FIXED INCOME GROUP PROJECT

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1 Part I

1.1 Bootstrapping OIS Discounting Rates

For the OIS discount rates, we are using the following methodology to find out.

$$PV_{fix}^{OIS} = PV_{flt}^{OIS}$$

$$D_0(0,6m) \times 6m \times OIS_{6m} = D_0(0,6m) \times \left[\left(1 + \frac{f_0}{360}\right)^{180} - 1 \right]$$

$$D_0(0,1y) \times OIS_{1y} = D_0(0,1y) \times \left[\left(1 + \frac{f_0}{360}\right)^{180} \left(1 + \frac{f_1}{360}\right)^{180} - 1 \right]$$

$$\vdots$$

$$(D_0(0,1y) \cdots + D_0(0,30y))OIS_{30y} = D_0(0,1y) \left[(1 + \frac{f_0}{360})^{180} (1 + \frac{f_1}{360})^{180} - 1 \right] \cdots + D_0(0,30y) \left[\left(1 + \frac{f_{30}}{360}\right)^{360} - 1 \right]$$

The above methodology can be simplified into the following equation:

$$(D_0(0,1y) + \cdots + D_0 0,T) \times OIS_T = 1 - D_0(0,T)$$

Let us use a simple example, tenor 6m to illustrate why it works this way.

$$D_0(0,6m) \times 6m \times OIS_{6m} = D_0(0,6m) \times \left[\left(1 + \frac{f_0}{360}\right)^{180} - 1 \right] \cdots (1)$$

$$D_0(0,6m) = \frac{1}{(1 + \frac{f_0}{360})^{180}} \cdots (2)$$

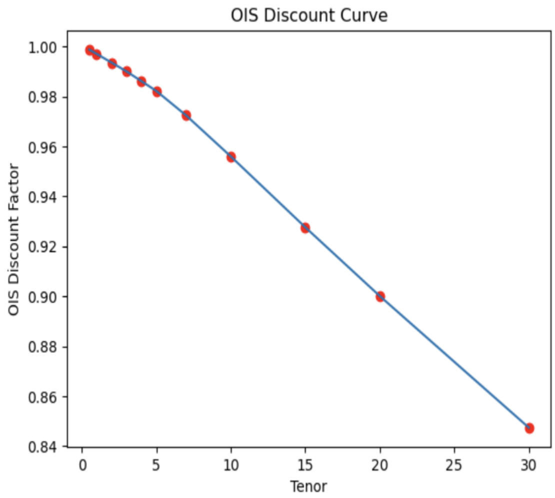
We substitute equation (2) into equation (1), we will get the following results:

$$D_0(0,6m) \times 6m \times OIS_{6m} = 1 - D_0(0,6m)$$

The same logic can be applied to the rest of the discounting rates calculation. As a result, The simplified methodology becomes the same as the LIBOR par swap rate calculation. Based on the methodology we derived, we build our codes to do the interpolation and get the OIS discount rates of the corresponding tenors. The graphs below are the OIS discounting rates results table and the visualization of it.

	Tenor	Product	OIS_Discount
0	0.5	OIS	0.998752
1	1.0	OIS	0.997009
2	2.0	OIS	0.993531
3	3.0	OIS	0.990015
4	4.0	OIS	0.986117
5	5.0	OIS	0.982184
7	7.0	OIS	0.972406
10	10.0	OIS	0.955977
15	15.0	OIS	0.927611
20	20.0	OIS	0.900076
30	30.0	OIS	0.847407

(a) OIS Discounting Rates



(b) Visualization

Figure 1: OIS DISCOUNTING RATES

1.2 Bootstrapping LIBOR Discount Factors

According to the project requirement, the swap market is collateralised. So, we used OIS discount rates from the part 1 to discount both the fix legs and floating legs semi-annually.

