MAE 5510 : Exercise Set 2 Solutions

Name ________________________

For the following problems, we will consider a version of the British Spitfire with the following geometric and aerodynamic characteristics:

$$S_w = 244 \text{ ft}^2, \qquad b_w = 36.83 \text{ ft}, \qquad C_{L_w,\alpha} = 4.62, \qquad \alpha_{L0_w} = -2.2°, \qquad C_{m_w} = -0.0527,$$
$$S_h = 31 \text{ ft}^2, \qquad b_h = 10.64 \text{ ft}, \qquad C_{L_h,\alpha} = 4.06, \qquad \varepsilon_e = 0.60, \qquad C_{m_h,\delta_e} = -0.55,$$
$$W = 8,375 \text{ lbf}, \qquad l_h - l_w = 18.16 \text{ ft}$$

Assume that the center of gravity lies at the quarter-chord of the main wing, the horizontal stabilizer has a symmetric airfoil, both the main wing and horizontal stabilizer are elliptic planforms, and that the main wing and horizontal stabilizer have zero twist.

2.1 From lifting-line theory, the lift coefficient produced on an elliptic wing with zero twist can be computed from

$$C_L = C_{L,\alpha}(\alpha - \alpha_{L0})_{\text{root}} \tag{1}$$

where the lift slope is

$$C_{L,\alpha} = \frac{\tilde{C}_{L,\alpha}}{[1 + \tilde{C}_{L,\alpha}/(\pi R_A)]} \tag{2}$$

Assuming the main wing has a thin airfoil, compute the lift slope of the main wing. Also compute the lift on the main wing at 5 deg angle of attack and a velocity of 200 mph at sea level.

Solution:

Because it has a thin airfoil, we can assume the section lift slope to be $\tilde{C}_{L,\alpha} = 2\pi$.
The aspect ratio is $R_A = \frac{b_w^2}{S_w} = \frac{36.83^2}{244} = 5.5592$.
Computing the lift slope using Eq. (2) yields

$$C_{L,\alpha} = \frac{2\pi}{[1 + 2\pi/(\pi 5.55922)]}$$
$$C_{L,\alpha} = 4.62079$$

Using this result in Eq. (1) gives

$$C_L = 4.62079(5 - (-2.2)) * \frac{\pi}{180}$$
$$C_L = 0.580666$$

Air density at sea level is $\rho = 0.0023769$ slug/ft^2. 200 mph is 293.333 fps.

$$L = 0.5\rho V^2 S_W C_L$$
$$L = (0.5)(0.0023769)(293.333^2)(244)(0.580666)$$
$$L = 14488.4 \text{ lbf}$$

2.2 Assuming the horozontal tail uses a thin airfoil, compute the lift slope of the horizontal tail. Also compute the lift on the horizontal tail at 5 deg angle of attack and a velocity of 200 mph at sea level.

Solution:

Because it has a thin airfoil, we can assume the section lift slope to be $\tilde{C}_{L,\alpha} = 2\pi$.
The aspect ratio is $R_A = \frac{b_h^2}{S_h} = \frac{10.64^2}{31} = 3.65192$.
Computing the lift slope using Eq. (2) yields

$$C_{L,\alpha} = \frac{2\pi}{[1 + 2\pi/(\pi 3.65192)]}$$
$$C_{L,\alpha} = 4.05981$$

Using this result in Eq. (1) gives

$$C_L = 4.05981(5 - (0)) * \frac{\pi}{180}$$
$$C_L = 0.354285$$

Air density at sea level is $\rho = 0.0023769$ slug/ft^2. 200 mph is 293.333 fps.

$$L = 0.5\rho V^2 S_W C_L$$
$$L = (0.5)(0.0023769)(293.333^2)(31)(0.354285)$$
$$L = 1123.1 \text{ lbf}$$

