Question 1

(1)

$F_0(T) = 105, T = 1.5, S_0 = 100, d = 0\%$

Let the risk-free rate is r.

$F_0(T) = S_0 * e^{r(T-t)}$

$105 = 100 * e^{r(1.5-0)}$

$$r = \frac{\ln(\frac{105}{100})}{1.5} = 0.0325$$

So, the implied risk-free rate is 3.25%.

(2)

$d = 2\%.$

$F_0(T) = S_0 * e^{(r-d)(T-t)}$

$105 = 100 * e^{(r-2\%)(1.5-0)}$

$$r = \frac{\ln(\frac{105}{100})}{1.5} + 2\% = 0.0525$$

With a dividend yield, the implied risk free is 5.25%.

Question 2

$F_0(1) = 105, F_0(2) = 109, S_0 = 100, d = 0\%$

$F_0(1) = S_0 * e^{r_1 * 1}$

$105 = 100 * e^{r_1 * 1}$

$$r_1 = \ln\left(\frac{105}{100}\right) = 0.0488 \approx 4.88\%$$

$F_0(2) = S_0 * e^{r_2 * 2}$

$109 = 100 * e^{r_2 * 2}$

$$r_2 = \ln\left(\frac{109}{100}\right) = 0.0431 \approx 4.31\%$$

According to the no-arbitrage interest rate parity, we have:

$e^{r_2 * 2} = e^{r_1 * 1} * e^{r_{1,2} * 1}$

$r_{1,2} = 0.03739 \approx 3.74\%$

Question 3

$F_0(1) = 105, S_0 = 100, R_f = 2\%, S_0 = 100, d = 1\%$

The no arbitrage price of this forward should be:

$$
\begin{aligned}
F_0(1) &= S_0 * e^{(r-d)(T-t)} \\
&= 100 * e^{(2\%-1\%)*1} \\
&= 100 * e^{1\%} \\
&= \$101.0050
\end{aligned}
$$

So, the no arbitrage price is $101.0050. Now, the forward is trading at $105. There is an arbitrage opportunity.

By doing the arbitrage, I will short the forward contract at price of $105. At the same time, I will borrow money $100 at interest rate 2% and long 1 share of stock and hold it until maturity. At maturity, I will sell the stock at $105 which is predetermined at the beginning of the contract to the counterpart
y. With $105 at hand, I will repay the $100 with the interest. Of course, we need to consider the dividend we have received from the stock. So, the overall risk-free profit I will have for one share of stock will be:

$$Profit = 105 - 101.0050 = \$3.9950$$

Question 4

(1)

1 month later,
$F_0(T) = 105, T = 1, S_0 = 100, d = 0\%, r_f = 2\%, S_{1/12} = 120$

$$
\begin{aligned}
PnL &= 100 * \left(120 * \ e^{(2\%)\left(\frac{11}{12}\right)} - 100 * e^{(2\%)1}\right) * e^{(-2\%)*\frac{11}{12}} \\
&= \$1983.31
\end{aligned}
$$

So, the PnL is $1983.31

(2)

$d = 5\%$

$$
\begin{aligned}
PnL &= 100 * \left(120 * \ e^{(2\%-5\%)\left(\frac{11}{12}\right)} - 100 * e^{(2\%)1}\right) * e^{(-2\%)*\frac{11}{12}} \\
&= \$1445.74
\end{aligned}
$$

Question 5

1. *American Call Option of underlying stock paying dividend*

For American call option of underlying stock paying dividend, it may be optimal to exercise early.

I will use **European call option** first to demonstrate the payoff and compare it with **American call option** later to prove that it may be optimal to exercise early for American call option with underlying stock paying dividends.

Portfolio A	A European Call Option C (Strike X) + $\frac{X}{(1+r)^T}$ (cash present value $PV(X)$)
Portfolio B	1 share of Stock $S - PV(DIV)$ (Stock paying dividend at maturity T will cause stock price to be lower today)

For portfolio A,

(1) At maturity T, when $S_T > X$, I will exercise the call option by using the cash X to buy the stock at strike price X. Now, the stock has the value of S_T

(2) At maturity T, when $S_T < X$, I will not exercise the call option and I will only have cash X.

For portfolio B,
I will have $S_T - DIV$ as my payoff for both $S_T > X$ and $S_T < X$ cases.

