Probabilistic Programming

Research Project

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Call By Push Value

Types

$$\begin{aligned}
\textbf{Val} &::= \mathbf{U}B \,|\, \Sigma_{i\in I} A_i \,|\, \textbf{unit} \,|\, A \times A \\
\textbf{Comp} &::= \mathbf{F}A \,|\, \Pi_{i\in I} B_i \,|\, \textbf{Val} \to \textbf{Comp} \\
\tau &::= \textbf{Val} \,|\, \textbf{Comp}
\end{aligned}$$

Expressions

$$\begin{aligned}
e ::= & \\
&|\lambda x.\, e \\
&|e'\, e \\
&|\textbf{produce}\, e \\
&|\textbf{thunk}\, e \\
&|\textbf{force}\, e \\
&|e\, \textbf{to}\, x.\, e \\
&|\textbf{let}\, x\, \textbf{be}\, e.\, e
\end{aligned}$$

Measure Theory

Definition (σ-algebra)**.** *A σ-algebra $\mathcal{B}$ on a set S is a collection of subsets of S*

- *containing the empty set $\varnothing$*
- *closed under complementation in S*
- *closed under countable union in S*

Definition (Measurable Space)**.** *Let S be a set and $\mathcal{B}$ be a σ-algebra on S. The pair $(S, \mathcal{B})$ is a* **measurable space***. The elements of $\mathcal{B}$ are called* **measurable sets** *of S.*

Remark (Probability)**.** *In probability theory, S can be thought of as the set of outcomes $\mathcal{B}$ can be thought as the set of events.*

Definition (Measurable Functions)**.** *Let $(S, \mathcal{B}_S)$ and $(T, \mathcal{B}_T)$ be measurable spaces. A function $f : S \to T$ is* **measurable** *if the inverse image*

$$f^{-1}(B) = \{x \in S | f(x) \in B\}$$

of every measurable subset $B \in \mathcal{B}_T$ is a measurable subset of S.

Definition (Countably Additive)**.** *A function $\mu : \mathcal{B} \to \mathbb{R}$ is* **countably additive** *if $\mathcal{A}$ is a countable set of pairwise disjoint events, then $\mu(\bigcup \mathcal{A}) = \sum_{A\in\mathcal{A}} \mu(A)$. Equivalently, if $A_0, A_1, A_2, \ldots$ is a countable collection of measurable sets such that $A_n \subseteq A_{n+1}$ for all $n \geq 0$, then $\lim_n \mu(A_n)$ exists and is equal to $\mu(\bigcup_n A_n)$.*

Definition (Signed Finite Measure)**.** *A* **signed finite measure** *on $(S, \mathcal{B})$ is a countably additive map*

$$\mu : \mathcal{B} \to \mathbb{R}$$

such that $\mu(\varnothing) = 0$.

Definition (Product Space)**.** *The* **product space** *of two measurable spaces $(S_1, \mathcal{B}_1)$ and $(S_2, \mathcal{B}_2)$ is $(S_1 \times S_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ where $S_1 \times S_2$ is the Cartesian product and $\mathcal{B}_1 \otimes \mathcal{B}_2 \triangleq \sigma(\{B_1 \times B_2 | B_1 \in \mathcal{B}_1,\, B_2 \in \mathcal{B}_2\})$.*

Probabilistic Imp

Syntax

$$
\begin{aligned}
d ::= &\, a \\
&\mid x \\
&\mid d \ \mathbf{op} \ d \\
\\
t ::= &\, d \\
&\mid \mathbf{coin}\ () \\
&\mid \mathbf{rand}\ () \\
&\mid t \ \mathbf{op} \ d \\
\\
b ::= &\, \mathbf{true} \\
&\mid \mathbf{false} \\
&\mid d == d \\
&\mid d < d \\
&\mid d > d \\
&\mid b \ \&\& \ b \\
&\mid b \,||\, b \\
&\mid !b \\
\\
e ::= &\, \mathbf{skip} \\
&\mid x := t \\
&\mid e; e \\
&\mid \mathbf{if}\ b\ \mathbf{then}\ e\ \mathbf{else}\ e \\
&\mid \mathbf{while}\ b\ \mathbf{do}\ e
\end{aligned}
$$

