Question 1

1) As the volatility tends to infinity, the price of a vanilla call option increases without bounds. On the other hand, the price of a vanilla put option tends to the price of the underlying asset as the volatility approaches infinity. This is because an infinitely volatile underlying asset makes it more likely that the option will end up in the money, and therefore more valuable.

2) As the volatility tends to 0, the price of a vanilla call option approaches its intrinsic value, which is the difference between the underlying asset's price and the option's strike price, capped at 0. The price of a vanilla put option will approach 0 as the volatility tends to 0, because the option will have no time value and therefore its price will only be influenced by its intrinsic value, which will be close to 0 if the underlying asset price is close to the strike price.

3) ***European Call Option:***

For the European call option, the upper bound of the option price will never exceed the underlying stock price. Otherwise, there will be arbitrage opportunity by selling a call option at price P and use the money to buy 100 shares of the stock at S. I will lock in the risk-free profit P-S. Thus, the upper bound will be:

$$C(X, 0) \leq S_0$$

For the lower bound, we can build up two portfolios A and B to analyze.

Portfolio	Value of the portfolio	Value of the Portfolio at Maturity	
		$S_T < X$	$S_T \geq X$
A	S_0	S_T	S_T
B	$C(X, 0) + Xe^{-rT}$	X	$S_T - X + X = S_T$

Portfolio A consist of a stock, worth S_0, at maturity, worth S_T. Portfolio B consist of an amount of cash with PV of $Xe^{-rT}$and a European call option with strike X. From the table, we can see that the value of portfolio A is for sure not more than that of B. So, the present value of A will not be worth more than B.

$$C(X, 0) + Xe^{-rT} \geq S_0$$
$$C(X, 0) \geq S_0 - Xe^{-rT}$$

So, the lower bound of European call option is

$$C(X, t) \geq \max\,(0, S_0 - Xe^{-r(T-t)})$$

So, the range of the European Call option will be:

$$\max\,(0, S_0 - Xe^{-r(T-t)}) \leq C(X, t) \leq S_0$$

European Put Option:

The upper bound of the European call option will be the strike present value of the strike X. This is because that the maximum profit for the buyer will be the strike price K – 0 when the underlying assets price is 0.

For the lower bound by using the same logic as above, the lower bound of a put option is

$$P(X,t) \geq \max\left(0, Xe^{-r(T-t)} - S_t\right)$$

So, the range of European put option will be:

$$\max\left(0, Xe^{-r(T-t)} - S_t\right) \leq P(X,t) \leq Xe^{-r(T-t)}$$

American Call option:

The upper bound of an American call option will be the same as that of European Call option.

$$c(X,t) \leq S_t$$

The lower bound of an American call option will be higher than that of European call option as American call option can be exercised earlier.

$$c(X,0) \geq C(X,0) \geq \max(0, S_0 - Xe^{-rT})$$

Moreover, considering a American call option can be exercised earlier at the strike price, so, if $c(X,0) < S_0 - X$, there will be arbitrage opportunity. Hence,

$$c(X,0) \geq \max\left(S_0 - X, 0\right)$$

As $S_0 - Xe^{-rT} \geq S_0 - X$, the lower bound of an American call option will be

$$c(X,0) \geq \max\left(0, S_0 - Xe^{-rT}\right)$$

The range of an American call option is:

$$\max\left(0, S_t - Xe^{-r(T-t)}\right) \leq c(X,t) \leq S_t$$

American put option:

Using the same logic, the range of an American put option will be:

$$\max\left(0, X - S_t\right) \leq p(X,t) \leq X$$

Combined Table:

Type	Upper Bound	Lower Bound
European Call	S_t	$\max\left(0, S_t - Xe^{-r(T-t)}\right)$
European Put	$Xe^{-r(T-t)}$	$\max\left(0, Xe^{-r(T-t)} - S_t\right)$
American Call	S_t	$\max\left(0, S_t - Xe^{-r(T-t)}\right)$
American Put	X	$\max\left(0, X - S_t\right)$

Question 2

Given the Vega formula, we can see that if we want to maximize Vega, we need to maximum the probability density function. As the probability density function is differentiated from Normal distribution cumulative equation, so when d1 = 0 will gives the highest density function. Hence,

$$d1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K} + (r - 1 + \frac{\sigma^2}{2}) \times T\right) = 0$$

Solve the equation by setting d1 =0, we have K as:

$$K = S_0 \times e^{(r-q+\frac{\sigma^2}{2})\times T}$$

We can also use python codes to search for the strike price. Commonly, the Vega is highest for at the money options, like the options with strike price nearer to the current stock price. This is because Vega is highest for the options with high sensitivity to changes in the underlying stock price. From the codes I have provided, the maximum Vega strike price is also near to the stock price I set.