2.3 Using MachUp 4, compute the lift produced on the main wing at 5 deg angle of attack and a velocity of 200 mph at sea level. Assume that the airfoil used on the main wing is thin and has a zero-lift angle of attack of −2.2°. The root chord of an elliptic wing can be computed from

$$c_{\text{root}} = \frac{4b}{\pi R_A}$$

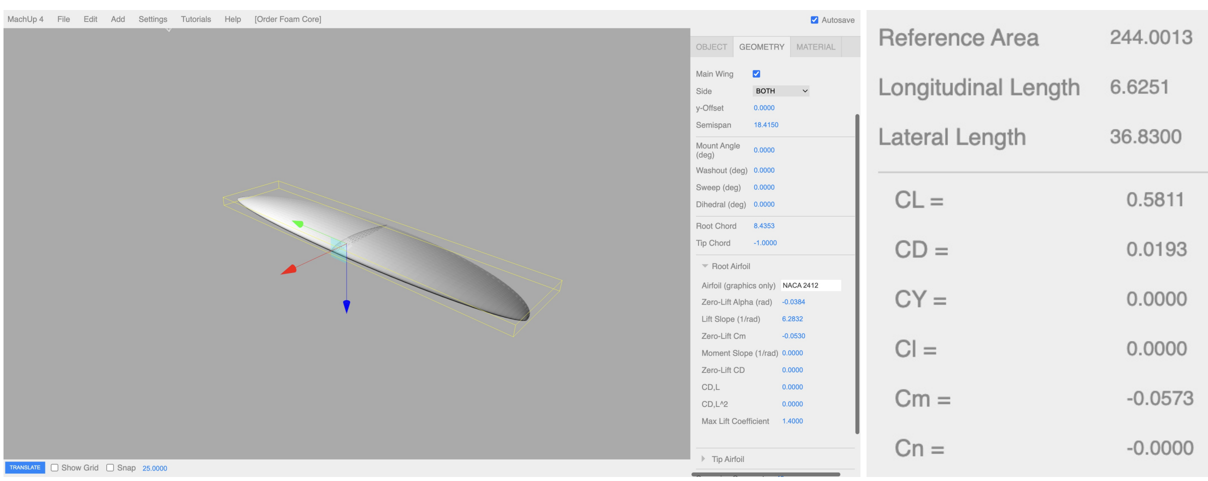
Compare this result to that in problem 2.1.

Solution:

The root chord and "Zero-Lift Alpha" and "Lift Slope" of the airfoil in radians are

$$c_{\text{root}} = \frac{4 * 36.83}{\pi(36.83^2/244)}$$
$$c_{\text{root}} = 8.43526$$
$$\alpha_{L0} = -2.2 * \frac{\pi}{180} = -0.038397$$
$$\tilde{C}_{L,\alpha} = 2 * \pi = 6.28319$$

The following shows how the geometry is defined for the main wing, the forces and moments analysis, and what the wing looks like modeled in MachUp 4.



The lift coefficient computed from MachUp 4 is $C_L = 0.5811$ for the main wing, which gives $L = 14499.2$ lbf. This is approximately a 11 lbf or 0.07 percent difference from the solution given in Problem 2.1. We would not expect either method used here to be accurate to within this error, so for practial purposes, they are giving the same solution.

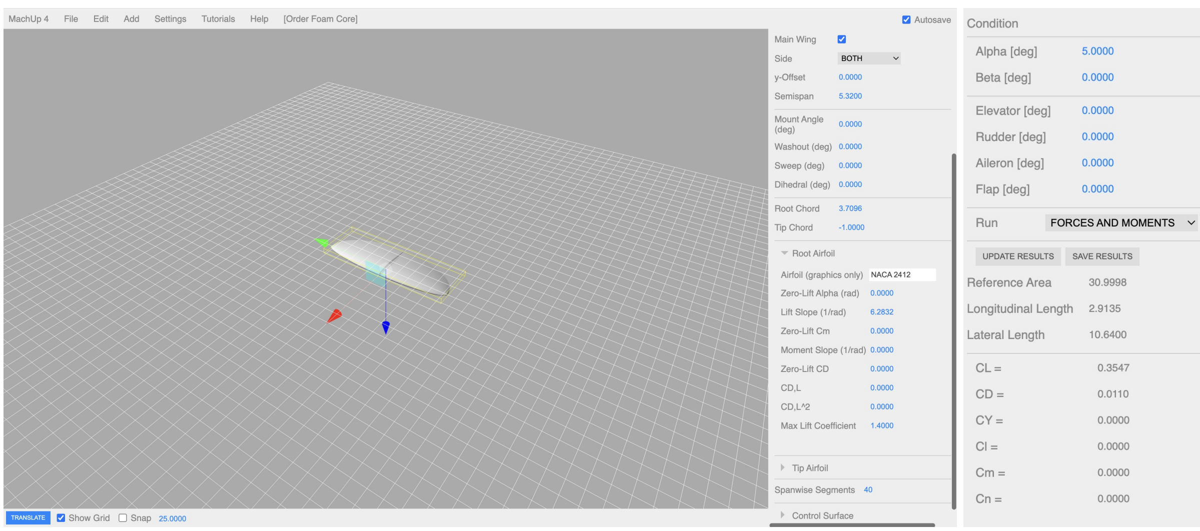
2.4 Using MachUp 4, compute the lift produced on the horizontal stabilizer at 5 deg angle of attack and a velocity of 200 mph at sea level. Compare this result to that in problem 2.3.

