不定积分

其实本来是上个学期的内容，但是由于上个学期期末太懒于是就没有总结这一部分，导致呢其实这部分也快忘得差不多了，于是乎先总结一下。 目前主要与三角函数有关

需要复习的内容

反（三角）函数求导

在此之前我们需要牢记六个反三角函数的定义域与值域，不然会在一些地方出现错误

反三角函数	定义域	值域
arcsinx	[-1,1]	[$-\frac{\pi}{2}$,$\frac{\pi}{2}$]
arccosx	[-1,1]	[0,π]
arctanx	($-\infty$, ∞)	($-\frac{\pi}{2}$,$\frac{\pi}{2}$)
arccscx	($-\infty$,-1]U[1,∞)	[$-\frac{\pi}{2}$,0)U(0,$\frac{\pi}{2}$]
arcsecx	($-\infty$,-1]U[1,∞)	[0,$\frac{\pi}{2}$)U(0,π]
arccotx	[$-\infty$, ∞]	(0,π)

为什么这么说呢，在推导后续三个反三角函数时就知道定义域与值域的重要性了

首先我们知道 $$ f(g(x))\ =\ x \ \frac{dg(x)}{dx}\ =\ \frac{1}{\frac{df(u)}{du}|*{u=g(x)}} $$ 所以对于三大反三角函数来说 $$ \frac{d(arcsinx)}{dx}\ = \ \frac{1}{\sqrt{1-x^2}} \\frac{d(arctanx)}{dx}\ = \ \frac{1}{1+x^2} \for\ the \ last\ three\ functions: \\frac{d(arcsecx)}{dx} = \frac{1}{\frac{dsecu}{du}*{u = arcsecx}} \ = \frac{1}{tanu \times secu} \here\ we\ know \ that\ u \in [0,\frac{\pi}{2})\cup(0,\pi],\ so: \tanu \times secu\ \ge 0 \ = \frac{1}{|x|\sqrt{x^2-1}} $$ 求出这几个后我们引入三个恒等式，其实可以很方便的得出其他的倒数啦 $$ arcsinx + arccosx = \frac{\pi}{2} \arcsecx + arccscx = \frac{\pi}{2} \arctanx + arccotx = \frac{\pi}{2} $$ 画个直角三角形即可证明，所以接下来的导数就可以用这几个恒等式推出： $$ \frac{d(arccscx)}{dx} = -\frac{1}{|x|\sqrt{x^2-1}} \\frac{d(arccotx)}{dx} = -\frac{1}{1+x^2} $$

三角函数和差化积与积化和差公式

在不定积分的计算中，积化和差公式是重要的，因为很多时候积化和差可以把三角函数的乘积/次方变为简单的加法 $$ sin\alpha sin\beta = \frac{1}{2}[sin(\alpha + \beta) + sin(\alpha - \beta)] \cos\alpha sin\beta = \frac{1}{2}[sin(\alpha + \beta) - sin(\alpha - \beta)] \cos\alpha cos\beta = \frac{1}{2}[cos(\alpha + \beta) + cos(\alpha - \beta)] \sin\alpha sin\beta = \frac{1}{2}[cos(\alpha - \beta) + cos(\alpha + \beta)] $$ 其实最重要的是看到三角函数的乘积或者是次方，要有意识的把他们用积化和差或者是倍角公式转换为和的形式

分部积分法

其实最需要总结的还是这一块，毕竟去年没有用也没有好好体会 $$ note\ u\ as\ u(x) \ note\ v\ as\ v(x) \\int udv = uv - \int vdu $$ 证明： $$ (uv)' = u'v + v'u \\int (uv)'dx = \int u'vdx + \int v'udx \\int udv + \int vdu = uv $$ 其实不难发现如果所求不定积分中有存在x的乘积式，那大概率要用到分部积分法了，而分部积分法的本