For the LIBOR discount rates, we used the following methodology:

$$PV_{fix}^{LIBOR} = PV_{flt}^{LIBOR}$$

$$D_0(0, 0.5y) \times 0.5 \times IRS_{0.5} = D_0(0, 0.5) \times \frac{\widetilde{D}(0,0) - \widetilde{D}(0, 0.5y)}{\widetilde{D}(0, 0.5y)}$$

$$(D_0(0, 0.5y) + D_0(0, 1y)) \times IRS_{1y} \times 0.5 = D_0(0, 0.5) \times \frac{\widetilde{D}(0,0) - \widetilde{D}(0, 0.5y)}{\widetilde{D}(0, 0.5y)} + D_0(0, 1y) \times \frac{\widetilde{D}(0, 0.5y) - \widetilde{D}(0, 1y)}{\widetilde{D}(0, 1y)}$$

$$\vdots$$

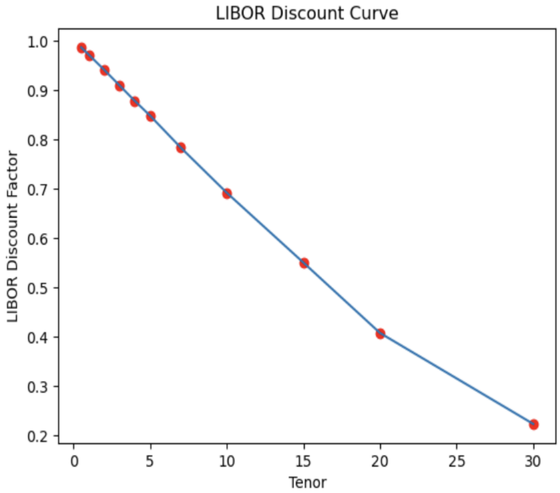
$$(D_0(0, 0.5) + \cdots + D_0(0, 30y)) \times IRS_{30y} \times 0.5 = D_0(0, 0.5) \times \frac{\widetilde{D}(0,0) - \widetilde{D}(0, 0.5y)}{\widetilde{D}(0, 0.5y)} + \cdots + D_0(0, 30y) \times \frac{\widetilde{D}(0, 29.5) - \widetilde{D}(0, 30)}{\widetilde{D}(0, 30)}$$

Where $\frac{\widetilde{D}(0,T-0.5)-\widetilde{D}(0,T)}{\widetilde{D}(0,T)}$ can be simplified to $1 + \frac{L(T-0.5y,T)}{2} - 1 = \frac{L(T-0.5y,T)}{2}$, T denotes the tenor 0.5y,1y,1.5y...30y

In a nutshell, we have transformed the expression of the present values used to be in the forward LIBOR rates discounting into multiple combinations of the LIBOR discounting factors. By doing so, instead of getting the LIBOR forward rates from the conventional methodology, we can now solely focus on calculating and interpolating the LIBOR discounting factors. As a result, the whole process will be very much simplified which made our codes neat and short.The graphs below are our LIBOR discount factors and visualisation of it.

	Tenor	OIS_Discount	Libor_DF
0	0.5	0.998752	0.987654
1	1.0	0.997009	0.972577
3	2.0	0.993531	0.942179
5	3.0	0.990015	0.910482
7	4.0	0.986117	0.878981
9	5.0	0.982184	0.848989
13	7.0	0.972406	0.784216
19	10.0	0.955977	0.692710
29	15.0	0.927611	0.551081
39	20.0	0.900076	0.408218
59	30.0	0.847407	0.223662

(a) LIBOR Discounting Rates



(b) Visualization

Figure 2: LIBOR DISCOUNTING RATES

1.3 Calculate The Forward Swaps

The following forward swaps will be calculated:

- $1y \times 1y, 1y \times 2y, 1y \times 3y, 1y \times 5y, 1y \times 10y$
- $5y \times 1y, 5y \times 2y, 5y \times 3y, 5y \times 5y, 5y \times 10y$
- $10y \times 1y, 10y \times 2y, 10y \times 3y, 10y \times 5y, 10y \times 10y$

