint d_m(a, x, y) \, \mathrm{d}a\mathrm{d}x\mathrm{d}y \end{aligned}$$

and average output per employed (i.e. productivity) is $\overline{y} = y/e$. Since wealth distribution affects the allocation of worker types $x$ to firm types $y$ (as we will show below), our model allows us to study how wealth inequality affects labor market sorting and labor misallocation, characterized by output loss or productivity loss relative to a benchmark economy (e.g. without search friction or with complete markets).

## 2.4 Algorithm

Consider grids $\{a_1, a_2, \ldots, a_{N_a}\}$ for asset, $\{x_1, x_2, \ldots, x_{N_x}\}$ for skills, $\left\{y_1, y_2, \ldots, y_{N_y}\right\}$ for skill requirements. Suppose they are equally spaced (probably assets are log-spaced) and $\Delta_a, \Delta_x, \Delta_y$ are the steps.

Since we only consider a stationary equilibrium, we normalize $d_x, d_y$ to be uniform on $[0,1]$.

1. Guess $\theta$ and $d_v(y_k)/v$

   We can start by guessing, for example, that $\theta = 0.8$ and $d_v(y_k)/v = 1/N_y$ for all $k = 1, ..., N_y$.

2. Guess bargaining solution for each pair $w(a_i, x_j, y_k)$.

   We can start from $w(a, x, y) = \gamma f(x, y)$, a fraction of the flow profit.

   Solve the worker's problem using the implicit method as in Achdou et al. (2022) (see Appendix II for details).

3. Calculate the stationary distribution of workers.

   Discretize the Kolmogorov forward (KF) equation using the upwind scheme. We can then write the system of KF equations compactly in matrix form:

   $$\mathbf{A}(\mathbf{W}^n)' d = 0$$

   and the scale of the worker density is pinned down by the fact that $d$ sums up to 1. $d_e(a, x, y)$ and $d_u(a, x)$. $d_v(y) = 1/N_y - \int d_e(a, x, y)\,\mathrm{d}a\mathrm{d}x$

4. Solve the firm's problem. The discretized HJB equation for firm is

   $$\rho J_i^{jk} = f^{jk} - w_i^{jk} - \delta J_i^{jk} + s_i^{jk,W} J_{a,i}^{jk}$$

   Update the value function by

   $$\begin{aligned}\frac{J^{l+1} - J^l}{\Delta} &= f^{jk} - w_i^{jk} + \mathbf{A}_{1,e} J^{l+1} - (\rho + \delta) J^{l+1} \\ \Rightarrow J^{l+1} &= \left[\left(\frac{1}{\Delta} + \rho + \delta\right) - \mathbf{A}_{1,e}\right]^{-1} \left(f^{jk} - w_i^{jk} + \frac{1}{\Delta} J^l\right)\end{aligned}$$

5. Update wage schedule according to the expression given by equation (A2) in Appendix I.2.

6. update $\theta$ and $dv$, go back to 1.

7. Derive other densities and update guess in step 1.

Detailed algorithm is shown in Appendix II.

# 3 Theoretical Results

## 3.1 Two Limiting Economies

Our model provides a unified framework of incomplete market and frictional sorting. It is a generalization that nests Krusell et al. (2010) and Shimer and Smith (2000).

If worker's preference is risk neutral, i.e., if the flow utility function $\mathfrak{u}$ is linear in consumption $c$, then our model becomes the frictional sorting model a la Shimer and Smith (2000). Alternatively, if workers have access to a full set of Arrow securities (i.e., complete market), the model becomes Shimer and Smith (2000). In either case, asset level will not affect optimal decision and becomes irrelvant to decision making.

If $\mathbb{X}$ and $\mathbb{Y}$ are singletons, then we have homogenous workers (in terms of the skills; workers are still heterogenous with respect to wealth) and firms. There is no sorting so to speak. The model becomes a standard Bewley (1977); Huggett (1993); Aiyagari (1994) type incomplete market model with Diamond-Mortensen-Pissarides search frictions. This has been explored by Krusell et al. (2010).

## 3.2 Wealth, Job Search and Wages

In this section we discuss several key implications from the model about the interactions between wealth, job search strategies and wages. These results will help us understand the mechanisms through which wealth affects labor market allocation and output, and how the model generates an endogenous joint distribution of wealth and wages.

**Proposition 1.** ***Precautionary Mismatch.*** *The matching set is wider for lower-asset holders. Fix worker type $x$ and job type $y$. If $a$ is the marginal wealth level such that the adjusted match surplus is exactly zero, then wealthier workers reject the match while poor workers accept the match. Formally, fix arbitrary $x$ and $y$, if $\hat{S}(a, x, y) = 0$, then $\hat{S}(a', x, y) < 0$ for any $a' > a$, and $\hat{S}(a'', x, y) > 0$ for any $a'' < a$.*

*Proof:* See Appendix I.4.

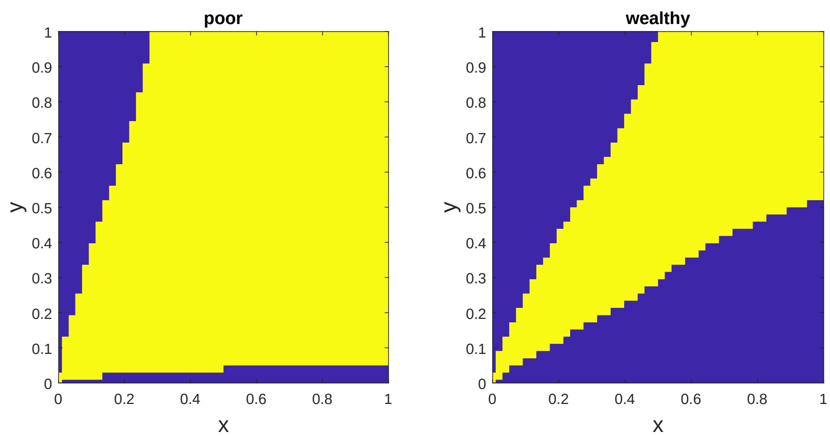
It is worth noting that this result does not rely on the properties of production function or sorting patterns. As long as workers have precautionary motives for self-insurance, matching sets will always be larger for lower-asset workers. In Figure 1, we demonstrate two cases of matching sets, depicted by the yellow areas. Panel (1a) shows matching sets when the production function is supermodular[10]. In this case, workers and firms are heterogeneous in

[10] $f(x, y) = f_0 + f_1 \left(x^\xi + y^\xi\right)^{1/\xi}$, $0 < \xi < 1$.

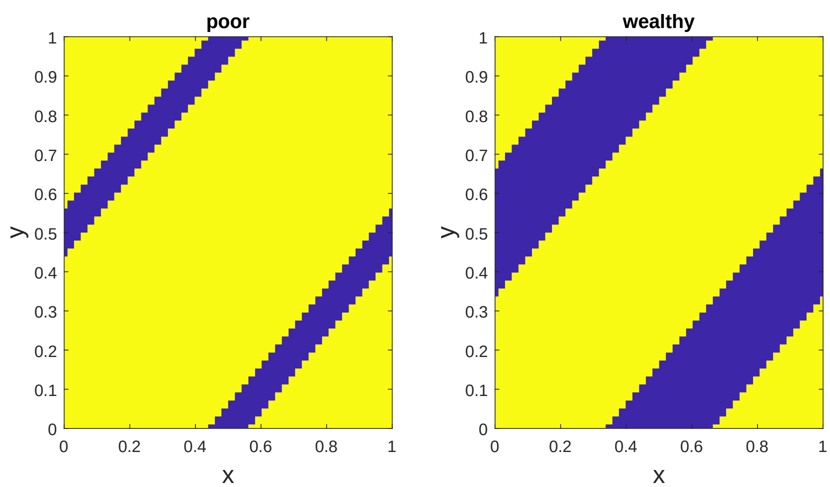
terms of the level of skills/skill requirements (i.e. vertical heterogeneity), and the allocation features positive assortative matching (PAM) so that matches between workers (horizontal axis) and firms (vertical axis) happen along the 45-degree line. Panel (1b) shows the case where the production function is circular [11], which means that workers and firms are located on a unit-circle and are heterogeneous only in terms of the type of skills/skill requirements but not levels (i.e. horizontal heterogeneity). In this case, matches happen where worker and firm skill types are close to each other on the unit-circle. In both cases however, there is a range of acceptable matches around the perfect matches due to search friction. The left figures of each panel show the matching sets for workers with the lowest wealth in our model, and the right figures correspond to workers with assets worth of 5 times yearly earnings. Indeed, low-wealth workers have wider matching sets regardless of the property of the matching functions.

Figure 1: Matching Sets

(a) Vertical Heterogeneity




(b) Horizontal Heterogeneity




**Proposition 2.** ***Wealth and Job Finding Rate.*** *Job finding rate* $\pi_{ue}(a, x)$ *is decreasing with*

[11] $f(x, y) = a - b \min\left(|x-y|, |1+x-y|, |1+y-x|\right)^2$.

*respect to wealth $a$.*

*Proof*: See Appendix I.5.

Proposition 2 is a direct corollary of Proposition 1: as wealth-poor workers have wider matching sets (given production type), they are more likely to find successful matches, and therefore have higher job finding rates.

**Proposition 3.** ***Wealth and Wages***. *Average wages $\bar{w}(a,x) = \int w(a,x,y)\,\Phi(a,x,y)\,\frac{d_v(y)}{y}dy$ conditional on finding a job is increasing in wealth $a$.*

We provide some intuitions rather than a formal proof for this result, and for conciseness we focus on the case with horizontal heterogeneity. Note, however, that the result is also true under other sorting patterns. In Figure 2, we plot the wage functions under a perfect match ($x = y$) in blue lines and a marginal match in red lines, which occurs at the edge of matching sets. For a marginal match, the worker and the firm are indifferent between accepting and staying unemployed/vacant.

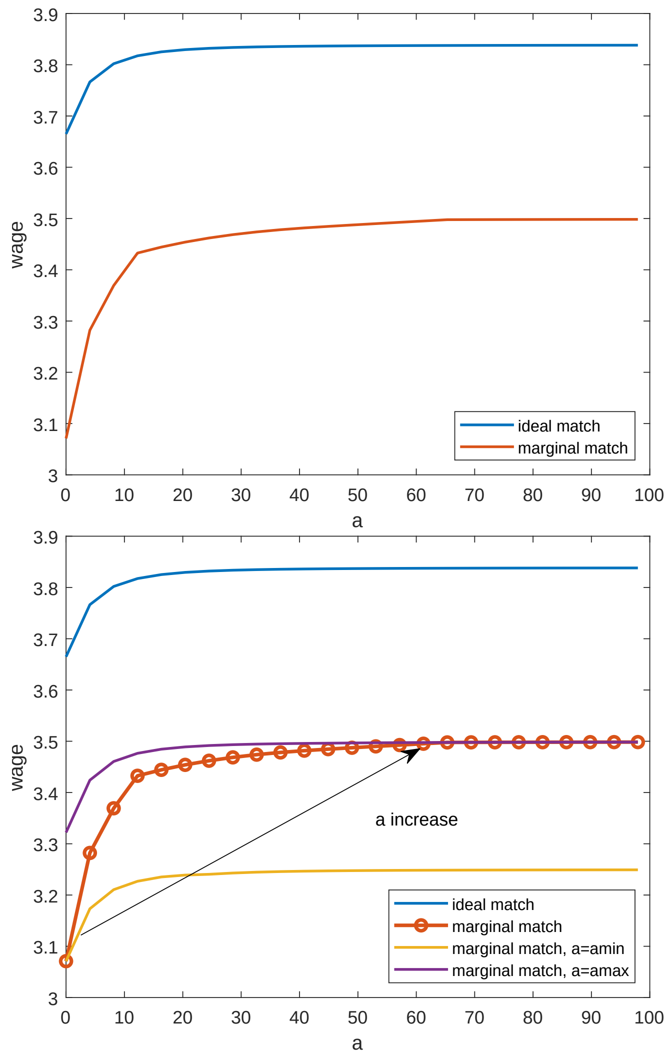
The top panel of Figure 2 shows that wages for both the perfect match and the marginal match increases with wealth. For perfect match, wage increases with wealth for the same reason as in Krusell, Mukoyama and Şahin (2010): the outside option, namely the value of being unemployed, increases quickly with wealth as precautionary motive dissipates, so that wealthier workers bargain for higher wages. For marginal match, wage increases for two reasons, as shown in the bottom panel of Figure 2. First, for any given match (purple or yellow line), wage increases with wealth due to Nash bargaining. Second, as workers become wealthier, their matching set shrinks, thus the marginal match gets closer to the perfect match. Therefore as wealth increases, wage not only increases along the marginal match wage curve but also across different marginal matches (the red dots correspond to points on the wage curves of different marginal matches). Since average wages lie in between the wage functions of the perfect match and the marginal match, we can conclude that average wages must also increase with wealth.

