Problem Set: Mathematical Foundations
SOLUTIONS

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1. (a)

$$\begin{aligned}\mathbf{v}\cdot\mathbf{v} &= 1\\ \beta^2(9+16) &= 1\\ \beta^2 &= \frac{1}{25}\\ \beta &= \pm\frac{1}{5}\end{aligned}$$

   (b) Let:

$$\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$$

   for some $a, b \in \mathbb{R}$
   Normality condition:

$$\begin{aligned}\mathbf{w}\cdot\mathbf{w} &= 1\\ a^2+b^2 &= 1\end{aligned}$$

   Orthogonality condition:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= 0\\ \beta(3a+4b) &= 0\end{aligned}$$

   Rearrange 2nd condition and sub in 1st:

$$\begin{aligned}b &= -3a/4\\ \implies a^2+\frac{9}{16}a^2 &= 1\\ \implies \frac{25}{16}a^2 &= 1\\ \implies a^2 &= \frac{16}{25}\\ \implies a &= \frac{4}{5}\\ \implies b &= \frac{-3}{5}\end{aligned}$$

So:

$$\mathbf{w} = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

(c) No, because $-\mathbf{w}$ is also orthonormal to $\mathbf{v}$

(d) Yes

(e) If $\mathbf{A}$ is invertible then $\det\mathbf{A} \neq 0$ by the Invertible Matrix Theorem.

$$\begin{aligned} \det\mathbf{A} &= (4.9) - \alpha.\alpha \\ &= 36 - \alpha^2 \\ \implies 36 - \alpha^2 &\neq 0 \\ \implies 36 &\neq \alpha^2 \\ \implies \alpha &\neq \pm 6 \end{aligned}$$

(f) If $\mathbf{A}$ is positive definite then its eigenvalues, $\{\lambda_i\}_{i=1}^{2}$, are positive.

Eigenvalues are satisfied by the following characteristic polynomial:

$$\begin{aligned} \det(\mathbf{A} - \lambda_i\mathbf{I}) &= 0 \\ \begin{vmatrix} 4 - \lambda_i & \alpha \\ \alpha & 9 - \lambda_i \end{vmatrix} &= 0 \\ (4 - \lambda_i)(9 - \lambda_i) - \alpha^2 &= 0 \\ (\lambda_i)^2 - 13\lambda_i + 36 - \alpha^2 &= 0 \\ \implies \lambda_i = \frac{13 \pm \sqrt{13^2 - 4(36 - \alpha^2)}}{2} \end{aligned}$$

So for $\lambda_i > 0$:

$$\begin{aligned} 13 - \sqrt{13^2 - 4(36 - \alpha^2)} &> 0 \\ 13^2 &> 13^2 - 4(36 - \alpha^2) \\ 36 &> \alpha^2 \\ \implies -6 < \alpha &< 6 \end{aligned}$$

(g) If $\mathbf{A}$ represents a covariance matrix then the diagonal elements give the variances, σ_1^2, σ_2^2:

$$\begin{aligned}{\sigma_1}^2 = 4 &\implies \sigma_1 = 2 \\ {\sigma_2}^2 = 9 &\implies \sigma_1 = 3\end{aligned}$$

The correlation, ρ, is given by the off-diagonal elements, so that:

$$\begin{aligned}\rho\sigma_1\sigma_2 &= \alpha \\ \rho &= \frac{\alpha}{\sigma_1\sigma_2} \\ &= \frac{-3}{2 \times 3} \\ &= -0.5\end{aligned}$$

(h) The shape will be an ellipse
The ellipse will be axis-aligned
The ratio of the semi-major to the semi-minor axis will be: 3:2

2. (a) Note that the form of $f(x)$ is:

$$f(x) = 2x^2 - 2(a_1 + a_2)x + (a_1^2 + a_2^2)$$

So, forming the Hessian:

$$\mathcal{H}(x) = 4$$

Thus the Hessian is positive definite.

So $f(x)$ is strictly convex

So the solution is globally optimal and unique (by properties of strictly convex functions)

(b) Note that $g(x)$ has a characteristic 'V' shape, with change of slope at b (a sketch would also be acceptable in order to demonstrate this)

State that $g(x)$ is convex, but not strictly convex

So the solution is globally optimal (by properties of convex functions).

But the solution is unique (by geometry).

(c) Note that $h(x)$ has a characteristic '_/' shape, with changes of slope at b_1 and b_2 (a sketch would also be acceptable in order to demonstrate this)

State that $h(x)$ is convex, but not strictly convex

So the solution is globally optimal (by properties of convex functions).

And the solution is not unique (by geometry).

