HOMEWORK 5

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To derive up-and-in-put option, $(K - S_T)^+ 1_{M_T^S \geq H}$*, we need to find the join distribution,* $P(Z_T \leq x, M_T^S \geq y)$*. Before we find the joint distribution of* Z_T *with drift, we find the joint distribution of* W_T *first. Then we use Girsanov's Theorem to find the* Z_T *with drift.*

$$
\begin{aligned}
P(W_T \leq x, M_T^S \geq y) &= P(W_T \leq x | M_T^S \geq y) P(M_T^S \geq y) \\
&= P(W_T \geq 2y - x | M_T^S \geq y) P(M_T^S \geq y) \\
&= P(M_T^S \geq y | W_T \geq 2y - x) P(W_T \geq 2y - x) \\
&= P(W_T \geq 2y - x)
\end{aligned}
$$

Let event A be:

$$P(Z_T \leq x, M_T^S \geq y)$$

$Z_t = vt + \sigma W_t$

And let $Z_t = \sigma B_t, B_t = \mu t + W_t,$ *and* $\mu = \frac{v}{\sigma}$

The probability A can be computed as:

$$
\begin{aligned}
P(A) &= E^P(1_A) \\
&= E^P\left(1_{\{Z_T \leq x, M_T^S \geq y\}}\right) \\
&= E^Q\left(1_{\{Z_T \leq x, M_T^S \geq y\}} \frac{dP}{dQ}\right) \quad as \quad \frac{dP_t}{dQ_t} = e^{\mu B_t - \frac{1}{2}\mu^2 t} \\
&= E^Q\left(1_{\{Z_T \leq x, M_T^S \geq y\}} \frac{dP}{dQ}\right) \\
&= E^Q\left(1_{\{Z_T \leq x, M_T^S \geq y\}} e^{\mu B_t - \frac{1}{2}\mu^2 t}\right) \\
&= E^Q\left(1_{\{Z_T \leq x, M_T^S \geq y\}} e^{\frac{v}{\sigma} Z_t - \frac{1}{2}\mu^2 t}\right) \\
&= E^Q\left(1_{\{Z_T \geq 2y - x\}}\right) e^{\left(\frac{v}{\sigma}(2y - Z_t) - \frac{1}{2}\mu^2 t\right)} \\
&= e^{\left(\frac{2vy}{\sigma^2}\right)} E^Q\left(1_{\{Z_T \geq 2y - x\}}\right) e^{\left(-\mu B_t - \frac{1}{2}\mu^2 t)\right)}
\end{aligned}
$$

We can regard the $e^{\left(-\mu B_t - \frac{1}{2}\mu^2 t)\right)}$ *as the Radon-Nikodym derivative which changes measure from Q to measure S.*

$X_t = \mu t + B_t$ *is a Brownian motion under S measure.*

$$\frac{dS_t}{dQ_t} = e^{\left(-\mu B_t - \frac{1}{2}\mu^2 t)\right)}$$

So, we now change measure from Q to S measure,

$$
\begin{aligned}
P(A) &= e^{\left(\frac{2vy}{\sigma^2}\right)} E^Q\left(1_{\{Z_T \geq 2y - x\}}\right) e^{\left(-\mu B_t - \frac{1}{2}\mu^2 t)\right)} \\
&= e^{\left(\frac{2vy}{\sigma^2}\right)} E^Q\left(1_{\{Z_T \geq 2y - x\}}\right) \frac{dS_t}{dQ_t}
\end{aligned}
$$

$$= e^{\left(\frac{2\upsilon y}{\sigma^2}\right)} E^S\left(1_{\{Z_T \geq 2y-x\}}\right)$$
$$= e^{\left(\frac{2\upsilon y}{\sigma^2}\right)} E^S\left(1_{\{\sigma B_t \geq 2y-x\}}\right)$$
$$= e^{\left(\frac{2\upsilon y}{\sigma^2}\right)} E^S\left(1_{\{\sigma(X_t-\mu t) \geq 2y-x\}}\right)$$
$$= e^{\left(\frac{2\upsilon y}{\sigma^2}\right)} E^S\left(1_{\{\sigma X_t \geq 2y-x+\upsilon t\}}\right)$$
$$= e^{\left(\frac{2\upsilon y}{\sigma^2}\right)} E^S\left(1_{\{\sigma X_t \geq 2y-x+\upsilon t\}}\right)$$
$$= e^{\left(\frac{2\upsilon y}{\sigma^2}\right)} \left(1 - \Phi\left(\frac{2y - x + \upsilon t}{\sigma\sqrt{t}}\right)\right)$$
$$= e^{\left(\frac{2\upsilon y}{\sigma^2}\right)} \left(\Phi\left(\frac{-2y + x - \upsilon t}{\sigma\sqrt{t}}\right)\right)$$