**Proposition 4.** ***Optimal Consumption Growth.*** *The Euler equations, which specify the optimal consumption growth paths for employed and unemployed workers, can be written as follows:*

$$\frac{\dot{c}^e}{c^e} = \frac{1}{\gamma}\left\{r - \rho + \omega_a + \sigma\left[\frac{u'(c^u)}{u'(c^e)} - 1\right]\right\}$$
$$\frac{\dot{c}^u}{c^u} = \frac{1}{\gamma}\left\{r - \rho - p(\theta)\int_{B(a,x)} \frac{d_v(y)}{v}\left[1 - \frac{u'(c^e)}{u'(c^u)}\right]\mathrm{d}y\right\}$$

*where arguments of $\omega(a,x,y), c^u(a,x,y), c^e(a,x,y)$ are suppressed for brevity. $B(a,x) := \{y : \Phi(a,x,y) = 1\}$ is the acceptance set of worker $(a,x)$.*

Figure 2: Wealth and Wages




*Proof*: See Appendix I.6.

The equations show us the reasons behind workers' saving decisions. First, there is a standard saving motive due to the difference in interest rate $r$ and the rate of time preference $\rho$. For employed workers, there are additional saving motives due to the effect of wealth on wages $w_a$, and the possibility of job loss. The term in square bracket corresponds to precautionary savings motive, which is particularly strong when wealth is low or current wages are high. For unemployed workers, there is a dis-saving motive due to the possibility of finding a job. Notably, the dis-saving motive depends on the worker's acceptance set $B(a, x)$. Everything else equal, a larger acceptance set induces unemployed workers to dis-save more. This term would be absent in models without endogenous job-finding strategies and two-sided heterogeneity.

These propositions show the reasons why our model generates an endogenous joint distribution of wealth and wages. Wealth affect wages by increasing workers' outside options and allowing job-searching workers to wait for better matches, while wages affect wealth by affecting the flow of income and saving rates. The joint distribution would be absent or degenerate in most existing models.

# 4 Data

Our empirical analysis is based on a selected worker panel from the 1979 National Longitudinal Survey of Youth (NLSY79), a nationally representative survey conducted on individuals 14-22 years old when first interviewed in 1979. We merge the NLSY79 work history and asset information with data from the Occupational Information Network (O*NET), an occupation-level data set with scores on the skill contents of 974 occupations, so that we have a matched worker-job data set with joint worker and job characteristics. In the sections below we provide a description of the data sources, how the measures of worker skills ($x$) and job skill requirements ($y$) are constructed, as well as some sample statistics.

## 4.1 Data Sources and Skill Measures

### NLSY79

We use the work history data from NLSY79 to construct a monthly panel, and focus on a cross-sectional sample of workers with no experience of serving in the military. We further exclude individuals who are already considered to be in the labor market at the beginning of the survey, where we consider an individual to be in the labor market if they work more than 30 hours per week or 1200 hours per year, or if they have finished their last schooling spell and started working. To minimize the effect of work experience gained during education on our estimation, we also exclude those who have more than 2 years of work experience before the end of his/her schooling spell.

Worker skill measures ($x$) are constructed using individual characteristics extracted from the test scores of the Armed Services Vocational Aptitude Battery (ASVAB), a special survey conducted by the US Departments of Defense and Military Services in 1980 that evaluates individuals in 10 categories. As the test was conducted before the majority of the respondents entered the labor market, we believe that our skill measure is mostly free from the endogeneity issue wherein jobs affect worker skills.

To construct skill bundles for workers, we follow a procedure similar to the one used in Lise and Postel-Vinay (2020) (see Appendix III.1 for details): we run principal component analysis (PCA) on the 10 individual-level ASVAB test scores and keep the first two principal components. We then construct worker skills along 2 dimensions, namely cognitive and manual skills by recombining the principal components so that they satisfy the following exclusion restrictions: (1) the ASVAB mathematicas knowledge score only loads on cognitive skill, and (2) the ASVAB automotive and shop information score only loads on manual skill. We then take the percentile ranks of the two principal components, so that the worker skill measures are distributed on a unit-length interval $[0, 1]$.

In addition to work history and test scores, we also obtain annual history on assets from NLSY79. Unfortunately, NLSY79 did not start extensively collecting assets information until 1985, when over half of the respondents had entered the labor market. Therefore our sample is heavily biased towards late-entrants when examining the relationship between liquid wealth and skill mismatch. We construct a measure of liquid wealth of individuals based on the sum of financial assets such as cash, deposit, mutual fund and money market accounts and other assets more than $500, net of debts that are not asset-backed. Since asset information is not updated in each round of survey for most respondents, we linearly interpolate the amount of assets for each individual to maximize the amount of information we can use in our empirical analysis.

### O*NET

The O*NET data contains ratings of importance and level on hundreds of specific aspects, called "descriptors", of each occupation. The descriptors can be summarized by 9 broad categories: skills, knowledge, abilities, work activities, work context, education levels required, job interests, work styles and work values. Following Lise and Postel-Vinay (2020), we keep the level ratings related to descriptors from the first 6 categories, which add up to over 200 descriptors for each occupation.

Similar to the procedure for worker skills construction, we reduce the descriptors to 2 dimensions using PCA and keep the first 2 components. Then, we recover cognitive and manual skill requirements by recombining the principal components in such a way that (1) the mathematics rating only loads on cognitive skill requirements, and (2) the mechanical knowledge rating only loads on manual skill requirements. We take the percentile ranks of the two principal components so that job skill requirements lie on a unit-length interval $[0, 1]$. Therefore, each job can be characterized by a bundle of skill requirements ($y$), in which a higher number in each dimension represents higher requirements of the corresponding skill.

For this paper, however, we only focus on sorting based on cognitive skills. While our structural model can easily account for multidimensional skill types in theory, solving and estimating such a model turns out to be computationally heavy.

## 4.2 Descriptive Statistics

### 4.2.1 Skill Measures and Sorting

Our selected sample includes 3,285 individuals with substantial heterogeneity in levels of education, ranging from no degree to PhD. Presumably, our measure of worker skills and job skill requirements should reflect their relative rankings and productivity in the sample respectively. An obvious way to examine this presumption is to see how the two measures vary by levels of education. Table 2 shows average worker cognitive skills and job cognitive skill requirements by highest degrees at the time of initial labor market entry. Both measures are normalized to a unit-length range $[0, 1]$, where a higher number represents higher cognitive skill/skill requirement.

Table 2: Average Worker Cognitive Skills and Job Skill Requirements, by Level of Education

| | High School | Some College | 2-yr College | 4-yr College | Masters | PhD |
|---|---|---|---|---|---|---|
| Worker Skill ($x$) | 0.388 | 0.456 | 0.543 | 0.684 | 0.714 | 0.770 |
| Job Skill Req ($y$) | 0.272 | 0.304 | 0.336 | 0.425 | 0.474 | 0.504 |
| Observations | 1053 | 233 | 402 | 500 | 323 | 99 |

*Note*: Both $x$ and $y$ are normalized to $[0, 1]$.

Comparison of the skill measures at the lowest and highest education levels (No Degree and PhD) shows that education seems to account for a substantial amount of worker skill heterogeneity, and a modest amount of job skill heterogeneity. It is perhaps not surprising to find that both worker skills (first row) and job skill requirements (second row) increase monotonically with level of education. Therefore at least with respect to skill differences across education groups, our skill measures are able to capture the relative ranking of workers and jobs, as well as positive sorting. However, some questions yet to be answered are whether we can identify sorting beyond sorting on education using the cognitive skill measures, and whether sorting is still positive after controlling for education groups. To answer this question, we show the correlation between job skill requirements and worker skills in Table 3, with and without controlling for worker educations.

We take one observation from each worker-employer match in the data and regress job cognitive skill requirements on worker cognitive skills. Column (1) shows that the correlation between worker skills and job skill requirements are 0.69, which is both statistically and economically significant. To isolate sorting on skills from sorting on education, we perform an additional regression controlling for dummies for years of education. After controlling for education, the remaining correlation is still large and highly significant at 0.51, suggesting a substantial amount

Table 3: Skill Sorting Over Occupations

| | Job Skill Req ($y$) | Job Skill Req ($y$) |
|---|---|---|
| Worker Skill ($x$) | 0.691*** | 0.513*** |
| | (0.020) | (0.027) |
| Education Level | No | Yes |
| Obs | 35616 | 35616 |

*Note*: Standard errors are clustered on occupation level. The sample is taken from the first observations of each worker-firm match.

of sorting on individual skills exists beyond sorting on education.

#### 4.2.2 Initial Liquid Wealth and Worker Characteristics

There are 1,114 individuals with valid information about liquid financial wealth at the time when they entered labor market. We define net liquid wealth as the value of financial assets such as cash, deposit, mutual fund and money market accounts net of debts that are not asset-backed. This measure is supposed to reflect assets that workers can access in a relatively short period of time. Table 4 shows the characteristics of workers upon labor market entry, where the workers are divided into quintiles according to their liquid wealth during the first month of work.

There are substantial heterogeneity in the level of liquid wealth upon labor market entry, ranging from $-7,971 in the lowest quintile to $31,830 in the highest quintile (in 1982 dollars), a difference of almost $40,000. Workers who enter the labor market with higher liquid wealth tend to have higher income, more education, higher age and higher parental income. The only exception is the lowest quintile, where weekly income, years of education, age and parental income are all higher than those in the quintile above. A likely explanation is that the lowest liquid wealth quintile could consist of individuals who borrow substantial amount of debt for their higher education, thereby lowering their initial wealth. Note that the age of labor market entry is highly upward biased (most workers enter labor market in early 20s) because NLSY79 didn't start collecting wealth information until 1985, when half of the sample were above 25. This means that later when we analyze the effect of initial wealth, our sample is biased towards late entrants.

In Section 3.2 we discussed several implications about wealth, job search and wages generated by the model. Now we use the merged NLSY79 and O*NET data to examine whether these implications are supported by empirical evidence.

Table 4: Worker Characteristics by Initial Wealth Quintile

| | Quintile 1 | Quintile 2 | Quintile 3 | Quintile 4 | Quintile 5 |
|---|---|---|---|---|---|
| Net Financial Assets (1000s) | -7.971 | 0.346 | 1.764 | 5.356 | 31.83 |
| Weekly Income | 233.8 | 195.2 | 193.3 | 255.8 | 301.4 |
| Years of Educ | 15.91 | 14.54 | 15.38 | 15.88 | 16.25 |
| Age | 27.48 | 27.09 | 26.98 | 27.68 | 29.13 |
| Male | 0.416 | 0.405 | 0.368 | 0.446 | 0.350 |
| PRTs Annual Income | 19874.5 | 18343.5 | 23147.3 | 25479.3 | 25623.4 |
| Observations | 202 | 200 | 204 | 202 | 203 |

*Note*: Liquid assets, weekly income and parents' annual income are in 1982 dollars.

### 4.3 Precautionary Mismatch

First, let us provide a formal definition of mismatch used for our empirical analysis.