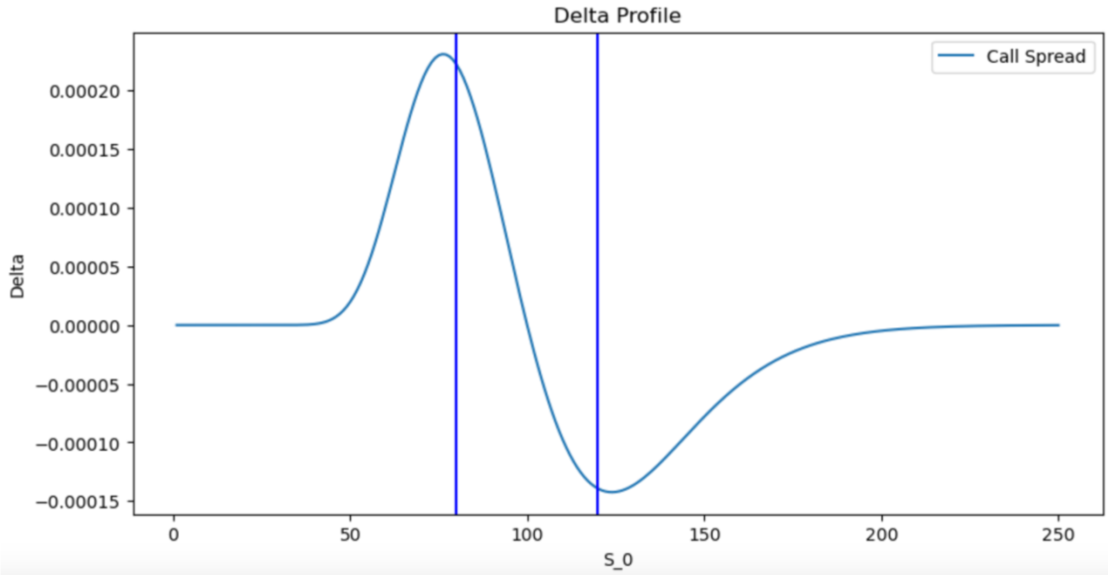
In [11]:

```
import scipy.optimize as opt
import math
import numpy as np

def black_scholes_vega(S0, K, r, q, T, sigma):
    d1 = (np.log(S0 / K) + (r - q + sigma**2 / 2) * T) / (sigma * np.sqrt(T))
    vega = S0 * math.exp(-q * T) * math.sqrt(T) * math.exp(-0.5 * d1**2) / math.sqrt(2 * math.pi)
    return vega

def vega_obj_func(K, S0, r, q, T, sigma):
    return -black_scholes_vega(S0, K, r, q, T, sigma)

def find_maximum_vega_strike(S0, r, q, T, sigma):
    res = opt.minimize_scalar(vega_obj_func, bounds=(0, S0), args=(S0, r, q, T, sigma), method='bounded')
    return res.x

S0 = 100 # current stock price
r = 0.05 # risk-free interest rate
q = 0.02 # dividend yield
T = 1 # maturity
sigma = 0.2 # volatility

maximum_vega_strike = find_maximum_vega_strike(S0, r, q, T, sigma)
maximum_vega_strike
```

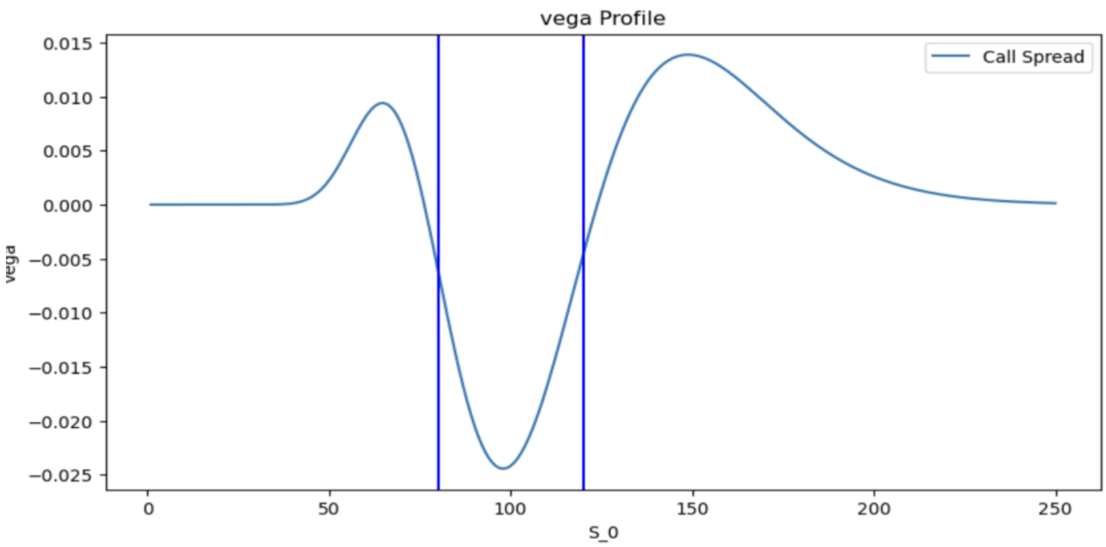
Out[11]: 99.9999921581193

Question 3

1) The digital option with a lower barrier L and an upper barrier U can be replicated using European call and put options. Here's how: Buy a European call option with strike K1 = U. This call option will pay out 1 if the underlying asset price at maturity ST is greater than U. Buy a European put option with strike K2 = L. This put option will pay out 1 if the underlying asset price at maturity ST is less than L. The sum of the payouts from the two options will replicate the payout of the digital option, since the call option will pay out if ST > U, and the put option will pay out if ST < L. By buying the call and put options, the portfolio will pay out 1 if L < ST < U, and 0 otherwise. This replicates the payout of the digital option. Note that the replication will not be perfect and there will be some deviation due to the price of the options and the fact that the digital option has a single point of discontinuity at ST = U, while the call and put options have a continuous payout.
2)



3)



Question 4

1) The delta of a European put option with the same maturity and strike as a European call option is equal to -1 + x. In other words, the delta of a put option is equal to the negative of the delta of a call option plus 1. This relationship holds because the two options combined represent a complete hedge against changes in the underlying asset price, so the sum of the delta of a call and put option with the same strike and maturity must
2) The Vega of a European put option with the same maturity and strike as a call option will have the same Vega y as that of a call option. As the volatility of the underlying asset is the same for both the call and put options, so they have the same Vega.