Probabilistic Programming

Research Project

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Call By Push Value

Types

$$\mathbf{Val} ::= \mathbf{U}B \,|\, \Sigma_{i \in I} A_i \,|\, \mathbf{unit} \,|\, A \times A$$
$$\mathbf{Comp} ::= \mathbf{F}A \,|\, \Pi_{i \in I} B_i \,|\, \mathbf{Val} \to \mathbf{Comp}$$
$$\tau ::= \mathbf{Val} \,|\, \mathbf{Comp}$$

Expressions

$$
\begin{aligned}
e ::= \\
&|\lambda x.\, e \\
&|e'\, e \\
&|\mathbf{produce}\, e \\
&|\mathbf{thunk}\, e \\
&|\mathbf{force}\, e \\
&|e\, \mathbf{to}\, x.\, e \\
&|\mathbf{let}\, x\, \mathbf{be}\, e.\, e
\end{aligned}
$$

Measure Theory

Definition (σ-algebra). *A σ-algebra $\mathcal{B}$ on a set S is a collection of subsets of S*

- *containing the empty set $\varnothing$*

- *closed under complementation in S*

- *closed under countable union in S*

Definition (Measurable Space). *Let S be a set and $\mathcal{B}$ be a σ-algebra on S. The pair $(S, \mathcal{B})$ is a* **measurable space**. *The elements of $\mathcal{B}$ are called* **measurable sets** *of S.*

Remark (Probability). *In probability theory, S can be thought of as the set of outcomes $\mathcal{B}$ can be thought as the set of events.*

Definition (Measurable Functions). *Let $(S, \mathcal{B}_S)$ and $(T, \mathcal{B}_T)$ be measurable spaces. A function $f : S \to T$ is* **measurable** *if the inverse image*
$$f^{-1}(B) = \{x \in S | f(x) \in B\}$$
of every measurable subset $B \in \mathcal{B}_T$ is a measurable subset of S.

Definition (Countably Additive). *A function $\mu : \mathcal{B} \to \mathbb{R}$ is* **countably additive** *if $\mathcal{A}$ is a countable set of pairwise disjoint events, then $\mu(\bigcup \mathcal{A}) = \sum_{A \in \mathcal{A}} \mu(A)$. Equivalently, if $A_0, A_1, A_2, \ldots$ is a countable collection of measurable sets such that $A_n \subseteq A_{n+1}$ for all $n \geq 0$, then $\lim_n \mu(A_n)$ exists and is equal to $\mu(\bigcup_n A_n)$.*

Definition (Signed Finite Measure). *A* **signed finite measure** *on $(S, \mathcal{B})$ is a countably additive map*

$$\mu : \mathcal{B} \to \mathbb{R}$$

such that $\mu(\varnothing) = 0$.

Definition (Product Space). *The* **product space** *of two measurable spaces $(S_1, \mathcal{B}_1)$ and $(S_2, \mathcal{B}_2)$ is $(S_1 \times S_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ where $S_1 \times S_2$ is the Cartesian product and $\mathcal{B}_1 \otimes \mathcal{B}_2 \triangleq \sigma(\{B_1 \times B_2 | B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\})$.*

Probabilistic Imp

Syntax

$$d ::= a$$
$$| \, x$$
$$| \, d \; \mathbf{op} \; d$$

$$t ::= d$$
$$| \, \mathbf{coin} \; ()$$
$$| \, \mathbf{rand} \; ()$$
$$| \, t \; \mathbf{op} \; d$$

$$b ::= \mathbf{true}$$
$$| \, \mathbf{false}$$
$$| \, d == d$$
$$| \, d < d$$
$$| \, d > d$$
$$| \, b \; \&\& \; b$$
$$| \, b \, || \, b$$
$$| \, !b$$

$$e ::= \mathbf{skip}$$
$$| \, x := t$$
$$| \, e; e$$
$$| \, \mathbf{if} \; b \; \mathbf{then} \; e \; \mathbf{else} \; e$$
$$| \, \mathbf{while} \; b \; \mathbf{do} \; e$$

