
Probabilistic Programming

Research Project

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Call By Push Value

Types

$$\begin{aligned}\mathbf{Val} &::= \mathbf{UB} \mid \Sigma_{i \in I} A_i \mid \mathbf{unit} \mid A \times A \\ \mathbf{Comp} &::= \mathbf{FA} \mid \Pi_{i \in I} B_i \mid \mathbf{Val} \rightarrow \mathbf{Comp} \\ \tau &::= \mathbf{Val} \mid \mathbf{Comp}\end{aligned}$$

Expressions

$$\begin{aligned}e &::= \\ &\mid \lambda x. e \\ &\mid e' e \\ &\mid \mathbf{produce} \, e \\ &\mid \mathbf{think} \, e \\ &\mid \mathbf{force} \, e \\ &\mid e \mathbf{to} \, x. e \\ &\mid \mathbf{let} \, x \mathbf{be} \, e. e\end{aligned}$$

Measure Theory

Definition (σ -algebra). A σ -algebra \mathcal{B} on a set S is a collection of subsets of S

- containing the empty set \emptyset
- closed under complementation in S
- closed under countable union in S

Definition (Measurable Space). Let S be a set and \mathcal{B} be a σ -algebra on S . The pair (S, \mathcal{B}) is a **measurable space**. The elements of \mathcal{B} are called **measurable sets** of S .

Remark (Probability). In probability theory, S can be thought of as the set of outcomes \mathcal{B} can be thought as the set of events.

Definition (Measurable Functions). Let (S, \mathcal{B}_S) and (T, \mathcal{B}_T) be measurable spaces. A function $f : S \rightarrow T$ is **measurable** if the inverse image

$$f^{-1}(B) = \{x \in S \mid f(x) \in B\}$$

of every measurable subset $B \in \mathcal{B}_T$ is a measurable subset of S .

Definition (Countably Additive). A function $\mu : \mathcal{B} \rightarrow \mathbb{R}$ is **countably additive** if \mathcal{A} is a countable set of pairwise disjoint events, then $\mu(\bigcup \mathcal{A}) = \sum_{A \in \mathcal{A}} \mu(A)$. Equivalently, if A_0, A_1, A_2, \dots is a countable collection of measurable sets such that $A_n \subseteq A_{n+1}$ for all $n \geq 0$, then $\lim_n \mu(A_n)$ exists and is equal to $\mu(\bigcup_n A_n)$.

Definition (Signed Finite Measure). A **signed finite measure** on (S, \mathcal{B}) is a countably additive map

$$\mu : \mathcal{B} \rightarrow \mathbb{R}$$

such that $\mu(\emptyset) = 0$.

Definition (Product Space). The **product space** of two measurable spaces (S_1, \mathcal{B}_1) and (S_2, \mathcal{B}_2) is $(S_1 \times S_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ where $S_1 \times S_2$ is the Cartesian product and $\mathcal{B}_1 \otimes \mathcal{B}_2 \triangleq \sigma(\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\})$.

Probabilistic Imp

Syntax

$$\begin{aligned} d ::= & a \\ & | x \\ & | d \textbf{ op } d \end{aligned}$$

$$\begin{aligned} t ::= & d \\ & | \textbf{coin } () \\ & | \textbf{rand } () \\ & | t \textbf{ op } d \end{aligned}$$

$$\begin{aligned} b ::= & \textbf{true} \\ & | \textbf{false} \\ & | d == d \\ & | d < d \\ & | d > d \\ & | b \&\& b \\ & | b || b \\ & | !b \end{aligned}$$

$$\begin{aligned} e ::= & \textbf{skip} \\ & | x := t \\ & | e; e \\ & | \textbf{if } b \textbf{ then } e \textbf{ else } e \\ & | \textbf{while } b \textbf{ do } e \end{aligned}$$

Small Step Semantics

$$\llbracket t \rrbracket : \mathbb{R}^n \times 0, 1^\omega \times [0, 1]^\omega \rightarrow \mathbb{R} \times 0, 1^\omega \times [0, 1]^\omega$$

$$\begin{aligned} \llbracket a \rrbracket & : (s, m, p) \triangleq (a, m, p) \\ \llbracket x_i \rrbracket & : (s, m, p) \triangleq (s(i), m, p) \\ \llbracket \textbf{coin } () \rrbracket & : (s, m, p) \triangleq (\textbf{hd } m, \textbf{tl } m, p) \\ \llbracket \textbf{rand } () \rrbracket & : (s, m, p) \triangleq (\textbf{hd } p, m, \textbf{tl } p) \\ \llbracket t \textbf{ op } t \rrbracket & : (s, m, p) \triangleq \end{aligned}$$