Solution:

The root chord and "Zero-Lift Alpha" and "Lift Slope" of the airfoil in radians are

$$c_{\text{root}} = \frac{4 * 10.64}{\pi(10.64^2/31)}$$
$$c_{\text{root}} = 3.70963$$
$$\alpha_{L0} = 0.0$$
$$\tilde{C}_{L,\alpha} = 2 * \pi = 6.28319$$

The following shows how the geometry is defined for the horizontal stabilizer, the forces and moments analysis, and what the wing looks like modeled in MachUp 4.

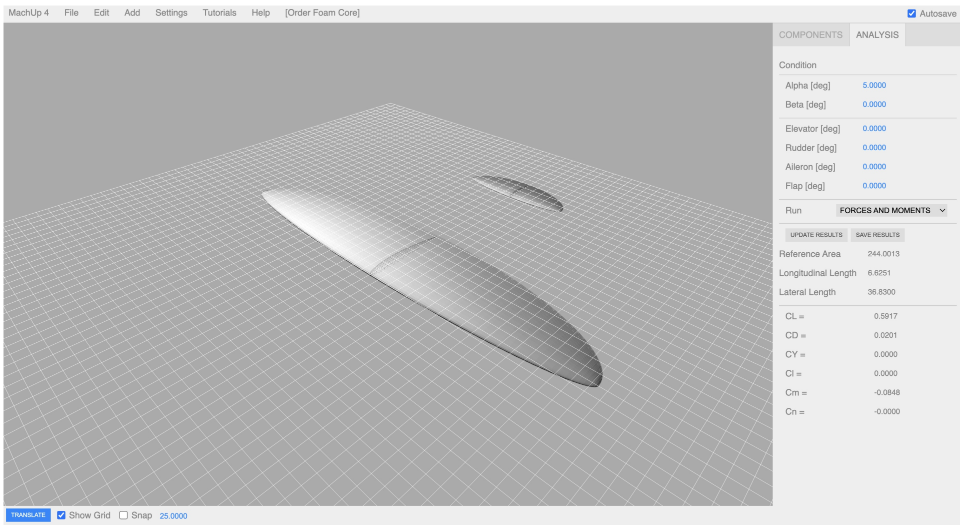


The lift coefficient computed from MachUp 4 is $C_L = 0.3547$, which gives $L = 1124.4$ lbf. This is approximately a 1.3 lbf or 0.12 percent difference from the solution given in Problem 2.2. We would not expect either method used here to be accurate to within this error, so for practical purposes, they are giving the same solution.

2.5 Using MachUp 4, compute the lift produced on the main wing and horizontal stabilizer at 5 deg angle of attack and a velocity of 200 mph at sea level when the horizontal stabilizer is placed a distance of $l_h - l_w = 18.16$ ft aft of the main wing. Compare this result to that in problems 2.1 – 2.4. Discuss your results.

Solution:

The geometry for each of these wings is the exact same as defined above. Except, instead of analyzing the wings in separate files, the combination is used with the main wing designated as 'Main Wing'. The total lift coefficient after running the analysis is 0.5917. as shown in the first figure.



However, this is the total lift coefficient for the complete aircraft. We are asked to break this down into the lift on the main wing and the lift on the tail. This can be done by selecting 'Save Results', which produces a JSON file with more information, shown in the second figure.

Because the JSON produces the CL for either side of the wing, it is required to double the right or left side to figure out the correct CL. In addition, the CL for the horizontal stabilizer must be re-normalized with the correct area.

A note on the lift on the horizontal stabilizer. This lift is nondimensionalized in the JSON file using the area of the main wing. Therefore to translate this to the actual lift on the horizontal stabilizer, we must multiply the following factor:

$$C_{L_h} = \frac{S_w}{S_h}\eta_h C_{L_h(MU)}$$
$$C_{L_h} = \frac{244}{31}0.0100135471$$
$$C_{L_h} = 0.07882$$

Table 1 shows the results from the different analyses.