Small Step Semantics

$$[\![t]\!] : \mathbb{R}^n \times {0,1}^\omega \times [0,1]^\omega \to \mathbb{R} \times {0,1}^\omega \times [0,1]^\omega$$

$$[\![a]\!] : (s, m, p) \triangleq (a, m, p)$$
$$[\![x_i]\!] : (s, m, p) \triangleq (s(i), m, p)$$
$$[\![\mathbf{coin} \; ()]\!] : (s, m, p) \triangleq (\mathbf{hd} \; m, \mathbf{tl} \; m, p)$$
$$[\![\mathbf{rand} \; ()]\!] : (s, m, p) \triangleq (\mathbf{hd} \; p, m, \mathbf{tl} \; p)$$
$$[\![t \; \mathbf{op} \; t]\!] : (s, m, p) \triangleq$$

Denotational Semantics

Definition. $T_B : \mathcal{M}\mathbb{R}^n \to \mathcal{M}\mathbb{R}^n$ *given by*

$$T_B = \lambda x. \, \mu(B \cap x)$$

$$[\![\mathbf{skip}]\!] \triangleq \mathrm{Id}_{\mathcal{M}\mathbb{R}^n}$$
$$[\![x_i := t]\!] \triangleq \mu \mapsto (F_t^i)_*(\mu)$$
$$[\![e_1; e_2]\!] \triangleq [\![e_2]\!] \circ [\![e_1]\!]$$
$$[\![\mathbf{if} \; b \; \mathbf{then} \; e_1 \; \mathbf{else} \; e_2]\!] \triangleq [\![e_1]\!] \circ T_{[\![b]\!]} + [\![e_2]\!] \circ T_{[\![b]\!]^C}$$
$$[\![\mathbf{while} \; b \; \mathbf{do} \; e]\!] \triangleq \bigvee_{n \geq 0} \tau^n(0) \qquad \text{where } \tau(S) = S \circ [\![e]\!] \circ T_{[\![b]\!]} + T_{[\![b]\!]^C}$$

Example 1

$$\llbracket \textbf{if } 1 == 1 \textbf{ then } x := 0 \textbf{ else } x := 1 \rrbracket = \llbracket x := 0 \rrbracket \circ T_{\llbracket 1==1 \rrbracket} + \llbracket x := 1 \rrbracket \circ T_{\llbracket 1==1 \rrbracket^C}$$

$$= \llbracket x := 0 \rrbracket \circ T_{\mathbb{R}} + \llbracket x := 1 \rrbracket \circ T_{\varnothing}$$

$$= \llbracket x := 0 \rrbracket \circ (\mu \mapsto \lambda x.\, \mu(x \cap \mathbb{R})) + \llbracket x := 1 \rrbracket \circ (\mu \mapsto \lambda x.\, \mu(x \cap \varnothing))$$

$$= (\mu \mapsto \mu(\mathbb{R})\delta_0) \circ (\mu \mapsto \mu) + (\mu \mapsto \mu(\mathbb{R})\delta_1) \circ (\mu \mapsto \lambda x.\, \mu(\varnothing))$$

$$= \mu \mapsto \mu(\mathbb{R})\delta_0 + \mu(\varnothing)\delta_1$$

$$\llbracket \textbf{if } 1 == 1 \textbf{ then } x := 0 \textbf{ else } x := 1 \rrbracket = \mu \mapsto \mu(\mathbb{R})\delta_0$$

Example 2

$$\llbracket x := 1; y := 2 \rrbracket = \llbracket y := 2 \rrbracket \circ \llbracket x := 1 \rrbracket$$

$$= (\mu \mapsto \lambda(B_1, B_2).\, \mu(B_1 \times \mathbb{R})\delta_2(B_2)) \circ (\mu \mapsto \lambda(B_1, B_2).\, \mu(\mathbb{R} \times B_2)\delta_1(B_1))$$

$$= \mu \mapsto \lambda(B_1, B_2).\, \mu(\mathbb{R}^2)\delta_1(B_1)\delta_2(B_2)$$

$$\llbracket y := 2; x := 1 \rrbracket = \llbracket x := 1 \rrbracket \circ \llbracket y := 2 \rrbracket$$

$$= (\mu \mapsto \lambda(B_1, B_2).\, \mu(\mathbb{R} \times B_2)\delta_1(B_1)) \circ (\mu \mapsto \lambda(B_1, B_2).\, \mu(B_1 \times \mathbb{R})\delta_2(B_2))$$

$$= \mu \mapsto \lambda(B_1, B_2).\, \mu(\mathbb{R}^2)\delta_1(B_1)\delta_2(B_2)$$

