Probabilistic Lambda Calculi

Research Project

Pedro Amorim and Eric Jackson

1 Abstract

In section 2, we introduce a probabilistic version of Call-By-Push Value. In section 3, we introduce two probabilistic lambda-calculi, λ_{IID} and λ_{PC}, which correspond to independent and identically distributed sampling and perfectly correlated sampling, respectively.

2 Probabilistic Call-By-Push-Value

2.1 Syntax

Types

CBPV types are given by $\tau = A \,|\, B$ where A and B are defined inductively according to the following rules:

$$\begin{aligned} A &::= U\underline{B} \,|\, A + A \,|\, A \times A \,|\, 1 \,|\, \mathbb{R} \\ \underline{B} &::= FA \,|\, \underline{B} \times \underline{B} \,|\, A \to \underline{B} \end{aligned}$$

Expressions

2.2 Static Semantics

$$\frac{}{\Gamma \vdash^c \mathsf{rand} : F\,\mathbb{R}}$$

2.3 Denotational Semantics

3 λ_{IID} and λ_{PC}

3.1 Syntax

4 Translating λ_{IID} and λ_{PC} to CBPV

Rather than defining denotational semantics for λ_{IID} and λ_{PC}, we will define a translation from each of these languages into CBPV. Then, we can use the CBPV semantics to generate denotational semantics.

4.1 Types

We begin with defining type translations from λ_{IID} and λ_{PC} to CBVP. These translations are essentially identical to the translations of CBN and CBV to CBPV presented in Levy's thesis.

λ_{IID}

$$\begin{aligned} [\![\mathbf{unit}]\!]_{IID} &\triangleq F1 \\ [\![\mathbb{R}]\!]_{IID} &\triangleq F\,\mathbb{R} \\ [\![\tau_1 \to \tau_2]\!]_{IID} &\triangleq (U\,[\![\tau_1]\!]_{IID}) \to [\![\tau_2]\!]_{IID} \\ [\![\tau_1 + \tau_2]\!]_{IID} &\triangleq F(U\,[\![\tau_1]\!]_{IID} + U\,[\![\tau_2]\!]_{IID}) \\ [\![\tau_1 \times \tau_2]\!]_{IID} &\triangleq [\![\tau_1]\!]_{IID} \times [\![\tau_2]\!]_{IID} \end{aligned}$$

λ_{PC}

$$\begin{aligned} [\![\mathbf{unit}]\!]_{PC} &\triangleq 1 \\ [\![\mathbb{R}]\!]_{PC} &\triangleq \mathbb{R} \\ [\![\tau_1 \to \tau_2]\!]_{PC} &\triangleq U([\![\tau_1]\!]_{PC} \to [\![F\tau_2]\!]_{PC}) \\ [\![\tau_1 + \tau_2]\!]_{PC} &\triangleq U\,[\![\tau_1]\!]_{PC} + [\![\tau_2]\!]_{PC} \\ [\![\tau_1 \times \tau_2]\!]_{PC} &\triangleq U(F\,[\![\tau_1]\!]_{PC} \times F\,[\![\tau_2]\!]_{PC}) \end{aligned}$$

4.2 Expressions

Now, we define translations from expressions in λ_{IID} and λ_{PC} to CBPV.

λ_{IID}

$$\begin{aligned}
\mathcal{T}[\![x]\!]_{IID} &\triangleq \textbf{force } x \\
\mathcal{T}[\![\lambda x.\, e]\!]_{IID} &\triangleq \lambda x.\, [\![e]\!]_{IID} \\
\mathcal{T}[\![\textbf{let} x = e_1 \textbf{ in } e_2]\!]_{IID} &\triangleq \textbf{let } x \textbf{ be } \mathcal{T}[\![e_1]\!]_{IID}.\, \mathcal{T}[\![e_2]\!]_{IID} \\
\mathcal{T}[\![e_1\, e_2]\!]_{IID} &\triangleq (\textbf{thunk } \mathcal{T}[\![e_2]\!]_{IID})`\, \mathcal{T}[\![e_1]\!]_{IID} \\
\mathcal{T}[\![\textbf{coin}]\!]_{IID} &\triangleq \textbf{coin} \\
\mathcal{T}[\![\textbf{rand}]\!]_{IID} &\triangleq \textbf{rand} \\
\mathcal{T}[\![\textbf{inl}_{\tau_1+\tau_2} e]\!]_{IID} &\triangleq \textbf{produce inl thunk } \mathcal{T}[\![e]\!]_{IID} \\
\mathcal{T}[\![\textbf{inr}_{\tau_1+\tau_2} e]\!]_{IID} &\triangleq \textbf{produce inr thunk } \mathcal{T}[\![e]\!]_{IID} \\
\mathcal{T}[\![\textbf{case } e_1 \textbf{of } e_2 | e_3]\!]_{IID} &\triangleq \mathcal{T}[\![e_1]\!]_{IID} \textbf{ to } z.\ \textbf{pm } z \textbf{ as } \{\textbf{inl } x. \mathcal{T}[\![e_2]\!]_{IID},\ \textbf{inr } x. \mathcal{T}[\![e_3]\!]_{IID}\} \\
\mathcal{T}[\![(e_1, e_2)]\!]_{IID} &\triangleq \\
\mathcal{T}[\![\#\mathbf{1}\ e]\!]_{IID} &\triangleq \\
\mathcal{T}[\![\#\mathbf{2}\ e]\!]_{IID} &\triangleq \\
\mathcal{T}[\![e_1 \textbf{ to } x \textbf{ in } e2]\!]_{IID} &\triangleq
\end{aligned}$$

