Basic Proof Methods

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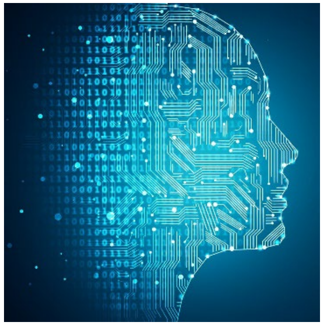

Outline

1. Why proofs?
2. Theorems and logic
3. Basic proof methods
    1. Direct proofs
    2. Proof by contradiction
    3. Proof by induction

Why proofs?

A data science perspective:

1. Many problems involving data may have multiple correct approaches and many incorrect approaches
   - Similar for proofs
2. Proofs provide a chance to review and deepen understandings of concepts and definitions
   - Unused or improperly used tools are not useful
3. New methods need to be evaluated
   - Empirical evaluations are necessary, but theoretical evaluations can save resources
4. Data science literature is very mathematical
   - You'll be a much better/useful (i.e. highly paid) data scientist if you can keep up with state of the art
5. Improve your problem solving skills

Theorems and logic

- Theorems may be stated like: if P then Q
  - P is the hypothesis
  - Q is the conclusion
  - We say P implies Q: $P \Rightarrow Q$
  - Not the same as Q implies P
- **Example**: If a book falls on Frank's head, his head will hurt.
- Suppose Frank's head hurts
  - Did a book fall on his head?
- Suppose Frank's head does not hurt
  - Did a book fall on his head?

Theorems and logic

- If $P \Rightarrow Q$, we say P is <u>sufficient</u> for Q
- However, P is not <u>necessary</u> for Q
  - Q may occur via some other way
- If Q is not true, then P must not be either

- P if and only if (iff) Q means $P \Rightarrow Q$ AND $Q \Rightarrow P$
  - Also written as $P \Leftrightarrow Q$
  - P is a necessary AND sufficient condition for Q

Basic proof methods

- There are many techniques for proofs
- Let's talk about a few of the most common approaches

1. Direct proofs
2. Proof by contradiction
3. Proof by induction

Direct proofs

- Typically start with the hypothesis, and use math techniques to demonstrate the conclusion
  - May use things like integration techniques/properties, linear algebra, Taylor series, definitions, etc.
- Often involves knowing the right tricks
- Best approach is to try several things
  - Using scratch paper is good for this
  - Mirrors the way things are discovered in the real world
  - Requires exploration, thought, and trial and error
- Experience shortens the time required in trying things out

Direct proofs

- **Example**: Prove that the sum of any two odd integers is even.
- What do we need?
  - Definition of an odd integer: $m = 2a + 1$, where a is some integer
  - Definition of an even integer: $r = 2k$, where k is some integer
- **Strategy**: define two odd integers m and n, add them together, and show that the result satisfies the definition of an even integer
  - Need to keep m and n general
- Proof on the board

Proof by counterexample

- Sometimes, we want to prove something is NOT true
- Often the easiest way to do this is to provide a counter example
- **Example**: Suppose I claim all students are lazy
- To disprove this, we just need to find one example of a student that is not lazy
  - Need to define “students” and “lazy”
- **Example**: For any positive integer m, if m is prime, them $m^2 + 4$ is also prime
  - The claim is not true
  - Use 2 (or any even number) as a counter example

Proof by contradiction

- **Goal**: prove that $P \Rightarrow Q$
- **Strategy**: assume P and assume that Q is NOT true, then show that these assumptions lead to a contradiction
- Good for proving uniqueness of a solution
  - Assume that there exists more than one solution, then show the contradiction or the solutions are equal
- **Example**: Prove that $\sqrt{2}$ is irrational
- What do we need?
  - Definition of an irrational number: a real number that is not rational
  - Definition of a rational number: a number that can be expressed as the ratio of two integers
  - We'll use contradiction by assuming $\sqrt{2}$ is rational

Proof by contradiction

- **Example**: Prove that $\sqrt{2}$ is irrational
- Assume $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = a/b$ with a, b integers. In fact, we can assume a and b have no common factors.
- Multiply both sides by b and square to get
$$2b^2 = a^2$$
- $\Rightarrow a^2$ is even. This implies a is even (how would you prove this?) so $a = 2m$ for some integer m. Thus
$$2b^2 = (2m^2) = 4m^2$$
$$\Rightarrow b^2 = 2m^2$$
- Thus b^2 and therefore b are even. Do we have a contradiction?
  - Yes, a and b have a common factor of 2

Proof by induction

- Probably won't show up as much, if at all, in this class
- Usually used to prove a relationship, often with integers
- **Strategy**: prove a base case is true, then show that a single step is also true
- **General example**: Suppose we want to prove that the relationship $S(n)$ is true for all natural numbers
  - First show the base case (typically $S(0)$ or $S(1)$) is true
  - Then assume that $S(n)$ is true.
  - Then show that this implies that $S(n+1)$ is also true

Proof by induction

- **Actual example**: Prove that the sum of the first n nonnegative integers is $\frac{n(n+1)}{2}$
  - I.e. show that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$
- Clearly true for $n = 0$ and $n = 1$
- Now assume that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$
- Need to show that $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$

Proof by induction

- Need to show that $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$
- We have that

$$\sum_{k=0}^{n+1} k = \left(\sum_{k=0}^{n} k\right) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$