Example 3

Claim.

$$\llbracket \textbf{ while } x == 0 \textbf{ do } x := \textbf{coin}() \rrbracket = \mu \mapsto \mu(\{0,1\})\delta_1 + \mu(- \cap \{0,1\}^C)$$

Proof. I will show by induction that for all $k \geq 1$,

$$\tau^k(0) = \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1$$

If we then take the limit as $k \to \infty$, we get

$$\mu \mapsto \mu(- \cap \{0\}^C) + \mu(\{0\})\delta_1$$

- **Base Case:** $\tau^1(0)$

$$\tau^1(0) = 0 \circ \llbracket x := \textbf{coin}() \rrbracket \circ T_{\{0\}} + T_{\{0\}^C}$$

$$= T_{\{0\}^C}$$

$$= \mu \mapsto \mu(- \cap \{0\}^C)$$

$$\tau^1(0) = \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(1-1)})\mu(\{0\})\delta_1$$

- **Inductive Case:** $\tau^{k+1}(0)$

$$\tau^{k+1}(0) = \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ \llbracket x := \textbf{coin}() \rrbracket \circ T_{\{0\}} + T_{\{0\}^C}$$

$$= \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ \mu \mapsto \mu(\mathbb{R})\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) \circ \mu \mapsto \mu(- \cap \{0\}) + T_{\{0\}^C}$$

$$= \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ \mu \mapsto \mu(\{0\})\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) + T_{\{0\}^C}$$

$$= \mu \mapsto \mu(\{0\})\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)(- \cap \{0\}^C) + (1 - 2^{-(k-1)})\mu(\{0\})\left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)(\{0\})\delta_1 + T_{\{0\}^C}$$

$$= \mu \mapsto \frac{1}{2}\mu(\{0\})\delta_1 + (1 - 2^{-(k-1)})\frac{1}{2}\mu(\{0\})\delta_1 + T_{\{0\}^C}$$

$$= \mu \mapsto \frac{1}{2}\mu(\{0\})\delta_1 + \frac{1}{2}\mu(\{0\})\delta_1 - 2^{-(k-1)-1}\mu(\{0\})\delta_1 + T_{\{0\}^C}$$

$$= \mu \mapsto \mu(\{0\})\delta_1 - 2^{-(k)}\mu(\{0\})\delta_1 + T_{\{0\}^C}$$

$$= \mu \mapsto \mu(- \cap \{0\}^C) + (1 - 2^{-((k+1)-1)})\mu(\{0\})\delta_1$$

QED

$$\llbracket x := 0; \ \textbf{while} \ x == 0 \ \textbf{do} \ x := \textbf{coin}() \rrbracket = \llbracket \textbf{while} \ x == 0 \ \textbf{do} \ x := \textbf{coin}() \rrbracket \circ \llbracket x := 0 \rrbracket$$

$$= \mu \mapsto \mu(\{0,1\})\delta_1 + \mu(- \cap \{0,1\}^C) \circ \mu \mapsto \mu(\mathbb{R})\delta_0$$

$$= \mu \mapsto (\mu(\mathbb{R})\delta_0)(\{0,1\})\delta_1 + (\mu(\mathbb{R})\delta_0)(- \cap \{0,1\}^C)$$

$$= \mu \mapsto \mu(\mathbb{R})\delta_1$$

$$\llbracket x := 0; \ \textbf{while} \ x == 0 \ \textbf{do} \ x := \textbf{coin}() \rrbracket = \llbracket \textbf{while} \ x == 0 \ \textbf{do} \ x := \textbf{coin}() \rrbracket \circ \llbracket x := 0 \rrbracket$$

$$= \mu \mapsto \mu(\{0,1\})\delta_1 + \mu(- \cap \{0,1\}^C)$$

Independently and Identically Distributed (λ_{IID})

