APS 1022

COMPUTATIONAL PROJECT

Financial Portfolio Optimization Models

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1.0 Introduction

This project implements and compares several different financial optimization models into a hypothetical investment universe that consist of 20 publicly traded companies. The ticker symbols for these 20 companies are F, CAT, DIS, MCD, KO, PEP, WMT, C, WFC, JPM, AAPL, IBM, PFE, JNJ, XOM, MRO, ED, T, VZ, and NEM.

1.1 Parameter Estimation

Monthly historical adjusted closing price data was collected for each stock from Dec 30, 2004, to Nov 30, 2008, from Yahoo Finance, each stored in their own respective .csv file. Sample means, variances, and covariances for each stock were computed using data from Dec 30, 2004, to Sept 30, 2008, in the form of a sample expected return vector and a sample covariance matrix. The 44 historical monthly returns were computed using equation 1 below.

$$r_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \tag{1}$$

Where $r_{i,t}$ represents the monthly return of asset i at month t, and $P_{i,t}$ represents the adjusted closing price of asset i at month t. The sample expected return vector and sample covariance matrix were computed using arithmetic means and covariances of the historical monthly returns of each asset as shown in equations 2 and 3 below.

$$\bar{r}_i = \frac{1}{T}\sum_{t=1}^{T} r_{i,t} \tag{2}$$

$$\sigma_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{i,t} - \bar{r}_i)(r_{j,t} - \bar{r}_j) \tag{3}$$

Where $\bar{r}_i$ is the sample mean return of asset i, T is the total number elapsed months in the data, which was taken as 44, and σ_{ij} represents the covariance between the sample mean returns of asset i and asset j.

The resulting sample mean returns, variances, and covariances are shown in Table 1, Table 2a, and Table 2b below.

Table 1 – Sample Expected Monthly Returns for Each Asset

	Mean Monthly Returns (%)
AAPL	3.349
C	-1.27
CAT	1.027
DIS	0.355
ED	0.459
F	-1.217
IBM	0.781
JNJ	0.404
JPM	1.067
KO	0.83
MCD	1.816
MRO	2.313
NEM	0.275
PEP	0.87
PFE	-0.109
T	1.055
VZ	0.364
WFC	1.011
WMT	0.538
XOM	1.27

Table 2a – Covariance Matrix Left Side

	AAPL	C	CAT	DIS	ED	F	IBM	JNJ	JPM	KO
AAPL	0.016427	0.00073921	0.0041106	0.0012686	0.00095747	0.001828	0.0024917	0.00049418	-0.00043317	0.0013916
C	0.00073921	0.0063456	0.00029277	0.00054521	0.00098914	0.0048206	0.00098861	0.00070546	0.004063	0.00068574
CAT	0.0041106	0.00029277	0.004228	0.0010376	-3.6107e-05	0.0013455	0.00081801	-0.00029791	-0.00041472	0.0004613
DIS	0.0012686	0.00054521	0.0010376	0.0020037	3.7645e-05	0.001288	0.00094125	0.00016293	-0.00028276	0.00054537
ED	0.00095747	0.00098914	-3.6107e-05	3.7645e-05	0.0014851	0.0010806	-9.004e-05	0.00031112	0.00075433	0.00039171
F	0.001828	0.0048206	0.0013455	0.001288	0.0010806	0.015166	0.0020424	0.00036999	0.0034214	0.00070858
IBM	0.0024917	0.00098861	0.00081801	0.00094125	-9.004e-05	0.0020424	0.0032098	0.00035114	0.00060735	0.00053506
JNJ	0.00049418	0.00070546	-0.00029791	0.00016293	0.00031112	0.00036999	0.00035114	0.00098623	0.00070863	0.00053258
JPM	-0.00043317	0.004063	-0.00041472	-0.00028276	0.00075433	0.0034214	0.00060735	0.00070863	0.0053945	0.00081586
KO	0.0013916	0.00068574	0.0004613	0.00054537	0.00039171	0.00070858	0.00053506	0.00053258	0.00081586	0.0013666
MCD	0.0040825	0.0013715	0.0012027	0.00085722	0.00075406	0.0022459	0.0014671	0.00054182	0.0010651	0.001088
MRO	0.005455	-0.00066669	0.0028653	0.00057342	0.00069717	0.00070863	0.00083558	-0.00046027	-0.0023297	0.00034093
NEM	0.0028794	-0.0011902	0.00129	-0.00096989	-0.00020844	-0.00037652	-3.9191e-06	-0.00085017	-0.0012781	-0.0006426
PEP	0.00090167	0.00060036	-2.3505e-05	0.00033958	0.00045719	-0.00029468	0.00010729	0.00067139	0.00086906	0.0008016
PFE	0.001025	0.0011799	0.00088189	8.6177e-05	0.00055917	0.0011259	-0.00036544	0.00020567	0.0013092	0.00034609
T	0.0038174	0.0010538	0.0018787	0.00099878	0.0004971	0.0010295	0.00085029	0.0003159	0.00016099	0.00075574
VZ	0.0033237	0.0011516	0.001476	0.00092289	0.00027856	0.0014783	0.00070357	0.00034259	8.1689e-05	0.0005963
WFC	-0.0026682	0.0035813	-0.0011085	-0.0006261	0.00064226	0.0019075	0.00049432	0.00047333	0.0047106	0.00025152
WMT	-0.00062409	0.00032532	-0.00022702	0.00017529	-0.00017825	0.0018646	0.00044196	0.00026012	0.00099914	-4.2557e-05
XOM	0.0040128	0.00061222	0.0017328	0.00017709	0.00053722	0.0022573	0.00064685	0.000143	-0.0004849	0.00055983

Table 2b – Covariance Matrix Right Side

MCD	MRO	NEM	PEP	PFE	T	VZ	WFC	WMT	XOM
0.0040825	0.005455	0.0028794	0.00090167	0.001025	0.0038174	0.0033237	-0.0026682	-0.00062409	0.0040128
0.0013715	-0.00066669	-0.0011902	0.00060036	0.0011799	0.0010538	0.0011516	0.0035813	0.00032532	0.00061222
0.0012027	0.0028653	0.00129	-2.3505e-05	0.00088189	0.0018787	0.001476	-0.0011085	-0.00022702	0.0017328
0.00085722	0.00057342	-0.00096989	0.00033958	8.6177e-05	0.00099878	0.00092289	-0.0006261	0.00017529	0.00017709
0.00075406	0.00069717	-0.00020844	0.00045719	0.00055917	0.0004971	0.00027856	0.00064226	-0.00017825	0.00053722
0.0022459	0.00070863	-0.00037652	-0.00029468	0.0011259	0.0010295	0.0014783	0.0019075	0.0018646	0.0022573
0.0014671	0.00083558	-3.9191e-06	0.00010729	-0.00036544	0.00085029	0.00070357	0.00049432	0.00044196	0.00064685
0.00054182	-0.00046027	-0.00085017	0.00067139	0.00020567	0.0003159	0.00034259	0.00047333	0.00026012	0.000143
0.0010651	-0.0023297	-0.0012781	0.00086906	0.0013092	0.00016099	8.1689e-05	0.0047106	0.00099914	-0.0004849
0.001088	0.00034093	-0.0006426	0.0008016	0.00034609	0.00075574	0.0005963	0.00025152	-4.2557e-05	0.00055983
0.0027403	0.0014496	-1.4916e-05	0.00060229	0.00063004	0.0012109	0.0011722	0.00035181	0.00014179	0.0012357
0.0014496	0.010096	0.0034701	-0.00029891	0.0013789	0.0015929	0.0016511	-0.002747	-0.0020314	0.0042684
-1.4916e-05	0.0034701	0.0075446	-0.0011283	0.00078689	-0.00058803	-0.00034301	-0.0015927	-0.0012447	0.0019236
0.00060229	-0.00029891	-0.0011283	0.0013567	0.00032687	0.00043057	0.00049664	0.00050614	-0.0001715	-0.00018019
0.00063004	0.0013789	0.00078689	0.00032687	0.0029252	0.00070391	0.0006053	0.0013252	-0.00071853	0.0010097
0.0012109	0.0015929	-0.00058803	0.00043057	0.00070391	0.0034009	0.0020913	-0.0006708	-0.00012219	0.0011126
0.0011722	0.0016511	-0.00034301	0.00049664	0.0006053	0.0020913	0.0025188	-0.00058654	-3.6711e-05	0.0010592
0.00035181	-0.002747	-0.0015927	0.00050614	0.0013252	-0.0006708	-0.00058654	0.0055624	0.0006575	-0.0010829
0.00014179	-0.0020314	-0.0012447	-0.0001715	-0.00071853	-0.00012219	-3.6711e-05	0.0006575	0.0020561	-0.00081432
0.0012357	0.0042684	0.0019236	-0.00018019	0.0010097	0.0011126	0.0010592	-0.0010829	-0.00081432	0.0037721

Where the diagonals of the covariance matrix represent each asset's variance of adjusted returns.

