Question 1

$r = 0, q^X = q^Y = q^Z = 0$

$Let\ A_T = \frac{X_T}{Y_T}$

$$\begin{aligned}
dA_T &= \frac{\partial A_T}{\partial X_T} dX_T + \frac{\partial A_T}{\partial Y_T} dY_T + \frac{1}{2}\left[\frac{\partial^2 A_T}{\partial {X_T}^2} d{X_T}^2 + \frac{\partial^2 A_T}{\partial {Y_T}^2} d{Y_T}^2 + 2 \times \frac{\partial^2 A_T}{\partial X_T \partial Y_T} dX_T dY_T\right] \\
&= \frac{1}{Y_T} dX_T - \frac{X_T}{(Y_T)^2} dY_T + \frac{1}{2} \times \left[0 + \frac{2X_T}{(Y_T)^3}(dY_T)^2 - \frac{2}{Y_T^2} dX_T dY_T\right] \\
&= \frac{1}{Y_T}[X_T \sigma^X dW_T^X] - \frac{X_T}{Y_T^2}[Y_T \sigma^Y dW_T^Y] + \frac{1}{2}\left[\frac{2X_T}{Y_T^3} \times Y_T^2 (\sigma^Y)^2 dt - \frac{2}{Y_T^2} \times X_T Y_T \sigma^X \sigma^Y \rho_{XY} dt\right] \\
&= \frac{X_T}{Y_T} \sigma^X dW_T^X - \frac{X_T}{Y_T} \sigma^Y dW_T^Y + \frac{X_T}{Y_T} (\sigma^Y)^2 dt - \frac{X_T}{Y_T} \sigma^X \sigma^Y \rho_{XY} dt
\end{aligned}$$

Hence, we will have the SDE for A_T:

$$\frac{dA_T}{A_T} = ((\sigma^Y)^2 - \sigma^X \sigma^Y \rho) dt + (\sigma^X dW_T^X - \sigma^Y dW_T^Y)$$

$$\sigma^X dW_T^X - \sigma^Y dW_T^Y = \sigma^A dW_T^A$$

$$\sigma^A = \sqrt{(\sigma^Y)^2 + (\sigma^X)^2 - 2 \times \sigma^X \sigma^Y \rho_{XY}}$$

$$q^A = \sigma^X \sigma^Y \rho_{XY} - (\sigma^Y)^2$$

$$\frac{dA_T}{A_T} = -q^A dt + \sigma^A dW_T$$

$$\frac{dZ_T}{Z_T} = \sigma^Z dW_T^Z$$

By using spread option finall payoff formula, we will have:

$$V_0 = A_0 e^{-q^A T} \Phi(d1) - Z_0 \Phi(d2)$$

Where:

$$\sigma = \sqrt{(\sigma^A)^2 + (\sigma^Z)^2}$$

$$d_{1,2} = \frac{\left(\ln\left(\frac{A_0}{Z_0}\right) + (-q^A \pm \frac{1}{2}\sigma^2) T\right)}{\sigma\sqrt{T}}$$

$Question\ 2$

$The\ pricing\ formula\ \max(X_T, Y_T)\ can\ be\ expressed\ as:$

$$Y_T + \max(X_T - Y_T, 0)$$

$Under\ risk\ neutral\ measure\ Q^*\ of\ a\ money\ account\ as\ numeraire, the\ payoff\ will\ be\ a\ martingle.$

$$\frac{V_0}{B_0} = E^*\left[\frac{V_T}{B_T}\right] = E^*\left[\frac{[Y_T + (X_T - Y_T)^+]}{B_T}\right]$$
$$V_0 = E^*\left[\frac{B_0}{B_T} \times [Y_T + (X_T - Y_T)^+]\right]$$
$$= E^*\left[\frac{B_0}{B_T} \times Y_T + \frac{B_0}{B_T}(X_T - Y_T)^+\right]$$
$$= E^*\left[\frac{B_0}{B_T} \times Y_T\right] + E^*\left[\frac{B_0}{B_T}(X_T - Y_T)^+\right]$$

$As\ \frac{dY_t}{Y_T} = (r - q^Y)dt + \sigma^Y dW_t^Y, so\ Y_T\ will\ follow\ a\ bronian\ motion\ as\ follow:$

$$Y_T = Y_0 \times e^{(r-q-\left(\frac{\sigma^Y}{2}\right)^2)T + \sigma^Y W_T^Y}$$