**Definition 4.1.** Mismatch measures

*Let $x_i$ denote the skill level of individual $i$, and $y_j$ denote the skill requirement of job $j$, then we define the* ***mismatch*** *between individual $i$ and job $j$ as*

$$m_{i,j} \equiv y_j - x_i \tag{15}$$

*$m_{i,j} > 0$ means that worker $i$ is under-qualified (or over-employed) for job $j$, and vice versa. We define the* ***magnitude of mismatch*** *between individual $i$ and job $j$ as*

$$mm_{i,j} = |m_{i,j}| \tag{16}$$

We normalize mismatch $m_{i,j}$ so that its average is 0. By doing so we implicitly assume that in aggregate, there is as much over-qualification (i.e. $m_{i,j} < 0$) as under-qualification ($m_{i,j} > 0$) in the labor market. We also re-scale the levels so that the mismatch measure has a unit standard deviation.

**Wealth and Mismatch**

We now document the relationship between workers' liquid wealth holdings and skill mismatch, which was discussed in Proposition 1. Figure 3a shows a binned scatter plot for those who have gone through an unemployment-to-employment (U2E) transition in the past month, where the vertical axis shows the standardized mismatch measure, and the horizontal axis shows workers' liquid wealth holdings in 1982 dollars. We include a set of aggregate and individual-level controls, including a quadratic function of age, work experience, occupation experience, race, gender, level of education, ASVAB scores, national unemployment rate, and the geographic region of the workers.

The fitted line represents the result from the OLS regression, which estimates that a $1,000 increase in liquid wealth is associated with a 0.006-standard-deviation drop in the magnitude of mismatch, and the coefficient is significant at the 0.01 level. This finding is consistent with the theoretical results from Proposition, which says that for wealthier workers, the set of acceptable jobs are smaller, leading to lower levels of mismatch. 1.

## 4.4 Wealth and Job Finding Rate

Next, we examine empirically whether the relationship between wealth and job finding rate, discussed in Proposition 2 of Section 3.2, also holds in the data. Figure 3d shows a binned scatter plot of the lengths of unemployment spells against workers' liquid wealth holdings, with the same set of controls as before. We only include observations at the beginning of each spell to avoid over-weighing long unemployment spells.

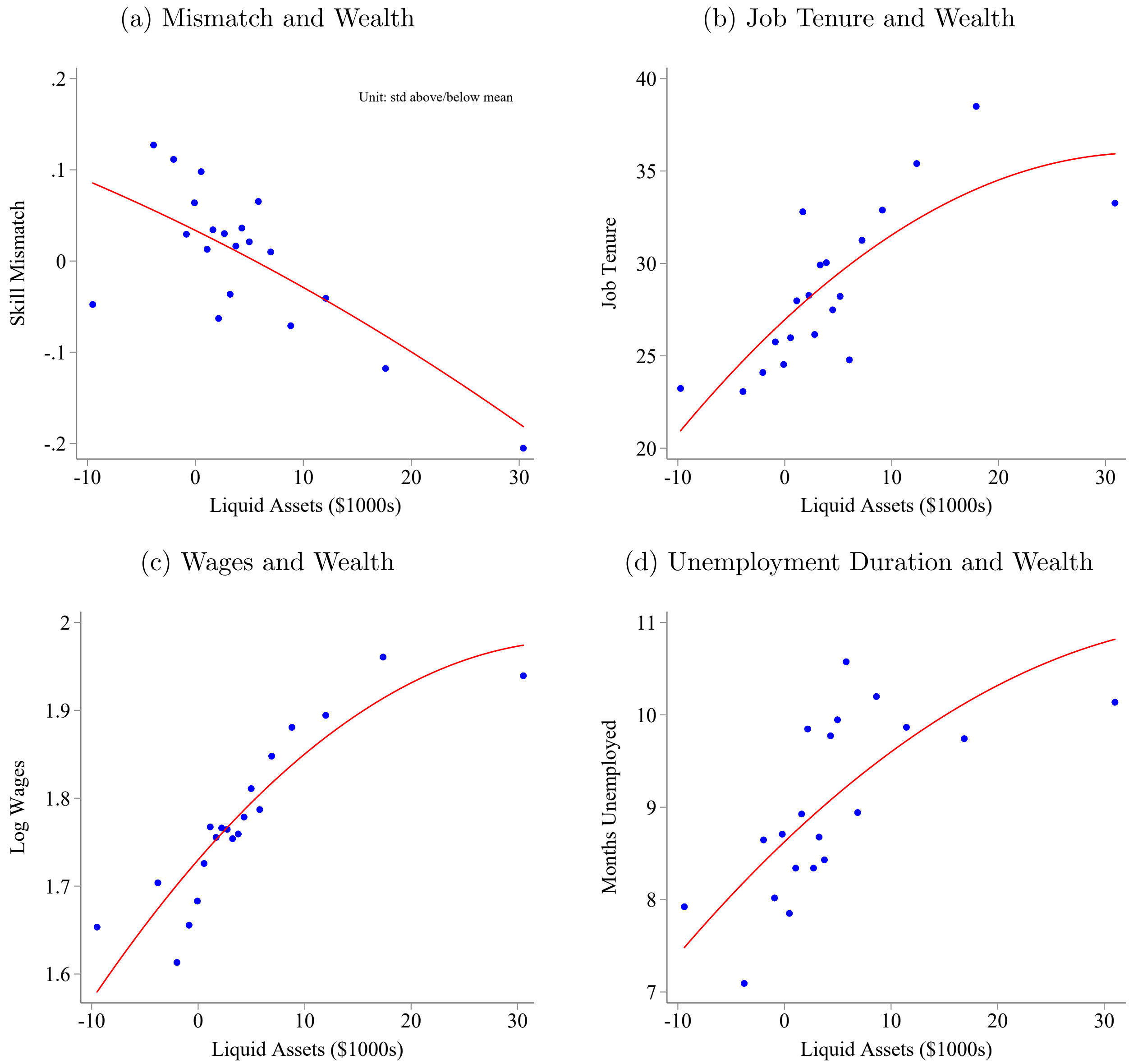
The OLS regression estimates that a $1,000 increase in workers' liquid wealth holdings is associated with a 0.07-month, i.e. a 2-day, increase in duration of unemployment, with the coefficient significant at the 0.01 level. This finding is consistent with Proposition 2, which says that wealthier workers take longer to find a job.

## 4.5 Wealth and Wages

Here we revisit Proposition 3 of Section 3.2 and check whether the model-implied relationship between wealth and wages holds in the data. Figure 3c shows a binned scatter plot of logged wages upon employment against workers' liquid wealth holdings, where the controls include wages from the previous job as well as the ones in the previous regressions.

The OLS regression estimates that a $1,000 increase in liquid wealth is associated with a 0.01% increase in wages upon re-employment, with the coefficient significant at the 0.01 level.

Figure 3: Labor Market Impacts of (Liquid) Wealth




Notes: Source: NLSY79. Controls include age, age squared, work experience, work experience squared, occupation experience, race, gender, education, ASVAB scores, national unemployment rate, region. The wage specification also controls for previous wages but the pattern is barely changed.

This means that the amount of wage dispersion created by wealth dispersion should be positive but small, which is supported by our model as well as Krusell et al. (2010).

# 5 Quantitative Exercises

Having shown that our model's key predictions are qualitatively consistent with data, we now need to estimate the model parameters in order to deliver quantitative results. It is particularly important to characterize the production function since it is the key object that determines the overall sorting pattern, the amount of wage dispersion and mismatch in equilibrium. Our strategy is to make some parametric assumptions about the production function, and discipline the parameters using moments of wage distribution since there is a tight link between them.

## 5.1 Parameterization

We adopt standard functional form assumptions to facilitate numerical analysis. We assume the flow utility function exhibits constant relative risk aversion (CRRA):

$$u\left(c\right)=\frac{c^{1-\gamma}}{1-\gamma}\quad,\quad\gamma>0.$$

The meeting function is assumed to take the Cobb-Douglas form:

$$M\left(u,v\right)=\chi u^{\alpha}v^{1-\alpha}.$$

Without loss of generality, worker and job types are normalized to be uniformly distributed. To see its generality, suppose the $\tilde{F}\left(\tilde{x}\right)$ and $\tilde{G}\left(\tilde{y}\right)$ are the cumulative density functions of the distribution of worker and job types, respectively, with a production function $\tilde{f}\left(\tilde{x},\tilde{y}\right)$. We could redefine a type according to its rank, i.e., $x:=\tilde{F}\left(\tilde{x}\right)$ and $y:=\tilde{G}\left(\tilde{y}\right)$, and rewrite the production function accordingly $f\left(x,y\right):=\tilde{f}\left(\tilde{F}^{-1}\left(x\right),G^{-1}\left(y\right)\right)$. The distribution of the rank-based type is thus uniform, as the CDF of any random variable is uniformly distributed between 0 and 1.[12] We specify a production function that induces positive assortative matching (PAM):

$$f\left(x,y\right)=f_0+f_1\left(x^{\xi}+y^{\xi}\right)^{\nu/\xi},\ 0<\xi<1 \tag{17}$$

[12]To see this, denote the transformed cumulative distribution functions as $F$ and $G$ such that $x\sim F$ and $y\sim G$. Consider an arbitrary $t\in[0,1]$. We have

$$F\left(t\right)=\mathbb{P}\left(x\leq t\right)=\mathbb{P}\left(\tilde{F}\left(\tilde{x}\right)\leq t\right)=\mathbb{P}\left(\tilde{x}\leq s,\ \text{for some } s\in\tilde{F}^{-1}\left(t\right)\right)=t.$$

Therefore $x\sim\mathcal{U}\left[0,1\right]$. Similarly, $y\sim\mathcal{U}\left[0,1\right]$.

$\xi$ controls the degree of complementarily between worker skills $x$ and job skill requirements $y$. A less positive $\xi$ leads to stronger complementarity. Empirical evidence in Hagedorn et al. (2017) supports PAM as a description of data. It is also intuitively correct to allow different workers and firms to have different levels of skills/skill requirements. $\nu$ controls the curvature of the production function across different skill levels. A larger $\nu$ extends the right tail of the production function and thus increases wage inequality between high- and low-skilled workers.

We allow home production $b$ to be a function of worker type $x$ in our model. In the quantitative exercise, we assume that $b(x)$ is a fraction of each worker type's lowest market output, i.e. $b(x) = b_0 f(x, 0)$ such that $b_0 \in [0, 1]$. This assumption makes sure that there are no inactive job and worker types in our model.

## 5.2 Calibration

We set the borrowing constraint at $\underline{a} = 0$ meaning that workers cannot have negative net worth. For computation, we use grids with 200 asset grid points, 20 worker types and 20 job types. We assume that in the stationary equilibrium, the vacancy posting cost is such that there is the same number of jobs as workers.

Our model is calibrated to match aggregate U.S. data on a quarterly frequency. Table 5 summarizes the parameter values. Some parameters are set as standard values in the literature, while others are calibrated internally in the model. In particular, we calibrate the PAM production function to match moments of wage dispersion in the data. We normalize $f_0$ to 1. Since $\xi$ affects the strength of sorting, we set its value to match the degree of frictional wage dispersion as measured by Hornstein et al. (2011). The frictional wage dispersion is a measure of wage variations that cannot be explained by worker characteristics and thus captures the size of matching sets. $\nu$ determines the curvature of the production function and is calibrated to match the Lorenz Curve of U.S. wage distribution. Then, we calibrate $f_1$ jointly with the home production parameter $b_0$ so that 1. on average home production is 40 percent of wages, following Shimer (2005), and 2. the 90th percentile of wages is 10 times the 10th percentile of wages, as calculated from the SCF.

We think that the most relevant form of wealth in our model should be liquid wealth (liquid assets including checking, saving and money market accounts net of liquid debts including credit card and personal loans), since illiquid wealth such as housing, pension and businesses cannot be easily withdrawn and would thus have no direct effect on consumption smoothing and households' precautionary motive. Therefore, we choose the discount rate $\rho$ to match the liquid wealth to earnings ratio calculated from the SCF.