1.2 Risk-Free Rate

The risk-free rate was defined as the average yearly rate of the 10 Year US Treasury Yield from 2005 to 2008, converted into an effective monthly rate. The reason for this is because Treasury Bonds are directly backed by the US government, so there is no risk of default aside from catastrophic events such as government or monetary collapse. The 10 Year Bond was chosen due to less fluctuation from shocks compared to shorter term bonds, and it was assumed that the investment horizon for these generated portfolios were relatively long term. The risk-free rate was computed to be 0.355% per month.

1.3 Risk Aversion and the Market Portfolio

The objective function for all mean variance optimizations in this project requires a risk aversion parameter. The risk aversion parameter λ, defines the level of certainty or uncertainty an investor is comfortable with taking, where an increasing λ is increasingly risk averse, and a decreasing λ is increasingly risk-seeking. The benchmark λ was taken as the same level of risk aversion as the market portfolio as it is an optimal portfolio according to the One-Fund Theorem, and it is sufficiently diversified. Equation 4 below defines λ identically to the same parameter in the Black-Litterman lecture slides.

$$\lambda = \frac{\mathrm{E}[\mathrm{r}_{\mathrm{mkt}}] - r_f}{\sigma_{mkt}^2} \tag{4}$$

Where $\mathrm{E}[\mathrm{r}_{\mathrm{mkt}}]$ represents the expected return of the market portfolio, and σ_{mkt}^2 represents the variance of the market portfolio. Since this project's investment universe was comprised of only the 20 selected stocks, the market portfolio was weighted based on the market capitalization of each of the 20 selected stocks. Equations 5 to 8 below define the necessary parameters required to create the market portfolio and solve for the risk aversion parameter defined in equation 4.

$$x_i = \frac{Market\ capitalization\ of\ asset\ i}{Market\ capitalization\ of\ the\ market} \tag{5}$$

$$E[r_{mkt}] = \mu^T * x_{mkt} \tag{6}$$
$$\sigma_{mkt}^2 = x_{mkt}^T * Q * x_{mkt} \tag{7}$$

Where x_i represents the weight of asset i in the market portfolio, μ^T represents a 1x20 vector of $\bar{r}_i$ shown in Table 1, x_{mkt} represents a 20x1 vector of x_i, and Q is the horizontal concatenation of Table 2a and Table 2b, defined as a 20x20 covariance matrix.

The market capitalizations of each asset were taken from Sept 30, 2008, as this date was assumed to be the present date of optimal portfolio creation. All market capitalizations were found on Bloomberg Terminal.

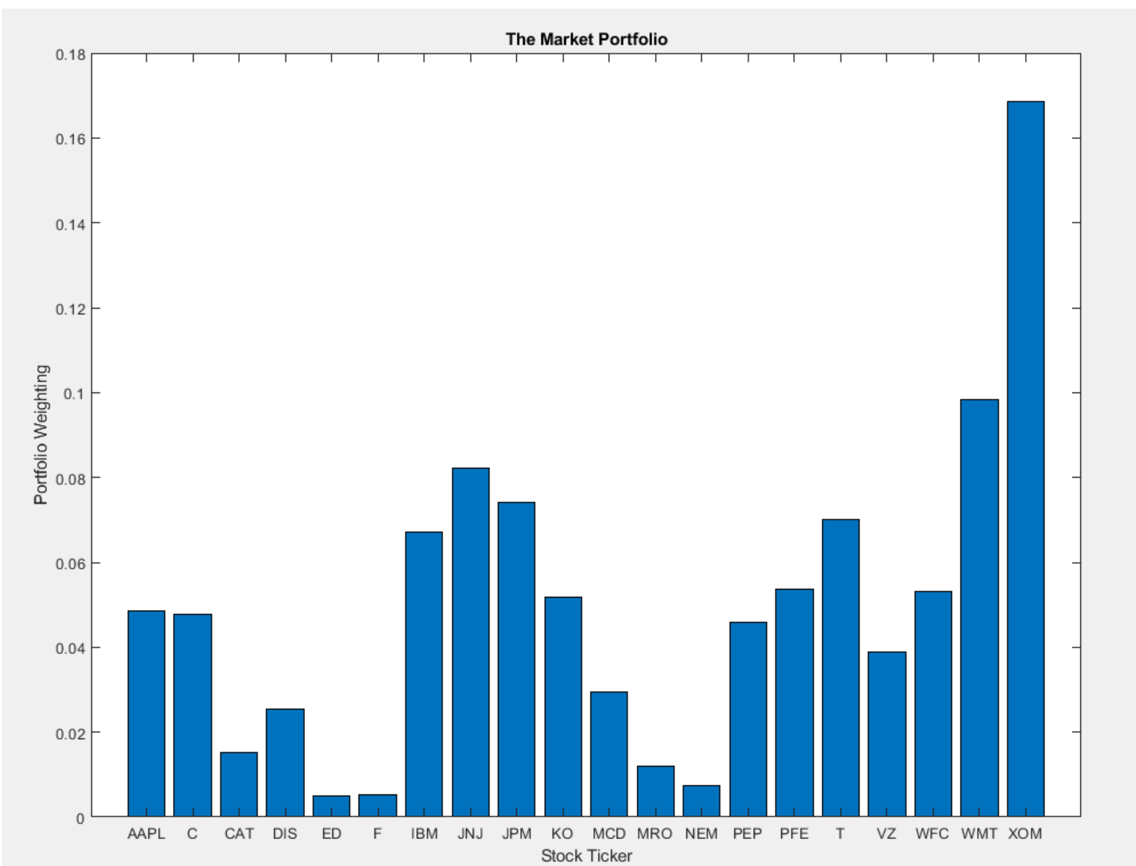
The computed expected market return, market portfolio variance, and risk aversion parameter is shown in Figure 1 as a MATLAB output below.

Figure 1 – Market Return, Market Variance, and Risk Aversion Parameter

```
Expected Market Return: 0.008566
Market Portfolio Variance: 0.000852
Risk Aversion Parameter: 5.888241
```

Figure 2 below shows the weighting of the market portfolio. Note that every stock has a non-zero investment because the market caps are non-zero. Hence, it is sufficiently diversified.

Figure 2 – The Market Portfolio



2.0 Portfolio Optimization

Three different types of optimization methods were implemented in optimal portfolio selection.

1. Mean-Variance Optimization (MVO)
2. Robust MVO (RMVO)
3. Risk-Parity (RP)

Where both the MVO and RMVO methods allowed short selling, but RP did not allow short selling. For the RMVO method, both the box and ellipsoidal models were computed.