质是利用莱布尼茨对于导数的表达式来消去不定积分中的dx项。并且含有三角函数以及自然指数函数都是分部积分法的常客

换元积分法积分策略

1. $\int \frac{dx}{x^2 - a^2}$： $$ Normally\ we\ would\ use\ subtitution, but: \\begin{aligned} &\int \frac{dx}{x^2 - a^2} \ = &\int \frac{dx}{(x-a)(x+a)} \ = &\int \frac{1}{2a} (\frac{1}{x-a} - \frac{1}{x+a})dx \ = &\frac{1}{2a}\int \frac{1}{x-a}d(x-a) - \frac{1}{2a}\int \frac{1}{x+a}d(x+a) \ = &\frac{1}{2a}(\ln|x-a| - \ln|x+a|) + C \end{aligned} $$
2. $\sin^m(x)\cos^n(x)$ $$ When\ m = 2k+1: \subtitute \cos(x)\ with\ u \\ \ \ \ \When\ n = 2k+1: \subtitute \sin(x)\ with\ u \\ \ \ \ \ \ \When\ both\ is\ odd,try\ to\ use: \ \sin^2x = \frac{1}{2}(1 - \cos(2x)) \ \cos^2x = \frac{1}{2}(1 + \cos(2x)) $$
3. $\tan^mx\sec^nx$ $$ When\ n = 2k: \subtitute\ u = \tan x \\ \ \ \ \ \When\ m = 2k - 1: \subtitute\ u = \sec x \\ \ \ \ \tan^2 x + 1 = \sec^2 x $$
4. $\int \sec x dx\ or\ \int \csc x dx$ $$ The\ easist\ way: \\begin{aligned} \frac{d\sec x}{dx} &= \sec x \tan x;\ \ \frac{d\tan x}{dx} = \sec^2x \\frac{d\csc x}{dx} &= \csc x \cot x;\ \ \ \frac{d\cot x}{dx} = -\csc^2x \end{aligned} \So: \\begin{aligned} \int \sec x dx&= \int \frac{\sec x (\sec x +\tan x)}{\sec x +\tan x}dx \ &= \int \frac{1}{\sec x+\tan x} d(\sec x + \tan x) \&= \ln |\sec x + \tan x| + C \\int \csc x dx&= \int \frac{\csc x (-\csc x +\cot x)}{-\csc x +\cot x}dx \ &= \int \frac{1}{-\csc x+\cot x} d(-\csc x + \cot x) \&= \ln |-\csc x + \cot x| + C \end{aligned} \ \ \ \ \ \Or\ we\ could\ consider: \ \sec x = \frac{\cos x}{\cos^2 x} \ \csc x = \frac{\sin x}{\sin^2 x} \So: \ \int \frac{\cos x}{\cos^2 x}dx = \int \frac{d\sin x}{1-\sin^2x}dx \which \ already\ be\ mentioned\ in \ 1 \\ \ \ \ \ \Especially\ for\ \csc x: \\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\begin{aligned} \int \csc x dx &= \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx \ &= \int \frac{d\frac{x}{2}}{\tan\frac{x}{2}\cos^2\frac{x}{2}}dx \ &= \int \frac{d\tan\frac{x}{2}}{\tan \frac{x}{2}}dx \ &= \ln |\tan\frac{x}{2}|+C \end{aligned} \ \ \ \ \So,\ we\ can\ know: \\tan\frac{x}{2} = \csc x - \cot x \ \frac{1-\cos x}{1+\cos x} = \csc x - \cot x \ \frac{1-\sin x}{1+\sin x} = \sec x + \tan x \ \frac{1+\tan x}{1+\tan x} = \sec x + \tan x $$
5. $\int \frac{x^2}{x^2+1}dx$ $$ \begin{aligned} &\int \frac{x^2}{x^2+1}dx \ = &\int \frac{x^2+1-1}{x^2+1}dx \ = &\int (1 - \frac{1}{1+x^2})dx \ = &x - \arctan x + C \end{aligned} $$

降次积分法

通常看到所求积分中含有幂指数，而幂指数是一个以字母代替的常量，此时我们需要用到降次积分法。其核心为推导系数为n时的表达式，通常表达式右边含有n-1或者n+1等。然后从这个表达式中再次剥离出n挪到等式左边。从而得到表达式。并且绝大多数情况下降次积分会与分步积分一起使用。