Table 5: Calibration

| Parameter | Value | Source |
|---|---|---|
| | *External Calibration* | |
| interest rate | $r = 0.005$ | annual interest rate 2% |
| relative risk aversion | $\gamma = 2$ | common parameterization |
| bargaining power | $\eta = 0.72$ | Shimer (2005) |
| matching elasticity | $\alpha = 0.72$ | Hosios condition |
| dissipation rate | $\delta = 0.0056$ | 45-year expected working life |
| separation rate | $\sigma = 0.0944$ | monthly job losing rate 0.034 |
| quitting opportunity | $\tilde{\sigma} = 3$ | monthly adjustment |
| | *Internal Calibration* | |
| discount rate | $\rho = 0.012$ | liquid wealth/annual earnings ratio 0.56 |
| production function | $f_0 = 1$ | normalization |
| home production | $b_0 = 0.75$ | avg $b(x)/w(x,y) = 40\%$ |
| production function | $f_1 = 1.78$ | wage 90-10 ratio = 10 |
| production function | $\xi = 0.8$ | frictional wage dispersion |
| | | Hornstein et al. (2011) |
| production function | $\nu = 3$ | U.S. wage Lorenz Curve Lise (2013) |
| matching efficiency | $\chi = 4.7$ | monthly job finding rate 0.45 |

*Notes*: The table summarizes the calibrated parameters and their sources.

## 5.3 Model vs. Data

### 5.3.1 Liquid Wealth Distribution

Although we do not target liquid wealth distribution in the calibration, our model endogenously generates a distribution of liquid wealth as a result of precautionary motives. One way to lend some credibility to the mechanisms in our model is to compare the model-generated distribution of liquid wealth against the data. A close fit with the data increases our confidence in the amount of precautionary motive generated by the model as well as the effects of redistributive policies that we will discuss later.

In SCF we define liquid wealth as the sum of all liquid assets (checking and saving account, money market account) net of all liquid debts (non-mortgage debts). We focus on households with non-negative liquid wealth and whose household heads are aged between 20 to 65 years old. We also exclude households at the top 10% of the liquid wealth distribution.

Figure 4: Liquid Wealth Distribution Model vs. Data

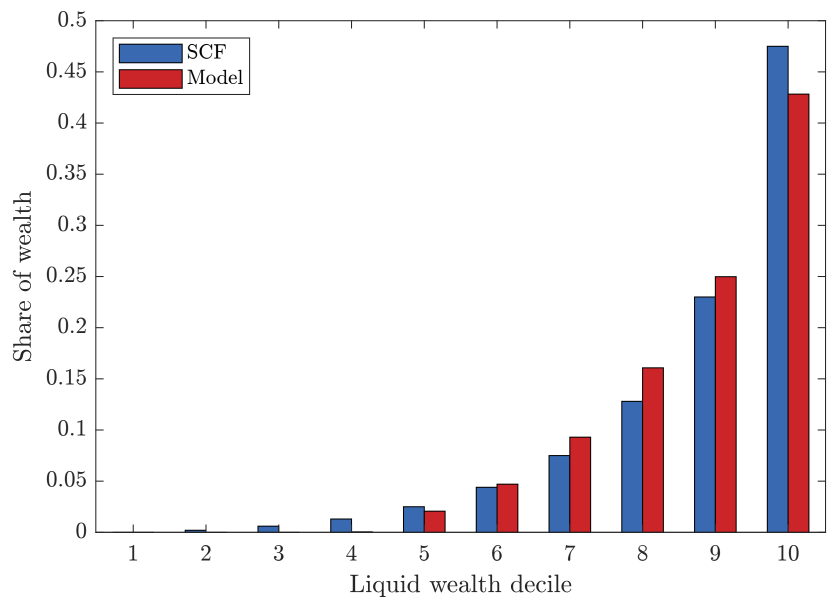
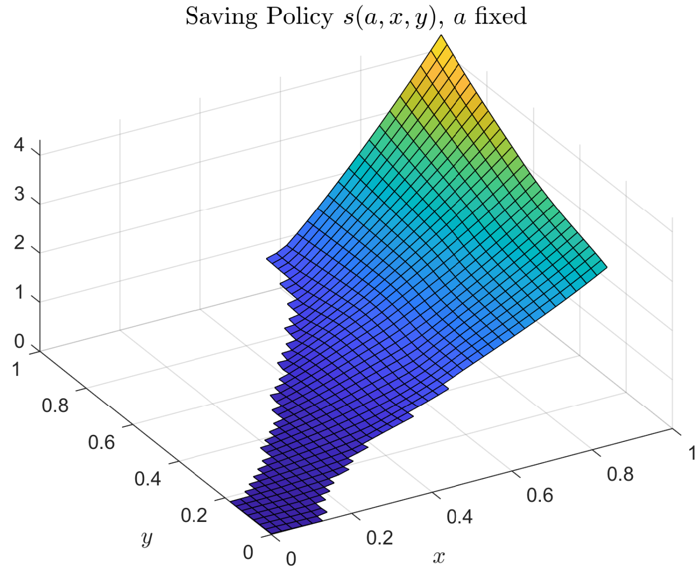



Figure (4) shows the share of liquid wealth accounted for by each quintile in the model (red) versus in the SCF data (blue). The model provides a good fit of the empirical liquid wealth distribution for the majority of the population.

The reason why the model generates substantial wealth dispersion lies in the saving rate heterogeneity which is caused by workers' varying degress of mismatch. Figure 5 shows the saving policy for different $(x, y)$ matches at an arbitrary wealth level. It shows that fixing wealth, higher $x$ workers tend to have higher savings flow, and workers at better-matched jobs tend to save more, as they are at the top of their job ladders.

Figure 5: Saving Rate Heterogeneity




#### 5.3.2 Model Elasticities

Another key feature of the data which lends credibility to our model is the rates at which job finding rates and wages upon re-employment vary with liquid wealth levels. These are important moments that reflect the strength of workers' precautionary motive, and how it affects their search decisions. We use monthly labor market history from NLSY79 to construct empirical moments. For job finding rates, we perform the following logit regression estimate

$$Pr(U2E_{i,m} = 1) = F\left(\alpha_1 + \beta_1 a_{i,m} + \mathbf{X'}_{i,m}\beta_{\mathbf{1},\mathbf{x}} + \epsilon_{1,i,m}\right)$$

where $F$ is the logit function, $i$ denotes individuals, $m$ denotes month, and $\mathbf{X}$ includes a variety of controls including a quadratic function of age, and quadratic function of work experience, race, gender, education, AFQT score, and national unemployment rate. The observations are taken during workers' unemployment spells.

For wages at re-employment, we perform the following OLS regression

$$\log w_{i,m} = \alpha_2 + \beta_2 a_{i,m} + \mathbf{X'}_{i,m}\beta_{\mathbf{2},\mathbf{x}} + \epsilon_{2,i,m} \quad \text{if } U2E_{i,m} = 1$$

where observations are taken at the months when workers experience a $U2E$ transition.

We perform corresponding regression estimates in the model where the controls are dummies for workers' skill types. Table (6) shows the comparison between the regression estimates from the model and NLSY79 respectively. The close alignment shows the the model is able to replicate the effect of precautionary motive on job finding rates and re-employment wages as seen in the data.

Table 6: Elasticities w.r.t $\log(a + \sqrt{1 + a^2})$

| | Model | NLSY79 | Obs. |
|---|---|---|---|
| U2E: $\beta_1$ | -0.193 | -0.115*** | 56,786 |
| | | (0.021) | |
| $\log(w)$: $\beta_2$ | 0.090 | 0.096*** | 5,508 |
| | | (0.006) | |

Standard errors in parentheses, * p<0.10, ** p<0.05, *** p<0.01

## 5.4 Aggregate Implications of Precautionary Mismatch

Now we are ready to answer the question: how large are the effects of precautionary mismatch on labor productivity and aggregate output? We tackle this question in two different ways.

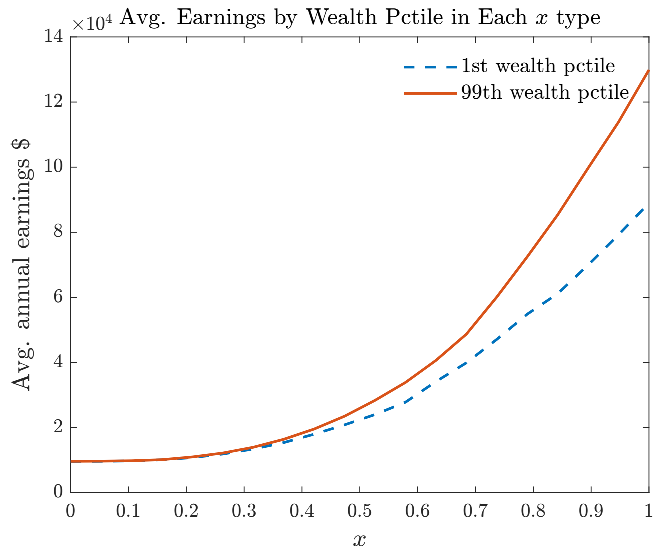
**Liquid Wealth and Labor Market Outcomes**

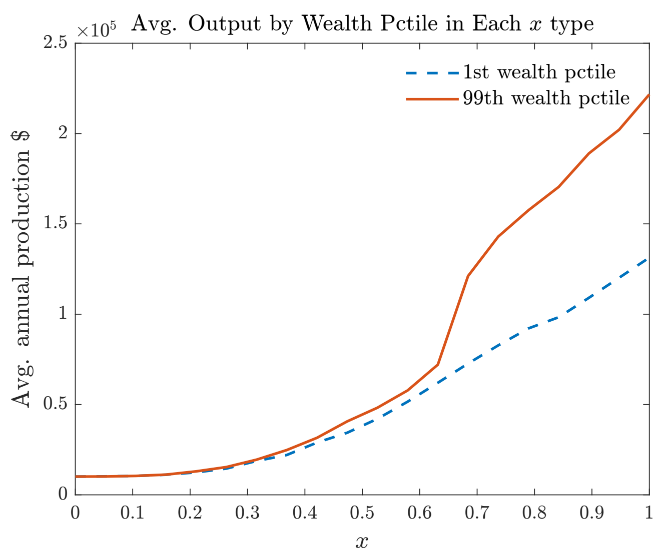
First, we ask the following question: how much does the increase in mismatch due to liquidity constraint affect wages and productivity? To answer this question, we compare average earnings and output of workers in the highest and lowest percentiles in terms of liquid wealth among their respective skill types. Figure 6 shows average (annualized) earnings and output of workers with different skill levels. The red solid line represents the outcomes of workers in the 99th percentile of the wealth distribution within their skill groups, and the blue dashed line represents those in the 1st wealth percentile within their skill groups.

We can see that there exists substantial earnings and productivity gap between the wealth-rich and the wealth-poor within skill groups, especially for skilled workers. The fact that liquidity-constrained workers earn and produce less than same-skilled workers who are not constrained suggests that they tend to match with lower ranking jobs due to precautionary motive. For highest-skilled workers ($x = 1$), the within-group earnings gap is 31.5% and the productivity gap is even larger at 40.8%.

This comparison indicates that there can be substantial productivity improvement if we can allocate the mismatched high-skilled workers to the right jobs. In the following exercise, we estimate the productivity gain if all employed workers are perfectly matched.

Figure 6: Within-group Earnings and Productivity Gap




**Precautionary Mismatch vs. Perfect Sorting**

In this exercise, we estimate the extent to which currently employed workers are misallocated across jobs. That is, given the set of workers that are employed and firms that are producing in equilibrium, we find the perfect sorting allocation rule that maximizes output, and then estimate the difference between the equilibrium and the maximal output.

To formalize the idea, let us introduce some notations here. Denote $E$ the total measure of employed workers and $s(x,y) := d_m(x,y)/E$ the density of matches $(x,y)$, so that $\int s(x,y)\,dxdy = 1$. The aggregate labor productivity is thus $y = \int f(x,y)\,s(x,y)\,dxdy$. Define employed worker density $s_e(x) = \int s(x,y)\,dy$ and producing job density $s_p(y) = \int s(x,y)\,dx$, so that $\int s_e(x)\,dx = \int s_p(y)\,dy = 1$.