Syntax

$$
\begin{aligned}
e ::=\ & x \\
& |\ () \\
& |\ \lambda x : \tau.\, e \\
& |\ \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 \\
& |\ e_1\, e_2 \\
& |\ r \\
& |\ e_1 \oplus e_2 \\
& |\ \mathbf{coin}() \\
& |\ \mathbf{rand}() \\
& |\ \mathbf{case}\ e_1\ \mathbf{of}\ e_2\,|\,e_3 \\
& |\ \mathbf{inl}_{\tau_1+\tau_2} e \\
& |\ \mathbf{inr}_{\tau_1+\tau_2} e \\
& |\ \#\mathbf{1}\ e \\
& |\ \#\mathbf{2}\ e \\
& |\ (e_1, e_2)
\end{aligned}
$$

$$
\begin{aligned}
\tau ::=\ & \mathbf{unit} \\
& |\ \mathbb{R} \\
& |\ \tau_1 \to \tau_2 \\
& |\ \tau_1 + \tau_2 \\
& |\ \tau_1 \times \tau_2 \\[4pt]
v ::=\ & \lambda x : \tau.\, e \\
& |\ r \\
& |\ (e, e)
\end{aligned}
$$

$$
\begin{aligned}
E ::=\ & [\cdot] \\
& |\ E\, e \\
& |\ E \oplus e \\
& |\ v \oplus E \\
& |\ (\mathbf{case}\ E\ \mathbf{of}\ e_2\,|\,e_3) \\
& |\ \mathbf{inl}_{\tau_1+\tau_2} E \\
& |\ \mathbf{inr}_{\tau_1+\tau_2} E \\
& |\ \#\ \mathbf{1}\ E \\
& |\ \#\ \mathbf{2}\ E
\end{aligned}
$$

Semantics

The small-step step semantics can be modeled by the relation on $(e \times \{0,1\}^\omega \times [0,1]^\omega) \times (e \times \{0,1\}^\omega \times [0,1]^\omega)$ defined below.

$$
\frac{\langle e,\, n,\, m \rangle \to \langle e',\, n',\, m' \rangle}{\langle E(e),\, n,\, m \rangle \to \langle E(e'),\, n',\, m' \rangle}\ \text{Context}
$$

$$
\frac{}{\langle \mathbf{rand}(),\, n,\, m \rangle \to \langle \mathbf{hd}\ m,\, n,\, \mathbf{tl}\ m \rangle}\ \text{Rand}
$$

$$
\frac{}{\langle (\lambda x : \tau.\, e)\, e_2,\, n,\, m \rangle \to \langle e\{e_2/x\},\, n,\, m \rangle}\ \beta\text{-Reduction}
$$

$$
\frac{}{\langle (\mathbf{case}\ \mathbf{inl}_{\tau_1+\tau_2} e\ \mathbf{of}\ e_2\,|\,e_3),\, n,\, m \rangle \to \langle e_2\, e,\, n,\, m \rangle}\ \text{Case-Left}
$$

$$
\frac{}{\langle \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2,\, n,\, m \rangle \to \langle e_2\{e_1/x\},\, n,\, m \rangle}\ \text{Let}
$$

$$
\frac{}{\langle (\mathbf{case}\ \mathbf{inr}_{\tau_1+\tau_2} e\ \mathbf{of}\ e_2\,|\,e_3),\, n,\, m \rangle \to \langle e_3\, e,\, n,\, m \rangle}\ \text{Case-Right}
$$

$$
\frac{r_1 \bar{\oplus} r_2 = r}{\langle r_1 \oplus r_2,\, n,\, m \rangle \to \langle r,\, n,\, m \rangle}\ \text{Bop}
$$

$$
\frac{}{\langle \#\mathbf{1}\ (e_1, e_2),\, n,\, m \rangle \to \langle e_1,\, n,\, m \rangle}\ \text{Proj-1}
$$

$$
\frac{}{\langle \mathbf{coin}(),\, n,\, m \rangle \to \langle \mathbf{hd}\ n,\, \mathbf{tl}\ n,\, m \rangle}\ \text{Coin}
$$

$$
\frac{}{\langle \#\mathbf{2}\ (e_1, e_2),\, n,\, m \rangle \to \langle e_2,\, n,\, m \rangle}\ \text{Proj-2}
$$