由于时间紧迫（博主太懒），于是奉上手写体的完全推导过程：

$$
\begin{aligned}
\int \cos^n x\,dx &= \int \cos^{n-1}x \cdot \cos x\,dx \\
&= \int \cos^{n-1}x\,d\sin x \\
&= \sin x\cos^{n-1}x - \int \sin x\,d\cos^{n-1}x \\
&= \sin x\cos^{n-1}x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx \\
&= \sin x\cos^{n-1}x + (n-1)\int (-\cos^n x + \cos^{n-2}x)\,dx
\end{aligned}
$$

$$\therefore \int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$
\begin{aligned}
\int \tan^n x\,dx &= \int \tan^{n-2}x\,(\sec^2 x - 1)\,dx \\
&= \int \tan^{n-2}x\,d\tan x - \int \tan^{n-2}x\,dx
\end{aligned}
$$

$$\therefore \int \tan^n x = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

$$
\begin{aligned}
\int (\ln x)^n\,dx &= x\ln^n x - \int x\,d(\ln x)^n \\
&= x\ln^n x - \int \frac{1}{x}\cdot x\,n\,(\ln x)^{n-1}\,dx \\
&= x\ln^n x - n\int (\ln x)^{n-1}\,dx
\end{aligned}
$$

$$\therefore \int (\ln x)^n\,dx = x\ln^n x - n\int (\ln x)^{n-1}\,dx$$

数列与级数

数列与级数基础

正项级数

正项级数(Nonnegative series)，指的是每一项都是正项或者是零的级数，首先我们来总结这种级数的审敛法(Test for convergent)

1. 正项级数收敛的充分必要条件(necessary and suffcient condition) $$ \sum_{n=1}^{\infty} u_n\ is\ convergent\ if\ and\ only\ if: \\lim_{n \rightarrow\infty}S_n = l\ (0\leq l < \infty) \Also\ it\ means\ {S_n}\ is\ convergent $$
2. 比较审敛法(comparsion test) $$ We\ note\ two\ nonnegative\ series: \\sum_{n=1}^{\infty}u_n\ and\ \sum_{n=1}^{\infty}v_n\ (u_n \leq v_n\ &\ n = 1,2,3...) \if\ \sum_{n=1}^{\infty}v_n\ is\ convergent\ then\ \sum_{n=1}^{\infty}u_n\ is\ convergent \if\ \sum_{n=1}^{\infty}u_n\ is\ divergent\ then\ \sum_{n=1}^{\infty}v_n\ is\ divergent \\ \ \ \We\ can\ also\ have\ an\ inference: \if\ \sum_{n=1}^{\infty}v_n\ is\ convergent \ and\ we\ have\ an\ integer\ N,\ when\ n \geq N\ and\ u_n\leq kv_n\ (k > 0) \then\ \sum_{n=1}^{\infty}u_n\ is\ convergent \if\ \sum_{n=1}^{\infty}v_n\ is\ divergent \ and\ we\ have\ an\ integer\ N,\ when\ n \geq N\ and\ u_n \geq kv_n\ (k > 0) \then\ \sum_{n=1}^{\infty}u_n\ is\ divergent \\ \ \ \ \ \ We\ can\ think\ about\ the\ proof\ of\ that\ inference: \ if\ \sum_{n=1}^{\infty}v_n\ is\ convergent\ then\ after\ removing\ finite\ terms\ it's\ still\ convergent \ also\ if\ we\ multiply\ constant\ number\ for\ each\ terms\ still\ convergent \ \ \ \
\So\ it's\ obvious\ for\ the\ first\ half\ part\ of\ the\ inference \ and\ as\ the\ same\ way\ for\ the\ last\ half\ part $$
3. 比较审敛法的极限形式(limit comparsion test) $$ We\ note\ two\ nonnegative\ series:
\\sum_{n=1}^{\infty}u_n\ and\ \sum_{n=1}^{\infty}v_n
\ \ \ \
\if\ \lim_{n\rightarrow\infty}\frac{u_n}{v_n} = l\ (0\leq l<\infty)\ and\ \sum_{n=1}^{\infty}v_n\ is\ convergent \ then\ \sum_{n=1}^{\infty}u_n\ is\ convergent \if\ \lim_{n\rightarrow\infty}\frac{u_n}{v_n} = l\ (l>0\ or\ l=\infty)\ and\ \sum_{n=1}^{\infty}v_n\ is\ divergent \ then\ \sum_{n=1}^{\infty}u_n\ is\ divergent $$
4. 比值审敛法(d'Alembert comparsion test) $$ We\ note\ \sum_{n=1}^{\infty}u_n\ as\ a\ nonnegative\ series \if\ \lim_{n\rightarrow\infty}\frac{u_{n+1}}{u_n} = \rho \if\ \rho < 1\ then\ it's\ convergent\ if\ \rho > 1\ then\ it's\ divergent \ need\ to\ clarify\ when\ \rho = 1\ we\ can't\ sure\ it's\ convergence $$
5. 根植审敛法(Cauchy's test) $$ We\ note\ \sum_{n=1}^{\infty}u_n\ as\ a\ nonnegative\ series \if\ \lim_{n\rightarrow\infty}\sqrt[n]{u_n} = \rho \ if\ \rho < 1\ then\ it's\ convergent\ if\ \rho > 1\ then\ it's\ divergent \ need\ to\ clarify\ when\ \rho = 1\ we\ can't\ sure\ it's\ convergence $$
6. 极限审敛法(limit test) $$ We\ note\ \sum_{n=1}^{\infty}u_n\ as\ a\ nonnegative\ series \if\ \lim_{n\rightarrow\infty}nu_n = l>0\ then\sum_{n=1}^{\infty}u_n\ is\ divergent \if\ \lim_{n\rightarrow\infty}n^pu_n = l\ (0 \leq l<\infty\ and\ p>1)\ then\sum_{n=1}^{\infty}u_n\ is\ convergent \ \ \ \ \We\ can\ prove\ it\ simply\ by\ the\ the\ comparsion\ test\ for\ convergent \comapre\ with\ the\ p-series: \\sum_{n=1}^{\infty} \frac{1}{n^p} \when\ 0<p \leq 1\ then\ it's\ divergent \when\ p>1\ then\ it's\ convergent $$