Small Step Semantics

$[\![t]\!] : \mathbb{R}^n \times 0, 1^\omega \times [0, 1]^\omega \to \mathbb{R} \times 0, 1^\omega \times [0, 1]^\omega$

$$
\begin{aligned}
[\![a]\!] : (s, m, p) &\triangleq (a, m, p) \\
[\![x_i]\!] : (s, m, p) &\triangleq (s(i), m, p) \\
[\![\mathbf{coin}\ ()]\!] : (s, m, p) &\triangleq (\mathbf{hd}\ m, \mathbf{tl}\ m, p) \\
[\![\mathbf{rand}\ ()]\!] : (s, m, p) &\triangleq (\mathbf{hd}\ p, m, \mathbf{tl}\ p) \\
[\![t\ \mathbf{op}\ t]\!] : (s, m, p) &\triangleq
\end{aligned}
$$

Denotational Semantics

Definition. $T_B : \mathcal{M}\mathbb{R}^n \to \mathcal{M}\mathbb{R}^n$ *given by*

$$T_B = \lambda x.\, \mu(B \cap x)$$

$$
\begin{aligned}
[\![\mathbf{skip}]\!] &\triangleq \mathrm{Id}_{\mathcal{M}\mathbb{R}^n} \\
[\![x_i := t]\!] &\triangleq \mu \mapsto (F^i_t)_*(\mu) \\
[\![e_1; e_2]\!] &\triangleq [\![e_2]\!] \circ [\![e_1]\!] \\
[\![\mathbf{if}\ b\ \mathbf{then}\ e_1\ \mathbf{else}\ e_2]\!] &\triangleq [\![e_1]\!] \circ T_{[\![b]\!]} + [\![e_2]\!] \circ T_{[\![b]\!]^C} \\
[\![\mathbf{while}\ b\ \mathbf{do}\ e]\!] &\triangleq \bigvee_{n \geq 0} \tau^n(0) \qquad \text{where } \tau(S) = S \circ [\![e]\!] \circ T_{[\![b]\!]} + T_{[\![b]\!]^C}
\end{aligned}
$$

Example 1

$$
\begin{aligned}
[\![\textbf{if } 1 == 1 \textbf{ then } x := 0 \textbf{ else } x := 1]\!] &= [\![x := 0]\!] \circ T_{[\![1==1]\!]} + [\![x := 1]\!] \circ T_{[\![1==1]\!]^C} \\
&= [\![x := 0]\!] \circ T_{\mathbb{R}} + [\![x := 1]\!] \circ T_{\varnothing} \\
&= [\![x := 0]\!] \circ (\mu \mapsto \lambda x.\, \mu(x \cap \mathbb{R})) + [\![x := 1]\!] \circ (\mu \mapsto \lambda x.\, \mu(x \cap \varnothing)) \\
&= (\mu \mapsto \mu(\mathbb{R})\delta_0) \circ (\mu \mapsto \mu) + (\mu \mapsto \mu(\mathbb{R})\delta_1) \circ (\mu \mapsto \lambda x.\, \mu(\varnothing)) \\
&= \mu \mapsto \mu(\mathbb{R})\delta_0 + \mu(\varnothing)\delta_1 \\
[\![\textbf{if } 1 == 1 \textbf{ then } x := 0 \textbf{ else } x := 1]\!] &= \mu \mapsto \mu(\mathbb{R})\delta_0
\end{aligned}
$$