Static Semantics

$$
\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}\ \text{Var}
$$

$$
\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'}\ \text{App}
$$

$$
\frac{}{\Gamma \vdash () : \mathbf{unit}}\ \text{Unit}
$$

$$
\frac{}{\Gamma \vdash r : \mathbb{R}}\ \text{Num}
$$

$$
\frac{\Gamma \vdash e : \tau'}{\Gamma \vdash (\lambda x : \tau.\, e) : \tau \to \tau'}\ \text{Fun}
$$

$$
\frac{}{\Gamma \vdash \mathbf{coin}() : \mathbb{R}}\ \text{Coin}
$$

$$
\frac{\Gamma \vdash e_1 : \tau \qquad \Gamma_{x:\tau} \vdash e_2 : \tau'}{\Gamma \vdash \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 : \tau'}\ \text{Let}
$$

$$
\frac{}{\Gamma \vdash \mathbf{rand}() : \mathbb{R}}\ \text{Rand}
$$

$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \qquad \Gamma \vdash e_2 : \tau_1 \to \tau \qquad \Gamma \vdash e_3 : \tau_2 \to \tau}{\Gamma \vdash \textbf{case } e_1 \textbf{ with } e_2 \,|\, e_3 : \tau} \;\textsc{Case}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#\,\textbf{1}\,e : \tau_1} \;\textsc{In-Left}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \textbf{inl}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2} \;\textsc{In-Left}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#\,\textbf{2}\,e : \tau_2} \;\textsc{In-Left}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \textbf{inr}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2} \;\textsc{In-Right}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \;\textsc{Pair}$$

Example 1

$$\langle (\lambda x.\, x + x)\, \textbf{coin}(),\, 1 :: 0 :: n,\, m \rangle \to \langle \textbf{coin}() + \textbf{coin}(),\, 1 :: 0 :: n,\, m \rangle$$
$$\to \langle 1 + \textbf{coin}(),\, 0 :: n,\, m \rangle$$
$$\to \langle 1 + 0,\, n,\, m \rangle$$
$$\to \langle 1,\, n,\, m \rangle$$

Example 2

$$\langle \textbf{let } x = \textbf{coin}() \textbf{ in } x + x,\, 1 :: 0 :: n,\, m \rangle \to \langle \textbf{coin}() + \textbf{coin}(),\, 1 :: 0 :: n,\, m \rangle$$
$$\to \langle 1 + \textbf{coin}(),\, 0 :: n,\, m \rangle$$
$$\to \langle 1 + 0,\, n,\, m \rangle$$
$$\to \langle 1,\, n,\, m \rangle$$

Perfectly Correlated (λ_{PC})

Syntax

$$v ::= \lambda x : \tau . e$$
$$| r$$
$$| (v, v)$$

$$e ::= x$$
$$| ()$$
$$| \lambda x : \tau . e$$
$$| \textbf{let } x = e_1 \textbf{ in } e_2$$
$$| e_1 \, e_2$$
$$| r$$
$$| e_1 \oplus e_2$$
$$| \textbf{coin}()$$
$$| \textbf{rand}()$$
$$| \textbf{case } e_1 \textbf{ of } e_2 \,|\, e_3$$
$$| \textbf{inl}_{\tau_1 + \tau_2} e$$
$$| \textbf{inr}_{\tau_1 + \tau_2} e$$
$$| \#\mathbf{1} \, e$$
$$| \#\mathbf{2} \, e$$
$$| (e_1, e_2)$$

$$\tau ::= \textbf{unit}$$
$$| \mathbb{R}$$
$$| \tau_1 \to \tau_2$$
$$| \tau_1 + \tau_2$$
$$| \tau_1 \times \tau_2$$

$$E ::= [\cdot]$$
$$| E \, e$$
$$| v \, E$$
$$| \textbf{let } x = E \textbf{ in } e_2$$
$$| E \oplus e$$
$$| v \oplus E$$
$$| (\textbf{case } E \textbf{ of } e_2 \,|\, e_3)$$
$$| \textbf{inl}_{\tau_1 + \tau_2} E$$
$$| \textbf{inr}_{\tau_1 + \tau_2} E$$
$$| \#\, \mathbf{1} \, E$$
$$| \#\, \mathbf{2} \, E$$
$$| (E, e)$$
$$| (v, E)$$