$Thus, under\ risk\ neutral\ meansure\ Q^*, the\ expectation\ of\ \frac{B_0}{B_T} \times Y_T\ will\ be:$

$$V_{0_1} = \ e^{-rT}E^*\left[Y_0 \times e^{(r-q-\left(\frac{\sigma^Y}{2}\right)^2)T + \sigma^Y W_T^Y}\right] = Y_0 \times e^{-q^Y T}$$

$V_{0_2} = E^*\left[\frac{B_0}{B_T}(X_T - Y_T)^+\right], under\ risk\ neutral\ measure, we\ cannot\ have\ a\ closed\ form\ solution, thus, we\ need\ to\ change\ measure\ to\ measure\ Y, with\ numeraire\ as\ Y_t e^{q^Y t}.$

$$V_{0_2} = E^*\left[\frac{B_0}{B_T}(X_T - Y_T)^+\right]$$
$$= E^*\left[\frac{B_0 Y_T}{B_T}\left(\frac{X_T}{Y_T} - 1\right)^+\right]$$
$$= \ E^*\left[\left(\frac{X_T}{Y_T} - 1\right)^+ \times \frac{\frac{Y_T e^{q^Y T}}{Y_0 e^{q^Y \times 0}}}{\frac{B_T}{B_0}} \times Y_0 e^{q^Y \times 0} \times e^{-q^Y T}\right]$$
$$= \ E^*\left[\left(\frac{X_T}{Y_T} - 1\right)^+ \times \frac{dQ^Y}{dQ^*} \times Y_0 \times e^{-q^Y T}\right]$$

$Use\ Radon\ Nikodym\ theorem, We\ will\ have\ a\ closed\ form\ formula\ for\ the\ payoff$

$$V_{0_2} = Y_0 \times e^{-q^Y T}\, E^Y \left[\left(\frac{X_T}{Y_T} - 1\right)^+\right]$$

By using call spread option payoff formula, we will have:

$$V_{0_2} = X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}\Phi(d_2)$$

Where:

$$d_{1,2} \ = \ d_{1,2} = \frac{(\ln\left(\frac{X_0}{Y_0}\right) + (q^Y - q^X \pm \frac{1}{2}\sigma^2)T)}{\sigma\sqrt{T}}$$

$$\sigma \ = \ \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X\sigma^Y\rho}$$

Therefore, by adding the two payoff function, we will have our final payoff as follow:

$$V_0 = V_{0_1} + V_{0_2}$$

$$V_0 = \ X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}\Phi(d_2) + Y_0 e^{-q^Y T}$$
$$= X_0 e^{-q^X T}\Phi(d_1) + Y_0 e^{-q^Y T}(1 - \Phi(d_2))$$

Question 3

The pricing formula $\min(X_T, Y_T)$ *can be expressed as:*

$$Y_T - \max(Y_T - X_T, 0)$$

Using the same logic as Question 2, *we will have the payoff funciton of* Y_T *as:*

$$V_{0_3} = \ e^{-rT} E^* \left[Y_0 \times e^{r-q-(\sigma^Y)^2 T + \sigma^Y W_T^Y}\right] = Y_0 \times e^{-q^Y T}$$

The payoff for the spread under measure X will be:

$$V_{0_4} = X_0 \times e^{-q^X T}\, E^X \left[\left(\frac{Y_T}{X_T} - 1\right)^+\right]$$

By using the spread option final payoff function, the payoff will be:

$$V_{0_4} = Y_0 e^{-q^Y T}\Phi(d_3) - X_0 e^{-q^X T}\Phi(d_4)$$

Where:

$$d_{3,4} = \frac{(\ln\left(\frac{Y_0}{X_0}\right) + (q^X - q^Y \pm \frac{1}{2}\sigma^2)T)}{\sigma\sqrt{T}}$$

$$\sigma \ = \ \sqrt{(\sigma^Y)^2 + (\sigma^X)^2 - 2\sigma^X\sigma^Y\rho}$$

$So, the\ payoff\ funciton\ for\ the\ \min(X_T, Y_T)\ will\ be:$

$$V_0 = Y_0 e^{-q^Y T} - Y_0 e^{-q^Y T}\Phi(d_3) + X_0 e^{-q^X T}\Phi(d_4)$$
$$= Y_0 e^{-q^Y T}\left(1 - \Phi(d_3)\right) + X_0 e^{-q^X T}\Phi(d_4)$$