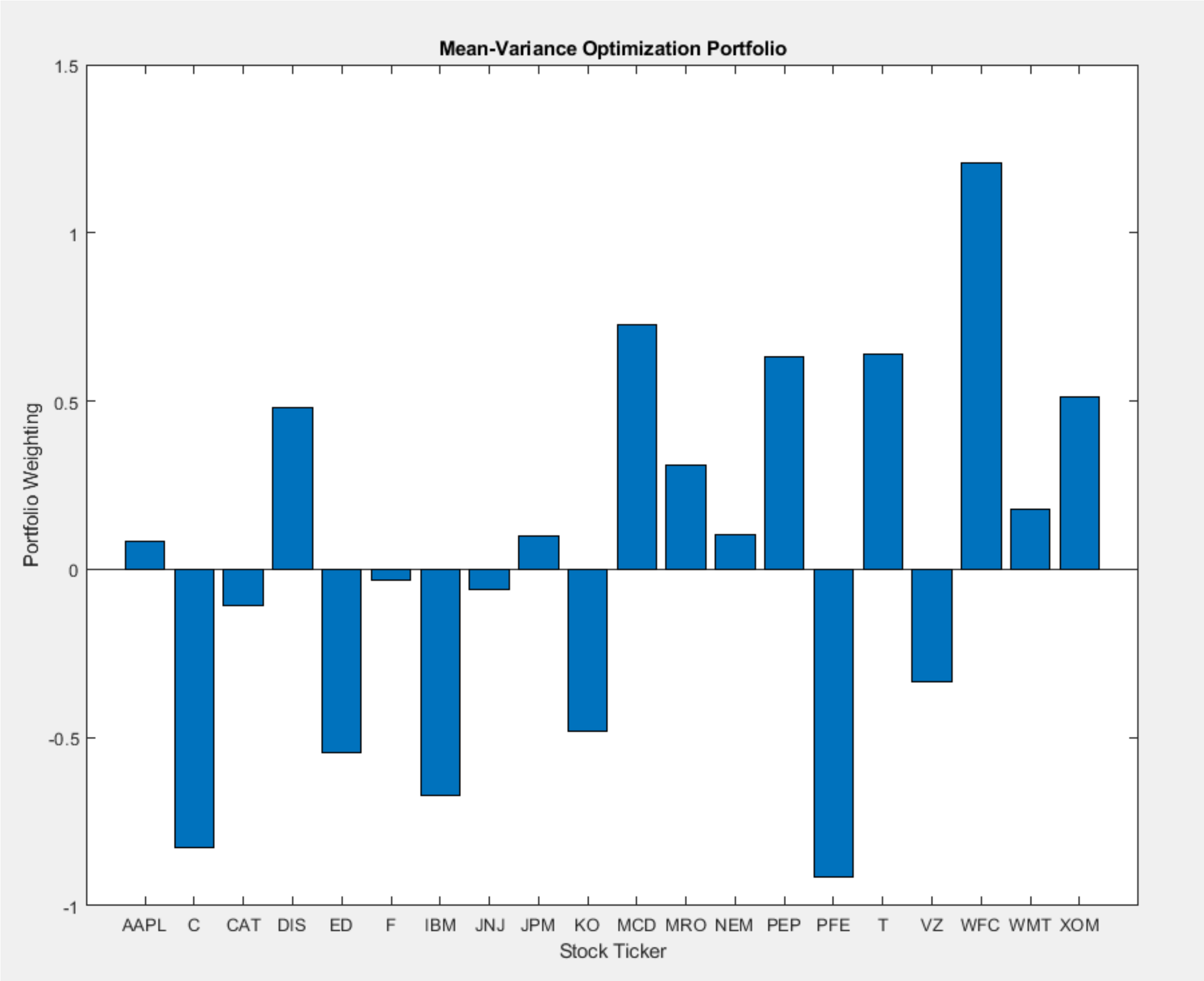
2.1 Mean-Variance Optimization

The MVO portfolio followed the quadratic programming optimization problem shown below.

$$\begin{aligned} \min_{x} \quad & \lambda x^T Q x - \mu^T x \\ s.t. \quad & 1^T x = 1 \end{aligned}$$

Where λ, Q, and μ were all previously calculated in section 1 of this report. Figure 3 below shows the weighting of the MVO portfolio.

Figure 3 – The Mean-Variance Optimization Portfolio



Some criticisms and limitations of the MVO method are that they may create insufficiently diversified portfolios, making them susceptible to idiosyncratic risk, they are reliant on expected returns rather than risk, which is difficult to reduce the standard errors of, and they do not account for standard errors, which makes them very sensitive to small changes in data.

2.2 Robust Mean-Variance Optimization (Box)

RMVO attempts to adjust for potential standard errors of expected returns by introducing an area of uncertainty around the estimated expected returns. This makes the optimization method more robust to small changes in expected returns and is more conservative than traditional MVO to account for potential standard errors.

One way to apply the RMVO method is to create a box of uncertainty around the estimated expected returns, which is demonstrated by the optimization function and constraints below.

$$\begin{array}{cc} \min\limits_{x,y} & \lambda x^T Q x - \mu^T x + \delta^T y \\ s.t. & 1^T x = 1 \\ & x - y \leq 0 \\ & -x - y \leq 0 \end{array}$$

Where δ is a vector of the maximum distance between each asset's expected return and the unknown true return of the asset that's assumed to be within the uncertainty box, and y is an auxiliary variable that represents the absolute value of x.

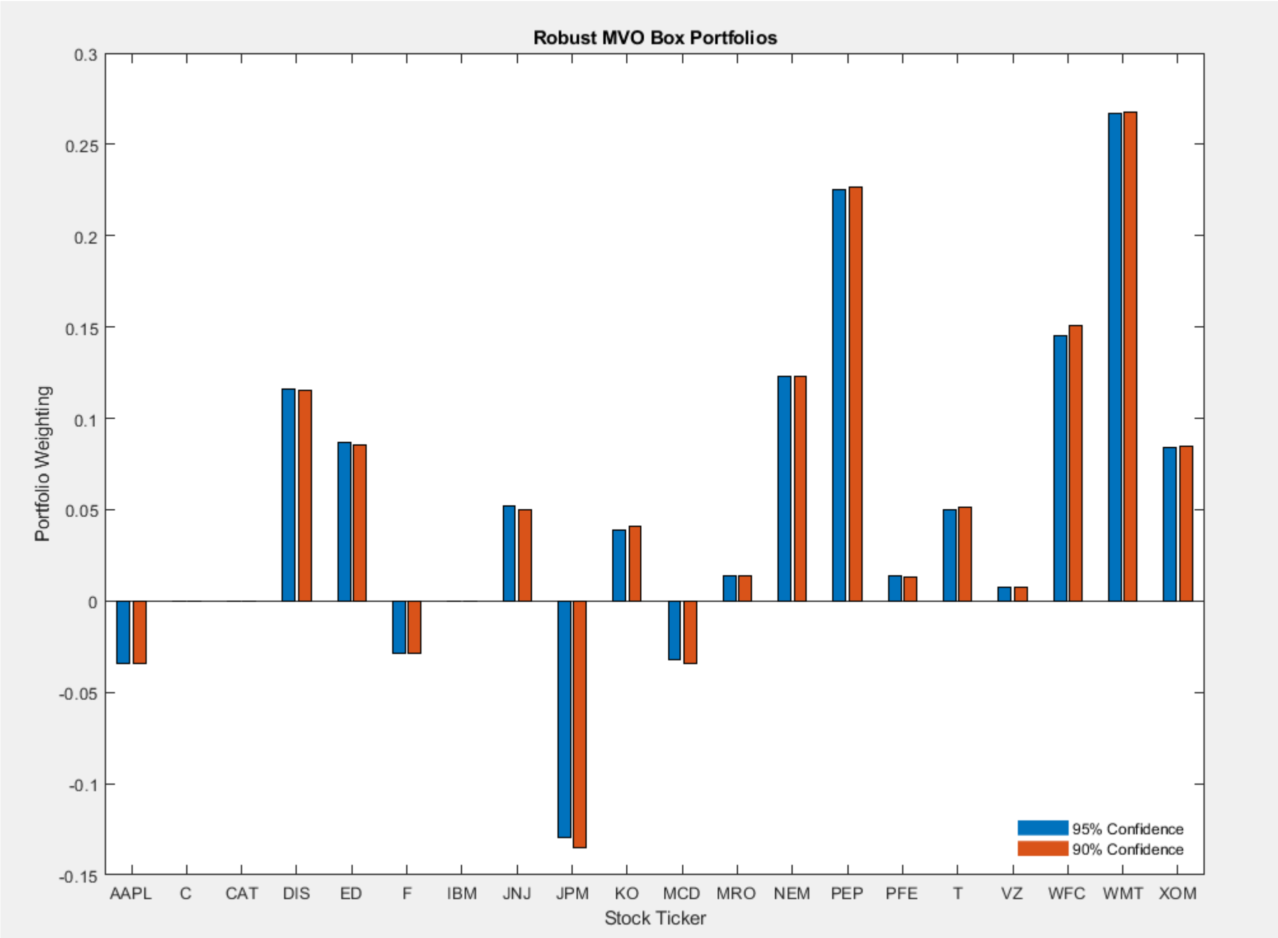
δ can be calculated using equation 8 below.