**Definition 5.1.** We define the perfect sorting allocation rule as the following:

$$\mathbb{M}\left(s_e\left(x\right), s_p\left(y\right)\right) = \max_m \int f\left(x, y\right) m\left(x, y\right) dxdy$$
$$\text{s.t.} \quad \int m\left(x, y\right) dy = s_e\left(x\right) \text{ and } \int m\left(x, y\right) dx = s_p\left(y\right)$$

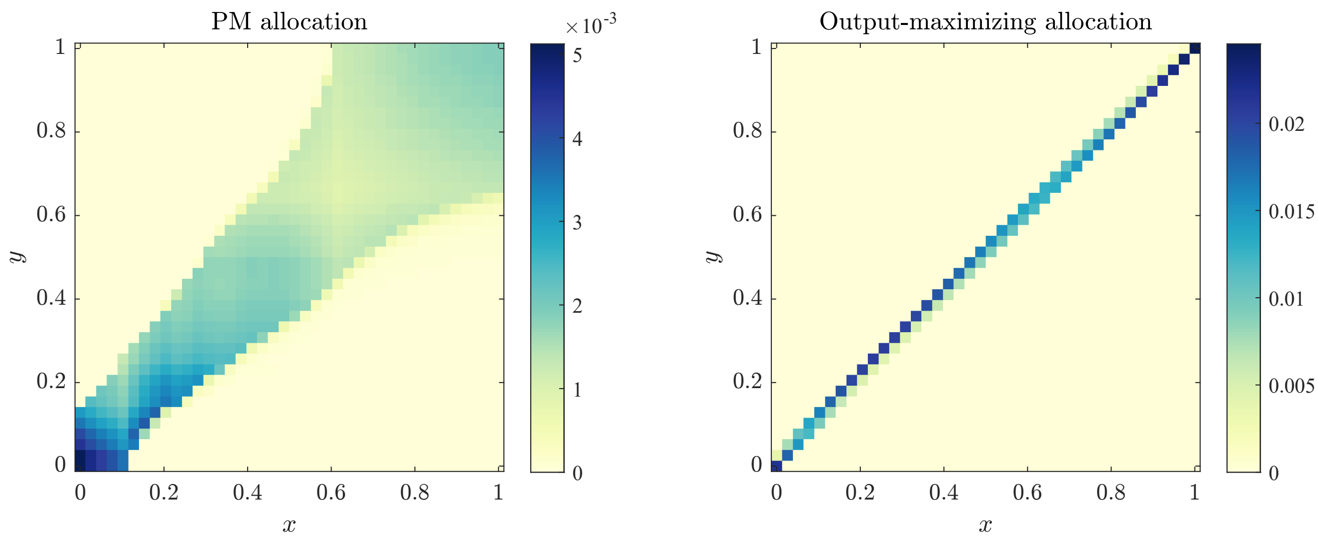
which is in essence a linear programming problem that can be easily solved numerically.

Then the output loss due to labor misallocation can be expressed as the difference between the total outputs under the two allocation rules:

$$\int f\left(x, y\right) \left[\mathbb{M}\left(s_e\left(x\right), s_p\left(y\right)\right) - s\left(x, y\right)\right] dxdy$$

Figure (7) shows a comparison between the equilibrium labor allocation (left panel) and the output-maximizing allocation (right panel), where the x-axes represents worker skill levels and the y-axes job skill requirement levels. Lighter color reflects higher density.

Figure 7: Equilibrium Labor Allocation vs. Output-maximizing Allocation




We can see that the output-maximizing allocation is formed tightly along the 45-degree line due to PAM, while the equilibrium allocation is more dispersed due to search frictions. Our calculation shows that the equilibrium level of aggregate output is 2.8% lower than the output from output-maximizing allocation.

## 5.5 Policy Experiment: Subsidy for Young Workers

In the baseline economy, workers enter the economy with zero liquid wealth and are thus subject to strong precautionary mismatch motive. These young workers form a big part of the bottom 1% in their respective skill groups' liquid wealth distributions, and in Section 5.4 we showed that they tend to have lower productivity. In light of this finding, we consider an experiment which provides all young workers entering the labor market some liquidity which is paid for by a lump-sum tax imposed on the whole population.

We assume that young workers receive an amount equal to the average liquid wealth level in the baseline economy, denoted by $\overline{w}^{PM}$, which is roughly half of average annual earnings according to our calculation from the SCF. In our calibration, $\delta = 1/180$ of the population get replaced by new workers every quarter. Therefore to balance the budget, the government needs to collect an amount equal to $\delta\overline{w}^{PM}$, which is roughly 0.28% of average annual earnings in the baseline equilibrium from each worker.

To get a sense of how long it would take to reach the new equilibrium were the government to implement the policy today, as well as what the new equilibrium looks like, we plot the transition path starting from the baseline equilibrium in Figure 8. The transition takes about 20 years, and the new equilibrium is characterized by a lower job finding rate and separation rate as young workers become more patient in picking the right match. Productivity increases, and so does the unemployment rate. As a result, the new steady state is characterized by a 0.1% drop in output. However, consumption goes up as workers receive higher wages and accumulate more wealth on their jobs.

Another effect of the subsidy policy that we expect is changes in the within-group earnings and productivity gap due to liquid wealth dispersion, as young workers are the wealth-poorest. In Figure 9, we plot the average (annualized) earnings and output for the workers in different wealth percentiles within their skill groups. The red solid line and the blue dashed line, which are the same as in Figure 6, represent the average earnings and output of workers in the top and bottom 1% of the within-group wealth distribution in the baseline economy, while the yellow dashed line shows those of the bottom 1% in the economy with subsidy. We can see that the subsidy policy reduces the within-group earnings and output gaps substantially, especially for skilled workers, as they can now wait long to find higher-ranking jobs. For the highest-skilled workers ($x = 1$), earnings gap shrinks by 30.8%, while productivity gap shrinks by 20.5%.

Figure 8: Transition Path to the Equilibrium with Subsidy to the Young

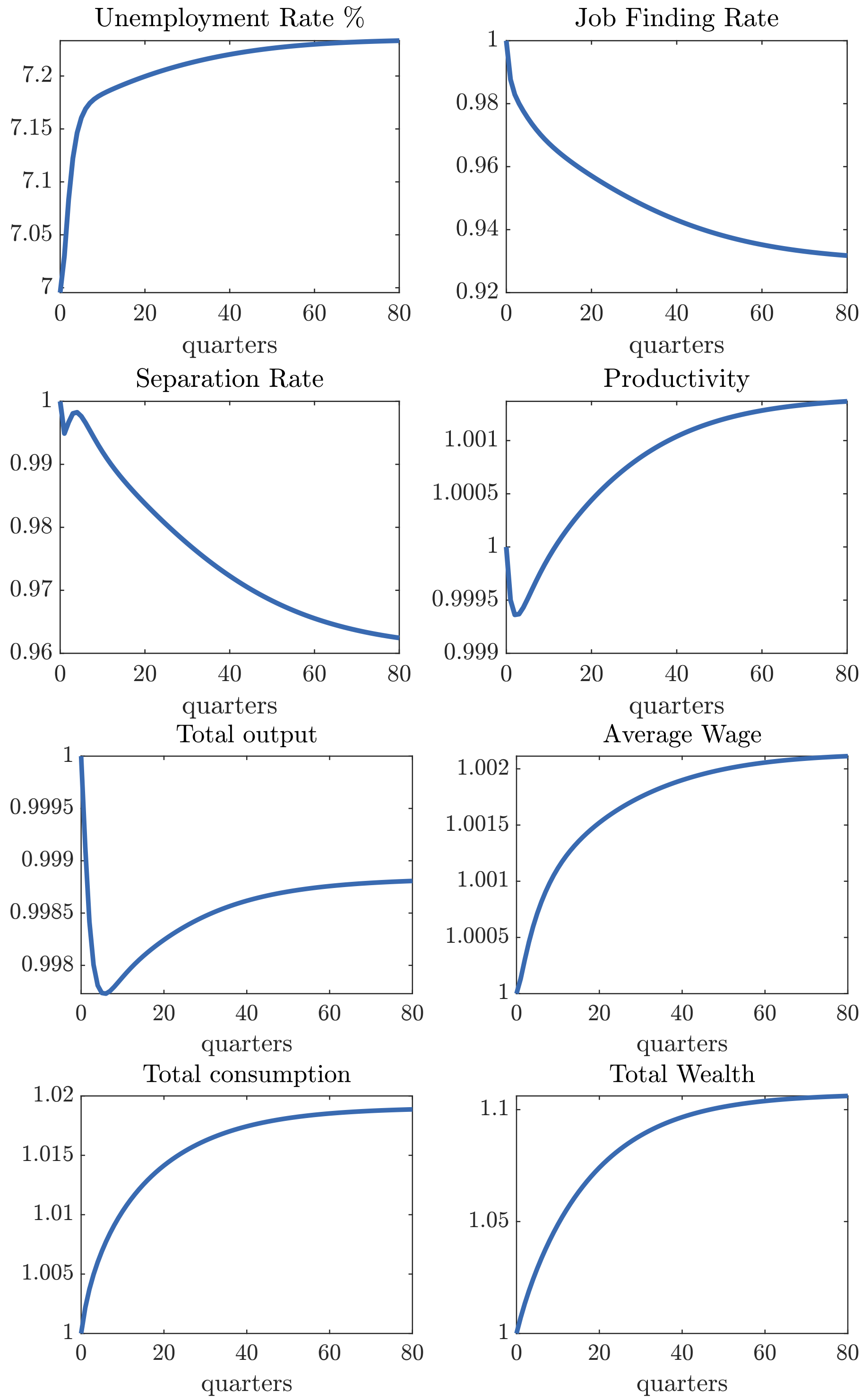
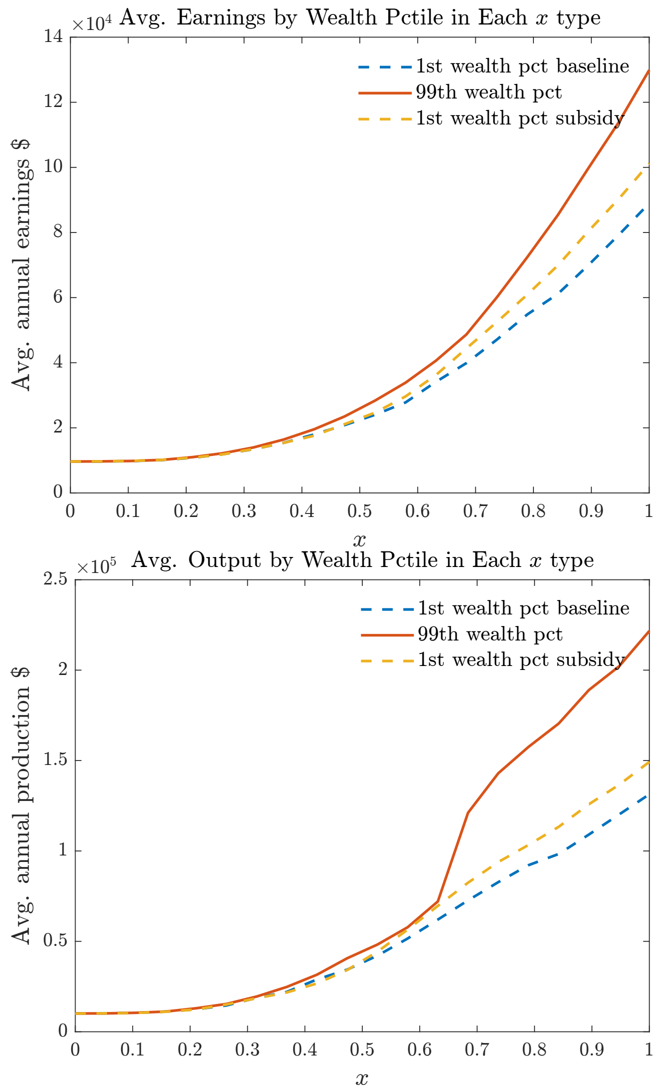


Figure 9: Within-group Earnings and Productivity Gap with Subsidy



# 6 Conclusion

In this paper we aim to study how household liquidity constraint affects the allocation of workers to firms. We develop a framework with two-sided heterogeneity, search frictions and incomplete markets, where workers need to adjust their job acceptance strategies to self-insure against unemployment shocks. Sorting occurs in equilibrium as workers and firms of different skill types need to mutually agree to form matches. We find both theoretically and empirically that under precautionary motive, wealth-poor workers speed up job search by accepting a wider range of jobs, which potentially come with lower wages and rankings that are either too high or too low relative to their own skill rankings.

