Radiative Transfer and Stellar Atmospheres, with notes on Stellar Structure and Shocks

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Notes partly based on a stellar astrophysics course taught by Em. Prof. John Davis at U. Sydney, and some ASTR377 notes of Prof. Mark Wardle

0 Basic Stellar Observations and Data

This section is either second year revision or material that will be covered in Orsola's lectures plus a bunch of other useful information. Other assumed knowledge includes:

- Introductory-level thermodynamics, including the use of the ideal gas equation, and a concept of temperature as an energy of $1/2k_BT$ per degree of freedom. We will also be using the concept of an *adiabatic change* i.e. PV^γ =const.
- Introductory-level atomic structure. The concept of pairs of spin-up and spin-down electrons, and the numbers of electrons per orbital for s, p and d orbitals. Some knowledge of solutions to Schroedinger's equation for the Hydrogen atom would also be useful.

0.1 Light from the Stars

It may be obvious but it is still worth making the point that our **ONLY** source of information about stars is the radiation we receive from them.

Our knowledge of stars has come from studies of:

- The spatial distribution of brightness.
  e.g. angular positions, sizes, shapes, clustering, etc. and the relative brightnesses etc. of the stars.
- The temporal variations in brightness and its spatial distribution.
  e.g. brightness variations due to eclipses, pulsations or explosions, and angular motions including parallaxes, orbital motions and angular variations in size due to pulsations or explosions.

- The spectral distribution of starlight.
  e.g. leading to chemical composition and physical conditions (such as temperature, pressure, rotation, etc.) and Doppler shifts of spectral lines revealing radial motion.
- The temporal variations in the spectral distribution of starlight.
  e.g. changing Doppler shifts of spectral lines due to orbital motion, pulsations, explosions, etc.

0.2 Stellar Distances

The distances to nearby stars are determined by measuring their parallax. The *parallax* of a star is the angle, measured in seconds of arc, subtended at the star by a baseline of 1 AU (AU = Astronomical Unit, the mean distance from the Earth to the Sun = 1.5×10^8 km).

The parallax is usually represented by the greek letter π.

A *parsec* is the distance at which a star has a parallax of 1 second of arc. Thus:

$$d = 1/\pi$$

where d is the distance in parsecs (pc) of a star with a parallax π.
Parallaxes can be measured directly for only a few hundred stars from the ground ($\sim$200 with an accuracy $< \pm 20\%$ by conventional methods). However, the HIPPARCOS satellite has determined the distances to some 20,000 stars to better than 10%, and the GAIA satellite (due for a late 2012 launch) promises to do much better.
Note that 1 pc = 3.26 light years = 206265 AU = 3.1×10^{13} km.

Examples of distances to well known stars are:

Proxima Centauri (nearest star)	1.31 pc
α Centauri	1.34 pc
α Canis Majoris (Sirius)	2.65 pc
α Carinae (Canopus)	60 pc
α Orionis (Betelgeuse)	200 pc
Diameter of Milky Way	25 kpc

0.3 Apparent Magnitudes

The apparent brightness of a star is measured on a logarithmic *magnitude* scale. The *apparent magnitude* m is defined by:

$$m = -2.5 log_{10} E + K$$

where E is the irradiance (flux in W.m^{-2}) received from the star and K is a constant that sets the zero point of the scale and which is defined by a set of standard stars.

NOTE the negative sign that makes fainter objects have larger magnitudes.

Magnitude differences are given by:

$$\Delta m = m_1 - m_2 = -2.5log_{10}(E_1/E_2)$$

NOTE that a magnitude difference of 5 corresponds to a brightness ratio of exactly 100. Note also that $\Delta m = 1$ corresponds to a brightness ratio of $\sqrt[5]{100} = 2.512 \sim 2.5$.

0.4 Magnitude Systems

There are several magnitude systems in use but the so-called UBV system is commonly used in stellar studies. It is a photo-electric system with passbands defined by photocathode responses and coloured glass filters (**U**ltraviolet, **B**lue and **V**isual). If s_λ represents the resultant spectral response function for one of the passbands and E_λ is the spectral irradiance from the star (Wm^{-2}nm^{-1}), then the magnitude for that passband is given by:

$$m_p = -2.5log_{10}\left[\int_0^\infty s_\lambda E_\lambda d\lambda\right] + K_p$$

where m_p and K_p are the magnitude and constant for passband p.

NOTE that the following symbols are used in place of p to represent the apparent magnitudes in the UBV system $V = m_V$, $B = m_B$ and $U = m_U$.
The UBV system has been extended to the red and infrared with additional bands labelled R, I, J, H, K, L, M and N. The effective wavelengths and spectral bandwidths for the passbands are listed in the following table.

Band	Effective Wavelength λ_{eff}	Bandwidth $\Delta\lambda$
U	350 nm	100 nm
B	430 nm	100 nm
V	550 nm	100 nm
R	640 nm	150 nm
I	790 nm	150 nm
J	1.25 μm	0.12 μm
H	1.66 μm	0.16 μm
K	2.22 μm	0.22 μm
L	3.45 μm	0.35 μm
M	4.65 μm	0.46 μm
N	10.3 μm	1.0 μm

Examples of V magnitudes are:

Object	V
The Sun	-26.5
Full Moon	-12.5
Sirius (brightest star)	-1.46
Unaided eye limit	$\sim +6$
5 m telescope (visual)	$\sim +20$

0.5 Colour Indices

The difference between two different colour magnitudes, such as $(B - V)$, for a given star is known as a *colour index*. Colour indices represent a measure of the broad shape of the spectral distribution of energy from a star and are an indicator of the temperature of the radiation field.

0.6 Bolometric Magnitudes and Bolometric Corrections

The magnitude representing the entire spectrum is known as the *bolometric magnitude* m_{bol}.

$$m_{bol} = -2.5 log_{10} \left[\int_0^\infty E_\lambda d\lambda \right] + constant$$

where $\int_0^\infty E_\lambda d\lambda$ is the bolometric flux.

The *bolometric correction* (BC) is defined as the difference between the bolometric and V magnitudes.

$$BC = m_{bol} - V$$

Also $BC = M_{bol} - M_V$ where M_{bol} and M_V are absolute magnitudes as defined in the next section.

0.7 Absolute Magnitudes

Since apparent magnitudes depend on distance they do not represent the relative *intrinsic* brightnesses of stars (for example, the Sun and α Centauri A are 'identical' stars with equal intrinsic brightnesses but $E_\odot / E_{\alpha Cen} \sim 10^{10}$).
The *absolute magnitude* M is defined as the apparent magnitude at a standard distance of 10 pc (in the absence of interstellar extinction). Thus, absolute magnitudes give a measure of the *true* relative brightnesses of stars and, using the inverse square law, are given by:

$$M = -2.5 log_{10}(E \times d^2/10^2) + constant$$

which can be written as:

$$M = m - 5 log_{10} d + 5$$

where d is the distance to the star in pc.

Examples of absolute magnitudes are:

Object	V	M_V	
The Sun	-26.5	$+4.79$	
Alpha Canis Majoris (Sirius)	-1.46	$+1.41$	
Alpha Orionis (Betelgeuse)	$+0.50$	-5.9	
Beta Orionis (Rigel)	$+0.12$	-7.0	($\sim$52,000$\times$Sun)

0.8 Distance Modulus

The *distance modulus* is the difference between the apparent and absolute magnitudes:

$$\text{Distance Modulus} = m - M = 5\ log_{10} d - 5$$

NOTE that m must be corrected for the effects of interstellar extinction before substitution in this equation.

0.9 Stellar Temperatures

The spectral distribution of light from a star is similar to that from a black body but there are significant differences. The temperature of a star may be specified in different ways:

- *Colour Temperature* T_c — this is the temperature of the black body that matches the *slope* of the observed stellar spectrum — either over a range of wavelengths or simply between two specific colours such as B and V. The colour temperature does not have a single characteristic value for a star since it depends on the wavelength range or colours used.

- *Brightness Temperature* T_b — this is the temperature of the black body that gives the same absolute flux per unit wavelength interval at a specific wavelength (or per hertz at a specific frequency) as the star. The brightness temperature for a given star only applies at the specified wavelength (or frequency).