Example 2

$$
\begin{aligned}
[\![x := 1; y := 2]\!] &= [\![y := 2]\!] \circ [\![x := 1]\!] \\
&= (\mu \mapsto \lambda(B_1, B_2).\, \mu(B_1 \times \mathbb{R})\delta_2(B_2)) \circ (\mu \mapsto \lambda(B_1, B_2).\, \mu(\mathbb{R} \times B_2)\delta_1(B_1)) \\
&= \mu \mapsto \lambda(B_1, B_2).\, \mu(\mathbb{R}^2)\delta_1(B_1)\delta_2(B_2)
\end{aligned}
$$

$$
\begin{aligned}
[\![y := 2; x := 1]\!] &= [\![x := 1]\!] \circ [\![y := 2]\!] \\
&= (\mu \mapsto \lambda(B_1, B_2).\, \mu(\mathbb{R} \times B_2)\delta_1(B_1)) \circ (\mu \mapsto \lambda(B_1, B_2).\, \mu(B_1 \times \mathbb{R})\delta_2(B_2)) \\
&= \mu \mapsto \lambda(B_1, B_2).\, \mu(\mathbb{R}^2)\delta_1(B_1)\delta_2(B_2)
\end{aligned}
$$

Example 3

Claim.

$$
[\![\textbf{ while } x == 0 \textbf{ do } x := \textbf{coin}()]\!] = \mu \mapsto \mu(\{0,1\})\delta_1 + \mu(- \cap \{0,1\}^C)
$$

Proof. I will show by induction that for all $k \geq 1$,

$$
\tau^k(0) = \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1
$$

If we then take the limit as $k \to \infty$, we get

$$
\mu \mapsto \mu(- \cap \{0\}^C) + \mu(\{0\})\delta_1
$$

- **Base Case:** $\tau^1(0)$

$$
\begin{aligned}
\tau^1(0) &= 0 \circ [\![x := \textbf{coin}()]\!] \circ T_{\{0\}} + T_{\{0\}^C} \\
&= T_{\{0\}^C} \\
&= \mu \mapsto \mu(- \cap \{0\}^C) \\
\tau^1(0) &= \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(1-1)})\mu(\{0\})\delta_1
\end{aligned}
$$

- **Inductive Case:** $\tau^{k+1}(0)$

$$
\begin{aligned}
\tau^{k+1}(0) &= \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ [\![x := \textbf{coin}()]\!] \circ T_{\{0\}} + T_{\{0\}^C} \\
&= \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ \mu \mapsto \mu(\mathbb{R}) \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) \circ \mu \mapsto \mu(- \cap \{0\}) + T_{\{0\}^C} \\
&= \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ \mu \mapsto \mu(\{0\}) \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) + T_{\{0\}^C} \\
&= \mu \mapsto \mu(\{0\}) \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\}) \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)(\{0\})\delta_1 + T_{\{0\}^C} \\
&= \mu \mapsto \frac{1}{2}\mu(\{0\})\delta_1 + (1 - 2^{-(k-1)})\frac{1}{2}\mu(\{0\})\delta_1 + T_{\{0\}^C} \\
&= \mu \mapsto \frac{1}{2}\mu(\{0\})\delta_1 + \frac{1}{2}\mu(\{0\})\delta_1 - 2^{-(k-1)-1}\mu(\{0\})\delta_1 + T_{\{0\}^C} \\
&= \mu \mapsto \mu(\{0\})\delta_1 - 2^{-(k)}\mu(\{0\})\delta_1 + T_{\{0\}^C} \\
&= \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-((k+1)-1)})\mu(\{0\})\delta_1
\end{aligned}
$$

QED

$$
\begin{aligned}
[\![x := 0;\ \textbf{while}\ x == 0\ \textbf{do}\ x := \textbf{coin}()]\!] &= [\![\ \textbf{while}\ x == 0\ \textbf{do}\ x := \textbf{coin}()]\!] \circ [\![x := 0]\!] \\
&= \mu \mapsto \mu(\{0,1\})\delta_1 + \mu(- \cap \{0,1\}^C) \circ \mu \mapsto \mu(\mathbb{R})\delta_0 \\
&= \mu \mapsto (\mu(\mathbb{R})\delta_0)(\{0,1\})\delta_1 + (\mu(\mathbb{R})\delta_0)(- \cap \{0,1\}^C) \\
&= \mu \mapsto \mu(\mathbb{R})\delta_1
\end{aligned}
$$