Denotational Semantics

Definition. $T_B : \mathcal{M}\mathbb{R}^n \rightarrow \mathcal{M}\mathbb{R}^n$ given by

$$T_B = \lambda x. \mu(B \cap x)$$

$$\begin{aligned} \llbracket \textbf{skip} \rrbracket & \triangleq \text{Id}_{\mathcal{M}\mathbb{R}^n} \\ \llbracket x_i := t \rrbracket & \triangleq \mu \mapsto (F_t^i)_*(\mu) \\ \llbracket e_1; e_2 \rrbracket & \triangleq \llbracket e_2 \rrbracket \circ \llbracket e_1 \rrbracket \\ \llbracket \textbf{if } b \textbf{ then } e_1 \textbf{ else } e_2 \rrbracket & \triangleq \llbracket e_1 \rrbracket \circ T_{\llbracket b \rrbracket} + \llbracket e_2 \rrbracket \circ T_{\llbracket b \rrbracket^c} \\ \llbracket \textbf{while } b \textbf{ do } e \rrbracket & \triangleq \bigvee_{n \geq 0} \tau^n(0) \end{aligned} \quad \text{where } \tau(S) = S \circ \llbracket e \rrbracket \circ T_{\llbracket b \rrbracket} + T_{\llbracket b \rrbracket^c}$$

Example 1

$$\begin{aligned}
\llbracket \text{if } 1 == 1 \text{ then } x := 0 \text{ else } x := 1 \rrbracket &= \llbracket x := 0 \rrbracket \circ T_{\llbracket 1 == 1 \rrbracket} + \llbracket x := 1 \rrbracket \circ T_{\llbracket 1 == 1 \rrbracket^C} \\
&= \llbracket x := 0 \rrbracket \circ T_{\mathbb{R}} + \llbracket x := 1 \rrbracket \circ T_{\emptyset} \\
&= \llbracket x := 0 \rrbracket \circ (\mu \mapsto \lambda x. \mu(x \cap \mathbb{R})) + \llbracket x := 1 \rrbracket \circ (\mu \mapsto \lambda x. \mu(x \cap \emptyset)) \\
&= (\mu \mapsto \mu(\mathbb{R})\delta_0) \circ (\mu \mapsto \mu) + (\mu \mapsto \mu(\mathbb{R})\delta_1) \circ (\mu \mapsto \lambda x. \mu(\emptyset)) \\
&= \mu \mapsto \mu(\mathbb{R})\delta_0 + \mu(\emptyset)\delta_1 \\
\llbracket \text{if } 1 == 1 \text{ then } x := 0 \text{ else } x := 1 \rrbracket &= \mu \mapsto \mu(\mathbb{R})\delta_0
\end{aligned}$$

Example 2

$$\begin{aligned}
\llbracket x := 1; y := 2 \rrbracket &= \llbracket y := 2 \rrbracket \circ \llbracket x := 1 \rrbracket \\
&= (\mu \mapsto \lambda(B_1, B_2). \mu(B_1 \times \mathbb{R})\delta_2(B_2)) \circ (\mu \mapsto \lambda(B_1, B_2). \mu(\mathbb{R} \times B_2)\delta_1(B_1)) \\
&= \mu \mapsto \lambda(B_1, B_2). \mu(\mathbb{R}^2)\delta_1(B_1)\delta_2(B_2) \\
\llbracket y := 2; x := 1 \rrbracket &= \llbracket x := 1 \rrbracket \circ \llbracket y := 2 \rrbracket \\
&= (\mu \mapsto \lambda(B_1, B_2). \mu(\mathbb{R} \times B_2)\delta_1(B_1)) \circ (\mu \mapsto \lambda(B_1, B_2). \mu(B_1 \times \mathbb{R})\delta_2(B_2)) \\
&= \mu \mapsto \lambda(B_1, B_2). \mu(\mathbb{R}^2)\delta_1(B_1)\delta_2(B_2)
\end{aligned}$$

Example 3

Claim.

$$\llbracket \text{while } x == 0 \text{ do } x := \text{coin}() \rrbracket = \mu \mapsto \mu(\{0, 1\})\delta_1 + \mu(- \cap \{0, 1\})^C$$