An important takeaway from our model is that while the precautionary mismatch motive is optimal from individual workers' standpoint, it tends to lower the allocative efficiency of the labor market as reallocation of mismatched workers to their respective perfectly-matched jobs increases overall productivity. This leaves open the room for policies aimed at providing better consumption insurance to impact labor market productivity.

To our knowledge, our model is among the first ones to introduce labor search with two-sided heterogeneity where workers are heterogeneous in two dimensions: skill and wealth level. A model with so much heterogeneity would in principle be rather difficult to compute, let alone to estimate. We overcome this challenge by casting our model in continuous time based on the technique introduced by Achdou et al. (2022), so that the problem boils down to solving systems of linear differential equations where we can take advantage of the sparsity of the resulting matrix to speed up computation. Furthermore, using continuous time representation we can also express wages using already-computed equilibrium objects, which further reduces the complexity of equilibrium computation.

We calibrate our model to match the U.S. economy and estimate that the loss of earnings and productivity due to liquidity constraints can be substantial, especially for skilled workers. If we could reallocated workers to perfectly matched jobs, total output would increase by about 3%. We conduct an experiment within our model in which we provide liquidity subsidy to young workers entering the labor market, who are the wealth-poorest both in the model and in the data. We find that the policy is successful at improving labor productivity by inducing young workers to wait for better matches, which in turn leads to higher wages and wealth accumulation. Future work should continue to explore the optimal policy to provide consumption insurance and improve labor market sorting, as well as the welfare implications of such policy.

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# I Mathematical Appendix

## I.1 Derivation of HJB Equations

Consider the discrete time problem with period length of $\Delta$

$$\begin{aligned}
W(a,x,y) = \max_c u(c)\Delta + \frac{1}{1+\rho\Delta}\Bigg\{ & \underbrace{\Delta\sigma U(a',x)}_{\text{exogenous separation}} + \\
& + (1-\Delta\sigma)\max\Big[W(a',x,y), \underbrace{U(a',x)}_{\text{endogeneous separation}}\Big]\Bigg\} \\
\text{s.t.} \quad a' = a + (ra + \omega(a,x,y) - c)\Delta &
\end{aligned}$$

Consider acceptable matches $W(a,x,y) \geq U(a,x)$. As $\Delta \to 0$, $a' \to a$, continuity will preserve that $W(a',x,y) \geq U(a',x)$.

We can take the max operator off for small $\Delta$

$$\begin{aligned}
W(a,x,y) = u(c^e)\Delta + \frac{1}{1+\rho\Delta}\Big\{&\Delta\sigma U(a + (ra+\omega-c^e)\Delta, x) \\
&+ (1-\Delta\sigma) W(a + (ra+\omega-c^e)\Delta, x, y)\Big\}
\end{aligned}$$

Multiply both sides by $(1+\rho\Delta)$, subtract $W$, and then divide them by $\Delta$

$$\begin{aligned}
\rho W(a,x,y) = u(c^e)(1+\rho\Delta) + &\frac{1}{\Delta}\left[W(a + (ra+\omega-c^e)\Delta, x, y) - W(a,x,y)\right] \\
&+ \sigma\left[U(a + (ra+\omega-c^e)\Delta, x) - W(a + (ra+\omega-c^e)\Delta, x, y)\right]
\end{aligned}$$

Take the limit $\Delta \to 0$,

$$\rho W(a,x,y) = u(c^e) + \underbrace{(ra+\omega-c^e)}_{\dot{a}} W_a(a,x,y) + \sigma\left[U(a,x) - W(a,x,y)\right]$$

Other value functions can be derived similarly.

## I.2 Nash Bargaining

To derive the wage setting, we start with the discrete time problem with period length of $\Delta$. The value for employed worker of type $x$ with asset $a$ that works at a job of type $y$ for an arbitrarily deviating flow wage $w$ (recognizing that in the following period the wage will go back to the equilibrium bargained wage) satisfies

$$\tilde{W}(w,a,x,y) = \max_c u(c)\Delta + \frac{1}{1+\rho\Delta}\{(1-\Delta\sigma)(1-\Delta\delta)W(a',x,y) + \Delta\sigma U(a',x) + \Delta\delta\cdot 0\}$$
$$\text{s.t.} \quad a' = a + (\tilde{r}a + w - c)\Delta,$$

where $\tilde{r} := r+\delta$ is the effective return. Denote the optimal consumption policy by $\tilde{c}^e(w,a,x,y)$. The Envelop condition delivers

$$\tilde{W}_w(w,a,x,y) = \frac{1}{1+\rho\Delta}\{(1-\Delta\sigma)(1-\Delta\delta)W_a(a+(\tilde{r}a+w-\tilde{c}^e)\Delta,x,y)\Delta + \Delta\sigma U_a(a+(\tilde{r}a+w-\tilde{c}^e)\Delta,x$$

Similarly, the value for such a producing job is

$$\tilde{J}(w,a,x,y) = f(x,y)\Delta - w\Delta + \frac{1}{1+r\Delta}\left[(1-\Delta(\sigma+\delta))J(a',x,y) + \Delta(\sigma+\delta)V(y)\right],$$

where $a' = a + (\tilde{r}a + w - \tilde{c}^e(w,a,x,y))\Delta$ is taken as given from the firm's point of view. From the Envelop theorem, we have

$$\tilde{J}_w(w,a,x,y) = -\Delta + \frac{1}{1+r\Delta}(1-\Delta(\sigma+\delta))J_a(a+(\tilde{r}a+w-\tilde{c}^e(w,a,x,y))\Delta,x,y)(1-\tilde{c}^e_w(w,a,x,y))\Delta.$$

Under Nash bargaining, the wage policy is determined by

$$\omega(a,x,y) = \arg\max_w \left[\tilde{W}(w,a,x,y) - U(a,x)\right]^\eta \left[\tilde{J}(w,a,x,y) - V(y)\right]^{1-\eta}.$$

The first order condition for the bargaining problem is

$$\eta\left(\tilde{J}(w,a,x,y) - V(y)\right)\tilde{W}_w(w,a,x,y) + (1-\eta)\left(\tilde{W}(w,a,x,y) - U(a,x)\right)\tilde{J}_w(w,a,x,y) = 0.$$

It is helpful to recognize that as $\Delta \to 0$,

$$\frac{\tilde{J}_w(w,a,x,y)}{\tilde{W}_w(w,a,x,y)} \to \frac{J_a(a,x,y)-1}{W_a(a,x,y)}.$$

This could be easily seen if one plugs in the expressions for $\tilde{J}_w$ and $\tilde{W}_w$ derived from the Envelop theorem.

Rewrite the Nash solution

$$\eta\frac{J(a,x,y) - V(y)}{1 - J_a(a,x,y)} = (1-\eta)\frac{W(a,x,y) - U(a,x)}{W_a(a,x,y)}$$

as

$$\eta\frac{rJ\left(a,x,y\right)+\left(\rho-r\right)J\left(a,x,y\right)-\rho V\left(y\right)}{1-J_{a}\left(a,x,y\right)}=\left(1-\eta\right)\frac{\rho W\left(a,x,y\right)-\rho U\left(a,x\right)}{W_{a}\left(a,x,y\right)}$$

Plug in the HJB equation of $rJ$ and $\rho W$:

$$\begin{aligned}\eta&\frac{f\left(x,y\right)-\omega\left(a,x,y\right)+\left(\tilde{r}a+\omega-\tilde{c}^{e}\right)J_{a}(a,x,y)+\left(\rho-r\right)J\left(a,x,y\right)-\rho V\left(y\right)}{1-J_{a}\left(a,x,y\right)}\\&=\left(1-\eta\right)\frac{u\left(c\right)+\left(\tilde{r}a+\omega-\tilde{c}^{e}\right)W_{a}\left(a,x,y\right)-\left(\rho+\delta\right)U\left(a,x\right)}{W_{a}\left(a,x,y\right)}\end{aligned}$$

Collecting terms, we obtain the following wage equation

$$\omega(a,x,y)=\eta\frac{f\left(x,y\right)+\left(\left(r+\delta\right)a-\tilde{c}^{e}\right)J_{a}(a,x,y)+\left(\rho-r\right)J\left(a,x,y\right)-\rho V\left(y\right)}{1-J_{a}\left(a,x,y\right)}\tag{A1}$$

$$-\left(1-\eta\right)\frac{u\left(\tilde{c}^{e}\right)+\left(\left(r+\delta\right)a-\tilde{c}^{e}\right)W_{a}\left(a,x,y\right)-\left(\rho+\delta\right)U\left(a,x\right)}{W_{a}\left(a,x,y\right)}\tag{A2}$$

## I.3 Additional Proofs

**Proposition 5.** $\lim_{\Delta\to 0}\tilde{c}^{e}_{w}\left(w,a,x,y;\Delta\right)=0$.

*Proof.* This is true because the optimal consumption policy is characterized by its first order condition

$$u'\left(\tilde{c}^{e}\right)=\frac{1}{1+\rho\Delta}\left\{\left(1-\Delta\sigma\right)W_{a}(a+\left(ra+w-\tilde{c}^{e}\right)\Delta,x,y)+\Delta\sigma U_{a}\left(a+\left(ra+w-\tilde{c}^{e}\right)\Delta,x\right)\right\}.$$

Notice that as $\Delta\to 0$, the limiting FOC becomes $\lim_{\Delta\to 0}u'\left(\tilde{c}^{e}\right)=W_{a}(a,x,y)$. Under mild technical conditions,

$$\lim_{\Delta\to 0}\frac{\partial\tilde{c}^{e}}{\partial w}\left(w,a,x,y;\Delta\right)=\frac{\partial}{\partial w}\lim_{\Delta\to 0}\tilde{c}^{e}\left(w,a,x,y;\Delta\right)=\frac{\partial}{\partial w}u'^{(-1)}\left(W_{a}(a,x,y)\right)=0.$$

□

## I.4 Proof of Proposition 1

*Proof.* From the discussion before, we know that Nash bargaining implies the following relationship for the adjusted match surplus could be written as

$$\frac{W(a,x,y)-U\left(a,x\right)}{W_{a}(a,x,y)}=\eta\hat{S}\left(a,x,y\right).$$

Worker optimization gives rise to the first order condition such that $W_a(a, x, y) = \mathfrak{u}'(c^e(a, x, y)) > 0$. Therefore, whether a match is formed or not, i.e., whether $\hat{S}(a, x, y) > 0$ is equivalent to whether $W(a, x, y) - U(a, x) > 0$.

Consider $a$ such that $W(a, x, y) - U(a, x) = 0$, i.e., a marginally acceptable match. Define $\Delta(a; x, y) := W(a, x, y) - U(a, x)$. Differentiate both sides with respect to wealth $a$:

$$\Delta_a = W_a - U_a = \mathfrak{u}'(c^e) - \mathfrak{u}'(c^u),$$

where the arguments are suppressed for simplicity. It is obvious that for acceptable matches, $c^e > c^u$. Since the flow utility exhibits the usual concavity property $\mathfrak{u}'' < 0$, it must be that $\Delta_a = \mathfrak{u}'(c^e) - \mathfrak{u}'(c^u) < 0$.