Basically, the calculations of the forward swap rates are the combination of the LIBOR discount factors and OIS discount factors. As the LIBOR discount factors are semi-annually based. We assumed that the forward swaps fix and floating legs payments are under semi-annual payment frequency. The methodology we used to calculate the forward swap rate is:

$$PV_{fix} = PV_{flt}$$

$$S_{1y,2y} \times 0.5 \times (D_0(0,1.5y) + D_0(0,2y)) = D_0(0,1.5y) \times \frac{\widetilde{D}(0,1y) - \widetilde{D}(0,1.5y)}{\widetilde{D}(0,1.5y)} + D_0(0,2.0y) \times \frac{\widetilde{D}(0,1.5y) - \widetilde{D}(0,2y)}{\widetilde{D}(0,2y)}$$

$$\vdots$$

$$S_{10y,20y} \times 0.5 \times \sum_{i=1.5}^{20} D_0(0,i) = D_0(0,1.5y) \times \frac{\widetilde{D}(0,1y) - \widetilde{D}(0,1.5y)}{\widetilde{D}(0,1.5y)} + \cdots + D_0(0,20y) \times \frac{\widetilde{D}(0,19.5y) - \widetilde{D}(0,20y)}{\widetilde{D}(0,20y)}$$

By using above methodology, we have our forward swap rates table below:

	1Y	2Y	3Y	5Y	10Y
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

Figure 3: Forward Swap Rates Table

2 Part II

2.1 Displaced Diffusion Model Calibration

For the DD model calibration, we use Black 1976 model as our reporting model for the implied volatility. Besides, we assumed the swap rate is a martingale process.

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}(t)$$

Our reporting model for the volatility of the swaptions is:

$$\begin{aligned} V_{n,N}^{Payer}(0) &= P_{n+1,N}(0)E^{n+1,N}[(S_{n,N}(T)-K)^+] \\ &= P_{n+1,N}(0)[S_{n,N}(0)\Phi(d_1) - K\Phi(d_2)] \end{aligned}$$

Where

$$d_1 = \frac{log\frac{S_{n,N}(0)}{K} + \frac{1}{2}\sigma_{n,N}T}{\sigma_{n,N}\sqrt{T}}, d_2 = d_1 - \sigma_{n,N}\sqrt{T}$$

The Displaced Diffusion model we are using is simply built upon the Black76 model:

$$dS_{n,N}(t) = \sigma_{n,N}[\beta S_{n,N}(t) + (1-\beta)S_{n,N}(0)]dW_t^{n,N}$$

The swaption price under the DD model will be:

$$V_{n,N}(0) = P_{n+1,N}(0)\text{Black76}\left(\frac{S_{n,N}(0)}{\beta}, K + \frac{1-\beta}{\beta}S_{n,N}(0), \sigma\beta, T\right)$$

We set the volatility for our DD model the same as the ATM in the swaption table. Then, we calibrate the beta to make the implied volatility derived from the benchmark model Black76 from the price of the DD model to fit the market volatility by least square method. We set our initial guess of beta as 0.3. Below are our beta and sigma tables.

	1Y	2Y	3Y	5Y	10Y
1Y	1.014612e-08	3.784358e-13	1.480932e-12	0.000002	0.000007
5Y	1.305340e-11	5.502228e-08	2.277269e-06	0.000143	0.055462
10Y	1.395105e-07	7.489483e-06	8.154979e-05	0.000001	0.001745

(a) Beta Calibration

	1Y	2Y	3Y	5Y	10Y
1Y	0.2250	0.2872	0.2978	0.2607	0.2447
5Y	0.2726	0.2983	0.2998	0.2660	0.2451
10Y	0.2854	0.2928	0.2940	0.2674	0.2437

(b) Sigma ATM

Figure 4: Beta and Sigma Calibration

2.2 SABR Calibration

We used the SABR model:

$$\begin{aligned} &\sigma_{\text{SABR}}(F_0, K, \alpha, \beta, \rho, \nu) \\ &= \frac{\alpha}{(F_0K)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\log^2\left(\frac{F_0}{K}\right) + \frac{(1-\beta)^4}{1920}\log^4\left(\frac{F_0}{K}\right) + \cdots\right\}} \\ &\times \frac{z}{x(z)} \times \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(F_0K)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(F_0K)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]T + \cdots\right. \end{aligned}$$

where

$$z = \frac{\nu}{\alpha}(F_0K)^{(1-\beta)/2}\log\left(\frac{F_0}{K}\right),$$

and

$$x(z) = \log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right].$$

to do the calibration to fit the market volatility. Our purpose is to minimize the errors between the market volatility and the SABR volatility. We used the same approach as part 1, least squares to minimize the errors to find the best fit parameters of α ρ and ν.