Independently and Identically Distributed ($\lambda_{\mathbf{IID}}$)

Syntax

$$\begin{aligned} e ::=\ & x \\ & |() \\ & |\lambda x : \tau .\, e \\ & |\mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 \\ & |e_1\, e_2 \\ & |r \\ & |e_1 \oplus e_2 \\ & |\mathbf{coin}() \\ & |\mathbf{rand}() \\ & |\mathbf{case}\ e_1\ \mathbf{of}\ e_2 \,|\, e_3 \\ & |\mathbf{inl}_{\tau_1+\tau_2} e \\ & |\mathbf{inr}_{\tau_1+\tau_2} e \\ & |\mathbf{\#1}\ e \\ & |\mathbf{\#2}\ e \\ & |(e_1, e_2) \end{aligned}$$

$$\begin{aligned} \tau ::=\ & \mathbf{unit} \\ & |\mathbb{R} \\ & |\tau_1 \to \tau_2 \\ & |\tau_1 + \tau_2 \\ & |\tau_1 \times \tau_2 \end{aligned}$$

$$\begin{aligned} v ::=\ & \lambda x : \tau .\, e \\ & |r \\ & |(e, e) \end{aligned}$$

$$\begin{aligned} E ::=\ & [\cdot] \\ & |E\, e \\ & |E \oplus e \\ & |v \oplus E \\ & |(\mathbf{case}\ E\ \mathbf{of}\ e_2 \,|\, e_3) \\ & |\mathbf{inl}_{\tau_1+\tau_2} E \\ & |\mathbf{inr}_{\tau_1+\tau_2} E \\ & |\#\ \mathbf{1}\ E \\ & |\#\ \mathbf{2}\ E \end{aligned}$$

Semantics

The small-step step semantics can be modeled by the relation on $(e \times \{0,1\}^\omega \times [0,1]^\omega) \times (e \times \{0,1\}^\omega \times [0,1]^\omega)$ defined below.

$$\frac{\langle e,\, n,\, m\rangle \to \langle e',\, n',\, m'\rangle}{\langle E(e),\, n,\, m\rangle \to \langle E(e'),\, n',\, m'\rangle}\ \textsc{Context}$$

$$\frac{}{\langle \mathbf{rand}(),\, n,\, m\rangle \to \langle \mathbf{hd}\ m,\, n,\, \mathbf{tl}\ m\rangle}\ \textsc{Rand}$$

$$\frac{}{\langle (\lambda x : \tau .\, e)\, e_2,\, n,\, m\rangle \to \langle e\{e_2/x\},\, n,\, m\rangle}\ \beta\text{-}\textsc{Reduction}$$

$$\frac{}{\langle (\mathbf{case}\ \mathbf{inl}_{\tau_1+\tau_2} e\ \mathbf{of}\ e_2 \,|\, e_3),\, n,\, m\rangle \to \langle e_2\, e,\, n,\, m\rangle}\ \textsc{Case-Left}$$

$$\frac{}{\langle \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2,\, n,\, m\rangle \to \langle e_2\{e_1/x\},\, n,\, m\rangle}\ \textsc{Let}$$

$$\frac{}{\langle (\mathbf{case}\ \mathbf{inr}_{\tau_1+\tau_2} e\ \mathbf{of}\ e_2 \,|\, e_3),\, n,\, m\rangle \to \langle e_3\, e,\, n,\, m\rangle}\ \textsc{Case-Right}$$

$$\frac{r_1 \bar{\oplus} r_2 = r}{\langle r_1 \oplus r_2,\, n,\, m\rangle \to \langle r,\, n,\, m\rangle}\ \textsc{Bop}$$

$$\frac{}{\langle \mathbf{\#1}\ (e_1, e_2),\, n,\, m\rangle \to \langle e_1,\, n,\, m\rangle}\ \textsc{Proj-1}$$

$$\frac{}{\langle \mathbf{coin}(),\, n,\, m\rangle \to \langle \mathbf{hd}\ n,\, \mathbf{tl}\ n,\, m\rangle}\ \textsc{Coin}$$

$$\frac{}{\langle \mathbf{\#2}\ (e_1, e_2),\, n,\, m\rangle \to \langle e_2,\, n,\, m\rangle}\ \textsc{Proj-2}$$