$$\delta_i = \varepsilon_1 \left(\Theta^{1/2}\right)_{ii} \tag{8}$$

Where ε_1 is the point at which x normalized to a mean of 0 and a variance of 1 provides a certain probabilistic guarantee, commonly known in statistics as the z-score. $\left(\Theta^{1/2}\right)_{ii}$ represents the standard error of expected returns of asset i. Where $\Theta = \frac{1}{T} diag(diag(Q))$, $\left(\Theta^{1/2}\right)_{ii} = \frac{\sigma_i}{\sqrt{T}}$, and $\left(\Theta^{1/2}\right)_{ij} = 0 \; for \; i \neq j$.

Figure 4 below shows the weighting of the RMVO box portfolios for given confidence levels of 90% and 95%.

Figure 4 – Robust MVO Box Portfolios for 90% and 95% Confidence



Compared to the MVO portfolio, this portfolio appears to be more diversified in long positions.

2.3 Robust Mean-Variance Optimization (Ellipsoid)

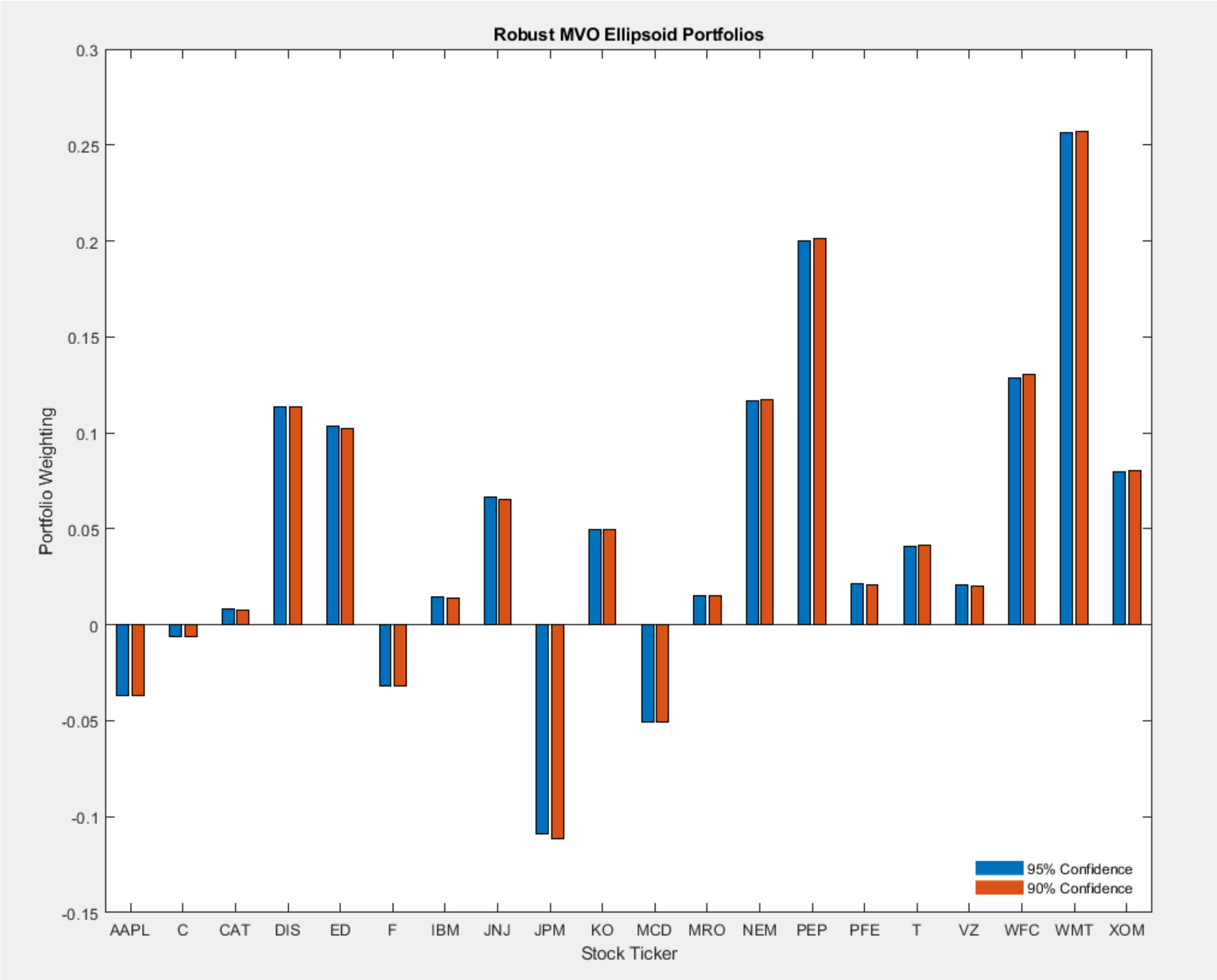
One potential concern of the box uncertainty is that the corners of the box can potentially create portfolios with less-than-optimal returns. To account for that, one can incorporate an ellipsoid uncertainty set rather than a box uncertainty.

$$\begin{aligned} \min_{x} \quad & \lambda x^T Q x - \mu^T x + \varepsilon_2 \left\| \Theta^{1/2} x \right\|_2 \\ s.t. \quad & 1^T x = 1 \end{aligned}$$

Where $\varepsilon_2 = \chi_n^2(\alpha)$ is similar to ε_1 for the box uncertainty as it is the distance between the estimated and true values, and $\left\| \Theta^{1/2} x \right\|_2$ is the Euclidean norm of the uncertainty, added as a penalty to the target returns.

Figure 5 shows the portfolios generated by the ellipsoidal RMVO for given confidence levels of 90% and 95%.

Figure 5 – Robust MVO Ellipsoid Portfolios for 90% and 95% Confidence



The results are very similar to the box RMVO portfolios. However, unlike the box RMVO portfolios, there are non-zero weights on all assets.