Proof. I will show by induction that for all $k \geq 1$,

$$\tau^k(0) = \mu \mapsto \mu(- \cap \{0\})^C + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1$$

If we then take the limit as $k \rightarrow \infty$, we get

$$\mu \mapsto \mu(- \cap \{0\})^C + \mu(\{0\})\delta_1$$

- **Base Case:** $\tau^1(0)$

$$\begin{aligned}
\tau^1(0) &= 0 \circ \llbracket x := \text{coin}() \rrbracket \circ T_{\{0\}} + T_{\{0\}^C} \\
&= T_{\{0\}^C} \\
&= \mu \mapsto \mu(- \cap \{0\})^C \\
\tau^1(0) &= \mu \mapsto \mu(- \cap \{0\})^C + (1 - 2^{-(1-1)})\mu(\{0\})\delta_1
\end{aligned}$$

- **Inductive Case:** $\tau^{k+1}(0)$

$$\begin{aligned}
\tau^{k+1}(0) &= \mu \mapsto \mu(- \cap \{0\})^C + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ \llbracket x := \text{coin}() \rrbracket \circ T_{\{0\}} + T_{\{0\}^C} \\
&= \mu \mapsto \mu(- \cap \{0\})^C + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ \mu \mapsto \mu(\mathbb{R}) \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \right) \circ \mu \mapsto \mu(- \cap \{0\}) + T_{\{0\}^C} \\
&= \mu \mapsto \mu(- \cap \{0\})^C + (1 - 2^{-(k-1)})\mu(\{0\})\delta_1 \circ \mu \mapsto \mu(\{0\}) \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \right) + T_{\{0\}^C} \\
&= \mu \mapsto \mu(\{0\}) \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \right) (- \cap \{0\})^C + (1 - 2^{-(k-1)})\mu(\{0\}) \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \right) (\{0\})\delta_1 + T_{\{0\}^C} \\
&= \mu \mapsto \frac{1}{2}\mu(\{0\})\delta_1 + (1 - 2^{-(k-1)})\frac{1}{2}\mu(\{0\})\delta_1 + T_{\{0\}^C} \\
&= \mu \mapsto \frac{1}{2}\mu(\{0\})\delta_1 + \frac{1}{2}\mu(\{0\})\delta_1 - 2^{-(k-1)-1}\mu(\{0\})\delta_1 + T_{\{0\}^C} \\
&= \mu \mapsto \mu(\{0\})\delta_1 - 2^{-(k)}\mu(\{0\})\delta_1 + T_{\{0\}^C} \\
&= \mu \mapsto \mu(- \cap \{0\})^C + (1 - 2^{-((k+1)-1)})\mu(\{0\})\delta_1
\end{aligned}$$

QED

$$\begin{aligned}
\llbracket x := 0; \textbf{while } x == 0 \textbf{ do } x := \textbf{coin}() \rrbracket &= \llbracket \textbf{while } x == 0 \textbf{ do } x := \textbf{coin}() \rrbracket \circ \llbracket x := 0 \rrbracket \\
&= \mu \mapsto \mu(\{0, 1\})\delta_1 + \mu(- \cap \{0, 1\}^C) \circ \mu \mapsto \mu(\mathbb{R})\delta_0 \\
&= \mu \mapsto (\mu(\mathbb{R})\delta_0)(\{0, 1\})\delta_1 + (\mu(\mathbb{R})\delta_0)(- \cap \{0, 1\}^C) \\
&= \mu \mapsto \mu(\mathbb{R})\delta_1
\end{aligned}$$

Independently and Identically Distributed (λ_{IID})

Syntax

$e ::= x$	$\tau ::= \mathbf{unit}$
$ ()$	$ \mathbb{R}$
$ \lambda x : \tau. e$	$ \tau_1 \rightarrow \tau_2$
$ \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2$	$ \tau_1 + \tau_2$
$ e_1 \ e_2$	$ \tau_1 \times \tau_2$
$ r$	$v ::= \lambda x : \tau. e$
$ e_1 \oplus e_2$	$ r$
$ \mathbf{coin}()$	$ (e, e)$
$ \mathbf{rand}()$	$E ::= []$
$ \mathbf{case} \ e_1 \ \mathbf{of} \ e_2 \mid e_3$	$ E \ e$
$ \mathbf{inl}_{\tau_1 + \tau_2} e$	$ E \oplus e$
$ \mathbf{inr}_{\tau_1 + \tau_2} e$	$ v \oplus E$
$ \#1 \ e$	$ (\mathbf{case} \ E \ \mathbf{of} \ e_2 \mid e_3)$
$ \#2 \ e$	$ \mathbf{inl}_{\tau_1 + \tau_2} E$
$ (e_1, e_2)$	$ \mathbf{inr}_{\tau_1 + \tau_2} E$
	$ \# \ 1 \ E$
	$ \# \ 2 \ E$