Table 1 Coefficient of Lift calculations for the 2 lifting surfaces.

Wing	Analytic C_L	(lbf)	MU4 C_L	(lbf)	MU4 Combined	(lbf)	% change MU
Main Wing	0.580666	14488.4	0.5811	14499.2	0.58170	14514.2	+0.1 %
Horizontal Stab.	0.354285	1123.1	0.3547	1124.4	0.0100	31.74	-97.18 %

The significant thing to note here is the change in lift on the horizontal stabilizer when both wings are considered. There was a **0.1 percent** increase in lift on the main wing, and a a **97.2 percent** decrease in lift on the horizontal stabilizer. This shows the effects of downwash and demonstrates how downwash cannot realistically be neglected even in preliminary analysis.

2.6 Using the simplified analysis for estimating the downwash, estimate the downwash on the horizontal as a function of the lift coefficient on the main wing.

Solution:

Using Equation 4.5.11 from the book, the downwash on the horizontal stabilizer can be approximated as a function of the lift coefficient on the main wing using the following equation

$$\varepsilon_d = \frac{\kappa_v \kappa_p \kappa_s}{\kappa_b} \frac{C_{L_w}}{R_{A_w}}$$

These kappa values can be approximated using the Figures 4.5.2-4.5.4 on pages 414-415. For an elliptic wing, $\kappa_v = 1.0$ and $\kappa_b = \pi/4$. Because we have no sweep, $\kappa_s = 1.0$. κ_p can be calculated using the position of the tail relative to the wing and the κ_b value.
The distance $x = 18.16$ and $y = 0$. Therefore the normalized x- and y-values are

$$\begin{aligned}
\bar{x} &= \frac{x}{b_w/2} \\
&= \frac{18.16}{36.83/2} \\
&= 0.986153 \\
\bar{y} &= \frac{y}{b_w/2} \\
&= \frac{0}{36.83/2} \\
&= 0.0
\end{aligned}$$

These can be used in Equation 4.5.6 to solve for κ_p.

$$\begin{aligned}
\kappa_p &= \frac{2\kappa_b^2}{\pi^2(\bar{y} + \kappa_b^2)} \left[1 + \frac{\bar{x}(\bar{x}^2 + 2\bar{y}^2 + \kappa_b^2)}{(\bar{x}^2 + \bar{y}^2)\sqrt{\bar{x}^2 + \bar{y}^2 + \kappa_b^2}} \right] \\
\kappa_p &= \frac{2(\pi/4)^2}{\pi^2(0 + (\pi/4)^2)} \left[1 + \frac{0.986153(0.986153^2 + 2 * 0^2 + (\pi/4)^2)}{(0.986153^2 + 0^2)\sqrt{0.986153^2 + 0^2 + (\pi/4)^2}} \right] \\
\kappa_p &= 0.461699
\end{aligned}$$

We can also compute the values for the axes in Figure 4.5.4 and use the plot to find an approximate value for κ_p.

$$\begin{aligned}
\frac{y}{\kappa_b b/2} &= 0.0 \\
\frac{x}{\kappa_b b/2} &= \frac{18.16}{(\pi/4)(36.83/2)} \\
&= 1.25561
\end{aligned}$$

This figure shows that $\kappa_p \approx 0.46$, which agrees with what we calculated above.
Using these values we can compute the downwash. As a note, the aspect ratio was found in problem 2.1 to be $R_A = 5.55922$.

$$\begin{aligned}
\varepsilon_d &= \frac{(1.0)(0.462)(1.0)}{(\pi/4)} \frac{C_{L_w}}{5.559} \\
&= 0.105744 C_{L_w}
\end{aligned}$$

This can be written in 2 ways:

$$\varepsilon_d(C_{L_w}) = 0.105744 C_{L_w}(\alpha)$$

or,

$$\varepsilon_d(\alpha) = \varepsilon_{d0} + \varepsilon_{d,\alpha}\alpha = 0.105744(C_{L_w}(0) + C_{L_w,\alpha}\alpha)$$

$$\varepsilon_d(\alpha) = 0.105744 C_{L_w 0} + 0.488537\alpha$$

2.7 Using the results of 2.6, find the mounting angle of the main wing and horizontal stabilizer required for the aircraft to be trim in steady-level flight at sea level at a velocity of 200 mph with zero elevator deflection and zero angle of attack. Compare your results to those from Problem 1.13. Discuss your results.