Now, we can start to price our up-and-input option.

$$\begin{aligned}
UIP &= N_0 E_0\left[\frac{(K - S_T)^+}{N_T} 1_{M_T^S \geq H}\right] \\
&= N_0 E_0\left[\frac{(K - S_T)}{N_T} 1_{S_T \leq K, M_T^S \geq H}\right] \\
&= N_0 E_0\left[\frac{(K - S_T)}{N_T} 1_{S_T \leq K, M_T^S \geq H}\right] \\
&= N_0 E_0\left[\frac{K}{N_T} 1_{S_T \leq K, M_T^S \geq H}\right] - N_0 E_0\left[\frac{S_T}{N_T} 1_{S_T \leq K, M_T^S \geq H}\right] \\
&= e^{-rT} K E_0^{\beta}\left[1_{S_T \leq K, M_T^S \geq H}\right] - S_0 e^{-qT} E_0^S\left[1_{S_T \leq K, M_T^S \geq H}\right]
\end{aligned}$$

$we\ set\ V_0^1 = e^{-rT} K E_0^{\beta}\left[1_{S_T \leq K, M_T^S \geq H}\right], we\ choose\ N_t = \beta_t = e^{rt}$
$we\ set\ V_0^2 = S_0 e^{-qT} E_0^S\left[1_{S_T \leq K, M_T^S \geq H}\right], we\ choose\ N_t = S_t e^{qt}$

$We\ set\ Z_t = \ln\left(\frac{S_t}{S_0}\right), x = \ln\left(\frac{K}{S_0}\right), y = \ln\left(\frac{H}{S_0}\right), \upsilon = \mu - \frac{1}{2}\sigma^2, hence, we\ have:$

$$P(Z_T \leq x, M_T^S \geq y) = P(S_T \leq K, M_T^S \geq H)$$

$$\begin{aligned}
V_0^1 &= e^{-rT} K E_0^{\beta}\left[1_{S_T \leq K, M_T^S \geq H}\right] \\
&= e^{-rT} K P(Z_T \leq x, M_T^S \geq y) \\
&= K e^{-rT}\left(e^{\left(\frac{2\upsilon y}{\sigma^2}\right)}\left(\Phi\left(\frac{-2y + x - \upsilon t}{\sigma\sqrt{t}}\right)\right)\right) \\
&= K e^{-rT}\left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(\frac{\ln\left(\frac{S_0 K}{H^2}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

$for\ V_0^2, under\ S\ measure, v = \mu + \frac{1}{2}\sigma^2, we\ have:$

$$\begin{aligned}
V_0^2 &= S_0 e^{-qT} E_0^S\left[1_{S_T \le K, M_T^S \ge H}\right] \\
&= S_0 e^{-qT} P(Z_T \le x, M_T^S \ge y) \\
&= S_0 e^{-qT}\left(e^{\left(\frac{2vy}{\sigma^2}\right)}\left(\Phi\left(\frac{-2y + x - vt}{\sigma\sqrt{t}}\right)\right)\right) \\
&= S_0 e^{-qT}\left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}+1} \Phi\left(\frac{\ln\left(\frac{S_0 K}{H^2}\right) - \left(\mu + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

Put it all together, the price of up-and-in-put is:

$$\begin{aligned}
V_0 &= V_0^1 + V_0^2 \\
&= K e^{-rT}\left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(\frac{\ln\left(\frac{S_0 K}{H^2}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\
&\qquad - S_0 e^{-qT}\left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}+1} \Phi\left(\frac{\ln\left(\frac{S_0 K}{H^2}\right) - \left(\mu + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$