The methodology we used is:

$$\begin{aligned}
\sigma_{\text{Mkt}}(K_1) - \text{SABR}(F, K_1, T, \alpha, 0.8, \rho, \nu) &= \epsilon_1 \\
\sigma_{\text{Mkt}}(K_2) - \text{SABR}(F, K_2, T, \alpha, 0.8, \rho, \nu) &= \epsilon_2 \\
&\vdots \\
\sigma_{\text{Mkt}}(K_n) - \text{SABR}(F, K_n, T, \alpha, 0.8, \rho, \nu) &= \epsilon_n
\end{aligned}$$

Our initial guess for the three parameters are 0.15,-0.4,0.6. The tables below are our results for the three parameters calibration.

```
SABR_Alpha
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	1Y	2Y	3Y	5Y	10Y
1Y	0.139061	0.184649	0.196849	0.178051	0.171073
5Y	0.166499	0.199385	0.209731	0.188195	0.172197
10Y	0.177136	0.193791	0.204197	0.192592	0.167843

(a) Alpha

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SABR_Rho
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	1Y	2Y	3Y	5Y	10Y
1Y	-0.633280	-0.525117	-0.482843	-0.414413	-0.264900
5Y	-0.584968	-0.546346	-0.547256	-0.493732	-0.392101
10Y	-0.544353	-0.539995	-0.542701	-0.534139	-0.443578

(b) Rho

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SABR_Nu
```

	1Y	2Y	3Y	5Y	10Y
1Y	2.049672	1.677407	1.438152	1.064907	0.778199
5Y	1.339284	1.061521	0.936034	0.680061	0.530337
10Y	1.005600	0.921104	0.862169	0.713666	0.599924

(c) Nu

Figure 5: SABR Parameters

2.3 Price Swaptions Using the SABR and Calibrated DD Model

First of all, we have interpolated the missing sigma and beta for DD model and nu, rho and alpha for the SABR model. Then,Based on our calibrated model parameters, we have got the following prices for the swaptions.

	Strikes	Displaced Diffusion	SABR
0	0.01	0.288142	0.289641
1	0.02	0.194936	0.198314
2	0.03	0.112326	0.115059
3	0.04	0.051345	0.051953
4	0.05	0.017366	0.021432
5	0.06	0.004106	0.010950
6	0.07	0.000651	0.006858
7	0.08	0.000067	0.004869

(a) $Payer2y \times 10y$

	Strikes	Displaced Diffusion	SABR
0	0.01	0.018985	0.020581
1	0.02	0.033904	0.039522
2	0.03	0.056649	0.061266
3	0.04	0.088980	0.089192
4	0.05	0.132050	0.128910
5	0.06	0.186136	0.186329
6	0.07	0.250582	0.259981
7	0.08	0.323971	0.342977

(b) Receiver 8y

Figure 6: Price of Payer and Receiver Swaptions

3 PART III

3.1 Calculate Present Values of Two Different CMS Products

We used finite difference method to define IRR functions.The payoff function $g(K)$ are simply the forward swap rates S. Therefore, the IRR, IRR first order differential equation and $h(K)$ second order differential equation are:

$$IRR(S) = \sum_{i=1}^{N\times m} \frac{\frac{1}{m}}{(1+\frac{S}{m})}$$

S denotes the forward swap rates and m is the payment frequency.

$$IRR^{'} = \frac{IRR(S+0.05\times S) - IRR(S-0.05\times S)}{2\times 0.05\times S}$$

$$IRR^{''} = \frac{IRR(S+0.05\times S) - 2\times IRR(S) + IRR(S-0.05\times S)}{(0.05\times S)^2}$$