交错级数

交错级数(alternating series)指各项正负交错的级数，其审敛法：

7. 莱布尼茨定理(Leibniz's theorem) $$ We\ note\ an\ alternating\ series\ as\ \sum_{n=1}^{\infty}(-1)^{n-1}u_n \if\ u_n \geq u_{n+1}\ (n = 1,2,3...) \and\ \lim_{n\rightarrow \infty}u_n = 0 \then\ \sum_{n=1}^{\infty}(-1)^{n-1}u_n\ is\ convergent \\ \ \ \ \We\ could\ discuss\ further\ about\ it's\ partial\ sum\ when\ it's\ convergent: \S_{2n} = (u_1 - u_2) + (u_3 - u_4) + ... +(u_{2n-1} - u_{2n}) \geq0

\S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - ...-(u_{2n-2} - u_{2n-1}) - u_{2n} \leq u_1 \S_{2n+1} = S_{2n} + u_{2n+1} \when\ n\ is\ large\ enough\ S_{2n+1} = S_{2n} + 0 = S_{2n} \So\ we\ could\ know\ \sum_{n=1}^{\infty} (-1)^{n-1}u_n \rightarrow s \leq u_1 \\ \ \ \ \ \Further,\ if\ we\ discuss\ remainder\ sum\ for\ S_{n}\ and\ note\ as\ R_{n}: \R_{n} = \sum_{i=n+1}^{\infty}u_i= \pm(u_{n+1} - u_{n+2}+...) \|R_{n}| = u_{n+1} - u_{n+2} + ... \so\ we\ find\ it's\ still\ a\ alternating\ series\ same\ as\ u_{n} \|R_{n}| \leq u_{n+1} $$

绝对收敛与条件收敛

我们定义一个普通级数：$\sum_{n=1}^{\infty}u_{n}$

绝对收敛(absolute convergence)：$\sum_{n=1}^{\infty}|u_{n}|$ 收敛

条件收敛(conditional convergence)：$\sum_{n=1}^{\infty}u_{n}$ 收敛，并且$\sum_{n=1}^{\infty}|u_{n}|$ 发散