λ_{PC}

$$\begin{aligned}
\mathcal{T}[\![x]\!]_{PC} &\triangleq \textbf{produce } x \\
\mathcal{T}[\![\lambda x.\, e]\!]_{PC} &\triangleq \textbf{produce thunk } \lambda x.\, [\![e]\!]_{PC} \\
\mathcal{T}[\![\textbf{let} x = e_1 \textbf{ in } e_2]\!]_{PC} &\triangleq \mathcal{T}[\![e_1]\!]_{PC} \textbf{ to } x.\, \mathcal{T}[\![e_2]\!]_{PC} \\
\mathcal{T}[\![e_1\, e_2]\!]_{PC} &\triangleq \mathcal{T}[\![e_2]\!]_{PC} \textbf{ to } x.\, \mathcal{T}[\![e_1]\!]_{PC} \textbf{ to } f.\, x`(\textbf{force } f) \\
\mathcal{T}[\![\textbf{coin}]\!]_{PC} &\triangleq \textbf{produce coin} \\
\mathcal{T}[\![\textbf{rand}]\!]_{PC} &\triangleq \textbf{produce rand} \\
\mathcal{T}[\![\textbf{inl}_{\tau_1+\tau_2} e]\!]_{PC} &\triangleq \mathcal{T}[\![e]\!]_{PC} \textbf{ to } z.\, \textbf{produce inl } z \\
\mathcal{T}[\![\textbf{inr}_{\tau_1+\tau_2} e]\!]_{PC} &\triangleq \mathcal{T}[\![e]\!]_{PC} \textbf{ to } z.\, \textbf{produce inr } z \\
\mathcal{T}[\![\textbf{case } e_1 \textbf{of } e_2 | e_3]\!]_{PC} &\triangleq \mathcal{T}[\![e_1]\!]_{PC} \textbf{ to } z.\ \textbf{pm } z \textbf{ as } \{\textbf{inl } x. \mathcal{T}[\![e_2]\!]_{PC},\ \textbf{inr } x. \mathcal{T}[\![e_3]\!]_{PC}\} \\
\mathcal{T}[\![(e_1, e_2)]\!]_{PC} &\triangleq \\
\mathcal{T}[\![\#\mathbf{1}\ e]\!]_{PC} &\triangleq \\
\mathcal{T}[\![\#\mathbf{2}\ e]\!]_{PC} &\triangleq \\
\mathcal{T}[\![e_1 \textbf{ to } x \textbf{ in } e2]\!]_{PC} &\triangleq
\end{aligned}$$

Theorem. *If* $\Gamma \vdash_{IID} e : \tau$ *then*

$$\Gamma \vdash [\![\, e]\!]_{IID} : [\![\, \tau \,]\!]_{IID}$$

and if $\Gamma \vdash_{PC} e : \tau$ *then*

$$\Gamma \vdash [\![\, e]\!]_{PC} : [\![\, \tau \,]\!]_{PC}$$

Proof. First, assume $\Gamma \vdash_{IID} e : \tau$. We proceed by induction on e.

- **Case:** $e = x$ ... TODO!
- **Case:** $e = \lambda x : \tau.\, e'$. By inspection, $\Gamma \vdash_{IID} e : \tau \to \tau'$ for some τ' such that $\Gamma_{x:\tau} \vdash_{IID} e' : \tau'$. Then $[\![\, e]\!]_{IID} = \lambda x.\, [\![\, e' \,]\!]_{IID}$.
- TODO!

Now, assume $\Gamma \vdash_{PC} e : \tau$. TODO! **QED**

Theorem. *If* $\Gamma \vdash_{IID} e : \tau_1 \times \tau_2$ *then* $[\![\, e \,]\!] = \mu_1 \times \mu_2$.

Proof. TODO! **QED**