By setting up above functions, we can proceed to calculate the $h^{''}$function.

$$h^{''} = \frac{(-IRR^{''}(S)\times S - 2\times IRR^{'}(S))}{IRR(S)^2} + \frac{2\times IRR^{'}(K)^2\times S}{IRR(K)^3}$$

Where $g(S) = S$, $g^{'}(S) = 1$, $g^{''}(S) = 0$

As CMS is a exotic product, so we need to use IRR swaptions to replicate its payoff. The model we used in our project is:

$$D(0,T)F + \int_0^F h^{''}(K)V^{rec}(K)dK + \int_F^\infty h^{''}(K)V^{pay}(K)dK$$

With the valuation formula we set up, now, we can use the model to value the following products.

- PV of a leg receiving CMS10y semi-annually over the next 5 years
- PV of a leg receiving CMS2y quarterly over the next 10 years

First, we have calculated the corresponding forward rates as our payoff $g(K)$. Besides, we used linear interpolation to get all the necessary ρ,ν and α to value both receiver and payer swaptions. Below is our data table.

	Start	Tenor	FS	DF	alpha	rho	nu	CMS
0	0.5Y	10Y	0.037845	0.998752	0.171130	-0.264831	0.777658	0.038037
1	1Y	10Y	0.038428	0.997009	0.171130	-0.264831	0.777658	0.038844
2	1.5Y	10Y	0.039020	0.995270	0.171793	-0.286815	0.742202	0.039691
3	2Y	10Y	0.039634	0.993531	0.172457	-0.308798	0.706746	0.040588
4	2.5Y	10Y	0.040200	0.991773	0.173120	-0.330782	0.671290	0.041458
5	3Y	10Y	0.040788	0.990015	0.173784	-0.352766	0.635834	0.042371
6	3.5Y	10Y	0.041412	0.988066	0.174447	-0.374750	0.600378	0.043340
7	4Y	10Y	0.042062	0.986117	0.175110	-0.396734	0.564922	0.044351
8	4.5Y	10Y	0.042831	0.984150	0.175774	-0.418717	0.529466	0.045507
9	5Y	10Y	0.043634	0.982184	0.176437	-0.440701	0.494010	0.046715

Figure 7: CMS Rates and Forward Rates Semi Annual Payments

By using the present value formula:

$$PV = 0.5 \times \sum_{i=0.5}^{5} D(0,i) \times CMS_{i,10y}$$

The present value we obtained is 0.2084, which is about 20.84%.

By using the same methodology, we have our forward swap rates and CMS rates table for the quarterly payment frequency over the next 10 years. As the table is too long, We only show the top 5 rows here for illustration purpose.

	Tenor	OIS_DF	Libor_DF	FS	alpha	rho	nu	CMS
0	0.25	0.998752	0.987654	0.027889	0.184649	0.184649	1.677407	0.027902
1	0.50	0.998752	0.987654	0.032178	0.184649	0.184649	1.677407	0.032216
2	0.75	0.997880	0.980116	0.032646	0.184649	0.184649	1.677407	0.032713
3	1.00	0.997009	0.972577	0.033122	0.184649	0.184649	1.677407	0.033225
4	1.25	0.996139	0.964977	0.033547	0.185570	0.185570	1.638914	0.033693

Figure 8: CMS Rates and Forward Rates Quarterly Payments

By using the present value formula:

$$PV = 0.5 \times \sum_{i=0.25}^{2} D(0,i) \times CMS_{i,10y}$$

The present value we obtained is 0.4081 which is about 40.81%. for the quarterly payments of CMS2y rates over the next 10 years.

3.2 Compare The Forward Rates With CMS Rates

To compare with the forward rates with semi-annual payment frequency, the CMS rates also need to be done in semi-annual manner by setting the m in IRR as $m = 2$. By using the CMS replication model we stated above, the CMS rate we obtained shows below.