8. 级数绝对收敛与级数收敛的关系 $$ if\ \sum_{n=1}^{\infty}u_{n}\ is\ absolute\ convergent\ then\ \sum_{n=1}^{\infty}u_{n}\ is\ convergent $$
9. 级数的柯西乘积(Cauchy's product) $$ We\ have\ two\ series\ \sum_{n=1}^{\infty}u_n\ and\ \sum_{n=1}^{\infty}v_n\ and\ Cauchy's product\ is\ C \C = \sum_{i=1}^{n}\sum_{j=1}^{i}u_jv_{n+1-j} \\ \ \ \ \Further,\ if\ both\ are\ absolute\ convergent\ and: \\sum_{n=1}^{\infty}u_n\rightarrow s\ and\ \sum_{n=1}^{\infty}v_n\rightarrow \sigma \the\ Cauchy's\ product\ is\ still\ absolute\ convergent\ which: \ C\rightarrow s\sigma $$

幂级数

首先是幂级数的定义(defination of power series) $$ \sum_{n=1}^{\infty}a_nx^n = a_0 + a_1x + a_2x^2+... \which\ a_n\ are\ called\ coefficients\ of\ the\ power\ series \\ \ \ \ \Also,\ we\ could\ note\ as\ a\ more\ general\ way: \\sum_{n=1}^{\infty}c_n(x-a)^n \and\ a\ function\ which\ domain\ is\ all\ the\ x\ makes\ series\ convergent: \f(x) = \sum_{n=1}^{\infty}c_n(x-a)^n\ or\ \sum_{n=1}^{\infty}a_nx^n \\ \ \ \ \ \Further,\ the\ first\ n\ terms\ we\ note\ as\ f_n(x) \\lim_{n\rightarrow\infty}f_n(x) = f(x) \And\ remainder\ sum\ r_n(x): \r_n(x) = s(x) - s_n(x) \\lim_{n\rightarrow\infty}r_n(x) = 0 $$ 值得一提的是根据课本上的定义，和函数$f(x)$的定义域是幂级数的收敛域，而幂级数其实还有发散域。接下来介绍一下幂级数的审敛定理：

1. 阿贝尔定理(Abel's theorem) $$ We\ have\ a\ series\ \sum_{n=0}^{\infty}a_nx^n\ it's\ convergent\ at\ x = x_0\ (x_0 \neq 0) \For\ |x|<|x_0|,\ the\ series\ will\ be\ absoluting\ convergent \We\ have\ a\ series\ \sum_{n=0}^{\infty}a_nx^n\ it's\ divergent\ at\ x = x_0\ (x_0 \neq 0) \For\ |x|>|x_0|,\ the\ series\ will\ be\ divergent \ \ \ \ \We\ could\ consider\ the\ proof\ of\ first\ part: \as\ it's\ convergent\ then: \lim_{n\rightarrow\infty}a_nx_0^n = 0 \also\ we\ could\ know\ there\ will\ definitely\ exist\ a\ number\ M: \|a_nx_0^n|\leq M \for\ |x| < |x_0|: \|a_nx^n| = |a_nx_0^n|\times |\frac{x^n}{x_0^n}|\leq M|\frac{x^n}{x_0^n}| \Obviously\ \sum_{n=0}^{\infty}M|\frac{x^n}{x_0^n}|\ is\ convergent\ so\ we\ prove\ the\ theorem \\ \ \ \ \For\ the\ second\ part\ we\ could\ use\ rebuttal\ method: \if\ for\ |x|>|x_0|\ the\ series\ is\ convergent,\ according\ to\ the\ first\ part: \when\ x = x_0\ should\ also\ be\ convergent\ which\ is\ opposite\ from\ the\ condition $$
2. 阿贝尔定理的推论: $$ For\ a\ power\ series\ \sum_{n=0}^{\infty}a_nx^n: \if\ it's\ not\ only\ be\ convergent\ either\ just\ on\ x=0\ or\ any\ x on\ the\ number\ axis \then\ there\ will\ definitely\ occur\ a\ positive\ number\ R: \when\ |x| < R\ the\ series\ is\ absoluting\ convergent; \when\ |x| > R\ the\ series\ is\ divergent; \when\ |x| = R\ we\ can't\ sure\ whether\ it's\ convergent\ or\ divergent. \\ \ \ \ \which\ R\ is\ called\ radius\ of\ convergence $$