- *Effective Temperature* T_e — this is defined as the temperature of the black body which would give the same total absolute energy flux (without regard to spectral distribution) as the star. It has a characteristic value for a given star.

$$\sigma T_e^4 = \int_0^\infty \mathcal{F}_\lambda d\lambda = \int_0^\infty \mathcal{F}_\nu d\nu \ \mathrm{Wm^{-2}}$$

where:
σ is the Stefan-Boltzmann constant = 5.7 $\times 10^{-8}$ $\mathrm{Wm^{-2}K^{-4}}$
$\mathcal{F}_\lambda$ is the stellar surface flux in $\mathrm{Wm^{-2}m^{-1}}$
$\mathcal{F}_\nu$ is the stellar surface flux in $\mathrm{Wm^{-2}Hz^{-1}}$

Note that:

$$\mathcal{F}_\lambda = \frac{4}{\theta^2} f_\lambda \ \mathrm{Wm^{-2}m^{-1}}$$

where:
θ is the stellar angular diameter in radians
f_λ is the received flux, corrected for interstellar extinction, in $\mathrm{Wm^{-2}m^{-1}}$

Thus we have:

$$\sigma T_e^4 = \frac{4}{\theta^2} \int_0^\infty f_\lambda d\lambda = \frac{4}{\theta^2} \int_0^\infty f_\nu d\nu \ \mathrm{Wm^{-2}}$$

Effective temperatures range from $\sim$ 2500 K to $\sim$ 35,000 K.

0.10 Stellar Luminosities

The *luminosity* L of a star is the total energy flux through the surface of the star and is given by:

$$L = 4\pi R^2 \int_0^\infty \mathcal{F}_\lambda d\lambda = 4\pi R^2 \sigma T_e^4 \text{ watts}$$

where R is the radius of the star.

Luminosities range from $\sim 10^{-3} L_\odot$ to $\sim 5 \times 10^5 L_\odot$.

0.11 Stellar Radii

Radii can be determined directly for stars of known distance if their angular diameters are measured (e.g. with SUSI!). This has been limited by the relatively small number of stars with accurate parallax measurements but the results from the astrometric satellite HIPPARCOS, that are now becoming available, will dramatically increase the number of accurately determined stellar radii.

Radii can also be determined from eclipsing binary stars for which spectroscopic data are also available and from spectroscopic binary stars which are also observed interferometrically (SUSI again).

0.12 Stellar Masses

Stellar masses are also determined from studies of binary stars since Kepler's third law, in general form, relates the masses of the component stars to the orbital period and the semi-major axis of the orbit of one star about the other.

Thus, masses can be determined for the component stars of visual binary systems of known parallax. For spectroscopic binaries, if lines from both components can be seen in the spectrum, the mass ratio can be determined. However, the individual masses cannot be found unless the orbital inclination is measured independently. The inclination can be determined if the binary star is also an eclipsing system or if it is observed interferometrically (e.g. with SUSI).

0.13 The Mass-Luminosity Relationship

For main-sequence stars there is a simple relationship between mass and luminosity — on a log-log plot the points lie close to a straight line so that:

$$L \propto M^\beta$$

where $\beta \approx 3.8$ on average.

For a given mass the luminosity is generally almost the same—except for white dwarfs. Surprisingly most giants and supergiants follow nearly the same mass-luminosity relation as main-sequence stars—see Figure 1-4. Red giants are an exception with generally smaller masses for a given luminosity.

0.14 Solar Data

Mass $M_\odot$	2.0×10^{30} kg
Radius $R_\odot$	7.0×10^{5} km
Effective Temperature $T_{e\odot}$	5770 K
Luminosity $L_\odot$	3.9×10^{26} W
Spectral Type	G2 V
Colour Index $(B-V)_\odot$	+0.62
Visual Magnitude $V_\odot$	−26.74
Absolute Visual Magnitude $M_{V\odot}$	+4.79
Absolute Bolometric Magnitude $M_{bol\odot}$	+4.72
Angular Diameter at 1 AU	1919 arcseconds

0.15 Spectral Classification

The spectral classification of stars is based on the relative strengths of features appearing in their spectra. The identification of spectral classes or spectral types by an apparently strange sequence of letters results from the way in which the scheme developed historically.

The scheme as it is now used has the following main classes (together with a traditional Rhyme from the days of male-dominated astronomy):

O	B	A	F	G	K	M
Oh	Be	A	Fine	Girl	Kiss	Me

The main spectral classes run from O to M. Each of the main spectral classes is subdivided from 0-9. Thus, the *spectral class* or *spectral type* of a star is given, for example, as A0, G2, M5 etc.

The spectral sequence is one of temperature, being hottest for class O and coolest for class M. The following table gives approximate values for $B-V$ and T_e for representative spectral types:

Sp.Type:	O5	B0	A0	F0	G0	K0	M0
T_e (K):	35,000	28,000	9,500	7,500	6,000	5,000	3500
$(B-V)$:	−0.31	−0.28	0.00	+0.27	+0.58	+0.9	+1.5
Jargon:	Early						Late

Beyond M there are different classes for giant stars (S stars that show ZrO bands rather than TiO bands, C stars, which are carbon-rich stars, and OH/IR stars) and for brown dwarfs (L, T and hypothetically, a Y class that includes Jupiter). Mass loss also creates a class which can be though of as beyond O, the Wolf-Rayet stars. All these additional classes are not part of the clear linear sequence defined by temperature.

0.16 The Hertzsprung-Russell Diagram

The original Hertzsprung-Russell Diagram was a plot of absolute magnitude M_V against spectral type for stars of known distance (i.e. for which M_V can be determined). This famous diagram has proved to be extremely important for stellar astrophysics and modern forms plot the positions of stars in the effective temperature-absolute magnitude or effective temperature- luminosity planes (or in an equivalent plane such as a colour-magnitude diagram).

These plots reveal that stars appear in grouped patterns which have been labelled the main sequence, giants, supergiants and so on. Physical associations of stars (globular and open clusters) have special forms of colour-magnitude diagrams whose significance will emerge when we discuss stellar evolution.

The H-R Diagram, when calibrated, contains information on the connections between the luminosity, surface temperature and radius of stars.

0.17 Luminosity Classes

The various groupings of stars in the H-R Diagram are classified into a range of *luminosity classes*. The classification is based on features of the stellar spectra which we will discuss later. The luminosity classes with examples are given below:

Symbol	Class	Examples	
		Star	Sp. Class
I	Supergiant - Ia	β Ori (Rigel)	B8 Ia
	Supergiant - Ib	α Ori (Betelgeuse)	M2 Ia-Iab
II	Bright Giants	α Car (Canopus)	F0 I-II
III	Giants	β Cru (Mimosa)	B0 III
IV	Subgiants		
V	Main Sequence (Dwarfs)	α CMa (Sirius)	A0 V
		The Sun	G2 V

0.18 Stellar Populations

The abundances of many elements in the atmospheres of the Sun and stars can be determined using spectroscopic techniques. The measured relative abundances are fairly constant from star to star but there are important differences which are mainly related to:

- Differences in initial composition due to evolution of the galaxy.
- Nuclear reactions within the star.
- Differential diffusion in the star (more rarely).

By studying the abundances of heavy elements in stars (C, N, O and above), astronomers have been able to identify ‘old’ and‘young’ populations of stars. The populations differ in space motions and in their colour-magnitude diagrams as well as in their abundances. The populations are labelled Population I and Population II. Population I stars are thought to have formed relatively recently from material enriched with heavy elements through previous cycles of star formation. They are found in open clusters, regions containing gas and dust as well as stars, and are typical of the spiral arms of our galaxy. Population II stars are older and were formed when the galactic gas was less enriched. They are found in globular clusters, the galactic halo and the central regions of the galaxy. Note that the division into two populations is not sharp and some stars are found with intermediate abundances.

1 Radiation and Some Thermodynamics

I introduce some statistical thermodynamic concepts important for astrophysics in this section because they are used throughout the course and it can be rather difficult to find all this information in one place. Typically, individual reference books are either too specific, or e.g. implicitly neglect relativistic effects because they are, after all, difficult to treat in a complete sense and are usually un-necessary. Nonetheless, some good reference material for this section is Sears and Salinger, "Thermodynamics, Kinetic Theory, and Statistical Thermodynamics", Cox & Giuli "Principles of Stellar Structure", Volumes 1 and 2. The Wikipedia articles are also typically very good, but have to be treated with caution because of vandalism risks.

The *only examinable part* of this section for Mike's lectures is the Planck function (Section 1.6). I include the rest of the notes as they will be useful for students going on to Masters, where statistical mechanics will be covered in much more detail.