	1Y	2Y	3Y	5Y	10Y
1Y	0.032007	0.033302	0.034108	0.035416	0.038802
5Y	0.039274	0.040468	0.040879	0.042379	0.046321
10Y	0.042189	0.044145	0.046311	0.050092	0.062396

Figure 9: CMS Rates

The forwards rates we have already obtained in Question 1.

	1Y	2Y	3Y	5Y	10Y
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

Figure 10: Forward Rates

We took the difference of the CMS rates and forward rates and visualized the phenomenon so that we can see the differences more comprehensively.

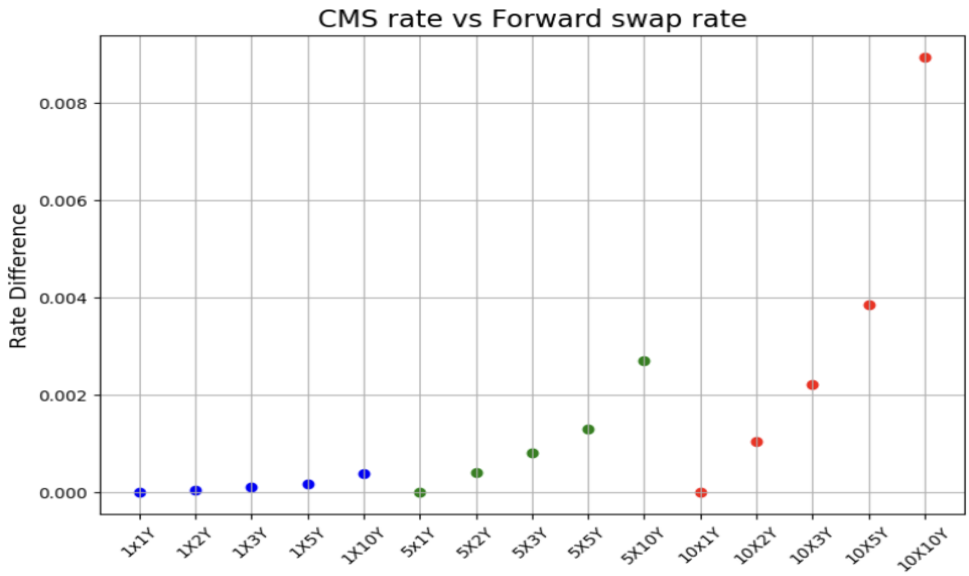

Figure 11: CMS Rates and Forward Rates Differences

There are minor differences between the CMS rates and forward swap rates. This is because the CMS is valued at a wrong time. The forward swap rate is determined today T_0 and payoff at the T_1. But the CMS rate is paid at T_1. So, if we were using the forward rate to value CMS product, we need some convexity corrections to calibrate the price of CMS.From the comparison table we have above, we can see that the longer the tenor, the higher the difference between the CMS rate product and forward rate. This may be due to the longer the tenor, we need more swaptions to do convexity corrections, which will cause higher truncation errors during the whole process, leading to a higher differences between the CMS rates and forward rates.

4 Part IV

4.1 Value A Decompounded Option

The decompounded option we need to evaluate is:

$$CMS10y^{\frac{1}{p}} - 0.04^{\frac{1}{q}}$$

Where $p = 4$ and $q = 2$. Now, the payoff function is no longer just swap rate S itself. Thus, we need to update the first order and second differential equations of the payoff function h in Question 3. The updated formulas will be:

$$h(K) = \frac{g(K)}{IRR(K)}$$

$$h^{'}(K) = \frac{IRR(K)g^{'}(K) - g(K)IRR^{'}(K)}{IRR(K)^2}$$

$$h^{''}(K) = \frac{IRR(K)g^{''}(K) - IRR^{''}(K)g(K) - 2 \times IRR^{'}(K)g^{'}(K)}{IRR(K)^2} + \frac{2 \times IRR^{'}(K)^2 g(K)}{IRR(K)^3}$$

In the payoff function, we can rewrite the payoff as:

$$g(F) = F^{\frac{1}{4}} - 0.04^{\frac{1}{2}} = F^{\frac{1}{4}} - 0.2$$

Thus, the first order difference of the payoff function will be:

$$g^{'}(F) = \frac{1}{4}F^{-\frac{3}{4}}$$

The second order difference of the payoff function will be:

$$g^{''}(F) = -\frac{3}{16}F^{-\frac{7}{4}}$$

Based on the static replication valuation formula we states in Question 3 and the updated payoff first and second differentiation functions, the present value of the payoff function $CMS10y^{\frac{1}{4}} - 0.04^{\frac{1}{2}}$ is 0.2241.