3. 有关收敛半径计算(radius of convergence) $$ For\ two\ terms\ a_n,a_{n+1}\ in\ \sum_{n=0}^{\infty}a_nx^n: \\lim_{n\rightarrow \infty}|\frac{a_{n+1}}{a_n}| = \rho \we\ have: \R=\left{ \begin{aligned} \frac{1}{\rho},\ \ \ &\rho \neq0 \ +\infty,\ \ \ &\rho = 0 \ 0,\ \ \ &\rho = +\infty \end{aligned} \right. \ \ \ \ \ \Consider\ |\frac{a_{n+1}x^{n+1}}{a_nx^n}| = |\frac{a_{n+1}}{a_n}|\times|x_n|: \\lim_{n\rightarrow\infty}|\frac{a_{n+1}x^{n+1}}{a_nx^n}| = \lim_{n\rightarrow\infty}|\frac{a_{n+1}}{a_n}|\times\lim_{n\rightarrow\infty}|x_n| = \rho\times |x_n| \then\ according\ to\ d'Alembert\ comparsion\ test\ of\ convergence: \when\ \rho\times |x_0|<1\ which\ means\ |x_0|<\frac{1}\rho\ the\ series\ is\ convergent \Very\ important: \the\ series\ here\ is\ \sum_{n=0}^{\infty}|a_nx^n| \\ \ \ \ \Further,\ according\ to\ the\ comparsion\ test: \a_nx^n\leq|a_nx^{n}| \so\ when\ |x_0|<\frac1\rho\ the\ \sum_{n=1}^\infty a_nx^n\ is\ convergent \when\ |x_0|>\frac1\rho\ the\ \sum_{n=1}^{\infty}|a_nx^n|\ is\ divergent,\ according\ to\ the\ former\ limit\ comparsion: \from\ a\ number\ n\ we\ have:\ |a_{n+1}|>|a_n| \so\ for\ both\ |a_n|\ and\ a_n\ will\ not\ approach\ to\ 0 \so\ \sum_{n=0}^\infty a_nx^n\ is\ divergent \ \ \ \ \Attention \ !\ !\ !\ : \when\ the\ series\ is\ \sum_{n=1}^\infty (-1)^{n-1}c_n (x-x_0)^n: \|x-x_0| < \frac{1}{\rho} \which\ x\in (-R,R)\ etc. \(proof\ refer\ to\ the\ defination) $$
4. 幂级数的计算 $$ We\ have\ two\ power\ series: \\sum_{n=0}^\infty a_nx^n\ and\ \sum_{n=0}^\infty b_nx^n \which\ the\ redius\ of\ convergence\ is\ respectively\ R\ and\ R' \\ \ \begin{aligned} Sum &= \sum_{n=0}^\infty (a_n+b_n)x^n \Difference &= \sum_{n=0}^\infty (a_n-b_n)x^n \Cauchy's\ Product &= \sum_{i=0}^{n}(x^i\sum_{j=0}^{i}a_jb_{n+1-j}) \end{aligned} \above\ radius\ of\ convergence\ is\ \min(R,R') \\ \ \ \ \Especially\ for\ division: \\frac{\sum_{n=0}^\infty a_nx^n}{\sum_{n=0}^\infty b_nx^n} = \sum_{n=0}^\infty c_nx^n \we\ know\ that: \\begin{aligned} a_0 &= b_0c_0 \a_1 &= b_1c_0+b_0c_1 \a_2 &= b_2c_0+b_1c_1+b_0c_2 \end{aligned} \\cdots \but\ we\ could\ not\ determine\ the\ radius\ of\ convergence\ of\ \sum_{n=0}^\infty c_nx^n $$
5. 幂级数的和函数的运算(derivatives and integrals of sum function) $$ We\ note\ \sum_{n=1}^\infty a_nx^n\ and\ s(x)\ as\ a\ power\ series\ and\ it's\ sum\ function \In\ its\ convergence\ domain\ I: \\int_0^x s(t)dt = \int_0^x\sum_{n=0}^\infty a_nt^n dt = \sum_{n=0}^\infty \int_0^x a_nt^ndt = \sum_{n=0}^\infty \frac{a_n}{n+1}x^{n+1} \the\ redius\ of\ convergence\ is\ same \\ \ \ \ \s'(x) = (\sum_{n=0}^\infty a_nx^n)' = \sum_{n=0}^\infty (a_nx^n)' = \sum_{n=0}^\infty na_nx^{n-1} \the\ redius\ of\ convergence\ is\ same $$