Static Semantics

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}\ \textsc{Var}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'}\ \textsc{App}$$

$$\frac{}{\Gamma \vdash () : \mathbf{unit}}\ \textsc{Unit}$$

$$\frac{}{\Gamma \vdash r : \mathbb{R}}\ \textsc{Num}$$

$$\frac{\Gamma \vdash e : \tau'}{\Gamma \vdash (\lambda x : \tau .\, e) : \tau \to \tau'}\ \textsc{Fun}$$

$$\frac{}{\Gamma \vdash \mathbf{coin}() : \mathbb{R}}\ \textsc{Coin}$$

$$\frac{\Gamma \vdash e_1 : \tau \qquad \Gamma_{x:\tau} \vdash e_2 : \tau'}{\Gamma \vdash \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 : \tau'}\ \textsc{Let}$$

$$\frac{}{\Gamma \vdash \mathbf{rand}() : \mathbb{R}}\ \textsc{Rand}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \qquad \Gamma \vdash e_2 : \tau_1 \to \tau \qquad \Gamma \vdash e_3 : \tau_2 \to \tau}{\Gamma \vdash \mathbf{case}\ e_1\ \mathbf{with}\ e_2 \,|\, e_3 : \tau}\ \textsc{Case}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathbf{inl}_{\tau_1+\tau_2} e : \tau_1 + \tau_2}\ \textsc{In-Left}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathbf{inr}_{\tau_1+\tau_2} e : \tau_1 + \tau_2}\ \textsc{In-Right}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#\ \mathbf{1}\ e : \tau_1}\ \textsc{In-Left}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#\ \mathbf{2}\ e : \tau_2}\ \textsc{In-Left}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}\ \textsc{Pair}$$

Example 1

$$\begin{aligned}
\langle (\lambda x.\, x + x)\, \mathbf{coin}(),\ 1 :: 0 :: n,\ m \rangle &\to \langle \mathbf{coin}() + \mathbf{coin}(),\ 1 :: 0 :: n,\ m \rangle \\
&\to \langle 1 + \mathbf{coin}(),\ 0 :: n,\ m \rangle \\
&\to \langle 1 + 0,\ n,\ m \rangle \\
&\to \langle 1,\ n,\ m \rangle
\end{aligned}$$

Example 2

$$\begin{aligned}
\langle \mathbf{let}\ x = \mathbf{coin}()\ \mathbf{in}\ x + x,\ 1 :: 0 :: n,\ m \rangle &\to \langle \mathbf{coin}() + \mathbf{coin}(),\ 1 :: 0 :: n,\ m \rangle \\
&\to \langle 1 + \mathbf{coin}(),\ 0 :: n,\ m \rangle \\
&\to \langle 1 + 0,\ n,\ m \rangle \\
&\to \langle 1,\ n,\ m \rangle
\end{aligned}$$

Perfectly Correlated ($\lambda_{\mathbf{PC}}$)