Semantics

The small-step step semantics can be modeled by the relation on $(e \times \{0, 1\}^\omega \times [0, 1]^\omega) \times (e \times \{0, 1\}^\omega \times [0, 1]^\omega)$ defined below.

$$\frac{\langle e, n, m \rangle \rightarrow \langle e', n', m' \rangle}{\langle E(e), n, m \rangle \rightarrow \langle E(e'), n', m' \rangle} \text{CONTEXT}$$

$$\frac{}{\langle \mathbf{rand}(), n, m \rangle \rightarrow \langle \mathbf{hd} \ m, n, \mathbf{tl} \ m \rangle} \text{RAND}$$

$$\frac{}{\langle (\lambda x : \tau. e) \ e_2, n, m \rangle \rightarrow \langle e\{e_2/x\}, n, m \rangle} \beta\text{-REDUCTION}$$

$$\frac{}{\langle (\mathbf{case} \ \mathbf{inl}_{\tau_1 + \tau_2} e \ \mathbf{of} \ e_2 \mid e_3), n, m \rangle \rightarrow \langle e_2 \ e, n, m \rangle} \text{CASE-LEFT}$$

$$\frac{}{\langle \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2, n, m \rangle \rightarrow \langle e_2\{e_1/x\}, n, m \rangle} \text{LET}$$

$$\frac{}{\langle (\mathbf{case} \ \mathbf{inr}_{\tau_1 + \tau_2} e \ \mathbf{of} \ e_2 \mid e_3), n, m \rangle \rightarrow \langle e_3 \ e, n, m \rangle} \text{CASE-RIGHT}$$

$$\frac{r_1 \oplus r_2 = r}{\langle r_1 \oplus r_2, n, m \rangle \rightarrow \langle r, n, m \rangle} \text{BOP}$$

$$\frac{}{\langle \#1 \ (e_1, e_2), n, m \rangle \rightarrow \langle e_1, n, m \rangle} \text{PROJ-1}$$

$$\frac{}{\langle \mathbf{coin}(), n, m \rangle \rightarrow \langle \mathbf{hd} \ n, \mathbf{tl} \ n, m \rangle} \text{COIN}$$

$$\frac{}{\langle \#2 \ (e_1, e_2), n, m \rangle \rightarrow \langle e_2, n, m \rangle} \text{PROJ-2}$$

Static Semantics

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{VAR}$$

$$\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \ e_2 : \tau'} \text{APP}$$

$$\frac{}{\Gamma \vdash () : \mathbf{unit}} \text{UNIT}$$

$$\frac{}{\Gamma \vdash r : \mathbb{R}} \text{NUM}$$

$$\frac{\Gamma \vdash e : \tau'}{\Gamma \vdash (\lambda x : \tau. e) : \tau \rightarrow \tau'} \text{FUN}$$

$$\frac{}{\Gamma \vdash \mathbf{coin}() : \mathbb{R}} \text{COIN}$$

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma_{x:\tau} \vdash e_2 : \tau'}{\Gamma \vdash \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 : \tau'} \text{LET}$$

$$\frac{}{\Gamma \vdash \mathbf{rand}() : \mathbb{R}} \text{RAND}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \quad \Gamma \vdash e_2 : \tau_1 \rightarrow \tau \quad \Gamma \vdash e_3 : \tau_2 \rightarrow \tau}{\Gamma \vdash \mathbf{case} \ e_1 \ \mathbf{with} \ e_2 \mid e_3 : \tau} \text{CASE}$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathbf{inl}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2} \text{IN-LEFT}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathbf{inr}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2} \text{IN-RIGHT}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \# \ \mathbf{1} \ e : \tau_1} \text{IN-LEFT}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \# \ \mathbf{2} \ e : \tau_2} \text{IN-LEFT}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \text{PAIR}$$