1.1 Particle in a Box

A free electron can be represented as the limit of the solutions of a particle in a three-dimensional infinite square well with side length L as $L- > \infty$. The electron wave-functions are the solutions to the Schröedinger equation:

$$\frac{ih}{2\pi}\frac{\partial}{\partial t}\Psi = -\frac{h^2}{8\pi^2 m}(\nabla^2 + V(\mathbf{r}))\Psi, \tag{1}$$

where the potential V is zero inside the box and infinite outside the box. This equation has eigenfunctions of the form $\Psi_E(\mathbf{x})\exp(-2\pi i E/h)$. The energy levels (eigenvalues) are given by:

$$E_{n_x,n_y,n_z} = \frac{h^2}{8mL^2}[n_x^2 + n_y^2 + n_z^2]. \tag{2}$$

In order to generalize this result to relativistic particles, we need to consider the momenta of the eigenstates rather than the energy:

$$p_{n_x,n_y,n_z} = \frac{h}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2}. \tag{3}$$

1.2 Photon in a Box

A photon in a box can be though of in a similar way to an electron in a box, except that in this case we use a boundary condition where the electric field (i.e. not the "wavefunction") equals zero at the walls of the box. In the x-direction, this permits waves of wavelength $\lambda = 2L/n_x$ only, where n_x is a quantum number in the sequence 1,2,3.. etc. The factor of 2 comes from the fact that only half a wave needs to fit in the box to match the $E = 0$ boundary conditions. The energy of these waves per photon is $E = hc/\lambda = hcn_x/L$, and the momenta is given by $p = E/c$.

It follows that the momenta of photons in the 3-dimensional box as a function of the quantum numbers n_x, n_y and n_z is therefore given by the equation:

$$p_{n_x,n_y,n_z} = \frac{h}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2}. \tag{4}$$

1.3 General Particle in a Box

We can see that both for massive and massless particles, the following equation holds:

$$p_{n_x,n_y,n_z} = \frac{h}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2}. \tag{5}$$

Indeed, this also holds in the relativistic limit (details beyond the scope of this course). If we consider $n_r = \sqrt{n_x^2 + n_y^2 + n_z^2}$, we can write this as:

$$\begin{aligned} p &= \frac{h}{2L} n_r \text{ , i.e.} & (6)\\ n_r &= \frac{2L}{h} p & (7)\end{aligned}$$

If we now write the total number of states in our box with momentum less than p as M_p, and consider the quadrant of a sphere in momentum space, we have:

$$\begin{aligned} \frac{dM_p}{dn_r} &= \frac{\pi}{2} n_r^2 & (8)\\ \frac{dM_p}{dp} &= \frac{dM_p}{dn_r} \times \frac{dn_r}{dp} & (9)\\ &= \frac{2\pi L^2 p^2}{h^2} \times \frac{2L}{h} & (10)\\ &= \frac{4\pi p^2 V}{h^3}. & (11)\end{aligned}$$

This equation is actually pretty intuitive... the number of states per unit volume per unit momentum-space volume ($4\pi p^2 dp$) is just equal to h^{-3}.

We will now write the total number of particles as N to distiguish it from the total number of states M_p. The total number of particles is given by by Bose-Einstein or Fermi-Dirac statistics according to the following equation (in both differential and integral form for clarity):

$$\begin{aligned} \frac{dN}{dM_p} &= g_i \frac{1}{e^{(-\mu+\epsilon(p)/kT)} \pm 1} & (12)\\ N &= g_i \int_0^\infty \frac{dM_p}{e^{(-\mu+\epsilon(p)/kT)} \pm 1}. & (13)\end{aligned}$$

Here g_i is the internal degeneracy factor (e.g. including spin degeneracy) and μ is the chemical potential. The fraction on the right hand side of Equation 12 is the occupation number, i.e. a term representing the probability that a particular state is occupied by a particle. This probability can be greater than 1 for Bose-Einstein statistics where many particles can occupy a single quantum state. The sign in the demoninator is "-1" for Bose-Einstein statistics and "+1" for Fermi-Dirac statistics. Maxwell-Boltzmann statistics apply whenever the "±1" can be neglected because the exponential term is so large. $\epsilon(p)$ is the particle energy as a function of momentum, which clearly must be the relativistic expression if we are to describe photons and electrons with the same expression.

We can change the variable of integration from dM_p to dp using Equation 11, and arrive at:

$$\frac{dn}{dp} = \frac{4\pi p^2 g_i}{h^3} \frac{1}{e^{(-\mu+\epsilon(p))/kT} \pm 1}. \tag{14}$$

$$\boxed{n = \frac{4\pi g_i}{h^3} \int_0^\infty \frac{p^2 dp}{e^{(-\mu+\epsilon(p))/kT} \pm 1}.} \tag{15}$$

Now n represents the particle density, i.e. $n = N/V$, and as V becomes large, the approximate expression for $\frac{dM_p}{dn_r}$ given above becomes exact for practical purposes. This equation is a general starting point for deriving the Planck radiation function, the Saha ionization equation, the equation of state of a white dwarf and a wealth of other important aspects of statistical equilibrium.

1.4 The Chemical Potential

The chemical potential μ in Equation 15 is most useful when there is a system like:

$$A + B \leftrightarrow C. \tag{16}$$

This system is only in equilibrium if:

$$\mu_A + \mu_B = \mu_C. \tag{17}$$

For those of you that remember details of your thermodynamics course, we usually consider fixed temperature and pressure in astrophysics, so the chemical potential is the change in Gibbs free energy per particle at fixed T and P.

1.5 Some Relativity

In relativity, some of the Newtonian identities about momentum, energy and velocity no longer work. In particular, energy as a function of momentum is:

$$\epsilon(p) = \sqrt{m^2c^4 + p^2c^2}. \tag{18}$$

Note that this includes a "rest-mass energy". In turn, this increases the chemical potential from non-relativistic considerations by mc^2, but is the most general way to write this equation (and is needed if you want to treat e.g. pair production of electrons and positrons at very high temperature). Sometimes it is convenient to consider the difference in energy between this an the rest-mass energy, i.e. $\epsilon'(p) = \epsilon(p) - mc^2$. Velocity as a function of momentum is:

$$v(p) = \frac{pc}{\sqrt{m^2c^2 + p^2}}. \tag{19}$$

1.6 Photons and the Planck Function

Where our particles are photons, $\mu = 0$ and we have a "-" sign in the denominator because photons obey Bose-Einstein statistics. A justification for $\mu = 0$ is that a speck of black dust in thermal equilibrium with the radiation field can absorb 1 photon and emit 2 photons. This is only possible if $\mu_\gamma = 0$. $g_i = 2$ because photons have two polarization states. The momentum of a photon is $p = h\nu/c$, and $dp = (h/c)d\nu$. The number density of photons then becomes:

$$n = \frac{8\pi}{c^3} \int_0^\infty \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1}. \tag{20}$$

Noting that $dn/d\nu$ is the integrand in this equation, and that $E = h\nu$ per photon, we can easily derive the following form of the Planck function:

$$\frac{dE}{d\nu} = \frac{8\pi h\nu^3}{c^3(e^{h\nu/kT} - 1)}. \tag{21}$$

This equation gives the energy density E as a function of frequency. Often we want to instead find the intensity of the emitted radiation, i.e. the power per unit solid angle per unit area per unit frequency (or wavelength). This is called the *specific intensity* I_ν. Considering a small solid angle Ω normal to an area A, we find that the energy passing through it is $\Omega AcE/4\pi$, resulting in:

$$B_\nu \equiv I_\nu = \frac{2h\nu^3}{c^2(e^{h\nu/kT} - 1)} \tag{22}$$

This is a rather complex equation, so there are two key limits that we can look at this equation in. The first is the Rayleigh-Jeans limit, where $h\nu$ is small compared with kT. In this limit,

$$B_\nu = \frac{2\nu^2 kT}{c^2}. \tag{23}$$

Note that this equation does not include Planck's constant, and indeed the original derivation was classical. In this approximation, the total energy per unit phase space is constant. A second limit is the Wien limit, where $h\nu$ is alrge compared with kT. In this limit, occupation numbers are much less than 1, and we can neglect that photons are Bosons. We arrive at:

$$B_\nu = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}. \tag{24}$$

Finally, the last names of dead physicists we must become familiar with are Stefan and Boltzmann. The Stefan-Boltzmann law integrates the Planck function over all frequencies, resulting in a relationship we will write as:

$$\frac{P}{A} = \pi \int_0^\infty B_\nu d\nu = \sigma T^4, \text{where} \tag{25}$$

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} \text{Wm}^{-2}\text{K}^{-4}. \tag{26}$$