Solution:

Assume that $\eta_h = 1.0$, which is a good preliminary design/analysis assumption.
The lift on the main wing at zero angle of attack is given by the following equation

$$\begin{aligned} C_{L_w 0} &= C_{L_w,\alpha}(\alpha_{0w} - \alpha_{L0w}) \\ C_{L_w 0} &= 4.62(\alpha_{0w} - -2.2\frac{\pi}{180}) \\ C_{L_w 0} &= 4.62(\alpha_{0w} + 0.038397) \end{aligned}$$

Therefore the equation for the downwash becomes

$$\begin{aligned} \varepsilon_d(\alpha) &= 0.105744(4.62(\alpha_{0w} + 0.038397)) + 0.488537\alpha \\ \varepsilon_d(\alpha) &= 0.488537(\alpha_{0w} + 0.038397) + 0.488537\alpha \end{aligned}$$

Using the equations we developed on the last HW set, plug in the information we have been given / have calculated.

Lift equation:

$$\begin{aligned} &\left[C_{L_w,\alpha} + \frac{S_h}{S_w}\eta_h C_{L_h,\alpha}(1 - \varepsilon_{d,\alpha})\right]\alpha + \left[\frac{S_h}{S_w}\eta_h C_{L_h,\alpha}\varepsilon_e\right]\delta_e \\ &\qquad = \frac{W\cos\gamma}{0.5\rho V_\infty^2 S_w} - \left[C_{L_w,\alpha}(\alpha_{0w} - \alpha_{L0w}) + \frac{S_h}{S_w}\eta_h C_{L_h,\alpha}(\alpha_{0h} - \alpha_{L0h} - \varepsilon_{d0})\right] \end{aligned}$$

$$\begin{aligned} &\left[4.62 + \frac{31}{244}(1.0)4.06(1 - 0.488537)\right](0) + \left[\frac{31}{244}(1.0)4.06 * 0.60\right](0) \\ &= \frac{8375}{0.5(2.3769 * 10^{-3})293.333^2 * 244} - [4.62(\alpha_{0w} - (-2.2\frac{\pi}{180})) \\ &\qquad + \frac{31}{244}(1.0)4.06(\alpha_{0h} - 0.0 - 0.488537(\alpha_{0w} + 0.038397)] \\ 0 &= 0.335654308 - [4.62\alpha_{0w} + 0.177395 + 0.51582\alpha_{0h} - 0.251997\alpha_{0w} - 0.009676] \\ 0.167935 &= 4.368\alpha_{0w} + 0.51582\alpha_{0h} \end{aligned}$$

Moment Equation:

$$\begin{aligned} &[-\frac{l_w}{\bar{c}_w}C_{L_w,\alpha} - \frac{S_h l_h}{S_w\bar{c}_w}\eta_h C_{L_h,\alpha}(1 - \varepsilon_{d,\alpha})]\alpha + \left[\frac{S_h\bar{c}_h}{S_w\bar{c}_w}\eta_h C_{m_h,\delta_e} - \frac{S_h l_h}{S_w\bar{c}_w}\eta_h C_{L_h,\alpha}\varepsilon_e\right]\delta_e \\ &= -[C_{m_w} + \frac{S_h\bar{c}_h}{S_w\bar{c}_w}\eta_h C_{m_h 0} - \frac{l_w}{\bar{c}_w}C_{L_w,\alpha}(\alpha_{0w} - \alpha_{L0w}) - \frac{S_h l_h}{S_w\bar{c}_w}\eta_h C_{L_h,\alpha}(\alpha_{0h} - \alpha_{L0h} - \varepsilon_{d0})] \\ &[-\frac{0}{\frac{244}{36.83}}4.62 - \frac{31 * 18.16}{244 * \frac{244}{36.83}}4.06(1 - 0.488537)](0) + [\frac{31 * \frac{31}{10.64}}{244\frac{244}{36.83}} * -0.55 - \frac{31 * 18.16}{244\frac{244}{36.83}}4.06 * 0.60](0) \\ &= -[-0.053 + \frac{31 * 31/10.64}{244 * 244/36.83}(0) - \frac{0}{244/36.83}4.62(\alpha_{0w} - (-2.2\frac{\pi}{180})) \\ &\qquad - \frac{31 * 18.16}{244 * 244/36.83}4.06(\alpha_{0h} - 0.488537(\alpha_{0w} + 0.038397))] \\ 0 &= -[-0.053 - \frac{31 * 18.16}{244 * 244/36.83}4.06(\alpha_{0h} - 0.488537(\alpha_{0w} + 0.038397))] \\ 0 &= 0.053 + 1.413922604\alpha_{0h} - 0.690754\alpha_{0w} - 0.0265229 \\ -0.026477 &= 1.413922604\alpha_{0h} - 0.690754\alpha_{0w} \end{aligned}$$