2.4 Risk-Parity Optimization

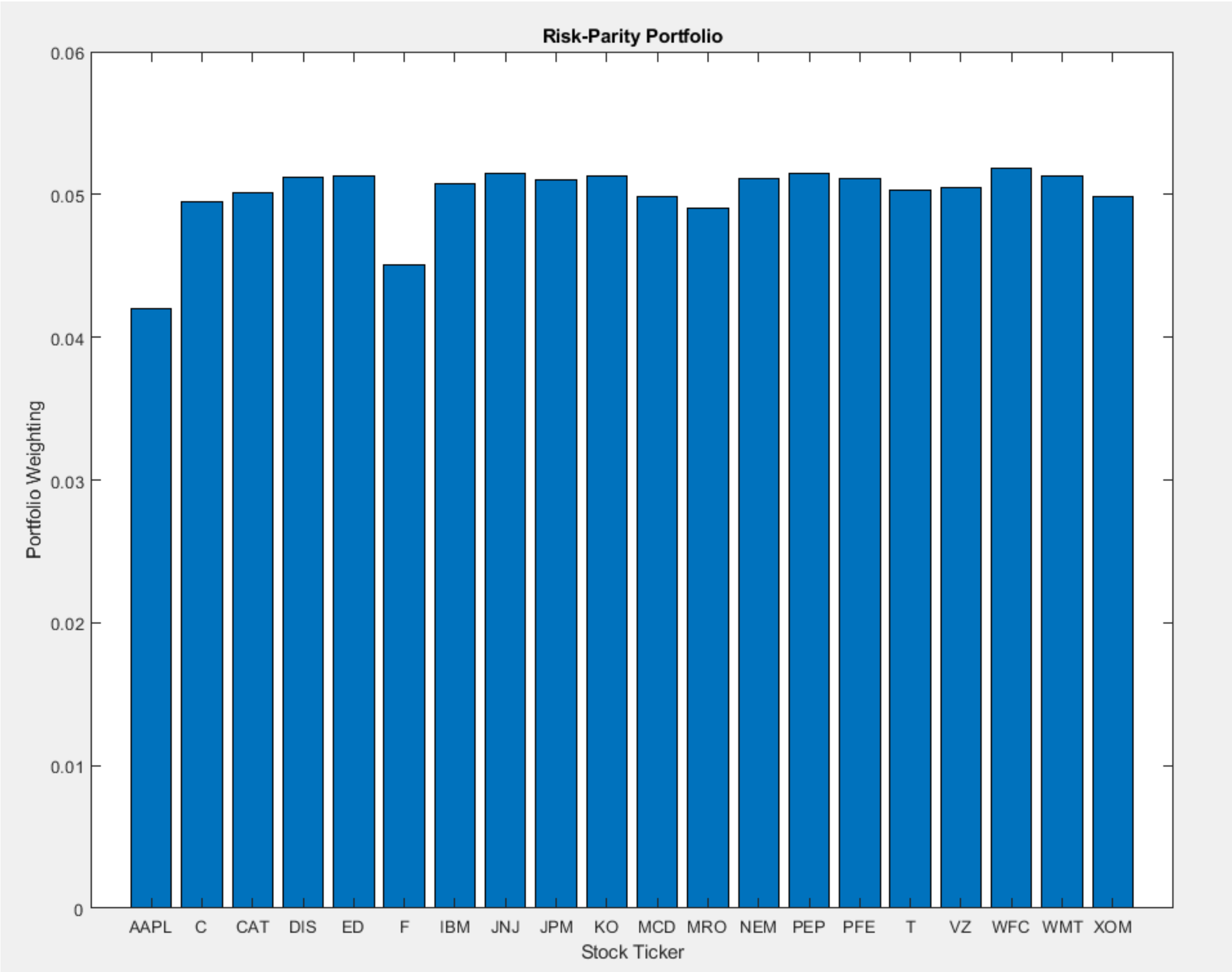
Risk-Parity (RP) Optimization takes a different approach than RMVO to solving the weaknesses of traditional MVO. Instead of being reliant on expected returns, which are highly erroneous, RP optimization only depends on variances and covariances, which can have reduced errors with increased frequency of data points. It is theoretically a well-rounded portfolio that does not provide the most optimal returns but can substantially outperform during economic troughs and recessions. RP optimizations follows the following optimization problem, and this portfolio assumes no short selling, as defined in the project requirements.

$$
\begin{aligned}
\min_{x,\theta} \quad & \sum_{i=1}^{n} (x_i(Qx)_i - \theta)^2 \\
s.t. \quad & 1^T x = 1 \\
& x \geq 0
\end{aligned}
$$

Where θ is an auxiliary variable used to optimize each asset's weighted risk contribution equally throughout the portfolio. From the objective function, it is evident that all assets must theoretically have a positive weight as they are all risky assets. To achieve equal risk contributions, every asset's weighting must be non-zero. However, this is a non-convex problem, meaning uniqueness cannot be guaranteed.

Figure 6 shows the generated optimal portfolio based on risk-parity optimization.

Figure 6 – Risk-Parity Portfolio



From observation of Figure 6, the portfolio seems to be similar to the equal-weighted portfolio. This could be due to the implementation of the risk-parity optimization having an initial guess of the equal-weighted portfolio. Since the problem is non-convex, it is possible to have multiple local optimal solutions, so the portfolio generated was the closest optimal solution to the equal-weighted portfolio.

2.5 Summary

Table 3 below summarizes the generated optimal weightings for each optimization method presented in sections 2.1 to 2.4, as well as the weighting for the market portfolio.

Table 3 – Summary of Optimal Weights

	Market Portfolio Weight	MVO Weight	RMVO Box 95 Weight	RMVO Box 90 Weight	RMVO Ellipsoid 95 Weight	RMVO Ellipsoid 90 Weight	Risk-Parity Weight
AAPL	0.048587	0.084677	-0.034135	-0.034118	-0.036691	-0.036633	0.042027
C	0.047671	-0.82852	-1.0519e-10	-1.1611e-10	-0.0063306	-0.006096	0.049506
CAT	0.015334	-0.10963	1.7298e-06	1.6303e-06	0.0082102	0.0078506	0.05009
DIS	0.025462	0.48173	0.11621	0.11546	0.11345	0.11346	0.051196
ED	0.0050119	-0.54568	0.086511	0.085673	0.10321	0.10239	0.051292
F	0.0052981	-0.032773	-0.028399	-0.02831	-0.031698	-0.031605	0.045041
IBM	0.067017	-0.67236	3.6228e-10	3.2299e-10	0.014796	0.014184	0.050751
JNJ	0.082297	-0.060186	0.052174	0.050031	0.066508	0.065307	0.051451
JPM	0.074233	0.099726	-0.12933	-0.13511	-0.10915	-0.11122	0.051002
KO	0.05175	-0.48017	0.038988	0.041104	0.049449	0.049464	0.051275
MCD	0.029329	0.72852	-0.031898	-0.034047	-0.050955	-0.050758	0.049864
MRO	0.012005	0.30894	0.013789	0.013802	0.015193	0.015133	0.049046
NEM	0.0075099	0.10406	0.12295	0.12316	0.11681	0.11714	0.051103
PEP	0.045768	0.63285	0.22543	0.22657	0.19989	0.20119	0.051466
PFE	0.053664	-0.91376	0.013897	0.013435	0.021123	0.020708	0.051106
T	0.070174	0.6386	0.050224	0.051446	0.040583	0.041237	0.0503
VZ	0.038876	-0.33329	0.0074266	0.0077652	0.020839	0.020449	0.050463
WFC	0.053162	1.2079	0.14531	0.15058	0.12876	0.1306	0.051874
WMT	0.098357	0.1768	0.26678	0.26772	0.25645	0.25717	0.051288
XOM	0.16849	0.51254	0.084068	0.084843	0.079556	0.080018	0.049857
Total	1	1	1	1	1	1	1

3.0 Project Questions

This section is dedicated to answering the questions asked in the project assignment.

3.1 Part A

Table 4 below summarizes the computed portfolio return, portfolio variance and standard deviation, and sharpe ratio for Oct 2008 returns, for all generated optimization models in section 2.

Table 4 – Part a) Answers

	Portfolio Return	Portfolio Variance	Portfolio Standard Deviation	Sharpe Ratio
Market Portfolio	-0.11482	0.00085173	0.029184	-4.056
MVO	0.064676	0.004479	0.066926	0.91333
RMVO Box 95	-0.12609	0.00072453	0.026917	-4.8162
RMVO Box 90	-0.12898	0.00071073	0.02666	-4.9712
RMVO Ellipsoid 95	-0.11159	0.00052692	0.022955	-5.0161
RMVO Ellipsoid 90	-0.11059	0.0005279	0.022976	-4.9676
Risk-Parity	-0.16202	0.00087851	0.02964	-5.5862

From observation of Table 4, the best performing portfolio was the MVO portfolio for Oct 2008. The MVO portfolio was the only portfolio with positive return and Sharpe ratio, but it also has by far the highest portfolio variance and standard deviation as well. Intuitively, this shows that the MVO portfolio is inherently more volatile than the other generated portfolios, meaning that returns and corrections are both steeper than the other portfolios, and the results for Oct 2008 ended up playing more in the favor than against the MVO portfolio.

It is logical that the positive or negative sign of portfolio return results in the same sign for the Sharpe ratio, as the formulation for Sharpe ratio is just the excess return normalized by the portfolio variance. Usually, Sharpe ratios are good observation parameters to identify risk-normalized returns, however, it is difficult to compare when returns are of opposing signs.