Sometimes it is more convenient to write the Planck function in terms of wavelength. Noting that $I_\lambda = I_\nu(d\nu/d\lambda)$ and $\nu = c/\lambda$, we have:

$$I_\lambda = \frac{2hc^2}{\lambda^5(e^{hc/\lambda kT} - 1)}. \tag{27}$$

1.7 Non-Relativistic, Non-Degenerate Electrons

In the non-relativistic approximation, we write $\epsilon(p) = p^2/2m$. Electrons are Fermions, so in principle Equation 15 has a "+" sign in the denominator. In the non-degenerate approximation, n_e is very small, so the chemical potential μ is large and negative, and the $+1$ in the denominator of Equation 15 is insignificant. This Equation then becomes:

$$n = \frac{4\pi g_i e^{\mu/kT}}{h^3} \int_0^\infty p^2 e^{-p^2/2mkT} dp. \tag{28}$$

$$\mu = -kT \log(\frac{4\pi g_i}{nh^3} \int_0^\infty p^2 e^{-p^2/2mkT} dp) \tag{29}$$

$$= -kT \log(\frac{g_i}{nh^3}(2\pi mkT)^{3/2}) \tag{30}$$

$$= -kT \log(\frac{g_i}{n}(\frac{2\pi mkT}{h^2})^{3/2}) \tag{31}$$

$$\frac{\mu}{kT} = \log(n) - \log(g_i) - \frac{3}{2}\log(\frac{2\pi mkT}{h^2}) \tag{32}$$

1.8 The Saha Equation

The system:

$$A^+ + e^- \leftrightarrow A \tag{33}$$

is in equilibrium when the sum of chemical potentials on the left is equal to the sum of chemical potentials on the right. Using Equation 32, and putting $m_A = m_{A+}$, this is equivalent to:

$$\log(n_{A+}) + \log(n_e) - \log(g_{A+}) - \log(g_e) - \frac{3}{2} - \log(\frac{2\pi mkT}{h^2}) = \log(n_{A+}) - \log(g_A) - \epsilon/kT. \tag{34}$$

Here, the ionization energy is ϵ, which e.g. enters this equation through the difference in rest mass energy between A and $A^+ + e^-$, if we were to write out the full relativistic expression for ϵ. Simplifying this equation, we arrive at the Saha equation:

$$\frac{n_{A+}n_e}{n_A} = 2\frac{g_{A+}}{g_A}(\frac{2\pi mkT}{h^2})^{3/2}e^{-\epsilon/kT}. \tag{35}$$

This equation can be generalized to the ith ionization state of the atom A, by replacing A+ with A_i and A with A_{i+1}.

1.9 Fully Degenerate Electron Gas

This will be covered in a much later lecture, but the basic thermodynamics belongs here in the notes. Where the Fermi energy is much greater than kT, we can write Equation 15 as:

$$n_e = \frac{8\pi}{h^3}\int_0^{p_{\max}} p^2 dp, \tag{36}$$

because the denominator in the integrand switches from 0 to ∞ over a very small range of p. Solving for $p_{\max}$, this then gives:

$$p_{\max}^3 = \frac{3n_e h^3}{8\pi}. \tag{37}$$

The equation of state for this fully degenerate electron gas is independent of T, so takes the form $P = f(\rho)$. The density ρ relates to the electron number density n_e by the simple relationship $\rho = m_{\mathrm{H}}\mu_e n_e$, where μ_e is the mass in atomic mass units per electron and m_{H} is one atomic mass unit (roughly equal to the hydrogen mass). For white dwarf stars, $\mu_e = 2$, because there is 1 proton and 1 neutron per electron for nuclei in the range ^{4}He to ^{16}O.

We can calculate the pressure simply by integrating the flow of momentum through a surface over all values of p. This integral is:

$$P_e = \frac{1}{3}\int_0^{p_{\max}} \frac{dn_e(p)}{dp} pv(p)dp, \text{where} \tag{38}$$

$$v(p) = \frac{pc}{\sqrt{p^2+m^2c^2}} \text{and} \tag{39}$$

$$\frac{dn_e}{dp} = \frac{8\pi p^2}{h^3} \text{...i.e.} \tag{40}$$

$$P_e = \frac{8\pi c}{3h^3}\int_0^{p_{\max}} \frac{p^4}{\sqrt{p^2+m^2c^2}}dp. \tag{41}$$

This integral can be trivially evaluated in the non-relativistic ($p_{\max} \ll mc$) and extreme-relativistic ($p_{\max} \gg mc$) limits.

In the non-relativistic limit,

$$P_e = \frac{8\pi}{3h^3 m}\frac{p_{\max}^5}{5} \tag{42}$$

$$= \frac{h^2}{60\pi^{2/3}m}(\frac{3\rho}{m_H\mu_e})^{5/3}. \tag{43}$$

In the extreme-relativistic limit,

$$P_e = \frac{8\pi c}{3h^3}\frac{p_{\max}^4}{4} \tag{44}$$

$$= \frac{ch}{24\pi^{1/3}}(\frac{3\rho}{m_H\mu_e})^{4/3}. \tag{45}$$

Note that this equation does not include the mass of the electron. This means that at high densities, all that matters for the equation of state is that electrons are spin 1/2 Fermions, and how much nuclear matter ($m_H\mu_e$) "belongs" to each electron.

Just for fun, lets evaluate Equation 41 for intermediate densities. It turns out that we still have an analytic solution, but I did need Mathematica to find it. Substituting $x = p/mc$, the integral becomes:

$$P_e = \frac{8\pi m^4 c^5}{3h^3}\int_0^{p_{\max}/mc}\frac{x^4 dx}{\sqrt{x^2+1}} \tag{46}$$

$$= \frac{\pi m^4 c^5}{3h^3}(y\sqrt{1+y^2}(2y^2-3)+3\sinh^{-1}(y)), \tag{47}$$

where $y = p_{\max}/mc$. Although this is an analytic function and is therefore convenient for computing numerical models, it is difficult to get much intuitive meaning from this rather complex equation. To compute numerical models, we actually only need:

$$\frac{dP_e}{d\rho} = \frac{8\pi m^4 c^5}{3h^3}\frac{h^3}{8\pi m^3 c^3 m_h \mu_e}\frac{y^2}{\sqrt{y^2+1}} \tag{48}$$

$$= \frac{m_e c^2 y^2}{3m_h\mu_e\sqrt{y^2+1}}. \tag{49}$$

2 Radiative Energy Transport

2.1 Introduction

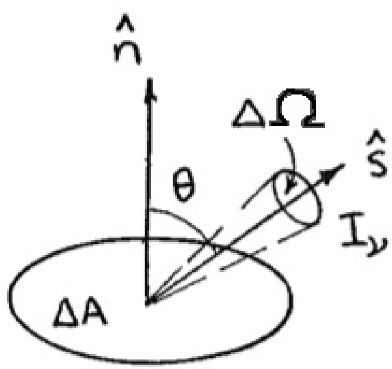
The transport of energy in the form of electromagnetic radiation is a key process in the spectral distribution of the radiation and in the energy balance of most astronomical objects — including stars. Before we discuss the transfer of radiation we must define some parameters.

2.2 Basic Definitions

2.2.1 Specific Intensity (I_ν)

The *specific intensity* I_ν is defined as the energy, per unit frequency interval, passing through unit area normal to the beam, into unit solid angle, per unit time.

NOTE that I_ν is a function of position, direction, time and frequency (or wavelength).



The elemental area ΔA in the diagram may lie on a radiating surface or it could lie in an imaginary surface inside a radiating gas. The above definition of I_ν can be written as

$$I_\nu = lim \frac{\Delta E_\nu}{\cos\theta\,\Delta A\,\Delta\Omega\,\Delta\nu\,\Delta t}$$

where ΔE_ν is the energy crossing area ΔA at an angle θ to the normal to ΔA, into solid angle $\Delta\Omega$, with frequencies between ν and $(\nu + \Delta\nu)$, in time Δt. In the limit we have:

$$dE_\nu = E_\nu\,d\nu = I_\nu\,\cos\theta\,dA\,d\Omega\,d\nu\,dt \tag{50}$$

where I_ν has units of $\mathrm{Jm^{-2}sr^{-1}Hz^{-1}s^{-1}}$ or $\mathrm{Wm^{-2}sr^{-1}Hz^{-1}}$. I_λ is defined in similar fashion to I_ν.

Note that the specific intensity along a given ray is invariant (constant) in the absence of emission or absorption — stellar surfaces appear equally bright for any distance.

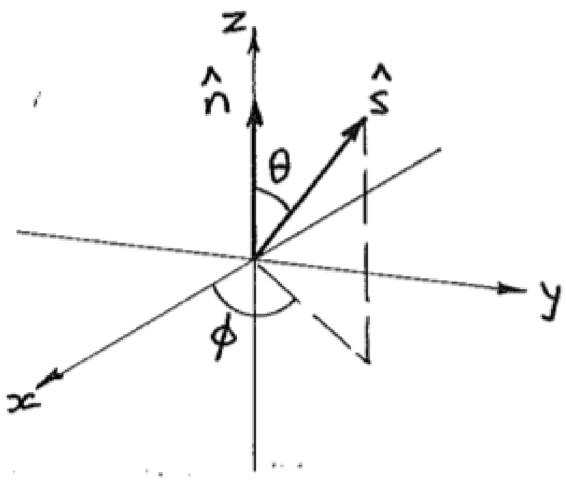
2.2.2 Mean Intensity (J_ν)

The *mean intensity* J_ν is the average of the intensity I_ν taken over all directions:

$$J_\nu = \frac{1}{4\pi} \oint I_\nu \, d\Omega \tag{51}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} I_\nu \sin\theta \, d\theta \, d\phi \tag{52}$$

Note that for an isotropic radiation field $J_\nu = I_\nu$.