Example 1

$$\begin{aligned} \langle (\lambda x. x + x) \ \mathbf{coin}(), 1 :: 0 :: n, m \rangle &\rightarrow \langle \mathbf{coin}() + \mathbf{coin}(), 1 :: 0 :: n, m \rangle \\ &\rightarrow \langle 1 + \mathbf{coin}(), 0 :: n, m \rangle \\ &\rightarrow \langle 1 + 0, n, m \rangle \\ &\rightarrow \langle 1, n, m \rangle \end{aligned}$$

Example 2

$$\begin{aligned} \langle \mathbf{let} \ x = \mathbf{coin}() \ \mathbf{in} \ x + x, 1 :: 0 :: n, m \rangle &\rightarrow \langle \mathbf{coin}() + \mathbf{coin}(), 1 :: 0 :: n, m \rangle \\ &\rightarrow \langle 1 + \mathbf{coin}(), 0 :: n, m \rangle \\ &\rightarrow \langle 1 + 0, n, m \rangle \\ &\rightarrow \langle 1, n, m \rangle \end{aligned}$$

Perfectly Correlated (λ_{PC})

Syntax

$e ::= x$	$v ::= \lambda x : \tau. e$
$ ()$	$ r$
$ \lambda x : \tau. e$	$ (v, v)$
$ \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2$	$E ::= [\cdot]$
$ e_1 \ e_2$	$ E \ e$
$ r$	$ v \ E$
$ e_1 \oplus e_2$	$ \mathbf{let} \ x = E \ \mathbf{in} \ e_2$
$ \mathbf{coin}()$	$ E \oplus e$
$ \mathbf{rand}()$	$ v \oplus E$
$ \mathbf{case} \ e_1 \ \mathbf{of} \ e_2 \mid e_3$	$ (\mathbf{case} \ E \ \mathbf{of} \ e_2 \mid e_3)$
$ \mathbf{inl}_{\tau_1 + \tau_2} e$	$ \mathbf{inl}_{\tau_1 + \tau_2} E$
$ \mathbf{inr}_{\tau_1 + \tau_2} e$	$ \mathbf{inr}_{\tau_1 + \tau_2} E$
$ \#1 \ e$	$ \#1 \ E$
$ \#2 \ e$	$ \#2 \ E$
$ (e_1, e_2)$	$ (E, e)$
$\tau ::= \mathbf{unit}$	$ (v, E)$
$ \mathbb{R}$	
$ \tau_1 \rightarrow \tau_2$	
$ \tau_1 + \tau_2$	
$ \tau_1 \times \tau_2$	

Semantics

The small-step semantics can be modeled by the relation on $(e \times \{0, 1\}^\omega \times [0, 1]^\omega) \times (e \times \{0, 1\}^\omega \times [0, 1]^\omega)$ defined below.

$$\frac{\langle e, n, m \rangle \rightarrow \langle e', n', m' \rangle}{\langle E(e), n, m \rangle \rightarrow \langle E(e'), n', m' \rangle} \text{CONTEXT}$$

$$\frac{}{\langle \mathbf{rand}(), n, m \rangle \rightarrow \langle \mathbf{hd} \ m, n, \mathbf{tl} \ m \rangle} \text{RAND}$$

$$\frac{}{\langle (\lambda x : \tau. e) v, n, m \rangle \rightarrow \langle e\{v/x\}, n, m \rangle} \beta\text{-REDUCTION}$$

$$\frac{}{\langle (\mathbf{case} \ \mathbf{inl}_{\tau_1 + \tau_2} v \ \mathbf{of} \ e_2 \mid e_3), n, m \rangle \rightarrow \langle e_2 v, n, m \rangle} \text{CASE-LEFT}$$

$$\frac{}{\langle \mathbf{let} \ x = v \ \mathbf{in} \ e_2, n, m \rangle \rightarrow \langle e_2\{v/x\}, n, m \rangle} \text{LET}$$

$$\frac{}{\langle (\mathbf{case} \ \mathbf{inr}_{\tau_1 + \tau_2} v \ \mathbf{of} \ e_2 \mid e_3), n, m \rangle \rightarrow \langle e_3 v, n, m \rangle} \text{CASE-RIGHT}$$

$$\frac{r_1 \bar{\oplus} r_2 = r}{\langle r_1 \oplus r_2, n, m \rangle \rightarrow \langle r, n, m \rangle} \text{BOP}$$

$$\frac{}{\langle \#1 \ (v_1, v_2), n, m \rangle \rightarrow \langle v_1, n, m \rangle} \text{PROJ-1}$$

$$\frac{}{\langle \mathbf{coin}(), n, m \rangle \rightarrow \langle \mathbf{hd} \ n, \mathbf{tl} \ n, m \rangle} \text{COIN}$$

$$\frac{}{\langle \#2 \ (v_1, v_2), n, m \rangle \rightarrow \langle v_2, n, m \rangle} \text{PROJ-2}$$