	$S_T > X$	$S_T < X$
Portfolio A Payoff	S_T	X
Portfolio B Payoff	$S_T - DIV$	$S_T - DIV$

From the payoff table above, we can see that payoff of portfolio A is always better than the payoff of portfolio B. By no-arbitrage theory, the value of portfolio A should be higher than the value of portfolio B.

$$C + PV(X) > S - PV(DIV)$$
$$C > S - PV(DIV) - PV(X)$$

For **American call option**, the value by exercising early will be:

$$c = \max(S - X, 0)$$

Thus, if I want to exercise early, I need to make sure:

$$S - X > S - PV(DIV) - PV(X)$$
$$PV(DIV) > X - PV(X)$$

Hence, if the present value of the dividend is lower than $X - PV(X)$, we should not exercise American call option early. If the present value of the dividend is greater than the value of $X - PV(X)$, it will be optimal to exercise American option early.

2. American call option of underlying stock without dividend payment

It is never optimal to exercise American call option without dividend payment early.
I will use same logic as part (1) to prove that. Two manually constructed portfolios are as followings:

Portfolio A	A European Call Option C (Strike X) + $\frac{X}{(1+r)^T}$ (cash present value $PV(X)$)
Portfolio B	1 share of Stock S

For portfolio A,

(3) At maturity T, when $S_T > X$, I will exercise the option by using the cash X to buy the stock at strike price K. Now, the stock has the value of S_T

(4) At maturity T, when $S_T < X$, I will not exercise the call option and I will only have cash X.

For portfolio B,
I will have S_T as my payoff for both $S_T > X$ and $S_T < X$ cases.

	$S_T > X$	$S_T < X$
Portfolio A Payoff	S_T	X
Portfolio B Payoff	S_T	S_T

Thus, from the table above, we can clearly see that the payoff of my portfolio A is always greater or equal to the payoff of the portfolio B at maturity. According to the no-arbitrage theory, the value of my portfolio A should be greater or equal to the value of portfolio B today.

$$C + PV(X) \geq S$$
$$C \geq \max\,(S - PV(X), 0)$$

As **American option** can exercise anytime, we can just replace the American call option in the contract A, the American call option payoff is:

$$c \geq \max\,(S - X, 0)$$

As $PV(X) < X, so\ S - PV(X) > S - X$. The payoff by exercising early is lower than holding to the maturity. Thus, without any dividend's payment, American option should never exercise early. The value of American call option is the same as the value of European call option. Instead, we can sell the call option to preserve the time value embedded in the option contract.

3. *American Put Option of underlying stock without dividend payment*

For American put option without dividend payment, it may be optimal to exercise early. This is because the profit the put option is limited, by exercising early, I can gain profit in advance and earn extra interest rate. By using the same logic, I will construct two portfolios as well to proof my conclusion.

Portfolio A	A European Put Option P (Strike X) + Holding 1 share Stock S
Portfolio B	$\frac{X}{(1+r)^T}$ (Cash present value $PV(X)$

For portfolio A,

(1) At maturity T, when $S_T > X$, I will not exercise the put option. I will only have the stock S_T.

(2) At maturity T, when $S_T < X$, I will exercise the put option by selling the stock at X.

For portfolio B,
I will have X as my payoff for both $S_T > X$ and $S_T < X$ cases.

	$S_T > X$	$S_T < X$
Portfolio A Payoff	S_T	X
Portfolio B Payoff	X	X

From the payoff table above, the payoff of Portfolio A is always greater than that of Portfolio B. By no-arbitrage theory, the present value of portfolio A should be greater or equal to that of portfolio B.

$$P + S \geq PV(X)$$
$$P \geq \max\,(PV(X) - S, 0)$$

Replace the European put option with American put option, then the value of American put option will be:

$$p \geq \max\,(X - S, 0)$$

As $X - S > PV(X) - S$, so it may be optimal to exercise early to earn some risk-free profit.

Question 6

The factor interest rate will cause the price of future contract and forward contract to be different. If the interest rate is constant, then the price of the forward and future contact will be the same.

However, if the interest movement and price of the future contact is positively correlated, then the future price will be higher than the forward price. This is because of the margin account of the future contract. When the price goes up, the interest rate will go up as well. The party with long position of the future contract will gain excess margin and he can withdraw the excess margin and deposit into his bank account with a higher interest rate. The short party will lose margin and when the margin is not enough, then short party needs to borrow money with a higher interest rate. Thus, the short party will short the future contract with a higher price to compensate the imbalance embedded in the contract.

Whereas, if the interest rate and future price is negatively correlated, then the future price will be lower than the forward price. When the future price goes up, the long party will gain excess margin, but long party can only deposit the money with a lower interest rate. However, the short party can borrow money with a lower interest rate to top up the margin account. When the future price moves down, the interest rate will be higher. The short party will benefit more from it as short party can withdraw the excess margin and deposit into bank account with a higher interest rate. Thus, the long party will demand the price of the future contract to be lower than the forward price.