We will be considering situations where I_ν is independent of the azimuthal angle ϕ and the geometry is plane-parallel (e.g. For most stars we can assume the atmosphere to be thin compared with the radius and for it to be planar). For this situation we can simplify equation (3) by putting $\mu = \cos\theta$ and taking z as the distance measured with respect to some chosen reference level in the plane-parallel geometry so that:

$$J_\nu(z) = \frac{1}{2} \int_{-1}^{+1} I_\nu(z, \mu) \, d\mu \tag{53}$$

2.2.3 Flux (F_ν)

The *flux* F_ν is defined as the net rate of energy flow across unit area (Wm^{-2}). It is a vector quantity.

$$\vec{F}_\nu = \oint I_\nu(z, \theta, \phi) \, \hat{s} \, d\Omega$$

For plane-parallel geometry, the flux in the direction of the normal $\hat{n}$, is given by:

$$
\begin{aligned}
F_\nu &= \vec{F}_\nu \cdot \hat{n} \\
&= \oint I_\nu(z,\theta)\, \hat{s}\cdot\hat{n}\, d\Omega \\
&= \oint I_\nu(z,\theta)\, cos\,\theta\, d\Omega \\
F_\nu &= 2\pi \int_{-1}^{+1} I_\nu(z,\mu)\,\mu\, d\mu \qquad (54)
\end{aligned}
$$

The term flux generally refers to the *magnitude* of the vector flux **but** we must be careful to distinguish it from the so-called *astrophysical flux* $F_{\text{astro}} = F/\pi$. We will not use this historical definition of flux in our lecture notes, but it is used in several textbooks (e.g. Mihalas, *Stellar Atmospheres*, which is otherwise a very good reference).

2.2.4 Moments of the Radiation Field

The nth moment over the radiation field is defined as:

$$M_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(z,\mu)\,\mu^n\, d\mu$$

Then the moment of order zero, the mean intensity, is:

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(z,\mu)\, d\mu \qquad (55)$$

The moment of order one, the Eddington flux, is:

$$H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(z,\mu)\,\mu\, d\mu \qquad (56)$$

The moment of order two, the so-called K-integral, is:

$$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(z,\mu)\,\mu^2\, d\mu \qquad (57)$$

Note that all the above quantities can be integrated over frequency to define corresponding quantities that apply for the entire spectrum.

2.2.5 Radiation Density (U_ν)

The *radiation density*, or *energy density* (U_ν), is the energy in the radiation field per unit volume and can be found by considering a small volume V through which radiation is passing. The energy per second entering the volume through the surface element dA, at an angle θ to the normal to dA, in solid angle dΩ, is given by $I_\nu\, dA\, \cos\theta\, d\Omega\, d\nu$. This energy is spread over a distance c so that, if the path length in the volume is l, the fraction of the energy within V is l/c. Also the increment of the volume occupied by the radiation is $dV = dA \cos\theta\, l$. Thus, the increment of energy within the volume V is given by:

$$dE_\nu = \frac{I_\nu\, dA\, \cos\theta\, d\Omega\, d\nu\, l}{c} = \frac{I_\nu\, d\Omega\, d\nu\, dV}{c}$$

The energy per unit volume contributed by the radiation in the solid angle $d\Omega$ is:

$$\frac{dE_\nu}{dV} = \frac{E_\nu d\nu}{dV} = \frac{I_\nu\, d\Omega\, d\nu}{c}$$

The radiation density U_ν is found by integrating over all solid angles:

$$U_\nu = \frac{1}{c} \oint I_\nu\, d\Omega = \frac{4\pi}{c}\, J_\nu \tag{58}$$

2.2.6 Radiation Pressure ($P_R(\nu)$)

The momentum transported by a radiation field across an area dA is found as follows. The momentum associated with a pencil of energy dE_ν is dE_ν/c. The component perpendicular to the surface dA gives the momentum per unit area per unit time as:

$$\begin{aligned} dP_R(\nu) &= \frac{1}{dA}\frac{dE_\nu\, \cos\,\theta}{c} \\ &= \frac{(I_\nu\, dA\, \cos\,\theta\, d\Omega)\cos\,\theta}{c\, dA} \\ dP_R(\nu) &= \frac{I_\nu\, \cos^2\,\theta\, d\Omega}{c} \end{aligned}$$

Integrating over all solid angles gives:

$$\begin{aligned} P_R(\nu) &= \frac{1}{c}\oint I_\nu\, \cos^2\,\theta\, d\Omega \\ P_R(\nu) &= \frac{4\pi}{c}\, K_\nu \end{aligned} \tag{59}$$

$P_R(\nu)$ is known as the *radiation pressure*. A more general form for this, without the assumption of plane-parallel geometry, requires the pressure to be a tensor. The components of this tensor include the rate of x-momentum flow through a surface with normal in the x-direction, the rate of y-momentum flow through a surface with normal in the x-direction, the rate of y-momentum flow through a surface with normal in the y-direction, and so on. For isotropic radiation, the radiation pressure tensor is a diagonal matrix with each element on the diagonal equal to $P_R(\nu)$.

2.3 Black Body Radiation

Black body radiation at a temperature T is isotropic with specific intensity B_ν $(= I_\nu)$ given by the Planck function:

$$B_\nu = \frac{2h\nu^3}{c^2}\,(e^{h\nu/kT} - 1)^{-1}$$

or:

$$B_\lambda = \frac{2hc^2}{\lambda^5}\,(e^{hc/\lambda kT} - 1)^{-1}$$

The frequency integrated intensity is related to temperature by:

$$\int_0^{+\infty} B_\nu\, d\nu = \int_0^{+\infty} B_\lambda\, d\lambda = \frac{\sigma T^4}{\pi}$$

It follows that for isotropic back body radiation the frequency-integrated moments are:

$$J = \sigma T^4/\pi; \qquad H = F = 0; \qquad K = \sigma T^4/3\pi; \qquad \text{and} \qquad P_R = 4\sigma T^4/3c$$

Note that $K/J = 1/3$. The ratio of K/J is sometimes referred to as the Eddington factor.

Example

Suppose $I(\mu) = I_o$ for $0 \leq \mu \leq 1$ and $I(\mu) = 0$ for $\mu < 0$ (This simple case might be considered as a schematic boundary condition at the surface of a star). Then we have:

$$J = \frac{1}{2}\,I_o \int_0^1 d\mu = \frac{1}{2}\,I_o$$

$$H = \frac{1}{2}\,I_o \int_0^1 \mu\, d\mu = \frac{1}{4}\,I_o = \frac{1}{2}\,J$$

and

$$K = \frac{1}{2}\,I_o \int_0^1 \mu^2\, d\mu = \frac{1}{6}\,I_o = \frac{1}{3}\,J$$

Again we have $K/J = 1/3$.

2.4 Extinction and Emission Coefficients

In general the energy in a beam of radiation passing through a medium will be modified by emission, absorption or scattering processes. Macroscopic coefficients are defined in order to take these processes into account but in the literature these take different forms and the symbols used also vary.

2.4.1 The Extinction Coefficient (α_ν, *sometimes* k_ν)

In these notes we define a macroscopic *extinction coefficient* (α_ν) to cover both scattering and true absorption processes (Note that it is often called the absorption coefficient). It is defined so that the energy removed from a beam of radiation by an elementary volume of cross-section dA and length ds is given by:

$$dE_\nu = \alpha_\nu \, I_\nu \, dA \, ds \, d\Omega \, d\nu \, dt$$

Note that the absorption and scattering components of the extinction may be treated separately by defining true absorption and scattering coefficients. Note that a *mass extinction coefficient* κ_ν is also used, particularly in stellar interior studies. If ρ is the density of the medium then $\rho\kappa_\nu = \alpha_\nu$. The units for α_ν are m^{-1} and, for κ_ν, $\mathrm{m}^2\mathrm{kg}^{-1}$.

