1. Consider the following vector and matrix:

$$\mathbf{v} = \beta \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 4 & \alpha \\ \alpha & 9 \end{bmatrix}$$

where $\alpha \in \mathbb{R}$

(a) What value(s) should β take so that $\mathbf{v}$ is normalised?

[3 marks]

(b) Find a vector, $\mathbf{w}$, which is orthonormal to $\mathbf{v}$?

[4 marks]

(c) Is such a vector unique? Explain.

[3 marks]

(d) Is $\mathbf{A}$ symmetric?

[2 marks]

(e) Under what conditions on α is $\mathbf{A}$ invertible?

[3 marks]

(f) If $\mathbf{A}$ is positive definite how is α constrained?

[3 marks]

(g) If $\mathbf{A}$ represents the covariance matrix associated with a bivariate Gaussian distribution, and $\alpha = -3$ then what is the correlation between the 2 random variables associated with that distribution?

[3 marks]

(h) If $\mathbf{A}$ represents the covariance matrix associated with a bivariate Gaussian distribution, and $\alpha = 0$, describe the *particular* shape of the 2-dimensional isocontours of probability traced out by this *particular* distribution.

[4 marks]

2. Discuss the nature of the optimality associated with minimisations of the following functions. For each consider whether we can be sure that a particular minimisation is globally or locally optimal and whether a particular minimisation is unique.

   (a)
   $$f(x) = \sum_{i=1}^{2} (x - a_i)^2$$
   where: $a_1, a_2 \in \mathbb{R}, a_1 \neq a_2$

   [5 marks]

   (b)
   $$g(x) = |x - b|$$
   where: $b \in \mathbb{R}$

   [5 marks]

   (c)
   $$h(x) = \sum_{i=1}^{2} |x - b_i|$$
   where: $b_1, b_2 \in \mathbb{R}, b_1 < b_2$

   [5 marks]

   (d)
   $$k(x) = \sum_{i=1}^{2} (x - a_i)^2 + \sum_{i=1}^{2} |x - b_i|$$
   where: $a_1, a_2, b_1, b_2 \in \mathbb{R}$ are all distinct.

   [5 marks]

3. Demonstrate, using Taylor's expansion:
   $$(1 - x) \leq e^{-x}$$

   [20 marks]

4. (a) Consider a dice-rolling game in which you are invited to roll a fair six-sided die, and are paid the same number of pounds as the outcome on the die.

   How much would you pay to play this game?

   [5 marks]

   (b) Consider a new dice-rolling game in which you are invited to roll a fair six-sided die, after which you are given the option to re-roll the die, after which the game ends, or to end the game immediately. Whenever the game ends you are paid the same number of pounds as the outcome on the final die roll.

   How much would you pay to play this game?

   [5 marks]

   (c) Finally, consider a new dice-rolling game in which you are invited to roll a fair six-sided die, after which you are given the option to re-roll the die or to end the game immediately. Should you choose to roll the die a second time you are then given a further option to re-roll the die a third time, after which the game ends, or to end the game after the second roll. Whenever the game ends you are paid the same number of pounds as the outcome on the final die roll.

   How much would you pay to play this game?

   [5 marks]

5. Consider the following probability density function (pdf):

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$.

(a) What is the name usually given to this pdf?

[2 marks]

(b) State the mode of $p(x)$.

[3 marks]

(c) Demonstrate that the stationary point of $p(x)$ is equal to the mode.

[5 marks]

6. Suppose that $\mathcal{X}$ is a discrete random variable, with outcomes 0 and 1, with a distribution characterised such that $\mathbb{P}(\mathcal{X} = 1) = \theta$.

(a) What is the mean of $\mathcal{X}$?

[3 marks]

(b) What is the variance of $\mathcal{X}$?

[3 marks]

(c) Assuming that a sequence of n samples are drawn iid from this distribution. State the log-likelihood function for this sequence.

[4 marks]

(d) Now assume that we observe one such sequence of outcomes: $\{1, 1, 0, 0, 1\}$. Use the technique of maximum likelihood estimation to infer a value for θ.

[5 marks]