Syntax

$$
\begin{aligned}
e ::= & \; x \\
& |() \\
& |\lambda x : \tau .\, e \\
& |\textbf{let } x = e_1 \textbf{ in } e_2 \\
& |e_1\, e_2 \\
& |r \\
& |e_1 \oplus e_2 \\
& |\textbf{coin}() \\
& |\textbf{rand}() \\
& |\textbf{case } e_1 \textbf{ of } e_2 \,|\, e_3 \\
& |\textbf{inl}_{\tau_1+\tau_2} e \\
& |\textbf{inr}_{\tau_1+\tau_2} e \\
& |\#\mathbf{1}\ e \\
& |\#\mathbf{2}\ e \\
& |(e_1, e_2)
\end{aligned}
$$

$$
\begin{aligned}
\tau ::= & \; \textbf{unit} \\
& |\mathbb{R} \\
& |\tau_1 \to \tau_2 \\
& |\tau_1 + \tau_2 \\
& |\tau_1 \times \tau_2
\end{aligned}
$$

$$
\begin{aligned}
v ::= & \; \lambda x : \tau .\, e \\
& |r \\
& |(v, v)
\end{aligned}
$$

$$
\begin{aligned}
E ::= & \; [\cdot] \\
& |E\, e \\
& |v\, E \\
& |\textbf{let } x = E \textbf{ in } e_2 \\
& |E \oplus e \\
& |v \oplus E \\
& |(\textbf{case } E \textbf{ of } e_2 \,|\, e_3) \\
& |\textbf{inl}_{\tau_1+\tau_2} E \\
& |\textbf{inr}_{\tau_1+\tau_2} E \\
& |\#\ \mathbf{1}\ E \\
& |\#\ \mathbf{2}\ E \\
& |(E, e) \\
& |(v, E)
\end{aligned}
$$

Semantics

The small-step step semantics can be modeled by the relation on $(e \times \{0,1\}^\omega \times [0,1]^\omega) \times (e \times \{0,1\}^\omega \times [0,1]^\omega)$ defined below.

$$\frac{\langle e,\, n,\, m\rangle \to \langle e',\, n',\, m'\rangle}{\langle E(e),\, n,\, m\rangle \to \langle E(e'),\, n',\, m'\rangle}\ \textsc{Context}$$

$$\frac{}{\langle \textbf{rand}(),\, n,\, m\rangle \to \langle \textbf{hd}\ m,\, n,\, \textbf{tl}\ m\rangle}\ \textsc{Rand}$$

$$\frac{}{\langle (\lambda x : \tau .\, e)\, v,\, n,\, m\rangle \to \langle e\{v/x\},\, n,\, m\rangle}\ \beta\text{-}\textsc{Reduction}$$

$$\frac{}{\langle (\textbf{case inl}_{\tau_1+\tau_2} v \textbf{ of } e_2 \,|\, e_3),\, n,\, m\rangle \to \langle e_2\, v,\, n,\, m\rangle}\ \textsc{Case-Left}$$

$$\frac{}{\langle \textbf{let } x = v \textbf{ in } e_2,\, n,\, m\rangle \to \langle e_2\{v/x\},\, n,\, m\rangle}\ \textsc{Let}$$

$$\frac{}{\langle (\textbf{case inr}_{\tau_1+\tau_2} v \textbf{ of } e_2 \,|\, e_3),\, n,\, m\rangle \to \langle e_3\, v,\, n,\, m\rangle}\ \textsc{Case-Right}$$

$$\frac{r_1 \bar{\oplus} r_2 = r}{\langle r_1 \oplus r_2,\, n,\, m\rangle \to \langle r,\, n,\, m\rangle}\ \textsc{Bop}$$

$$\frac{}{\langle \#\mathbf{1}\ (v_1, v_2),\, n,\, m\rangle \to \langle v_1,\, n,\, m\rangle}\ \textsc{Proj-1}$$

$$\frac{}{\langle \textbf{coin}(),\, n,\, m\rangle \to \langle \textbf{hd}\ n,\, \textbf{tl}\ n,\, m\rangle}\ \textsc{Coin}$$

$$\frac{}{\langle \#\mathbf{2}\ (v_1, v_2),\, n,\, m\rangle \to \langle v_2,\, n,\, m\rangle}\ \textsc{Proj-2}$$