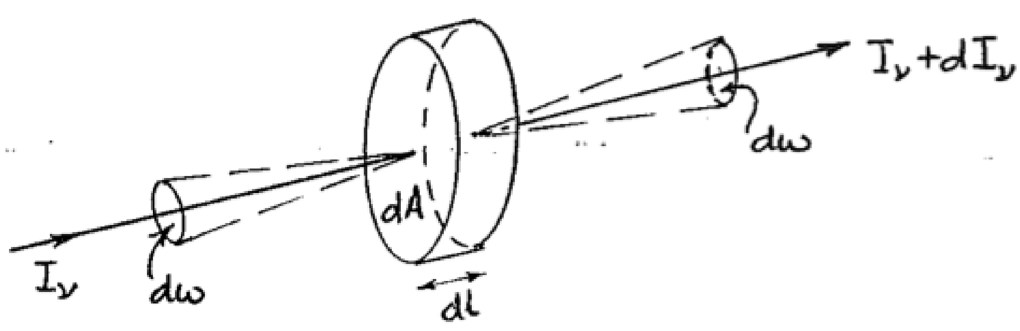
2.4.2 The Emission Coefficient (j_ν)

The *emission coefficient* j_ν, which has units of $\mathrm{W\,sr^{-1}Hz^{-1}m^{-3}}$, is defined so that the energy added to a beam is given by:

$$dE_\nu = j_\nu \, dA \, ds \, d\Omega \, d\nu \, dt$$

Note that if scattering is present j_ν will contain a 'scattering' component. Note also that j_ν is sometimes used to represent a *mass emission coefficient* with units of $\mathrm{W\,sr^{-1}Hz^{-1}kg^{-1}}$. The emission coefficient defined above is equal to the mass emission coefficient multiplied by the density of the medium. In these notes the emission coefficient as defined above will be used.

2.5 The Equation of Radiative Transfer



If we consider the transport of energy by a pencil of radiation through a small element of a medium, the energy *change* ($\Delta E_\nu \, d\nu$) in passing through the element must simply equal the energy emitted by the element minus the energy absorbed by it.

$$\begin{aligned} \text{We have from Equation (1):} \quad \Delta E_\nu d\nu &= dI_\nu \, dA \, d\Omega \, d\nu \, dt \\ \text{and we require:} \quad \Delta E_\nu d\nu &= j_\nu \, dA \, ds \, d\Omega \, d\nu \, dt - \alpha_\nu \, I_\nu \, dA \, ds \, d\Omega \, d\nu \, dt \end{aligned}$$

$$\text{so:} \qquad \frac{dI_\nu}{ds} = j_\nu - \alpha_\nu \, I_\nu \tag{60}$$

For plane-parallel geometry, with $ds = dz/\mu$, we have:

$$\mu \, \frac{dI_\nu}{dz} = j_\nu - \alpha_\nu \, I_\nu \tag{61}$$

We now define two important parameters, the *optical depth* τ_ν and the *source function* S_ν. Unfortunately, there are two sign conventions for the optical depth. For general astrophysical contexts, the optical depth is typically defined as:

$$d\tau'_\nu \equiv \alpha_\nu \, dz \tag{62}$$

However, these notes are designed primarily for stellar atmospheres, so we will use the following sign convention throughout:

$$d\tau_\nu \equiv - \alpha_\nu \, dz \tag{63}$$

The negative sign appears so that, with z increasing outwards in a stellar atmosphere, the optical depth increases inwards.

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu} \tag{64}$$

With these two definitions we can now write the *equation of radiative transfer* in its standard form for stellar atmospheres:

$$\mu \, \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \tag{65}$$

For applications in other areas of astrophysics, the signs on the right hand side are reversed, because of the differing sign convention for optical depth. This equation provides the basis for the calculation of radiation transport and its solution is generally carried out subject to specific boundary conditions. For example, a case of particular interest which we will come to is that of a medium with a boundary on one side but extending effectively to infinity on the other — the semi-infinite atmosphere.

2.6 Solution of the Equation of Radiative Transfer

2.6.1 Simple Examples of Radiative Transfer

1. Uniform medium in thermodynamic equilibrium

In thermodynamic equilibrium the radiation does not change with time or position. Therefore, from equation (65), we require $dI_\nu/d\tau_\nu = 0$ which gives:

$$I_\nu = S_\nu$$

It follows that:

$$S_\nu = I_\nu = B_\nu = \frac{2h\nu^3}{c^2}\,(e^{h\nu/kT} - 1)^{-1}$$

Also, from equation (61) we get:

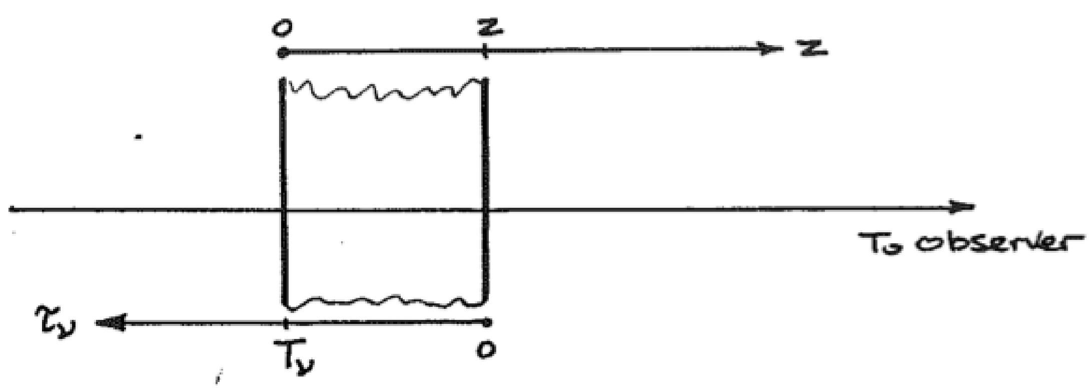
$$j_\nu = \alpha_\nu\, I_\nu = \alpha_\nu\, B_\nu = \text{rate of thermal emission} \quad \text{— Kirchhoff's Law}$$

2. No material present

From equation (61) for no material present (i.e. $\alpha_\nu = j_\nu = 0$):

$$\frac{dI_\nu}{ds} = 0 \;\text{ or }\; I_\nu = \text{constant}$$

Now consider a slab of material of geometrical thickness Z and optical depth T_ν as shown in the diagram below.



3.

Emission only

Equation (13) for $\alpha_\nu = 0$ becomes:

$$\mu\,\frac{dI_\nu}{dz} = j_\nu$$

$$\text{and emergent intensity is} \quad I_\nu(\mu) = \frac{1}{\mu}\int j_\nu\, dz + I_{0\nu}$$

where $I_{0\nu}$ is the incident specific intensity.

4. Absorption only

Equation (13) for $j_\nu = 0$ becomes:

$$\mu \frac{dI_\nu}{dz} = -\alpha_\nu I_\nu$$

and, for a finite slab the emergent radiation is given by:

$$\begin{aligned} I_\nu(\mu) &= I_{0\nu} \exp\left(-\frac{1}{\mu}\int_0^Z \alpha_\nu \, dz\right) \\ I_\nu(\mu) &= I_{0\nu} \exp(-T_\nu/\mu) \end{aligned}$$

2.6.2 Formal Solution of the Equation of Radiative Transfer

For convenience we will omit the subscript ν from equation (65) so that it becomes:

$$\mu \frac{dI}{d\tau} = I - S \tag{66}$$

Multiplying through by $\exp(-\tau/\mu)$ gives:

$$\mu \frac{dI}{d\tau} e^{-\tau/\mu} = I \, e^{-\tau/\mu} - S \, e^{-\tau/\mu}$$

$$\text{or} \qquad \frac{d(I \, e^{-\tau/\mu})}{d\tau} = -\frac{S \, e^{-\tau/\mu}}{\mu}$$

If we consider a medium with optical depth ranging from τ_1 to τ_2 we have:

$$\begin{aligned} \left| I \, e^{-\tau/\mu} \right|_{\tau_1}^{\tau_2} &= -\int_{\tau_1}^{\tau_2} S(\tau) \, e^{-\tau/\mu} \frac{d\tau}{\mu} \\ I(\tau_1, \mu) &= I(\tau_2, \mu) \, e^{-(\tau_2 - \tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S(\tau) \, e^{-(\tau - \tau_1)/\mu} \frac{d\tau}{\mu} \end{aligned} \tag{67}$$

Thus the intensity at optical depth τ_1 is equal to the intensity at optical depth τ_2 attenuated by the exponential factor, which represents the absorption between optical depths τ_2 and τ_1, plus the emission between τ_2 and τ_1 attenuated between the integration point and τ_1.

Examples

Example 1: A homogeneous medium for which α_ν and j_ν, and hence S, are independent of position. We have for this case:

$$I(\tau_1, \mu) = I(\tau_2, \mu) \, e^{-(\tau_2 - \tau_1)/\mu} + S \left[1 - e^{-(\tau_2 - \tau_1)/\mu}\right]$$

Now consider the two extreme cases of very small and very large optical thicknesses:

Case 1: $\Delta\tau \ll 1$

$$I(\tau_1, \mu) = I(\tau_2, \mu)\,\left[1 - \frac{(\tau_2 - \tau_1)}{\mu}\right] + S\,\frac{(\tau_2 - \tau_1)}{\mu} \qquad \text{OPTICALLY THIN CASE}$$

In this case there is no significant absorption of incident intensity.