Solve the system of equations to solve for α_{0h} and α_{0w}.

$$\begin{aligned}
0.167935 &= 0.51582\alpha_{0h} + 4.368\alpha_{0w} \\
-0.026477 &= 1.413922604\alpha_{0h} - 0.690754\alpha_{0w} \\
\alpha_{0w} &= 0.03844 \text{ rad} = 2.20247 \text{ deg} \\
\alpha_{0h} &= 0.0000536 \text{ rad} = 0.00307 \text{ deg}
\end{aligned}$$

This shows that compared to the mounting angles found in 1.13, the main wing mounting angle increased slightly, while the horizontal tail mounting angle decreased. α_{0h} actually decreased significantly from -2.147 deg to 0.00307 deg, while the main wing stayed the same at 2.202. This demonstrates the significant effect that downwash can have.

2.8 Compute the aircraft static margin. Compare your results to those from problem 1.14. Discuss your results.

Solution:

The static margin can be computed using equation 4.4.8 in the book.

$$\frac{l_{np}}{\bar{c}_w} = -\frac{C_{m,\alpha}}{C_{L,\alpha}}$$

Use the equation for the pitch stability derivative developed above

$$\begin{aligned}
C_{m,\alpha} &= -\frac{l_w}{\bar{c}_w}C_{L_w,\alpha} - \frac{S_h l_h}{S_w \bar{c}_w}\eta_h C_{L_h,\alpha}(1 - \varepsilon_{d,\alpha}) \\
C_{m,\alpha} &= -\frac{0}{4}.62 - \frac{31 * 18.16}{244 * 244/36.83}4.06(1 - 0.488537) \\
C_{m,\alpha} &= -0.723168
\end{aligned}$$

Take the derivative of the lift equation with respect to α and compute.

$$\begin{aligned}
C_{L,\alpha} &= C_{L_w,\alpha} + \frac{S_h}{S_w}\eta_h C_{L_h,\alpha}(1 - \varepsilon_{d,\alpha}) \\
C_{L,\alpha} &= 4.62 + \frac{31}{244}4.06(1 - 0.488537) \\
C_{L,\alpha} &= 4.88382
\end{aligned}$$

Now compute the static margin for the aircraft.

$$-\frac{C_{m,\alpha}}{C_{L,\alpha}} = -\frac{-0.723168}{4.88382} = 0.148074 = 14.81 \text{ percent}$$

This is a significant change in static margin from 27.5 percent to 14.8 percent. This shows that downwash effects cannot be neglected in aircraft analysis because they have major effects on the aerodynamics and performance of the aircraft.

2.9 If the main wing and horizontal stabilizer both have zero mounting angles, compute the angle of attack and elevator deflection required to trim the aircraft in a steady climb at an altitude of 5,000 ft and a climb angle of 20 deg at a speed of 200 mph. Include the effects of downwash and compare your results to those from Problem 1.15. Discuss your results.

Solution:

Using the equations we developed, plug in the information we have given/calculated.

$$\begin{bmatrix} C_{L,\alpha} & C_{L,\delta_e} \\ C_{m,\alpha} & C_{m,\delta_e} \end{bmatrix} \begin{Bmatrix} \alpha \\ \delta_e \end{Bmatrix} = \begin{bmatrix} C_L - C_{L0} \\ -C_{m0} \end{bmatrix}$$