4.2 Valuation of a Caplet

As the payoff function is $(CMS10y^{\frac{1}{p}} - 0.04^{\frac{1}{q}})^{+}$, $p = 4$, $q = 2$, we can treat it as a CMS caplet. The payoff function can be further simplified into:

$$CMS^{\frac{1}{4}} > 0.2$$

$$CMS > 0.2^4$$

$$CMS > 0.0016 = L$$

Therefore, this payoff function is a CMS caplet with a strike L equal to 0.0016. So, we can use following model to value the payoff function:

$$\begin{aligned} CMSCaplet &= D(0,T)\int_L^{\infty} gKf(K)dK \\ &= D(0,T)(F-L) + h^{'}(L)V^{pay}(L) + \int_L^{\infty} h^{''}(K)V^{pay}dK \end{aligned}$$

By using the formula we stated above, we calculated the value this payoff function which is 0.2883.

5 Appendix

5.1 Visualization of DD and SABR Calibrations to Market Volatility

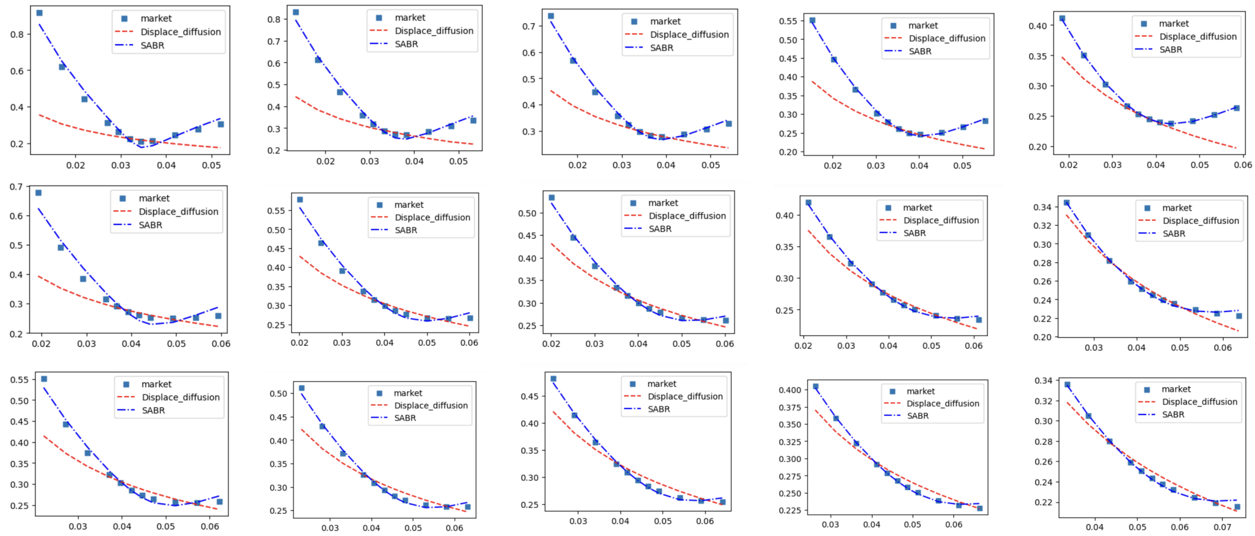

Figure 12: DD and SABR Model Calibrations

The table denotes the expiry and tenor of the swaps for the above visualization graphs.

1y x 1y	1y x 2y	1y x 3y	1y x 4y	1y x 5y
5y x 1y	5y x 2y	5y x 3y	5y x 5y	5y x 10y
10y x 1y	10y x 2y	10y x 3y	10y x 5y	10y x 10y