Case 2: $\Delta\tau \gg 1$

$$I(\tau_1, \mu) = S \qquad \text{OPTICALLY THICK CASE}$$

In this case $I(\tau_2)$ is heavily attenuated and only emission for $\tau < 1$ is significant.

NOTE that the optical thickness (optical depth) is a dimensionless parameter which gives a measure of the attenuation (e.g. an optical thickness of $\tau = 1$ corresponds to an attenuation by a factor of e and for $\tau = 5$ the attenuation factor is e^5 or approximately 150).

Example 2: Semi-infinite atmosphere with $\tau_1 = 0$ and $\tau_2 = \infty$ and source function a linear function of optical depth given by $S = S_0 + S_1\tau$.

Using equation (67) it can be shown that the emergent intensity is:

$$I(0, \mu) = S_0 + S_1\,\mu$$

This basically shows that the emerging radiation is characteristic of optical depth unity along the line of sight.

Example 3: An arbitrary point inside a semi-infinite atmosphere with no radiation incident externally on the boundary.

We distinguish between ingoing and outgoing radiation at the arbitrary point which we will take to have optical depth represented by τ_*.

For outgoing radiation $\mu \geq 0$ and equation (67) becomes:

$$I(\tau_*, \mu) = \lim_{\tau_2 \to \infty} I(\tau_2, \mu)\, e^{-(\tau_2 - \tau_*)/\mu} + \int_{\tau_*}^{\infty} S(\tau)\, e^{-(\tau - \tau_*)/\mu}\,\frac{d\tau}{\mu}$$

$$\text{or} \quad I(\tau_*, \mu) = \int_{\tau_*}^{\infty} S(\tau)\, e^{-(\tau - \tau_*)/\mu}\,\frac{d\tau}{\mu} \qquad (0 \leq \mu \leq 1) \qquad (68)$$

For ingoing radiation $\mu \leq 0$ and equation (67) becomes:

$$\text{or} \quad I(\tau_*, \mu) = -\int_{0}^{\tau_*} S(\tau)\, e^{(\tau_* - \tau)/\mu}\,\frac{d\tau}{\mu} \qquad (-1 \leq \mu \leq 0) \qquad (69)$$

Equations(68) and (69) represent a complete solution of the transfer equation if the source function is known. We will discuss the source function in more detail later.

Example 4: An arbitrary point where we make the assumption of local thermodynamic equilibrium and a constant temperature.

We use the Kirchoff law for thermal emission and integrate over all frequencies to ensure that energy in = energy out. Assuming that all emission is absorption and not scattering:

$$\begin{aligned}
\text{Power Absorbed}/V &= \int d\nu \int d\Omega I_\nu \alpha_\nu \\
&= 4\pi \int_0^\infty J_\nu \alpha_\nu d\nu \\
\text{Power Radiated}/V &= 4\pi \int_0^\infty j_\nu d\nu \\
&= 4\pi \int_0^\infty S_\nu \alpha_\nu d\nu \\
&= 4\pi \int_0^\infty B_\nu \alpha_\nu d\nu
\end{aligned} \tag{70}$$

If the system is in radiative equilibrium, then the power absorbed equals the power radiated:

$$\int_0^\infty J_\nu \alpha_\nu d\nu = \int_0^\infty B_\nu(T_d) \alpha_\nu d\nu. \tag{71}$$

Example 5: Material in the circumstellar enviroment of a cool giant star, with an extinction coefficient $\alpha \propto \nu$. What is its temperature, assuming a 3000 K black-body?

This example was gone through in detail in the lecture and is an excellent exercise (e.g. it has been asked in past exams). Equate energy in to energy out like in example 4 and you should reach:

$$T_d = T_*(\frac{R_*}{2r_d})^{0.4},$$

where R_* is the stellar radius, T_* is the stellar temperature and r_d is the distance of the dust from the star. Sometimes, a *dilution factor* corresponding to the ratio of the solid angle of the star as seen by the dust to 4π is defined (i.e. $W = \pi R_*^2/4r_d^2$). This results in:

$$T_d = T_* W^{0.2}.$$

For example, if we are observing dust at 10 μm, which is the Planck function maximum for 300 K radiation, what is the radius corresponding to this temperature for a 10 $R_\odot$ star of temperature 30,000 K? Answer: about 10,000 AU, or 0.1 pc, larger than the typical distance between stars in clusters.

Example 6: (Advanced Only) If the equation of hydrostatic equilibrium is generalized to accelerating matter, it becomes:

$$\frac{\partial^2 r}{\partial t^2} = -g - \frac{1}{\rho}\frac{\partial P}{\partial r}$$

Assume that gas pressure can be neglected in a tenuous dusty wind, and derive the acceleration of the dust based on the mass absorption coefficient κ_ν and the stellar luminosity L_ν.

We start by considering the second term in the above equation for the case of radiation pressure:

$$\begin{aligned}
\frac{1}{\rho}\frac{dP}{dr} &= \frac{1}{\rho}\int_0^\infty \alpha_\nu \frac{dP}{d\tau_\nu} d\nu \\
&= \frac{4\pi}{c}\int_0^\infty \frac{d}{d\tau}\left(\frac{1}{2}\int_{-1}^1 I_\nu \mu^2 d\mu\right)\kappa_\nu d\nu \\
&= \frac{2\pi}{c}\int_0^\infty \left(\int_{-1}^1 \frac{dI_\nu}{d\tau}\mu^2 d\mu\right)\kappa_\nu d\nu \\
&= \frac{2\pi}{c}\int_0^\infty \left(\int_{-1}^1 (B_\nu - I_\nu)\mu d\mu\right)\kappa_\nu d\nu \\
&= -\frac{4\pi}{c}\int_0^\infty H_\nu \kappa_\nu d\nu.
\end{aligned}$$

Each equation above involved the following identities/methods: $d\tau_\nu = +\alpha_\nu dr$, with a τ scale *increasing* with increasing r; radiation pressure $P = 4\pi K/c$ and mass absorption coefficient $\kappa_\nu = \alpha_\nu/\rho$; algebraic rearrangement; the equation of radiative transfer and LTE (i.e. $B_\nu = S_\nu$); definition of H_ν and an integration of an odd function $= 0$. Next, we note that the stellar luminosity is just the flux integrated over the surface area of the sphere of radius r:

$$\begin{aligned}
L_\nu &= 4\pi r^2 F_\nu \\
&= (4\pi)^2 r^2 H_\nu.
\end{aligned}$$

Finally, we put all this togther in to the equation in the question and arrive at:

$$\frac{\partial^2 r}{\partial t^2} = \frac{1}{r^2}\left(\frac{1}{4\pi c}\int_0^\infty \kappa_\nu L_\nu d\nu - GM\right).$$

For a typical solar-mass star with full dust condensation at solar metallicity ($\kappa \sim 1\mathrm{cm}^2\mathrm{g}^{-1}$), acceleration is positive if $L > 10^4 \mathrm{L}_\odot$. This is only applicable to the very last stages of the life of the sun (the assymptotic giant branch, at the end of the Mira variable stage) when it sheds its outer layers and becomes a white dwarf.

3 Stellar Atmospheres

3.1 Introduction

The atmosphere of a star is defined as those layers near enough to the surface of the star that some photons from them can escape. In other words, the atmosphere is as deep as one can see into the star and it corresponds to a few mean free paths for the photons emitted by the star.

Actual atmospheres are complex and contain a number of significantly different layers or zones which reflect differences in the major processes that transfer mass and energy through the stellar atmosphere. For many stars, the major zones are as follows (moving from the inside outwards):

- Photosphere Energy transfer dominated by radiation. For the Sun the photosphere is ~ 400 km thick.
- Chromosphere and Corona Dissipation of non-radiative energy fluxes heats the gas above the radiative equilibrium temperature (see below).

  Chromosphere — if the density is high the temperature rise due to a given rate of heating will be small since the added energy can be radiated away efficiently — producing a chromosphere. For the Sun the chromosphere is $\sim 10^4$ km thick with a temperature of $\sim 15{,}000$ K.

  Corona — if the density is low the temperature rise may be very large producing a corona. The solar corona extends for $> 10^6$ km with electron temperatures $\sim 2 \times 10^6$ K.
- Stellar Wind Many, if not all stars lose mass through a phenomenon known as the stellar wind.

In these lectures we will concentrate on the photosphere.