$$\begin{bmatrix} C_{L_w,\alpha} + \frac{S_h}{S_w}\eta_h C_{L_h,\alpha}(1-\varepsilon_{d,\alpha}) & \frac{S_h}{S_w}\eta_h C_{L_h,\alpha}\varepsilon_e \\ -\frac{l_w}{\bar{c}_w}C_{L_w,\alpha} - \frac{S_h l_h}{S_w \bar{c}_w}\eta_h C_{L_h,\alpha}(1-\varepsilon_{d,\alpha}) & \frac{S_h \bar{c}_h}{S_w \bar{c}_w}\eta_h C_{m_h,\delta_e} - \frac{S_h l_h}{S_w \bar{c}_w}\eta_h C_{L_h,\alpha}\varepsilon_e \end{bmatrix} \begin{Bmatrix} \alpha \\ \delta_e \end{Bmatrix}$$
$$= \begin{bmatrix} \frac{W\cos\gamma}{0.5\rho V_\infty^2 S_w} - [C_{L_w,\alpha}(\alpha_{0w} - \alpha_{L0w}) + \frac{S_h}{S_w}\eta_h C_{L_h,\alpha}(\alpha_{0h} - \alpha_{L0h} - \varepsilon_{d0})] \\ -[C_{m_w} + \frac{S_h \bar{c}_h}{S_w \bar{c}_w}\eta_h C_{m_h 0} - \frac{l_w}{\bar{c}_w}C_{L_w,\alpha}(\alpha_{0w} - \alpha_{L0w}) - \frac{S_h l_h}{S_w \bar{c}_w}\eta_h C_{L_h,\alpha}(\alpha_{0h} - \alpha_{L0h} - \varepsilon_{d0})] \end{bmatrix}$$

Note that ε_{d0} is calculated as follows (starting with $C_{L_w 0}$:

$$\begin{aligned} C_{L_w 0} &= C_{L_w,\alpha}(\alpha_{0w} - \alpha_{L0w}) \\ C_{L_w 0} &= 4.62(\alpha_{0w} - (-2.2\frac{\pi}{180})) \\ C_{L_w 0} &= 4.62(\alpha_{0w} + 0.038397) \end{aligned}$$

Since the problem specifies zero mounting angles (for both the horizontal stabilizer and the wing):

$$\begin{aligned} C_{L_w 0} &= 4.62(0.038397) \\ C_{L_w 0} &= 0.17739 \end{aligned}$$

Now, multiply $C_{L_w 0}$ by the coefficient you solved for in 2.6. This gives you the downwash when alpha is zero (see the beggining of the solution for 2.7 for clarification)!

$$\begin{aligned} \varepsilon_{d0} &= 0.105744 C_{L_w 0} \\ \varepsilon_{d0} &= 0.018758 \end{aligned}$$

$$\begin{bmatrix} 4.62 + \frac{31}{244}4.06(1-0.488537) & \frac{31}{244}4.06 * 0.60 \\ -\frac{0}{244/36.83}4.62 - \frac{31*18.16}{244*244/36.83}4.06(1-0.488537) & \frac{31*31/10.64}{244*244/36.83}(-0.55) - \frac{31*18.16}{244*244/36.83}4.06*0.60 \end{bmatrix} \begin{Bmatrix} \alpha \\ \delta_e \end{Bmatrix}$$
$$= \begin{bmatrix} \frac{8375\cos 20}{0.5(2.048*10^{-3})293.333^2 244} - [4.62(-(-2.2\frac{\pi}{180})) + \frac{31}{244}(1)4.06(-0.018758)] \\ -[-0.053 + \frac{31*31/10.64}{244*244/36.83}(0) - \frac{0}{244/36.83}4.62(-(-2.2\frac{\pi}{180})) - \frac{31*18.16}{244*244/36.83}4.06(-0.018758)] \end{bmatrix}$$

$$\begin{bmatrix} 4.88382 & 0.309492 \\ -0.723168 & -0.879084 \end{bmatrix} \begin{Bmatrix} \alpha \\ \delta_e \end{Bmatrix} = \begin{bmatrix} 0.366066 - 0.1677192696 \\ 0.02648 \end{bmatrix}$$

$$\begin{Bmatrix} \alpha \\ \delta_e \end{Bmatrix} = \begin{bmatrix} 0.04486 \\ -0.06702 \end{bmatrix}$$

Thus,

$$\alpha = 0.04486 \text{ rad} = 2.57 \text{ deg}$$
$$\delta_e = -0.06702 \text{ rad} = -3.84 \text{ deg}$$

Note that the angle of attack to trim the aircraft is nearly identical to that from Problem 1.15. However, the elevator deflection is significantly different.

2.10 Using MachUp 4, compute the global inviscid lift, drag, and pitching-moment coefficients about the origin for the aircraft at angles of attack of 4, 5, and 6 deg. Use zero mounting angle for both wings.