(d) Note that $k(x) = f(x) + h(x)$, and that the sum of a convex function and a strictly convex function is strictly convex.

So the solution is globally optimal (by properties of convex functions)

And the solution is unique (by properties of strictly convex functions)

3. **Consider** $x \geq 1$:

$$(1-x) \leq 0 < e^{-x}$$

Consider $0 < x < 1$:

Binomial expansion, valid for $|x| < 1$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

Taylor expansion of e^x:

$$e^x = 1 + x + \frac{1}{2}x^2 + + \frac{1}{3!}x^3 + ...$$

For positive x this implies:

$$e^x < \frac{1}{1-x}$$
$$\implies (1-x) < e^{-x}$$

Consider $x = 0$:

$$\implies (1-x) = 1 = e^0 = e^{-x}$$

Consider $x < 0$:

Let $y = -x > 0$:

$$\implies (1-x) = (1+y)$$

And:

$$\implies e^{-x} = e^y = 1 + y + \frac{1}{2}y^2 + + \frac{1}{3!}y^3 + ...$$

So, since $y > 0$:

$$(1+y) < e^y$$
$$\implies (1-x) < e^{-x}$$

4. (This solution assumes a linear, risk-neutral, utility function througout).

   (a)

$$\begin{aligned}
\mathbb{E}[1\text{ roll}] &= \frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \frac{1}{6}\times 3 + \frac{1}{6}\times 4 + \frac{1}{6}\times 5 + \frac{1}{6}\times 6 \\
&= \frac{1}{6}\times(1+2+3+4+5+6) \\
&= 3.5
\end{aligned}$$

   (b) Assume 1st roll takes place at $t=0$, 2nd roll takes place at $t=1$

$$\begin{aligned}
\mathbb{E}[\text{2nd roll at } t=1] &= \frac{1}{6}\times(1+2+3+4+5+6) \\
&= 3.5
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\text{1st roll at } t=0] &= \frac{1}{6}\times\big(\max(1,\mathbb{E}[\text{2nd roll at } t=1]) + \max(2,\mathbb{E}[\text{2nd roll at } t=1]) \\
&\qquad + \max(3,\mathbb{E}[\text{2nd roll at } t=1]) + \max(4,\mathbb{E}[\text{2nd roll at } t=1]) \\
&\qquad + \max(5,\mathbb{E}[\text{2nd roll at } t=1]) + \max(6,\mathbb{E}[\text{2nd roll at } t=1])\big) \\
&= \frac{1}{6}\times(3.5+3.5+3.5+4+5+6) \\
&= 4.25
\end{aligned}$$

   (c) Assume 1st roll takes place at $t=0$, 2nd roll takes place at $t=1$, 3rd roll takes place at $t=2$

$$\begin{aligned}
\mathbb{E}[\text{3rd roll at } t=2] &= \frac{1}{6}\times(1+2+3+4+5+6) \\
&= 3.5
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\text{2nd roll at } t=1] &= \frac{1}{6}\times\big(\max(1,\mathbb{E}[\text{3rd roll at } t=2]) + \max(2,\mathbb{E}[\text{3rd roll at } t=2]) \\
&\qquad + \max(3,\mathbb{E}[\text{3rd roll at } t=2]) + \max(4,\mathbb{E}[\text{3rd roll at } t=2]) \\
&\qquad + \max(5,\mathbb{E}[\text{3rd roll at } t=2]) + \max(6,\mathbb{E}[\text{3rd roll at } t=2])\big) \\
&= \frac{1}{6}\times(3.5+3.5+3.5+4+5+6) \\
&= 4.25
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\text{1st roll at } t=0] &= \frac{1}{6}\times\big(\max(1,\mathbb{E}[\text{2nd roll at } t=1]) + \max(2,\mathbb{E}[\text{2nd roll at } t=1]) \\
&\qquad + \max(3,\mathbb{E}[\text{2nd roll at } t=1]) + \max(4,\mathbb{E}[\text{2nd roll at } t=1]) \\
&\qquad + \max(5,\mathbb{E}[\text{2nd roll at } t=1]) + \max(6,\mathbb{E}[\text{2nd roll at } t=1])\big) \\
&= \frac{1}{6}\times(4.25+4.25+4.25+4.25+5+6) \\
&= 4.67
\end{aligned}$$

5. (a) (Univariate) Normal or Gaussian distribution

(b) μ

(c)

$$
\begin{aligned}
\frac{dp}{dx} &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \times \left(-\frac{2(x-\mu)}{2\sigma^2}\right) \\
&= 0 \\
\implies -\frac{2(x-\mu)}{2\sigma^2} &= 0 \\
\implies x &= \mu
\end{aligned}
$$