函数的展开

写在最前

下文 “3.拉格朗日余项” 的证明提到了积分第一中值定理，其表述为：

函数$f(x)$在闭区间[a,b]上连续，$g(x)$在[a,b]上不变号且可积，则在闭区间上至少存在一点ξ: $$ \int_{a}^{b}f(x)g(x)dx = f(\xi)\int_a^b g(x)dx $$ 事实上这个区间也可以是开区间，证明过于复杂，链接如下：https://zh.m.wikipedia.org/zh-hans/%E7%A7%AF%E5%88%86%E7%AC%AC%E4%B8%80%E4%B8%AD%E5%80%BC%E5%AE%9A%E7%90%86

展开的推导

1. 函数的泰勒展开(Taylor expand) $$ We\ have\ a\ power\ series\ \sum_{n=0}^\infty a_n(x-x_0)^n\ which\ is\ convergent\ in\ U(x_0) \then\ we\ know\ it's\ sum\ function: \ \begin{aligned} f(x) &= \sum_{n=0}^\infty (x-x_0)^n \f^{(n)}(x) &= \sum_{i=n}^\infty i!a_i(x-x_0)^{i-n} \end{aligned} \ \begin{aligned} when\ x = x_0: \f^{(n)}(x_0) = n!a_n \end{aligned} \\ \ \begin{aligned} thus\ for\ n=0,1,2\cdots \end{aligned} \a_n = \frac{1}{n!}f^{(n)}(x_0) $$

2. 函数的麦克劳林级数(McLaughlin's series) $$ We\ know\ in\ Taylor\ expand: \f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\ \ x\in(-R,R) \when\ x_0 = 0: \f(x) = \sum_{n=0}^\infty\frac{f^{(n)}(0)}{n!}x^n\ \ x\in(-R,R) $$
3. 函数泰勒展开的拉格朗日余项(Lagrange remainder) $$ \begin{aligned} f(x) &= \sum_{i=0}^\infty \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i \&= \sum_{i=0}^n \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i + R_n(x) \end{aligned} \Get\ the\ n\th\ dervative: \f^{(n)}(x) = f^{(n)}(x_0) + R_n^{(n)}(x) \and\ according\ to\ the\ Lagrange's\ mean\ value\ theorem: \f^{(n+1)}(\theta x)(x-x_0) = f^{(n)}(x) - f^{(n)}(x_0) \so: \\begin{aligned} R_n^{(n)}(x) &= f^{(n+1)}(\theta x)(x-x_0) \R_n^{(n-1)}(x) &= \int_{0}^{x} R_n^{(n)}(x) \ &= \int_{0}^{x}f^{(n+1)}(\theta x)(x-x_0) \end{aligned} \according\ to\ First\ mean\ value\ theorem\ for\ definite\ integrals: \R_n{(x)} = \frac{f^{(n+1)}(\theta x)(x-x_0)^{n+1}}{(n+1)!} $$

常见函数的展开与策略

我们经常会要求求某一价的函数展开式，即x的幂系数，此时我们要善于利用目标函数的展开式，目标函数导数的函数展开式（通过不定积分可以得到目标函数），目标函数不定积分的表达式（通过求导可以得到目标函数）。不要拘泥于目标函数，尤其是展开式不含所求阶时。下面我们总结函数的展开式：