Static Semantics

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{VAR}$$

$$\frac{\Gamma \vdash e : \tau'}{\Gamma \vdash (\lambda x : \tau. e) : \tau \rightarrow \tau'} \text{FUN}$$

$$\frac{}{\Gamma \vdash () : \mathbf{unit}} \text{UNIT}$$

$$\frac{\Gamma \vdash e_1 : \tau \quad \Gamma_{x:\tau} \vdash e_2 : \tau'}{\Gamma \vdash \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 : \tau'} \text{LET}$$

$$\begin{array}{c}
\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'} \text{APP} \\
\\
\frac{}{\Gamma \vdash r : \mathbb{R}} \text{NUM} \\
\\
\frac{}{\Gamma \vdash \mathbf{coin}() : \mathbb{R}} \text{COIN} \\
\\
\frac{}{\Gamma \vdash \mathbf{rand}() : \mathbb{R}} \text{RAND} \\
\\
\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \quad \Gamma \vdash e_2 : \tau_1 \rightarrow \tau \quad \Gamma \vdash e_3 : \tau_2 \rightarrow \tau}{\Gamma \vdash \mathbf{case } e_1 \mathbf{ with } e_2 | e_3 : \tau} \text{CASE}
\end{array}
\qquad
\begin{array}{c}
\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathbf{inl}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2} \text{IN-LEFT} \\
\\
\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathbf{inr}_{\tau_1 + \tau_2} e : \tau_1 + \tau_2} \text{IN-RIGHT} \\
\\
\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \# \mathbf{1} e : \tau_1} \text{IN-LEFT} \\
\\
\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \# \mathbf{2} e : \tau_2} \text{IN-LEFT} \\
\\
\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \text{PAIR}
\end{array}$$

Example 1

$$\begin{aligned}
\langle (\lambda x. x + x) \mathbf{coin}(), 1 :: 0 :: n, m \rangle &\rightarrow \langle (\lambda x. x + x) 1, 0 :: n, m \rangle \\
&\rightarrow \langle 1 + 1, 0 :: n, m \rangle \\
&\rightarrow \langle 2, 0 :: n, m \rangle
\end{aligned}$$

Example 2

$$\begin{aligned}
\langle \mathbf{let } x = \mathbf{coin}() \mathbf{ in } x + x, 1 :: 0 :: n, m \rangle &\rightarrow \langle \mathbf{let } x = 1 \mathbf{ in } x + x, 0 :: n, m \rangle \\
&\rightarrow \langle 1 + 1, 0 :: n, m \rangle \\
&\rightarrow \langle 2, 0 :: n, m \rangle
\end{aligned}$$

CBN to CBV

$$\begin{aligned}\mathcal{T}[[x]] &\triangleq x(\lambda y. y) \\ \mathcal{T}[[r]] &\triangleq r \\ \mathcal{T}[[\lambda x. e]] &\triangleq \lambda x. \mathcal{T}[[e]] \\ \mathcal{T}[[\mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2]] &\triangleq \mathbf{let} \ x = (\lambda z. \mathcal{T}[[e_1]]) \ \mathbf{in} \ \mathcal{T}[[e_2]] \\ \mathcal{T}[[e_1 \ e_2]] &\triangleq \mathcal{T}[[e_1]](\lambda z. \mathcal{T}[[e_2]]) \\ \mathcal{T}[[\mathbf{coin}]] &\triangleq \mathbf{coin} \\ \mathcal{T}[[e_1 + e_2]] &\triangleq \mathcal{T}[[e_1]] + \mathcal{T}[[e_2]]\end{aligned}$$

CBV to CBN

$$\begin{aligned}\mathcal{T}[[x]] &\triangleq \\ \mathcal{T}[[r]] &\triangleq r \\ \mathcal{T}[[\lambda x. e]] &\triangleq \\ \mathcal{T}[[\mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2]] &\triangleq \\ \mathcal{T}[[e_1 \ e_2]] &\triangleq \\ \mathcal{T}[[\mathbf{coin}]] &\triangleq\end{aligned}$$