Semantics

The small-step step semantics can be modeled by the relation on $(e \times \{0, 1\}^\omega \times [0, 1]^\omega) \times (e \times \{0, 1\}^\omega \times [0, 1]^\omega)$ defined below.

$$\frac{\langle e,\, n,\, m \rangle \to \langle e',\, n',\, m' \rangle}{\langle E(e),\, n,\, m \rangle \to \langle E(e'),\, n',\, m' \rangle} \ \text{Context}$$

$$\frac{}{\langle \textbf{rand}(),\, n,\, m \rangle \to \langle \textbf{hd } m,\, n,\, \textbf{tl } m \rangle} \ \text{Rand}$$

$$\frac{}{\langle (\lambda x : \tau . e) \, v,\, n,\, m \rangle \to \langle e\{v/x\},\, n,\, m \rangle} \ \beta\text{-Reduction}$$

$$\frac{}{\langle (\textbf{case inl}_{\tau_1 + \tau_2} v \textbf{ of } e_2 \,|\, e_3),\, n,\, m \rangle \to \langle e_2 \, v,\, n,\, m \rangle} \ \text{Case-Left}$$

$$\frac{}{\langle \textbf{let } x = v \textbf{ in } e_2,\, n,\, m \rangle \to \langle e_2\{v/x\},\, n,\, m \rangle} \ \text{Let}$$

$$\frac{}{\langle (\textbf{case inr}_{\tau_1 + \tau_2} v \textbf{ of } e_2 \,|\, e_3),\, n,\, m \rangle \to \langle e_3 \, v,\, n,\, m \rangle} \ \text{Case-Right}$$

$$\frac{r_1 \bar{\oplus} r_2 = r}{\langle r_1 \oplus r_2,\, n,\, m \rangle \to \langle r,\, n,\, m \rangle} \ \text{Bop}$$

$$\frac{}{\langle \#\mathbf{1} \, (v_1, v_2),\, n,\, m \rangle \to \langle v_1,\, n,\, m \rangle} \ \text{Proj-1}$$

$$\frac{}{\langle \textbf{coin}(),\, n,\, m \rangle \to \langle \textbf{hd } n,\, \textbf{tl } n,\, m \rangle} \ \text{Coin}$$

$$\frac{}{\langle \#\mathbf{2} \, (v_1, v_2),\, n,\, m \rangle \to \langle v_2,\, n,\, m \rangle} \ \text{Proj-2}$$

Static Semantics

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \ \text{Var}$$

$$\frac{\Gamma \vdash e : \tau'}{\Gamma \vdash (\lambda x : \tau . e) : \tau \to \tau'} \ \text{Fun}$$

$$\frac{}{\Gamma \vdash () : \textbf{unit}} \ \text{Unit}$$

$$\frac{\Gamma \vdash e_1 : \tau \qquad \Gamma_{x:\tau} \vdash e_2 : \tau'}{\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 : \tau'} \ \text{Let}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'} \text{ App}$$

$$\frac{}{\Gamma \vdash r : \mathbb{R}} \text{ Num}$$

$$\frac{}{\Gamma \vdash \mathbf{coin}() : \mathbb{R}} \text{ Coin}$$

$$\frac{}{\Gamma \vdash \mathbf{rand}() : \mathbb{R}} \text{ Rand}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \qquad \Gamma \vdash e_2 : \tau_1 \to \tau \qquad \Gamma \vdash e_3 : \tau_2 \to \tau}{\Gamma \vdash \mathbf{case}\ e_1\ \mathbf{with}\ e_2 \,|\, e_3 : \tau} \text{ Case}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathbf{inl}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2} \text{ In-Left}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathbf{inr}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2} \text{ In-Right}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#\ \mathbf{1}\ e : \tau_1} \text{ In-Left}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#\ \mathbf{2}\ e : \tau_2} \text{ In-Left}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \text{ Pair}$$

Example 1

$$\langle (\lambda x.\, x + x)\, \mathbf{coin}(),\ 1 :: 0 :: n,\ m \rangle \to \langle (\lambda x.\, x + x)\, 1,\ 0 :: n,\ m \rangle$$
$$\to \langle 1 + 1,\ 0 :: n,\ m \rangle$$
$$\to \langle 2,\ 0 :: n,\ m \rangle$$