One reason that the MVO portfolio outperformed the other portfolios is because the MVO portfolio took a significant short position in Citigroup ($C), by being over 80% short on the stock, which contributed the most to the portfolio's returns for Oct 2008. This is just one example of many large short positions in the MVO portfolio, either taken due to negative expected returns or spread trades to raise capital from underperforming assets and placing the capital into assets with greater expected returns. However, it is generally abnormal for S&P 500 stocks to have negative returns for extended time periods aside from major corrections like the Great Recession. If the S&P 500 broke the trend from the previous few months and bottomed out within Oct 2008, the large short positions would have severely underperformed. Therefore, a non-negligible portion of the outperformance of the MVO portfolio can be attributed to luck and uncertainty. Table 5 shows the breakdown of the returns for each stock in each portfolio.

Table 5 – Return Breakdown for Each Asset in Each Portfolio

	Market Portfolio Return	MVO Return	RMVO Box 95 Return	RMVO Box 90 Return	RMVO Ellipsoid 95 Return	RMVO Ellipsoid 90 Return	Risk-Parity Return
AAPL	-0.0025948	-0.0045222	0.001823	0.0018221	0.0019595	0.0019564	-0.0022444
C	-0.015944	0.27711	3.5184e-11	3.8837e-11	0.0021174	0.0020389	-0.016558
CAT	-0.0055059	0.039365	-6.2111e-07	-5.8537e-07	-0.002948	-0.0028188	-0.017985
DIS	-0.0039658	-0.07503	-0.0181	-0.017982	-0.01767	-0.017672	-0.0079739
ED	4.1998e-05	-0.0045726	0.00072493	0.00071791	0.00086486	0.00085802	0.00042981
F	-0.0030668	0.018971	0.016438	0.016387	0.018348	0.018295	-0.026072
IBM	-0.013746	0.13791	-7.4309e-11	-6.625e-11	-0.0030349	-0.0029094	-0.01041
JNJ	-0.0094318	0.0068977	-0.0059796	-0.005734	-0.0076223	-0.0074847	-0.0058967
JPM	-0.0086632	-0.011638	0.015094	0.015768	0.012738	0.012979	-0.0059521
KO	-0.0083248	0.077243	-0.0062719	-0.0066123	-0.0079546	-0.0079571	-0.0082484
MCD	-0.001792	-0.044514	0.001949	0.0020803	0.0031135	0.0031014	-0.0030468
MRO	-0.003243	-0.083452	-0.0037247	-0.0037283	-0.004104	-0.0040879	-0.013249
NEM	-0.0023945	-0.033181	-0.039203	-0.039269	-0.037245	-0.03735	-0.016294
PEP	-0.008932	-0.12351	-0.043995	-0.044217	-0.03901	-0.039265	-0.010044
PFE	-0.0021245	0.036174	-0.00055016	-0.00053188	-0.00083623	-0.00081978	-0.0020232
T	-0.0028904	-0.026303	-0.0020687	-0.002119	-0.0016716	-0.0016985	-0.0020718
VZ	-0.0029318	0.025134	-0.00056006	-0.00058559	-0.0015715	-0.0015421	-0.0038055
WFC	-0.0049295	-0.11201	-0.013474	-0.013963	-0.01194	-0.01211	-0.00481
WMT	-0.0067005	-0.012045	-0.018175	-0.018239	-0.01747	-0.017519	-0.003494
XOM	-0.0076804	-0.023363	-0.0038321	-0.0038674	-0.0036264	-0.0036474	-0.0022726
Total	-0.11482	0.064676	-0.12609	-0.12898	-0.11159	-0.11059	-0.16202

Generally, the market portfolio is treated as a benchmark for the equities market in general. Since the market portfolio had a negative portfolio return and a negative Sharpe ratio for the month of Oct 2008, it is evident that the sentiment for equities investors on the economy for that month was on the decline as monthly returns were significantly worse than the risk-free rate.

Further analysis of Table 4 shows that the RMVO portfolios were roughly in-line with the market portfolio on all parameters in Table 4. Compared with MVO, the RMVO results make sense because RMVO is a conservatively modified approach to MVO that reduces potential risk and reward to account for standard errors of mean and variance. Additionally, the RMVO ellipsoidal portfolio had strictly better results than the RMVO box portfolio, with lower losses as well as lower portfolio variance. This was assumed to be due to the corners of the uncertainty box creating potentially sub-optimal portfolios, while the ellipsoidal uncertainty area is more uniform in terms of distance from the MVO portfolio. The difference between 90% and 95% certainty appears to be negligible as the portfolio weights, return, and variance results were near identical for each RMVO portfolio. This was assumed to be due to normal distributions having small tails and the granularity of the nature of portfolio optimization being not too sensitive to changes in finer details.

The results for the risk-parity portfolio were surprising as it was the worst performing portfolio in terms of return and was the second riskiest portfolio after MVO. Theoretically, the risk-parity portfolio should be a risk-averse portfolio that does not provide substantial gains or losses. This could be due to the limitation that short selling was not allowed for the risk-parity portfolio, which would have been advantageous especially in a declining market. Additionally, being zoomed in on a single month within a large-scale economic contraction can provide skewed results which could smooth out over longer period analyses.

3.2 Part B

Table 6 below summarizes the computed portfolio return, portfolio variance and standard deviation, and sharpe ratio for Nov 2008 returns, for all generated optimization models in section 2.

Table 6 – Part b) Answers

	Portfolio Return	Portfolio Variance	Portfolio Standard Deviation	Sharpe Ratio
Market Portfolio	-0.037881	0.00085173	0.029184	-1.4197
MVO	0.24347	0.004479	0.066926	3.5848
RMVO Box 95	0.0061118	0.00072453	0.026917	0.095151
RMVO Box 90	0.0016155	0.00071073	0.02666	-0.072586
RMVO Ellipsoid 95	-0.0063564	0.00052692	0.022955	-0.4316
RMVO Ellipsoid 90	-0.0056414	0.0005279	0.022976	-0.40007
Risk-Parity	-0.024671	0.00087851	0.02964	-0.95215

From observation of Table 6, some trends mentioned in section 3.1 remained consistent. All variances and standard deviations were identical to part a, and the MVO portfolio outperformed again as the overall market took another decline, and again, the short selling of Citigroup was by far the largest contributor to this outperformance. However, the declines across the board were not as bad as Oct 2008, signalling a reduction in the lowering of tides of the overall market.

This time, all portfolios outperformed the market portfolio. Since the market portfolio is weighted by market capitalization and the risk-parity portfolio is relatively equally weighted, it is evident that the smaller market capitalization assets in general outperformed the larger market capitalization assets for Nov 2008.

When comparing the RMVO portfolios, the box portfolios outperformed the ellipsoid portfolios for Nov 2008. This accounted for the discrepancy observed for Oct 2008 and may have invalidated the argument that the corners of the box uncertainty created sub-optimal portfolios in section 3.1 for this specific portfolio. Further analysis would be required for confirmation.

From a relative perspective, the RMVO box portfolio with 95% confidence outperformed the same method with 90% confidence by approximately 4 times for Nov 2008. However, from an absolute perspective, both portfolios were relatively flat with minimal gains, and the difference in returns were nominal enough that it could not rule out the difference in performance being due to skew and uncertainty. Further analysis would be required to more accurately define the effects of choosing 95% confidence over 90% confidence.

3.3 Part C

Using the same optimization equations as defined in section 2, for all methods, efficient frontiers were created by varying the magnitude of the risk aversion parameters for optimization from Dec 30, 2004, to Sept 30, 2008. The risk aversion parameters for all methods were assumed to have a lower limit of 50, and an upper limit of 500, with 25 total risk aversion parameter values equally spaced between the lower and upper limits.