Independently and Identically Distributed Semantics

$$\frac{\langle e, n, m \rangle \rightarrow \langle e', n', m' \rangle}{\langle E(e), n, m \rangle \rightarrow \langle E(e'), n', m' \rangle} \text{CONTEXT}$$

$$\frac{}{\langle \mathbf{rand}, n, m \rangle \rightarrow \langle \mathbf{hd} \ m, n, \mathbf{tl} \ m \rangle} \text{RAND}$$

$$\frac{}{\langle (\lambda x : \tau. e) e_2, n, m \rangle \rightarrow \langle e\{e_2/x\}, n, m \rangle} \beta\text{-REDUCTION}$$

$$\frac{}{\langle \langle \mathbf{case} \ \mathbf{inl}_{\tau_1+\tau_2} e \ \mathbf{of} \ e_2 \mid e_3 \rangle, n, m \rangle \rightarrow \langle e_2 \ e, n, m \rangle} \text{CASE-LEFT}$$

$$\frac{}{\langle \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2, n, m \rangle \rightarrow \langle e_2\{e_1/x\}, n, m \rangle} \text{LET}$$

$$\frac{}{\langle \langle \mathbf{case} \ \mathbf{inr}_{\tau_1+\tau_2} e \ \mathbf{of} \ e_2 \mid e_3 \rangle, n, m \rangle \rightarrow \langle e_3 \ e, n, m \rangle} \text{CASE-RIGHT}$$

$$\frac{r_1 \bar{\oplus} r_2 = r}{\langle r_1 \oplus r_2, n, m \rangle \rightarrow \langle r, n, m \rangle} \text{BOP}$$

$$\frac{}{\langle \#1 \ (e_1, e_2), n, m \rangle \rightarrow \langle e_1, n, m \rangle} \text{PROJ-1}$$

$$\frac{}{\langle \#2 \ (e_1, e_2), n, m \rangle \rightarrow \langle e_2, n, m \rangle} \text{PROJ-2}$$

$$\frac{}{\langle \mathbf{coin}, n, m \rangle \rightarrow \langle \mathbf{hd} \ n, \mathbf{tl} \ n, m \rangle} \text{COIN}$$

$$\frac{}{\langle v_1 \ \mathbf{to} \ x \ \mathbf{in} \ e_2, n, m \rangle \rightarrow \langle e_2\{v_1/x\}, n, m \rangle} \text{TO-PC}$$

Big Step

$$\frac{\langle e_1, n, m \rangle \Downarrow \langle \lambda x. e'_1, n', m' \rangle \quad \langle e'_1\{e_2/x\}, n', m' \rangle \Downarrow \langle v, n'', m'' \rangle}{\langle e_1 \ e_2, n, m \rangle \Downarrow \langle v, n'', m'' \rangle} \beta\text{-REDUCTION}$$

$$\frac{\langle e_2\{e_1/x\}, n, m \rangle \Downarrow \langle v, n', m' \rangle}{\langle \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2, n, m \rangle \Downarrow \langle v, n', m' \rangle} \text{LET}$$

$$\frac{\langle e_1, n, m \rangle \Downarrow \langle r_1, n', m' \rangle \quad \langle e_2, n', m' \rangle \Downarrow \langle r_2, n'', m'' \rangle \quad r_1 \bar{\oplus} r_2 = r}{\langle e_1 \oplus e_2, n, m \rangle \Downarrow \langle r, n'', m'' \rangle} \text{BOP}$$

$$\frac{}{\langle \mathbf{coin}, n, m \rangle \Downarrow \langle \mathbf{hd} \ n, \mathbf{tl} \ n, m \rangle} \text{COIN}$$

$$\frac{}{\langle \mathbf{rand}, n, m \rangle \Downarrow \langle \mathbf{hd} \ m, n, \mathbf{tl} \ m \rangle} \text{RAND}$$

Denotational

$$\begin{aligned} \llbracket r \rrbracket &\triangleq \delta_r \\ \llbracket \lambda x. e \rrbracket &\triangleq x \rightarrow \llbracket e \rrbracket \\ \llbracket \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 \rrbracket &\triangleq \llbracket e_2\{e_1/x\} \rrbracket \\ \llbracket e_1 \ e_2 \rrbracket &\triangleq \llbracket e_1 \rrbracket \circ \llbracket e_2 \rrbracket \\ \llbracket \mathbf{coin} \rrbracket &\triangleq \frac{1}{2}(\delta_0 + \delta_1) \\ \llbracket \mathbf{rand} \rrbracket &\triangleq \lambda \end{aligned}$$