Example 2

$$\langle \mathbf{let}\ x = \mathbf{coin}()\ \mathbf{in}\ x + x,\ 1 :: 0 :: n,\ m \rangle \to \langle \mathbf{let}\ x = 1\ \mathbf{in}\ x + x,\ 0 :: n,\ m \rangle$$
$$\to \langle 1 + 1,\ 0 :: n,\ m \rangle$$
$$\to \langle 2,\ 0 :: n,\ m \rangle$$

CBN to CBV

$$\mathcal{T}[\![x]\!] \triangleq x(\lambda y.\, y)$$

$$\mathcal{T}[\![r]\!] \triangleq r$$

$$\mathcal{T}[\![\lambda x.\, e]\!] \triangleq \lambda x.\, \mathcal{T}[\![e]\!]$$

$$\mathcal{T}[\![\textbf{let } x = e_1 \textbf{ in } e_2]\!] \triangleq \textbf{let } x = (\lambda z.\, \mathcal{T}[\![e_1]\!]) \textbf{ in } \mathcal{T}[\![e_2]\!]$$

$$\mathcal{T}[\![e_1\, e_2]\!] \triangleq \mathcal{T}[\![e_1]\!](\lambda z.\, \mathcal{T}[\![e_2]\!])$$

$$\mathcal{T}[\![\textbf{coin}]\!] \triangleq \textbf{coin}$$

$$\mathcal{T}[\![e_1 + e_2]\!] \triangleq \mathcal{T}[\![e_1]\!] + \mathcal{T}[\![e_2]\!]$$

CBV to CBN

$$\mathcal{T}[\![x]\!] \triangleq$$

$$\mathcal{T}[\![r]\!] \triangleq r$$

$$\mathcal{T}[\![\lambda x.\, e]\!] \triangleq$$

$$\mathcal{T}[\![\textbf{let } x = e_1 \textbf{ in } e_2]\!] \triangleq$$

$$\mathcal{T}[\![e_1\, e_2]\!] \triangleq$$

$$\mathcal{T}[\![\textbf{coin}]\!] \triangleq$$

Independently and Identically Distributed Semantics

$$\frac{\langle e,\, n,\, m\rangle \to \langle e',\, n',\, m'\rangle}{\langle E(e),\, n,\, m\rangle \to \langle E(e'),\, n',\, m'\rangle} \text{ CONTEXT}$$

$$\frac{}{\langle \mathbf{rand},\, n,\, m\rangle \to \langle \mathbf{hd}\ m,\, n,\, \mathbf{tl}\ m\rangle} \text{ RAND}$$

$$\frac{}{\langle(\lambda x:\tau.\, e)\, e_2,\, n,\, m\rangle \to \langle e\{e_2/x\},\, n,\, m\rangle} \beta\text{-REDUCTION}$$

$$\frac{}{\langle(\mathbf{case}\ \mathbf{inl}_{\tau_1+\tau_2}e\ \mathbf{of}\ e_2\,|\,e_3),\, n,\, m\rangle \to \langle e_2\, e,\, n,\, m\rangle} \text{ CASE-LEFT}$$

$$\frac{}{\langle(\mathbf{case}\ \mathbf{inr}_{\tau_1+\tau_2}e\ \mathbf{of}\ e_2\,|\,e_3),\, n,\, m\rangle \to \langle e_3\, e,\, n,\, m\rangle} \text{ CASE-RIGHT}$$

$$\frac{}{\langle \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2,\, n,\, m\rangle \to \langle e_2\{e_1/x\},\, n,\, m\rangle} \text{ LET}$$

$$\frac{}{\langle \#\mathbf{1}\ (e_1,e_2),\, n,\, m\rangle \to \langle e_1,\, n,\, m\rangle} \text{ PROJ-1}$$

$$\frac{r_1\bar\oplus r_2 = r}{\langle r_1 \oplus r_2,\, n,\, m\rangle \to \langle r,\, n,\, m\rangle} \text{ BOP}$$

$$\frac{}{\langle \#\mathbf{2}\ (e_1,e_2),\, n,\, m\rangle \to \langle e_2,\, n,\, m\rangle} \text{ PROJ-2}$$

$$\frac{}{\langle \mathbf{coin},\, n,\, m\rangle \to \langle \mathbf{hd}\ n,\, \mathbf{tl}\ n,\, m\rangle} \text{ COIN}$$

$$\frac{}{\langle v_1\ \mathbf{to}\ x\ \mathbf{in}\ e_2, n, m\rangle \to \langle e_2\{v/x\}, n, m\rangle} \text{ TO-PC}$$