Therefore, for any $a' > a$ we will have $\hat{S}(a', x, y) < 0$ and for any $a'' < a$ we will have $\hat{S}(a'', x, y) > 0$. □

## I.5 Proof of Proposition 2

*Proof.* Consider $a > a'$. From Proposition 1 we know that $\Phi(a, x, y) \subset \Phi(a', x, y)$. Therefore the job finding rate of the worker of type $x$ with wealth $a$ is

$$\begin{aligned}\pi_{ue}(a, x) &= p(\theta) \int \frac{d_v(y)}{v} \Phi(a, x, y) \mathrm{d}y \\ &\leq p(\theta) \int \frac{d_v(y)}{v} \Phi(a', x, y) \mathrm{d}y \\ &= \pi_{ue}(a', x)\end{aligned}$$

□

## I.6 Proof of Proposition 4

*Proof.* Total differentiating $W_a(a, x, y)$, we have

$$\mathsf{d}W_a(a, x, y) = W_{aa}(a, x, y)\, \mathsf{d}t$$

Apply the Envelope theorem to employed value $W(a, x, y)$ with respect to $a$,

$$\rho W_a(a, x, y) = \sigma [U_a(a, x) - W_a(a, x, y)] + \dot{a} W_{aa}(a, x, y) + [r + \omega_a(a, x, y)] W_a(a, x, y).$$

Note that $W_a(a,x,y) = u'(c^e(a,x,y))$ and $U_a(a,x) = u'(c^u(a,x))$ by FOCs

$$u''(c^e)\,\mathsf{d}c^e = (\rho - r - \omega_a)\,u'(c^e)\,\mathsf{d}t - \sigma\left[u'(c^u) - u'(c^e)\right]\mathsf{d}t$$

Rearrange

$$\underbrace{-\frac{u''(c^e)\,c^e}{u'(c^e)}}_{\text{relative risk aversion}} \cdot \underbrace{\frac{\mathsf{d}c^e/\mathsf{d}t}{c^e}}_{\text{consumption growth}} = r - \rho + \omega_a + \sigma\left[\frac{u'(c^u)}{u'(c^e)} - 1\right]$$

Similarly, total differentiating $U_a(a,x)$, we have

$$\mathsf{d}U_a(a,x) = U_{aa}(a,x)\left[ra + b - c^u\right]\mathsf{d}t$$

Apply the Envelope theorem to unemployed value $U(a,x)$ with respect to $a$

$$\rho U_a(a,x) = p(\theta)\int \frac{d_v(y)}{v}\left[W_a(a,x,y) - U_a(a,x)\right]^+ \mathsf{d}y + \dot{a}U_{aa}(a,x) + rU_a(a,x)$$

Plugging in FOCs

$$u''(c^u)\,\mathsf{d}c^u = (\rho - r)\,u'(c^u)\,\mathsf{d}t - p(\theta)\int_{B(a,x)} \frac{d_v(y)}{v}\left[u'(c^e) - u'(c^u)\right]\mathsf{d}y\mathsf{d}t$$

Rearrange

$$-\frac{u''(c^u)\,c}{u'(c^u)} \cdot \frac{\mathsf{d}c^u/\mathsf{d}t}{c} = r - \rho + p(\theta)\int_{B(a,x)} \frac{d_v(y)}{v}\left[\frac{u'(c^e)}{u'(c^u)} - 1\right]\mathsf{d}y$$

$\square$

# II Algorithmic Appendix

## II.1 HJB Equations

Rewrite $W(a,x,y)$ as the employed value, and $U(a,x)$ as the unemployed value. The HJB equations are $\rho W(w,a,x,y) -> c_w$

$$\rho W(a,x,y) = \max_c u(c) + \delta[U(a,x) - W(a,x,y)] + (ra + w(a,x,y) - c) W_a(a,x,y)$$
$$\rho U(a,x) = \max_c u(c) + p(\theta) \sum_k \frac{d_v(k)}{v} [W(a,x,y_k) - U(a,x)]^+ + (ra + b(x) - c) U_a(a,x)$$

with the first order conditions $u'(c) = W_a(a,x,y)$ and $u'(c) = U_a(a,x)$ respectively. The FD approximation to the HJB equations are

$$\rho W(a_i,x_j,y_k) = u(c_{i,j,k}) + \delta[U(a_i,x_j) - W(a_i,x_j,y_k)] + (ra_i + w(a_i,x_j,y_k) - c_{i,j,k}) W_{a,}(a_i,x_j,y_k) \tag{A3}$$

$$\rho U(a_i,x_j) = u(c_{i,j}) + p(\theta) \sum_k \frac{d_v(k)}{v} [W(a_i,x_k,y_k) - U(a_i,x_j)]^+ + (ra_i + b(x_j) - c) U_a(a_i,x_k) \tag{A4}$$

## II.2 Upwind Scheme

To compute the HJB equations, we need to approximate the derivatives of value functions numerically. Here we follow Achdou et al. (2022) and use the upwind scheme. The idea is to basically use the forward difference approximation whenever savings policy is positive, and backward difference whenever savings is negative.

Define the forward difference and backward difference as

$$W_{a,F}(a_i,x_j,y_k) = \frac{W(a_{i+1},x_j,y_k) - W(a_i,x_j,y_k)}{\Delta_a}$$

$$W_{a,B}(a_i,x_j,y_k) = \frac{W(a_i,x_j,y_k) - W(a_{i-1},x_j,y_k)}{\Delta_a}$$
$$\bar{W}_a(a_i,x_j,y_k) = u'(ra_i + w(a_i,x_j,y_k))$$

We use the "upwind scheme". From the first order condition we can get $c = (u')^{-1} W_a(a,x,y)$. Define

$$s^W_{i,j,k,F} = ra_i + w(a_i,x_j,y_k) - (u')^{-1}(W_{a,F}(a_i,x_j,y_k))$$

$$s^W_{i,j,k,B} = ra_i + w\left(a_i, x_j, y_k\right) - \left(u'\right)^{-1}\left(W_{a,B}\left(a_i, x_j, y_k\right)\right)$$

and approximate the derivative as follows

$$W_a\left(a_i, x_j, y_k\right) = W_{a,B}\left(a_i, x_j, y_k\right)\mathbf{1}_{\left\{s^W_{i,j,k,B}<0\right\}} + W_{a,F}\left(a_i, x_j, y_k\right)\mathbf{1}_{\left\{s^W_{i,j,k,F}>0\right\}} + \bar{W}_a\left(a_i, x_j, y_k\right)\mathbf{1}_{\left\{s^W_{i,j,k,F}<0<s^W_{i,j,k,B}\right\}}. \tag{A5}$$

Since $W$ is concave in $a$, we have $s^W_{i,j,k,F} < s^W_{i,j,k,B}$, then at some point $i$ we have $s^W_{i,j,k,F} < 0 < s^W_{i,j,k,B}$, in which case we set savings to 0. Plugging the expression (A5) into the discretized HJB equation (A3), then the HJB equation can be written as

$$\rho W_i^{jk} = u\left(c_i^{jk}\right) + \delta\left[U_i^j - W_i^{jk}\right] + \underbrace{\frac{W_{i+1}^{jk} - W_i^{jk}}{\Delta_a}}_{W_{a,F}} s_{i,F}^{jk,W+} + \underbrace{\frac{W_i^{jk} - W_{i-1}^{jk}}{\Delta_a}}_{W_{a,B}} s_{i,B}^{jk,W-}$$

In matrix notation

$$\rho W_i^{jk} = u\left(c_i^{jk}\right) + \delta\left[U_i^j - W_i^{jk}\right] + \frac{1}{\Delta_a}\begin{bmatrix} -s_{i,B}^{jk,W-}, & s_{i,B}^{jk,W-} - s_{i,F}^{jk,W+}, & s_{i,F}^{jk,W+} \end{bmatrix}\begin{bmatrix} W_{i-1}^{jk} \\ W_i^{jk} \\ W_{i+1}^{jk} \end{bmatrix} \tag{A6}$$

Similarly define

$$\rho U_i^j = u\left(c_i^j\right) + p\left(\theta\right)\sum_k \frac{d_v\left(k\right)}{v}\left[W_i^{jk} - U_i^j\right]^+ + \frac{1}{\Delta_a}\begin{bmatrix} -s_{i,B}^{j,U-}, & s_{i,B}^{j,U-} - s_{i,F}^{j,U+}, & s_{i,F}^{j,U+} \end{bmatrix}\begin{bmatrix} U_{i-1}^j \\ U_i^j \\ U_{i+1}^j \end{bmatrix} \tag{A7}$$

## II.3 Implicit method

Let $\mathbf{W}$ denote the vector that stacks all value functions together. The implicit method updates the value functions in the following way:

$$\frac{1}{\Delta}\left(\mathbf{W}^{n+1} - \mathbf{W}^n\right) + \rho\mathbf{W}^{n+1} = \tilde{\mathbf{u}}\left(\mathbf{W}^n\right) + \mathbf{A}\left(\mathbf{W}^n\right)\mathbf{W}^{n+1}$$

which gives

$$\left(\left(\rho + \frac{1}{\Delta}\right)\mathbf{I} - \mathbf{A}\left(\mathbf{W}^n\right)\right)\mathbf{W}^{n+1} = \tilde{\mathbf{u}}\left(\mathbf{W}^n\right) + \frac{1}{\Delta}\mathbf{W}^n$$

$$\Rightarrow \mathbf{W}^{n+1} = \left(\left(\rho + \frac{1}{\Delta}\right)\mathbf{I} - \mathbf{A}\left(\mathbf{W}^n\right)\right)^{-1}\left(\tilde{\mathbf{u}}\left(\mathbf{W}^n\right) + \frac{1}{\Delta}\mathbf{W}^n\right)$$

Stack the value $\mathbf{W}$ where we first loop over assets $a_1, ..., a_{Na}$, then over worker skills $x_1, .., x_{N_x}$, and then finally over firm type $y_1, ..., y_{N_y}$ in the outer loop.

The matrix $\mathbf{A}(\mathbf{W}^n)$ has three components: one with respect to asset accumulation (the last terms of equations (A6) and (A7)), another with respect to job separation $\delta\left[U_i^j - W_i^{jk}\right]$, and the last one with respect to job matching $p(\theta)\sum_k \frac{d_v(k)}{v}\left[W_i^{jk} - U_i^j\right]^+$, which we denote as $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ respectively, then $\mathbf{A}(\mathbf{W}^n) = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3$ such that

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{A}_{1e} & 0 \\ 0 & \mathbf{A}_{1u} \end{bmatrix}$$

$$\mathbf{A}_{1e} = \begin{bmatrix} \beta_1^{11,W} & \gamma_1^{11,W} & 0 & \cdots & 0 \\ \alpha_2^{11,W} & \beta_2^{11,W} & \gamma_2^{11,W} & 0 & \cdots \\ 0 & \alpha_3^{11,W} & \beta_3^{11,W} & \gamma_3^{11,W} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \alpha_{Na}^{NxNy,W} & \beta_{Na}^{N_xN_y,W} \end{bmatrix}, \mathbf{A}_{1u} = \begin{bmatrix} \beta_1^{1,U} & \gamma_1^{1,U} & 0 & \cdots & 0 \\ \alpha_2^{1,U} & \beta_2^{1,U} & \gamma_2^{1,U} & 0 & 0 \\ 0 & \alpha_3^{1,U} & \beta_3^{1,U} & \gamma_3^{1,U} & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \beta_{Na}^{N_x,U} & \gamma_{N_a}^{N_x,U} \end{bmatrix}$$

where

$$\begin{aligned} \alpha_i^{jk,W} &= \frac{-s_{i,B}^{jk,W-}}{\Delta_a} \\ \beta_i^{jk,W} &= \frac{s_{i,B}^{jk,W-} - s_{i,F}^{jk,W+}}{\Delta_a} \\ \gamma_i^{jk,W} &= \frac{s_{i,F}^{jk,W+}}{\Delta_a} \end{aligned}$$

and analogously for the unemployed coefficients.

$$
\mathbf{A}_2 = \begin{bmatrix}
-\delta & 0 & \cdots & 0 & & & & & & & & & 0 & \delta & 0 & \cdots & 0 \\
0 & -\delta & 0 & 0 & & & & & & & & & \vdots & 0 & \delta & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & & & & & & & & & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -\delta & & & & & & & & & \vdots & 0 & \cdots & 0 & \delta \\
\vdots & \ddots & \ddots & \ddots & \ddots & & & & & & & & \vdots & \delta & 0 & \cdots & 0 \\
\vdots & & & & & \ddots & & & & & & & \vdots & 0 & \delta & 0 & 0 \\
\vdots & & & & & & \ddots & & & & & & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & & & & & & & \ddots & & & & & \vdots & 0 & \cdots & 0 & \delta \\
\vdots & & & & & & & & & \ddots & & & \vdots & \delta & 0 & \cdots & 0 \\
\vdots & & & & & & & & & & \ddots & & \vdots & 0 & \delta & 0 & 0 \\
\vdots & & & & & & & & & & & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\delta & 0 & \cdots & 0 & \delta \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}
$$

where each diagonal submatrix corresponds to a loop over asset states $a_1, ..., a_{N_a}$ and worker skills $x_1, ..., x_{N_x}$. The bottom part is a matrix of $N_1 \times N_2$ zeros where $N_1 = N_a \times N_x$ and $N_2 = N_a \times N_x \times (N_y + 1)$.

$$
\mathbf{A}_3 = p(\theta) \times \begin{bmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\mathbf{A}_{31} & \mathbf{A}_{32} & \cdots & \mathbf{A}_{3N_y} & \mathbf{A}_{3N_y+1}
\end{bmatrix}
$$

where

$$
\mathbf{A}_{3k} = \begin{bmatrix}
d_v^k \mathbb{1}_1^{1k} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & d_v^k \mathbb{1}_2^{1k} & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & d_v^k \mathbb{1}_{N_a}^{1k} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & d_v^k \mathbb{1}_{N_a}^{N_x k}
\end{bmatrix}
$$

$$
\mathbb{1}_i^{jk} = \begin{cases} 1 & \text{if } U(a_i, x_j, y_k) > W(a_i, x_j) \\ 0 & \text{otherwise} \end{cases}
$$

and

$$
\mathbf{A}_{3N_y+1} = \begin{bmatrix} -\sum_k d_v^k \nVdash_1^{1k} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -\sum_k d_v^k \nVdash_2^{1k} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\sum_k d_v^k \nVdash_{N_a}^{1k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots\cdots & \cdots\cdots & \cdots\cdots & 0 & -\sum_k d_v^k \nVdash_{N_a}^{N_x k} \end{bmatrix}
$$

and the top part is a matrix of $N_1 \times N_2$ zeros where $N_1 = N_a \times N_x \times N_y$ and $N_2 = N_a \times N_x \times (N_y + 1)$.