Solution:

Angle of Attack [deg]	C_L	C_{D_i}	C_m
4.0	0.5059	0.0146	-0.0719
5.0	0.5917	0.0201	-0.0848
6.0	0.6779	0.0263	-0.0981

2.11 Using the results of problem 2.10, find the location of the aerodynamic center (x_{ac}, y_{ac}) using the general relations for the aerodynamic center. Compare your results to that which would be obtained from the simplified analysis in Problem 2.8. Discuss your findings.

The general relations for Aerodynamic Center are given by equations 4.8.29-4.8.30:

$$\bar{x}_{ac} = \frac{C_{A,\alpha}C_{m_0,\alpha,\alpha} - C_{m0,\alpha}C_{A,\alpha,\alpha}}{C_{N,\alpha}C_{A,\alpha,\alpha} - C_{A,\alpha}C_{N,\alpha,\alpha}} c_{\text{ref}}$$
$$\bar{y}_{ac} = \frac{C_{N,\alpha}C_{m_0,\alpha,\alpha} - C_{m0,\alpha}C_{N,\alpha,\alpha}}{C_{N,\alpha}C_{A,\alpha,\alpha} - C_{A,\alpha}C_{N,\alpha,\alpha}} c_{\text{ref}}$$

Solution:

Begin by translating the Lift and Drag components to normal and axial forces using the following relations given in Equations 4.8.24 and 4.8.25:

$$C_N = C_L \cos\alpha + C_D \sin\alpha$$
$$C_A = C_D \cos\alpha - C_L \sin\alpha$$

Therefore the table we developed in problem 2.10 becomes

Angle of Attack [deg]	C_A	C_N	C_{m0}
4.0	-0.02073	0.5057	-0.0719
5.0	-0.03155	0.5912	-0.0848
6.0	-0.0447	0.6769	-0.0981

We can then use a central-difference approximation to estimate the derivatives for C_A, C_N, and C_{m0}. The general form for this approximation is given by:

$$C_{X,\alpha} = \frac{C_X(\alpha + \Delta\alpha) - C_X(\alpha - \Delta\alpha)}{2\Delta\alpha}$$
$$C_{X,\alpha,\alpha} = \frac{C_X(\alpha + \Delta\alpha) - 2C_X(\alpha) + C_X(\alpha - \Delta\alpha)}{\Delta\alpha^2}$$

Using these equations, solve for the first and second derivatives with respect to α for C_A, C_N, and C_{m0}. NOTE: be aware that the subtraction in the numerator is subtracting **function** outputs, not a multiplication of angles.
C_A about 5 deg:

$$C_{A,\alpha} = \frac{C_A(6.0) - C_A(4.0)}{2(\pi/180)}$$
$$C_{A,\alpha} = \frac{-0.0447 - (-0.02073)}{2(\pi/180)}$$
$$C_{A,\alpha} = -0.6869$$
$$C_{A,\alpha,\alpha} = \frac{-0.0447 - 2 * -0.03155 - 0.02073}{(\pi/180)^2}$$
$$C_{A,\alpha,\alpha} = -7.6693$$

Following a similar method for C_N and C_{m0} yields:

$$C_{N,\alpha} = 4.9059$$
$$C_{N,\alpha,\alpha} = 0.7259$$
$$C_{m0,\alpha} = -0.7506$$
$$C_{m0,\alpha,\alpha} = -1.3131$$

These can then be put into equation 4.8.28-4.8.30 to solve for the aerodynamic center. Note that the reference chord is the average chord, which can also be found on MachUp4 as the "Longitudinal Length", in this case 6.625 ft.

$$\bar{x}_{ac} = 0.8662 \text{ [behind quarter-chord]}$$
$$\bar{y}_{ac} = 1.0523 \text{ [down]}$$

Of note, MachUp4 can also be used to compute the aerodynamic center. Doing this for this aircraft at 5 degrees gives:

$$\bar{x}_{ac} = -0.8006$$
$$\bar{y}_{ac} = -1.5232$$

which is close to the values we calculated for the x-axis, but pretty different for the y axis. Note that because the axes are defined the opposite in MachUp4, these values are negative, but have very similar magnitudes. Now let's briefly compare what would be obtained using the simplified analysis in Problem 2.8. This simplified analysis is given by

$$\bar{x}_{ac} = -c_{ref}\frac{C_{m,\alpha}}{C_{L,\alpha}} = 0.9810$$
$$\bar{y}_{ac} = 0.0$$

This is considerably different from what was computed with the general relations above.