Perfectly Correlated Semantics

$$\frac{\langle e, n, m \rangle \rightarrow \langle e', n', m' \rangle}{\langle E(e), n, m \rangle \rightarrow \langle E(e'), n', m' \rangle} \text{CONTEXT}$$

$$\frac{}{\langle \mathbf{let} \ x = v \ \mathbf{in} \ e_2, n, m \rangle \rightarrow \langle e_2\{v/x\}, n, m \rangle} \text{LET}$$

$$\frac{}{\langle (\lambda x : \tau. e) v, n, m \rangle \rightarrow \langle e\{v/x\}, n, m \rangle} \beta\text{-REDUCTION}$$

$$\frac{r_1 \bar{\oplus} r_2 = r}{\langle r_1 \oplus r_2, n, m \rangle \rightarrow \langle r, n, m \rangle} \text{BOP}$$

$$\begin{array}{c}
\overline{\langle \mathbf{coin}(), n, m \rangle \rightarrow \langle \mathbf{hd} \ n, \mathbf{tl} \ n, m \rangle} \text{ COIN} \\
\overline{\langle \mathbf{rand}(), n, m \rangle \rightarrow \langle \mathbf{hd} \ m, n, \mathbf{tl} \ m \rangle} \text{ RAND} \\
\overline{\langle (\mathbf{case} \ \mathbf{inl}_{\tau_1 + \tau_2} v \ \mathbf{of} \ e_2 \mid e_3), n, m \rangle \rightarrow \langle e_3 \ v, n, m \rangle} \text{ CASE-RIGHT} \\
\overline{\langle \#1 \ (v_1, v_2), n, m \rangle \rightarrow \langle v_1, n, m \rangle} \text{ PROJ-1} \\
\overline{\langle \#2 \ (v_1, v_2), n, m \rangle \rightarrow \langle v_2, n, m \rangle} \text{ PROJ-2} \\
\overline{\langle (\mathbf{case} \ \mathbf{inl}_{\tau_1 + \tau_2} v \ \mathbf{of} \ e_2 \mid e_3), n, m \rangle \rightarrow \langle e_2 \ v, n, m \rangle} \text{ CASE-LEFT} \\
\overline{\langle e_1 \ \mathbf{to} \ x \ \mathbf{in} \ e_2, n, m \rangle \rightarrow \langle e_2 \{(\lambda y. e_1) () / x\}, n, m \rangle} \text{ TO-IID}
\end{array}$$

Big Step

$$\begin{array}{c}
\frac{\langle e_1, n, m \rangle \Downarrow \langle \lambda x. e'_1, n', m' \rangle \quad \langle e_2, n', m' \rangle \Downarrow \langle v, n'', m'' \rangle \quad \langle e'_1 \{v/x\}, n'', m'' \rangle \Downarrow \langle v', n''', m''' \rangle}{\langle e_1 \ e_2, n, m \rangle \Downarrow \langle v', n'', m'' \rangle} \beta\text{-REDUCTION} \\
\frac{\langle e_1, n, m \rangle \Downarrow \langle v, n', m' \rangle \quad \langle e_2 \{v/x\}, n', m' \rangle \Downarrow \langle v', n'', m'' \rangle}{\langle \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2, n, m \rangle \Downarrow \langle v', n'', m'' \rangle} \text{ LET} \\
\frac{\langle e_1, n, m \rangle \Downarrow \langle r_1, n', m' \rangle \quad \langle e_2, n', m' \rangle \Downarrow \langle r_2, n'', m'' \rangle \quad r_1 \bar{\oplus} r_2 = r}{\langle e_1 \oplus e_2, n, m \rangle \Downarrow \langle r, n'', m'' \rangle} \text{ BOP} \\
\overline{\langle \mathbf{coin}, n, m \rangle \Downarrow \langle \mathbf{hd} \ n, \mathbf{tl} \ n, m \rangle} \text{ COIN} \\
\overline{\langle \mathbf{rand}, n, m \rangle \Downarrow \langle \mathbf{hd} \ m, n, \mathbf{tl} \ m \rangle} \text{ RAND}
\end{array}$$

Denotational

$$\begin{array}{l}
\llbracket r \rrbracket \triangleq \delta_r \\
\llbracket \lambda x. e \rrbracket \triangleq \\
\llbracket \mathbf{coin} \rrbracket \triangleq \\
\llbracket e_1 \ e_2 \rrbracket \triangleq
\end{array}$$