Static Semantics

$$\frac{\Gamma \vdash e : \tau'}{\Gamma \vdash (\lambda x : \tau .\, e) : \tau \to \tau'}\ \textsc{Fun}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}\ \textsc{Var}$$

$$\frac{\Gamma \vdash e_1 : \tau \qquad \Gamma_{x:\tau} \vdash e_2 : \tau'}{\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 : \tau'}\ \textsc{Let}$$

$$\frac{}{\Gamma \vdash () : \textbf{unit}}\ \textsc{Unit}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'}\ \textsc{App}$$

$$\frac{}{\Gamma \vdash r : \mathbb{R}}\ \textsc{Num}$$

$$\frac{}{\Gamma \vdash \mathbf{coin}() : \mathbb{R}}\ \textsc{Coin}$$

$$\frac{}{\Gamma \vdash \mathbf{rand}() : \mathbb{R}}\ \textsc{Rand}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \qquad \Gamma \vdash e_2 : \tau_1 \to \tau \qquad \Gamma \vdash e_3 : \tau_2 \to \tau}{\Gamma \vdash \mathbf{case}\ e_1\ \mathbf{with}\ e_2 \,|\, e_3 : \tau}\ \textsc{Case}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathbf{inl}_{\tau_1+\tau_2} e : \tau_1 + \tau_2}\ \textsc{In-Left}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathbf{inr}_{\tau_1+\tau_2} e : \tau_1 + \tau_2}\ \textsc{In-Right}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#\ \mathbf{1}\ e : \tau_1}\ \textsc{In-Left}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#\ \mathbf{2}\ e : \tau_2}\ \textsc{In-Left}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}\ \textsc{Pair}$$

Example 1

$$\begin{aligned}
\langle (\lambda x.\, x + x)\, \mathbf{coin}(),\ 1 :: 0 :: n,\ m\rangle &\to \langle (\lambda x.\, x + x)\, 1,\ 0 :: n,\ m\rangle \\
&\to \langle 1 + 1,\ 0 :: n,\ m\rangle \\
&\to \langle 2,\ 0 :: n,\ m\rangle
\end{aligned}$$

Example 2

$$\begin{aligned}
\langle \mathbf{let}\ x = \mathbf{coin}()\ \mathbf{in}\ x + x,\ 1 :: 0 :: n,\ m\rangle &\to \langle \mathbf{let}\ x = 1\ \mathbf{in}\ x + x,\ 0 :: n,\ m\rangle \\
&\to \langle 1 + 1,\ 0 :: n,\ m\rangle \\
&\to \langle 2,\ 0 :: n,\ m\rangle
\end{aligned}$$

CBN to CBV

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &\triangleq x(\lambda y.\, y) \\
\mathcal{T}[\![r]\!] &\triangleq r \\
\mathcal{T}[\![\lambda x.\, e]\!] &\triangleq \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![\textbf{let } x = e_1 \textbf{ in } e_2]\!] &\triangleq \textbf{let } x = (\lambda z.\, \mathcal{T}[\![e_1]\!]) \textbf{ in } \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![e_1\, e_2]\!] &\triangleq \mathcal{T}[\![e_1]\!](\lambda z.\, \mathcal{T}[\![e_2]\!]) \\
\mathcal{T}[\![\textbf{coin}]\!] &\triangleq \textbf{coin} \\
\mathcal{T}[\![e_1 + e_2]\!] &\triangleq \mathcal{T}[\![e_1]\!] + \mathcal{T}[\![e_2]\!]
\end{aligned}
$$

CBV to CBN

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &\triangleq \\
\mathcal{T}[\![r]\!] &\triangleq r \\
\mathcal{T}[\![\lambda x.\, e]\!] &\triangleq \\
\mathcal{T}[\![\textbf{let } x = e_1 \textbf{ in } e_2]\!] &\triangleq \\
\mathcal{T}[\![e_1\, e_2]\!] &\triangleq \\
\mathcal{T}[\![\textbf{coin}]\!] &\triangleq
\end{aligned}
$$