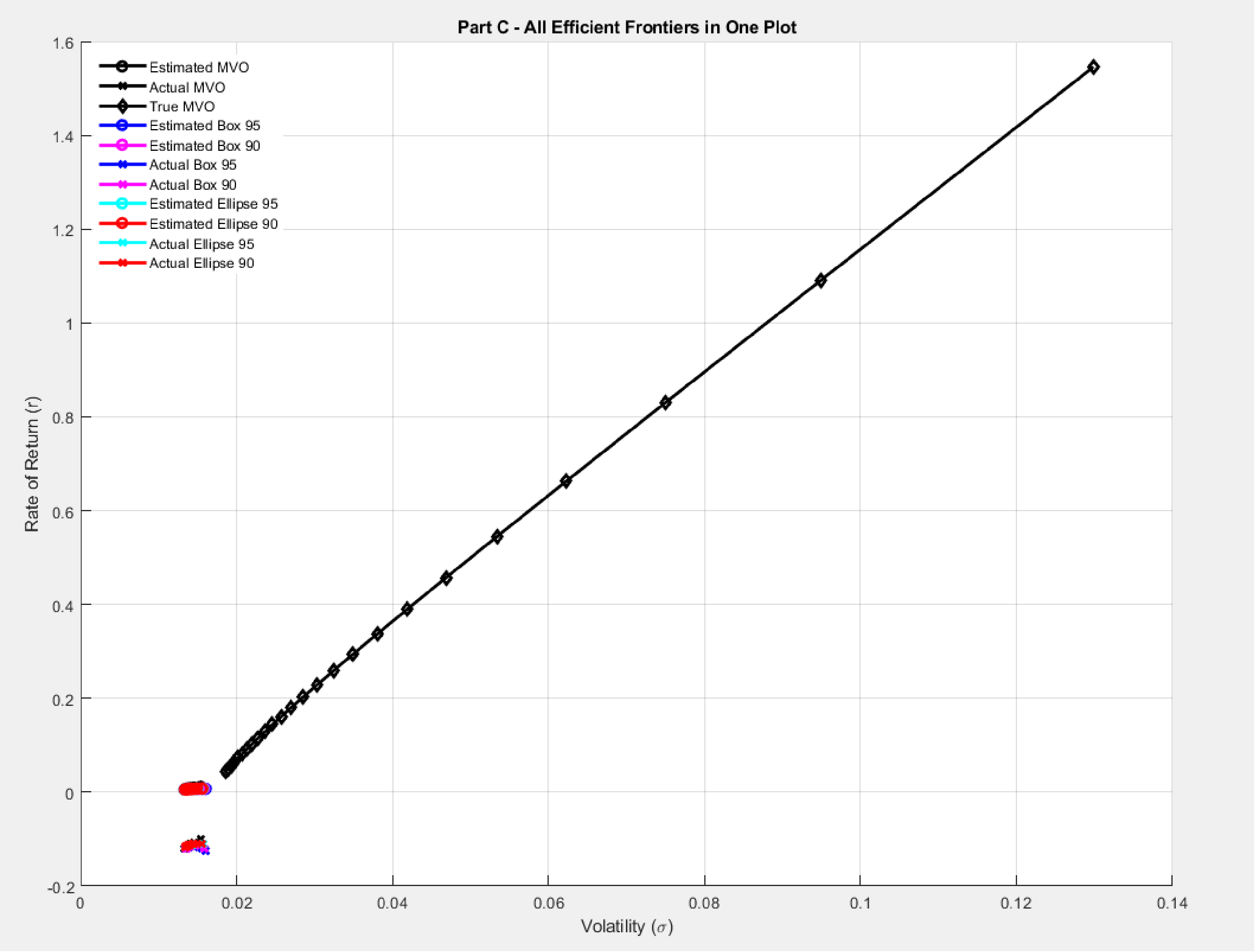
Three different forms of efficient frontiers were generated for question part c: estimated efficient frontier, actual efficient frontier, and true efficient frontier. The returns for the estimated efficient frontier were generated using optimization given expected asset returns using data from Dec 30, 2004, to Sept 30, 2008, the returns for the true efficient frontier were generated using optimization given realized asset returns for Oct 2008, and the returns for the actual efficient frontier were generated using the optimal weights computed for the estimated efficient frontier multiplied by the realized asset returns for Oct 2008. All three efficient frontier types were generated using all 25 total risk aversion parameters defined above. Table 7 summarizes all the efficient frontiers generated as per the project problem statement for part c.

Table 7 – List of Efficient Frontiers Generated

Optimization Method	Estimated Efficient Frontier	True Efficient Frontier	Actual Efficient Frontier
MVO	x	x	x
RMVO Box 95	x		x
RMVO Box 90	x		x
RMVO Ellipsoidal 95	x		x
RMVO Ellipsoidal 90	x		x

A total of 11 efficient frontiers were generated and plotted. Figure 7 shows the plot of all the efficient frontiers plotted on the same graph.

Figure 7 – All Efficient Frontiers in One Plot



From observation of Figure 7, it is evident that the estimated efficient frontiers all became grouped together as shown in the upper cluster of circular markers in the bottom left corner of the graph, the actual efficient frontiers all became grouped together as shown in the lower cluster of crossed markers in the bottom left corner of the graph, and the true efficient frontier stretched much larger magnitudes of return and volatility compared to the estimated and actual frontiers.

It makes sense that the true efficient frontier using MVO spans through much larger magnitudes of return and volatility because since the returns are already known with certainty, and short selling is allowed, one could theoretically achieve infinite returns by short selling all the assets with negative returns and using the cash to buy all the assets with positive returns. The measured volatility would not matter because all returns are known with certainty anyway. Since the risk-aversion parameter was bounded with a lower limit of 50 and an upper limit of 500, infinite returns were not achieved, but as the risk aversion parameter approaches 0, the true rate of return and volatility should both approach infinity. This was the reason why the lower limit of the risk-aversion parameter was set at a number as high as 50 because smaller lower bounds would result in true efficient frontiers eclipsing any estimated or actual results.

Table 8 demonstrates the difference between the expected asset returns and the true asset returns for Oct 2008.

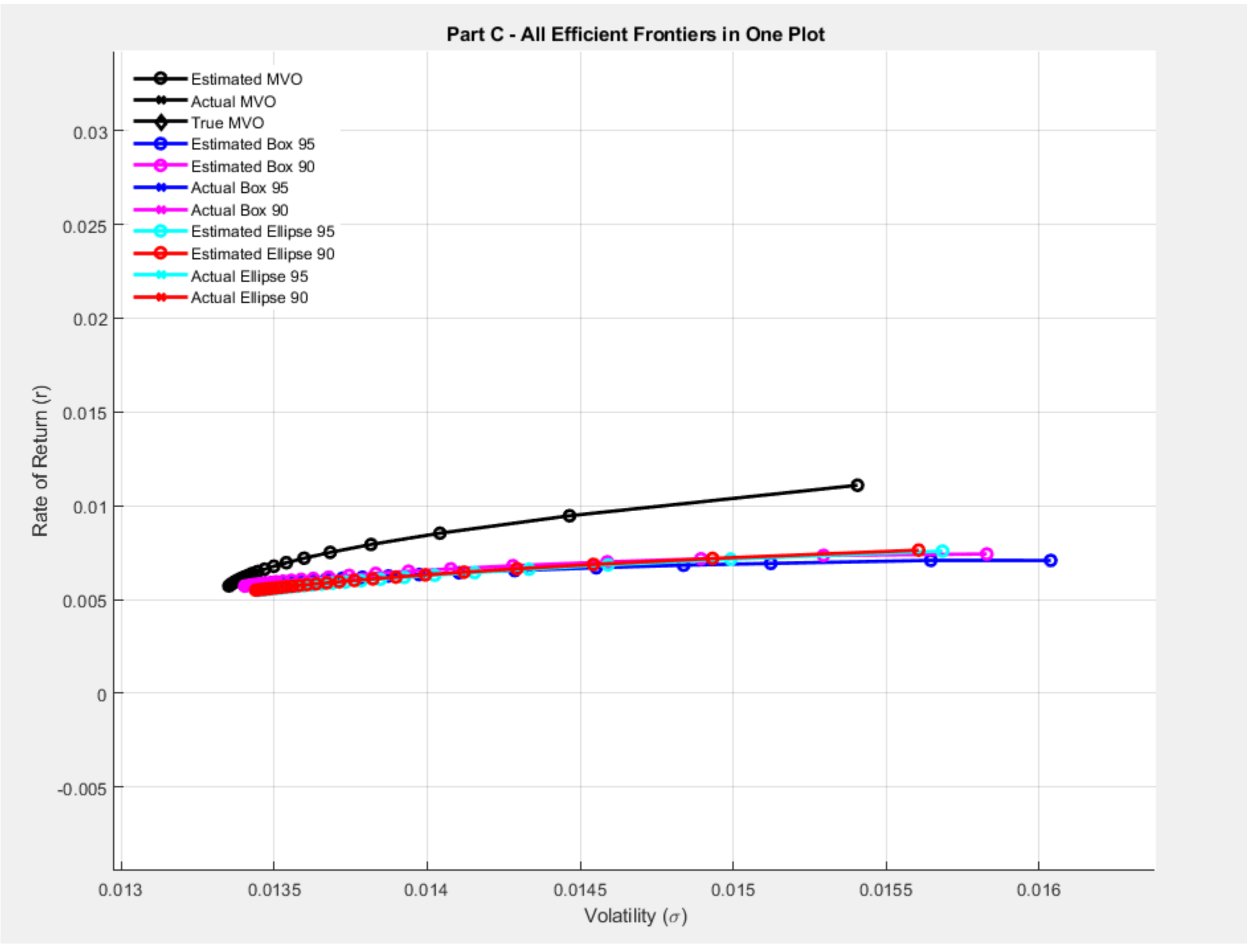
Table 8 – Expected Asset Returns vs True Asset Returns