Denotational

$$[\![r]\!] \triangleq \delta_r$$

$$[\![\lambda x.\, e]\!] \triangleq x \to [\![e]\!]$$

$$[\![\mathbf{let}\ x = e_1\ \mathbf{in}\ e_2]\!] \triangleq [\![e_2\{e_1/x\}]\!]$$

$$[\![e_1\ e_2]\!] \triangleq [\![e_1]\!] \circ [\![e_2]\!]$$

$$[\![\mathbf{coin}]\!] \triangleq \frac{1}{2}(\delta_0 + \delta_1)$$

$$[\![\mathbf{rand}]\!] \triangleq \lambda$$

Perfectly Correlated Semantics

$$\frac{\langle e,\, n,\, m\rangle \to \langle e',\, n',\, m'\rangle}{\langle E(e),\, n,\, m\rangle \to \langle E(e'),\, n',\, m'\rangle} \text{ CONTEXT}$$

$$\frac{}{\langle \mathbf{rand}(),\, n,\, m\rangle \to \langle \mathbf{hd}\ m,\, n,\, \mathbf{tl}\ m\rangle} \text{ RAND}$$

$$\frac{}{\langle(\lambda x:\tau.\, e)\, v,\, n,\, m\rangle \to \langle e\{v/x\},\, n,\, m\rangle} \beta\text{-REDUCTION}$$

$$\frac{}{\langle(\mathbf{case}\ \mathbf{inl}_{\tau_1+\tau_2}v\ \mathbf{of}\ e_2\,|\,e_3),\, n,\, m\rangle \to \langle e_2\, v,\, n,\, m\rangle} \text{ CASE-LEFT}$$

$$\frac{}{\langle(\mathbf{case}\ \mathbf{inr}_{\tau_1+\tau_2}v\ \mathbf{of}\ e_2\,|\,e_3),\, n,\, m\rangle \to \langle e_3\, v,\, n,\, m\rangle} \text{ CASE-RIGHT}$$

$$\frac{}{\langle \mathbf{let}\ x = v\ \mathbf{in}\ e_2,\, n,\, m\rangle \to \langle e_2\{v/x\},\, n,\, m\rangle} \text{ LET}$$

$$\frac{}{\langle \#\mathbf{1}\ (v_1,v_2),\, n,\, m\rangle \to \langle v_1,\, n,\, m\rangle} \text{ PROJ-1}$$

$$\frac{r_1\bar\oplus r_2 = r}{\langle r_1 \oplus r_2,\, n,\, m\rangle \to \langle r,\, n,\, m\rangle} \text{ BOP}$$

$$\frac{}{\langle \#\mathbf{2}\ (v_1,v_2),\, n,\, m\rangle \to \langle v_2,\, n,\, m\rangle} \text{ PROJ-2}$$

$$\frac{}{\langle \mathbf{coin}(),\, n,\, m\rangle \to \langle \mathbf{hd}\ n,\, \mathbf{tl}\ n,\, m\rangle} \text{ COIN}$$

$$\frac{}{\langle e_1\ \mathbf{to}\ x\ \mathbf{in}\ e_2, n, m\rangle \to \langle e_2\{(\lambda y.\, e_1)\,(\lambda z.\, z)/x\}, n, m\rangle} \text{ TO-IID}$$

Denotational

$$\llbracket r \rrbracket \triangleq \delta_r$$
$$\llbracket \lambda x.\, e \rrbracket \triangleq$$
$$\llbracket \mathbf{coin} \rrbracket \triangleq$$
$$\llbracket e_1\ e_2 \rrbracket \triangleq$$

$$\llbracket r \rrbracket \triangleq \delta_r$$
$$\llbracket \lambda x.\, e \rrbracket \triangleq$$

$$\llbracket e_1\ e_2 \rrbracket \triangleq$$