QUESTION 4

$In\ order\ to\ find\ the\ US\ stock\ process\ in\ SGD\ risk\ neutral\ measure, we\ use\ \beta_t^d\ as\ our\ numeraire.$

$$\frac{X_t Y_t}{\beta_t^d} = E_t^{\beta_t^d}\left[\frac{X_T Y_T}{\beta_T^d}\right]$$

$$d\beta_t^d = r^d \beta_t^d dt$$

$Let\ L_T = \frac{X_T Y_T}{\beta_T^d}, then\ the\ L_t should\ be\ risk\ neutral\ under\ SGD\ measure.$

$We\ assume\ that\ under\ SGD\ risk\ neutral\ measure, USD\ follows:$

$$\frac{dY_t}{Y_t} = \mu dt + \sigma^Y d\overline{W_t}^Y$$

$Where\ d\overline{W_t}^Y\ is\ a\ brownian\ motion\ under\ SGD\ risk\ neutral\ measure.$

$By\ Ito's\ formula:$

$$dL_T = \frac{\partial L_T}{\partial T} dT + \frac{\partial L_T}{\partial X_T} dX_T + \frac{\partial L_T}{\partial Y_T} dY_T + \frac{\partial^2 L_T}{\partial X_T Y_T} dX_T Y_T$$

$Rearrange, we\ have:$

$$\frac{dL_T}{L_T} = (-r^d + r^d - r^f + \mu + \sigma^X \sigma^Y \rho_{XY})dt + \sigma^Y dW_t^Y + \sigma^X dW_t^X$$

$In\ order\ for\ the\ L_T\ to\ be\ a\ martingale, the\ drift\ term\ must\ be\ zero, thus:$

$$-r^d + r^d - r^f + \mu + \sigma^X \sigma^Y \rho_{XY} = 0$$
$$\mu = r^f - \sigma^X \sigma^Y \rho_{XY}$$

$As\ there\ is\ dividend\ yield\ q^Y, thus, under\ SGD\ risk\ neutral\ measure, Y_t\ follows:$

$$dY_t = (r^f - q^Y - \sigma^X \sigma^Y \rho_{XY}) Y_t dt + \sigma^Y Y_t d\overline{W_t}^Y$$

$As\ S_0 and\ Y_0\ both\ are\ constant, so, the\ payoff\ will\ be$

$$max(\overline{S_T} - \overline{Y_T}, 0)$$
$$\overline{S_T} = \frac{S_T}{S_0}, \overline{Y_T} = \frac{Y_T}{Y_0}$$
$$\frac{d\overline{S_T}}{\overline{S_T}} = (r^d - q^S)dt + \sigma^S dW_t^S$$
$$\frac{d\overline{Y_t}}{\overline{Y_t}} = (r^f - q^Y - \sigma^S\sigma^Y\rho_{XY})dt + \sigma^Y d\overline{W_t}^Y$$
$$= (r^d - r^d + r^f - q^Y - \sigma^S\sigma^Y\rho_{XY})dt + \sigma^Y d\overline{W_t}^Y$$

$$\frac{d\overline{Y_t}}{\overline{Y_t}} = \left(r^d - \overline{q^Y}\right)dt + \sigma^Y d\overline{W_t}^Y$$

$Where$:

$$\overline{q^Y} = r^d - r^f + q^Y + \sigma^S\sigma^Y\rho_{XY}$$

$By\ applying\ the\ spread\ option\ formula, we\ have$:

$$\overline{S_0}e^{-q^S T}\Phi(d_1) - \overline{Y_0}e^{-\overline{q^Y}T}\Phi(d_2)$$
$$d_{1,2} = d_{1,2} = \frac{(\ln\left(\frac{\overline{S_0}}{\overline{Y_0}}\right) + (\overline{q^Y} - q^S \pm \frac{1}{2}\sigma^2)T)}{\sigma\sqrt{T}}$$
$$\sigma = \sqrt{(\sigma^S)^2 + (\sigma^Y)^2 - 2\sigma^S\sigma^Y\rho_{SY}}$$