	Expected Asset Returns (%)	True Asset Returns (%)
AAPL	3.3494	-5.3405
C	-1.2696	-33.447
CAT	1.0266	-35.906
DIS	0.35545	-15.575
ED	0.4593	0.83796
F	-1.2175	-57.885
IBM	0.78128	-20.511
JNJ	0.40426	-11.461
JPM	1.067	-11.67
KO	0.83027	-16.087
MCD	1.8157	-6.1102
MRO	2.3134	-27.013
NEM	0.27536	-31.885
PEP	0.87046	-19.516
PFE	-0.10945	-3.9588
T	1.055	-4.1189
VZ	0.36366	-7.5413
WFC	1.0106	-9.2726
WMT	0.53788	-6.8125
XOM	1.2705	-4.5583

As observed from Table 8, the absolute values of the true asset returns were significantly larger than the absolute values of the expected asset returns. Since the objective functions for both MVO and RMVO methods attempted to minimize portfolio variance while maximizing portfolio returns (i.e. penalizing the minimization of portfolio returns), and true asset returns were significantly greater than expected asset returns, the maximizing of portfolio returns far outweighed the minimization of portfolio variance for the true efficient frontier compared to all other generated efficient frontiers. As a result, the objective function for the true efficient frontier was far more risk-seeking than the objective functions for all other efficient frontiers and requires a much larger risk averse parameter to achieve the same level of volatility as the other efficient frontiers.

Figure 7a is a narrowed down view of Figure 7, focused only on the estimated efficient frontiers.

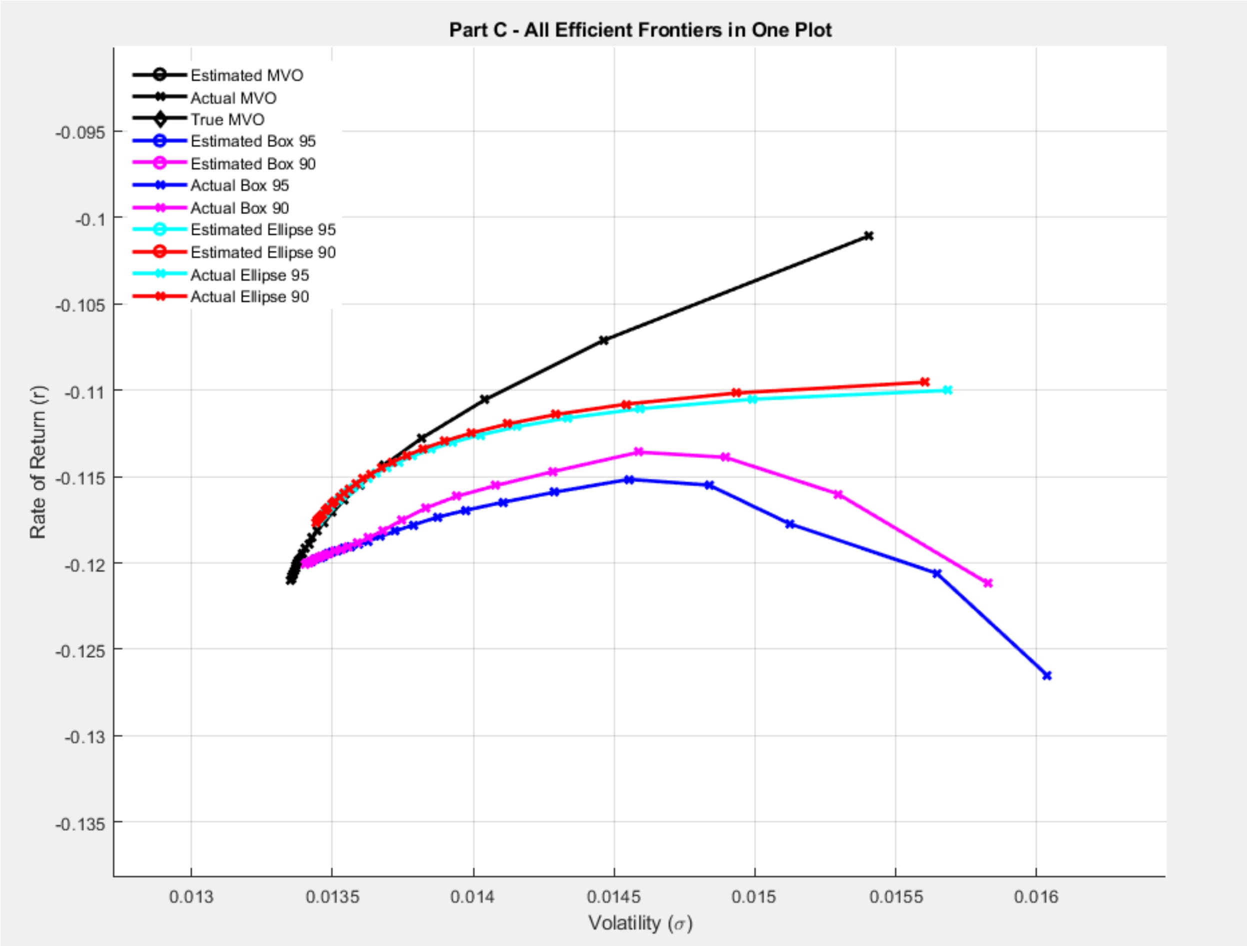
Figure 7a – All Efficient Frontiers in One Plot: Zoomed in on Estimated Efficient Frontiers



It is evident that from Figure 7a, the estimated MVO has higher expected returns compared to all the RMVO efficient frontiers. This is logical because the concept of RMVO is aimed at reducing the expected return uncertainty errors by penalizing return constraints. All the RMVO estimated efficient frontiers are overlapping because they were all created with comparable confidence levels. Additionally, the box and ellipsoidal robust optimization methods were designed to reach similar conclusions as they are just two methods of addressing the same issues with traditional MVO.

Figure 7b is a narrowed down view of Figure 7, focused only on the actual efficient frontiers.

Figure 7b – All Efficient Frontiers in One Plot: Zoomed in on Actual Efficient Frontiers



As observed in Figure 7b, the actual MVO frontier outperformed all the RMVO efficient frontiers. This is consistent with the observation of MVO outperforming all other portfolios in part a (section 3.1).

It was also observed that the actual efficient frontiers for the ellipsoidal RMVO method outperformed the actual efficient frontiers for the box RMVO method. This could be attributed to the corners of the box uncertainty being weak points for the box RMVO method.