Alternatively, we loop over $j$ and $k$

$$
\rho W_i^{jk} = u\left(c_i^{jk}\right) + \delta\left[U_i^j - W_i^{jk}\right] + \frac{1}{\Delta_a}\left[\begin{array}{ccc} -s_{i,B}^{jk,W-}, & s_{i,B}^{jk,W-} - s_{i,F}^{jk,W+}, & s_{i,F}^{jk,W+} \end{array}\right]\begin{bmatrix} W_{i-1}^{jk} \\ W_i^{jk} \\ W_{i+1}^{jk} \end{bmatrix}
$$

$$
\mathbf{W}_{jk}^{n+1} = \left(\left(\rho + \frac{1}{\Delta}\right)\mathbf{I} - \mathbf{A}\left(\mathbf{W}_{jk}^n\right)\right)^{-1}\left(\tilde{\mathbf{u}}\left(\mathbf{W}^n\right)_{jk}^W + \frac{1}{\Delta}\mathbf{W}_{jk}^n\right)
$$

and then loop over $j$

$$
\mathbf{U}_j^{n+1} = \left(\left(\rho + \frac{1}{\Delta}\right)\mathbf{I} - \mathbf{A}\left(\mathbf{U}_j^n\right)\right)^{-1}\left(\tilde{\mathbf{u}}\left(\mathbf{W}^n\right)_j^U + \frac{1}{\Delta}\mathbf{U}_j^n\right)
$$

and then stack

$$
\mathbf{W}^{n+1} = \begin{pmatrix} \mathbf{W}_{11}^{n+1} \\ \mathbf{W}_{12}^{n+1} \\ \vdots \\ \mathbf{W}_{N_x N_y}^{n+1} \\ \mathbf{U}_1^{n+1} \\ \mathbf{U}_{12}^{n+1} \\ \vdots \\ \mathbf{U}_{N_x N_y}^{n+1} \end{pmatrix}
$$

This case is easy to code because $\mathbf{A}$ is standard (although $\tilde{\mathbf{u}}$ is new, but this is straightforward). However, the loop may slowdown the algorithm.

## II.4 Stationary Density

Recall the Kolmogorov Forward (KF) equations for density:

$$
\begin{aligned}
0 &= -\frac{\partial}{\partial a}\left[s_e\left(a,x,y\right)d_m\left(a,x,y\right)\right] - \delta d_m\left(a,x,y\right) + p\left(\theta\right)\frac{d_v\left(y\right)}{v}\Phi\left(a,x,y\right)d_u\left(a,x\right) \\
0 &= -\frac{\partial}{\partial a}\left[s_u\left(a,x\right)d_u\left(a,x\right)\right] - \int p\left(\theta\right)\frac{d_v\left(y\right)}{v}\Phi\left(a,x,y\right)d_u\left(a,x\right)dy + \int \delta d_m\left(a,x,y\right)dy
\end{aligned}
$$

together with the condition that density integrates to 1:

$$
1 = \int_{\underline{a}}^{\infty} d_m\left(a,x,y\right)dadxdy + \int_{\underline{a}}^{\infty} d_u\left(a,x\right)dadx
$$

as well as

$$
\begin{aligned}
d_x &= \int_{\underline{a}}^{\infty} d_m\left(a,x,y\right)dady + \int_{\underline{a}}^{\infty} d_u\left(a,x\right)da \\
d_y &= \int_{\underline{a}}^{\infty} d_m\left(a,x,y\right)dadx + d_v\left(y\right)
\end{aligned}
$$

which can be discretized as

$$
\begin{aligned}
0 &= -\frac{\partial}{\partial a}\left[s_i^{jk,W}d_i^{jk,W}\right] - \delta d_i^{jk,W} + p\left(\theta\right)\frac{d_v^k}{v}\mathbb{1}_i^{jk}d_i^{j,U} \\
0 &= -\frac{\partial}{\partial a}\left[s_i^{j,U}d_i^{j,U}\right] - p\left(\theta\right)\sum_{k=1}^{N_y}\frac{d_v^k}{v}\mathbb{1}_i^{jk}d_i^{j,U} + \delta\sum_{k=1}^{N_y}d_i^{jk,W}
\end{aligned}
$$

and

$$
\begin{aligned}
1 &= \sum_{i=1}^{N_a}\sum_{j=1}^{N_x}\sum_{k=1}^{N_y}d_i^{jk,W}\Delta_a\Delta_x\Delta_y + \sum_{i=1}^{N_a}\sum_{j=1}^{N_x}d_i^{j,U}\Delta_a\Delta_x \qquad \text{(A8)} \\
d_x^j &= \sum_{i=1}^{N_a}\sum_{k=1}^{N_y}d_i^{jk,W}\Delta_a\Delta_y + \sum_{i=1}^{N_a}d_i^{j,U}\Delta_a \\
d_y^k &= \sum_{i=1}^{N_a}\sum_{j=1}^{N_x}d_i^{jk,W}\Delta_a\Delta_x + d_v^k
\end{aligned}
$$

## II.5 Upwind Scheme

For the derivatives, we again use the forward scheme

$$
\begin{aligned}
0 &= -\frac{s_{i,F}^{jk,W+} d_i^{jk,W} - d_{i-1}^{jk,W} s_{i-1,F}^{jk,W+}}{\Delta_a} - \frac{d_{i+1}^{jk,W} s_{i+1,B}^{jk,W-} - d_i^{jk,W} s_{i,B}^{jk,W-}}{\Delta_a} - \delta d_i^{jk,W} + p\left(\theta\right) \frac{d_v^k}{v} \nVdash_i^{jk} d_i^{j,U} \\
0 &= -\frac{s_{i,F}^{j,U+} d_i^{j,U} - d_{i-1}^{j,U} s_{i-1,F}^{j,U+}}{\Delta_a} - \frac{d_{i+1}^{j,U} s_{i+1,B}^{j,U-} - d_i^{j,U} s_{i,B}^{j,U-}}{\Delta_a} - p\left(\theta\right) \sum_{k=1}^{N_y} \frac{d_v^k}{v} \nVdash_i^{jk} d_i^{j,U} + \delta \sum_{k=1}^{N_y} d_i^{jk,W}
\end{aligned}
$$

Collecting terms, we have

$$
\begin{aligned}
0 &= d_{i-1}^{jk,W} \alpha_{i-1}^{jk,W} + d_i^{jk,W} \beta_i^{jk,W} + d_{i+1}^{jk,W} \gamma_{i+1}^{jk,W} + p\left(\theta\right) \frac{d_v^k}{v} \nVdash_i^{jk} d_i^{j,U} \\
0 &= d_{i-1}^{j,U} \alpha_{i-1}^{j,U} + d_i^{j,U} \beta_i^{j,U} + d_{i+1}^{j,U} \gamma_{i+1}^{j,U} + \delta \sum_{k=1}^{N_y} d_i^{jk,W}
\end{aligned}
$$

where

$$
\begin{cases}
\alpha_{i-1}^{jk,W} & = \frac{s_{i-1,F}^{jk,W+}}{\Delta_a} \\
\beta_i^{jk,W} & = \frac{s_{i,B}^{jk,W-} - s_{i,F}^{jk,W+}}{\Delta_a} - \delta \\
\gamma_i^{jk,W} & = -\frac{s_{i+1}^{jk,W-}}{\Delta_a}
\end{cases}
\quad
\begin{cases}
\alpha_{i-1}^{j,U} & = \frac{s_{i-1,F}^{j,U+}}{\Delta_a} \\
\beta_i^{j,U} & = \frac{s_{i,B}^{j,U-} - s_{i,F}^{j,U+}}{\Delta_a} - p\left(\theta\right) \sum_{k=1}^{N_y} \frac{d_v^k}{v} \nVdash_i^{jk} \\
\gamma_i^{j,U} & = -\frac{s_{i+1}^{j,U-}}{\Delta_a}
\end{cases}
$$

Let $\mathbf{d}$ be the stacked vector of densities (arranged in the same order as $\mathbf{W}$), then the KF equations expressed using the upwind scheme can be written as

$$
\mathbf{A}^T \mathbf{d} = 0 \tag{A9}
$$

where $\mathbf{A}^T$ is the same matrix that was defined in Section II.3.

To solve the problem of equation (A9) subject to the constraints (A8), we can do the following. Fix (1) either $d_i^{jk,W}$ or $d_i^{j,U}$ to be 0.1 (or any other non-zero number) for arbitrary $(i, j, k)$; (2) then solve the system for some $\tilde{d}$ and then to renormalize

$$
d_i^{jk,W} = \tilde{d}_i^{jk,W} / \left( \sum_{i=1}^{N_a} \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \tilde{d}_i^{jk,W} \Delta_a \Delta_x \Delta_y + \sum_{i=1}^{N_a} \sum_{j=1}^{N_x} \tilde{d}_i^{j,U} \Delta_a \Delta_x \right)
$$

and

$$
d_i^{j,U} = \tilde{d}_i^{j,U} / \left( \sum_{i=1}^{N_a} \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \tilde{d}_i^{jk,W} \Delta_a \Delta_x \Delta_y + \sum_{i=1}^{N_a} \sum_{j=1}^{N_x} \tilde{d}_i^{j,U} \Delta_a \Delta_x \right).
$$

# III Data Appendix

## III.1 Construction of Worker and Firm Types

This section describes the method to construct multi-dimensional worker skills and job skill requirements, used by Lise and Postel-Vinay (2020).

We create 2-dimensional worker skill bundles and job skill requirement bundles using a data set combining NLSY79 job history and O*NET, following Lise and Postel-Vinay (2020).

For jobs, we

- match weekly NLSY79 job record to O*NET data which contains measures of a variety of job skill descriptors
- take the first 2 principal components of these measures in the panel
- recombine the 2 principal components so that they satisfy the following exclusion restrictions: (1) the *mathematics* measure only reflects cognitive skill requirements; (2) the *mechanical knowledge* scores only reflects manual skill requirements
- normalize the skill requirements so that each component lies in $[0, 1]$

For workers, we

- use all 10 components of individual ASVAB test scores and a measure of health (BMI)
- take the first 2 principal components of these measures
- recombine them so that (1) the ASVAB *mathematics knowledge* score only reflects cognitive skills; (2) the ASVAB *automotive and shop information* score only reflects manual skills
- normalize the skill measures so that each component lies in $[0, 1]$

In the analysis above, we only use the first component, i.e. cognitive skill/skill requirement.