The theory of stellar atmospheres (photospheres) relates T_e (the effective temperature), g (the acceleration due to gravity) and the abundances of the elements to the spectral distribution of the light emitted by a star (i.e. the continuum intensity as a function of wavelength and the profiles of the spectral lines).

3.2 Model Stellar Atmospheres

Theoretical models of stellar atmospheres (photospheres) are constructed for given values of the effective temperature (T_e), acceleration due to gravity (g) and the abundances of the elements — PLUS physics PLUS assumptions. The output is a numerical model which gives values for the run of physical variables with depth in the atmosphere and for the emergent spectrum. Generally a model predicts the continuum and the detailed profiles of the spectral lines are treated separately.

Assumptions in Modelling Stellar Atmospheres

Several simplifying assumptions are made in constructing a model stellar atmosphere:

1. Plane-Parallel Geometry It is assumed that atmospheres are stratified into homogeneous plane-parallel layers. This implies that the thickness of the atmosphere $\ll$ the radius of the star.

2. Steady State and LTE Pulsations, shocks, etc. are ignored and it is assumed that the transfer equation and atomic occupation numbers do not depend on time.

   For a gas in thermodynamic equilibrium all the microscopic changes occurring will be in 'detailed balance' and the densities of the microscopic states can be predicted using the relations due to Maxwell, Boltzmann and Saha:

$$\begin{aligned}
\text{Maxwell}: \quad N_e(v)\,dv &= N_e\,4\pi v^2 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) dv \\
\text{Boltzmann}: \quad \frac{N_2}{N_1} &= \frac{g_2}{g_1} \exp\left(-\frac{E_{12}}{kT}\right) \\
\text{Saha}: \quad \frac{N_e N_i}{N_o} &= \frac{2U_i}{U_o} \frac{(2\pi mkT)^{3/2}}{h^3} \exp\left(-\frac{E}{kT}\right)
\end{aligned}$$

   A stellar atmosphere is not strictly in thermodynamic equilibrium because there is a net flow of radiation energy outwards (i.e. there is a temperature gradient). The simplifying assumption of Local Thermodynamic Equilibrium (LTE) is made — it is assumed that the occupation numbers for the microscopic states, the opacity (absorption), the emission and, in fact, all the thermodynamic properties of a small volume of material in the atmosphere, are the same as the thermodynamic equilibrium values for the *local* values of the temperature T.

3. Hydrostatic Equilibrium Hydrostatic equilibrium describes the condition where the forces due to the pressure gradient and gravity are balanced. In a plane-parallel model atmosphere the condition of *hydrostatic equilibrium* is given by:

$$\frac{dP}{dz} = -\rho\, g$$

   where P is the total pressure (gas, radiation, etc.), ρ is the density and g is the acceleration due to gravity (called the surface gravity by astrophysicists).

   The radiation pressure can be neglected except for the very hottest stars. Generally, hydrodynamic pressure (due to convection) may also be ignored — for the Sun it is $\sim 1\%$ of the gas pressure.

4. Radiative Equilibrium Through most of the interior and all of the atmosphere of a star there is no energy production by nuclear reactions — only transport of energy. When all the energy is transported by radiation we have what is called *radiative equilibrium*. Convection can generally be ignored - as we'll see in future lectures, stellar atmospheres are never unstable to convection in the photospheric (i.e. near-surface) regions.

   For plane-parallel geometry, radiative equilibrium requires that the total radiative flux is constant with depth:

$$0 = \frac{dF}{dz} = 4\pi \frac{dH}{dz} = 4\pi \int_0^\infty \alpha_\nu [J_\nu - S_\nu]\, d\nu \tag{72}$$

$$\text{so} \quad \int_0^\infty \alpha_\nu\, J_\nu\, d\nu = \int_0^\infty \alpha_\nu\, S_\nu\, d\nu \tag{73}$$

   If LTE also holds, the source function S_ν is equal to B_ν.

5. No External Incident Radiation This assumption is valid except for close binary systems.

6. Semi-Infinite Extent

7. No hydrodynamic or magnetic effects

3.3 The Gray Atmosphere

3.3.1 Introduction

The basic properties of stellar atmospheres can be illustrated using a simple model that is remarkably close to reality. A *gray atmosphere* uses the simplifying assumption that the absorption is independent of frequency (i.e. $\alpha_\nu = \alpha$) with the result that it does not depend on the detailed physics of the state of the gas. In practice, the absorption throughout the most important wavelength regions of moderate temperature stars is dominated by the negative hydrogen ion H^- and its absorption coefficient is almost independent of frequency. Thus, a gray atmospheric model is a much better representation of real stars than might be expected at first sight.

3.3.2 A Gray Model

The analysis for a gray atmosphere, assuming LTE and radiative equilibrium, proceeds as follows:

- Given that α is independent of frequency it follows, from equation (63), that τ is also independent of frequency. The transfer equation (65), integrated over frequency, becomes:

$$\mu \frac{dI}{d\tau} = I - S \tag{74}$$

For radiative equilibrium, since $S = J$ from equation (73), we have:

$$\mu \frac{dI}{d\tau} = I - J$$

- Integration of equation (74) over μ gives the zero order moment of the equation:

$$\frac{dH}{d\tau} = J - J = 0 \qquad \text{and H is constant}$$

- The first order moment of the equation is obtained by multiplying by $\mu \, d\mu$ and integrating:

$$\frac{dK}{d\tau} = H \quad (= \text{constant})$$

$$\text{Hence:} \quad K(\tau) = H\tau + C = \frac{F}{4\pi}\tau + C$$

where C is a constant of integration.

- If we now adopt the Eddington approximation ($3K = J$) and substitute for K we get:

$$J(\tau) = \frac{3F}{4\pi}\tau + 3C$$

- To evaluate the constant C we adopt the approximation that the intensity at the surface is independent of the angle of emergence (i.e. that the radiation field is uniform over the outward facing hemisphere and zero over the inward facing hemisphere) so that $H = J(0)/2$. Also, at the stellar surface, $\tau = 0$.

$$\text{So:} \quad C = \frac{2H}{3} = \frac{F}{6\pi}$$

$$\text{and} \quad J(\tau) = \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right) \tag{75}$$

- For LTE the source function $S = B = \sigma T^4/\pi$ so that for radiative equilibrium ($S = J$) we have $J(\tau) = \sigma T(\tau)^4/\pi$. Recalling the definition of effective temperature, we also have $F = \sigma T_e^4$. Thus, equation (75) can be written:

$$J(\tau) = S(\tau) = \frac{\sigma T(\tau)^4}{\pi} = \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right) = \frac{3}{4}\frac{\sigma T_e^4}{\pi}\left(\tau + \frac{2}{3}\right) \tag{76}$$

(77)

There are a number of points that can be made about equation (77). In a gray atmosphere:

- Equation (77) gives the run of temperature with optical depth.
- The fourth power of the temperature in the atmosphere varies linearly with optical depth.
- The temperature at the surface is given by $T(\tau=0)^4 = T_e^4/2$. Thus the surface temperature is predicted to be $0.841 T_e$—the exact solution of the gray atmosphere problem leads to $0.811 T_e$ (exact solution: not using the Eddington approximation or $H = J(0)/2$)
- At $\tau = 1$ the temperature is given by $T(\tau{=}1)^4 = 5T_e^4/4$ or $T(\tau{=}1) \approx 1.057 T_e$.
- $T(\tau) = T_e$ for $\tau = 2/3$ — in other words, the 'effective depth' of continuum formation is $\tau = 2/3$. A photon emitted at this depth has a probability of exp(-2/3) $\approx$ 0.5 of emerging from the surface.

  Also for $\tau = 2/3$, recalling that flux is constant with depth for radiative equilibrium, substitution in equation (76) gives:

$$F(\tau{=}0) = S(\tau{=}2/3) \qquad \text{- the Eddington-Barbier relation}$$

3.3.3 Limb Darkening

We can calculate the angular dependence of the emergent intensity using equation (67) — the solution to the equation of radiative transfer — which we recall here:

$$I(\tau_1, \mu) = I(\tau_2, \mu)\, e^{-(\tau_2 - \tau_1)/\mu} + \int_{\tau_1}^{\tau_2} S(\tau)\, e^{-(\tau - \tau_1)/\mu}\, \frac{d\tau}{\mu}$$

For our semi-infinite gray atmosphere we set $\tau_1 = 0$ and $\tau_2 = \infty$ and, substituting for $S(\tau)$ from equation (76) we get:

$$\begin{aligned} I(0,\mu) &= \int_0^\infty \frac{3}{4\pi} F \left(\tau + \frac{2}{3}\right) e^{-\tau/\mu}\, \frac{d\tau}{\mu} \\ I(0,\mu) &= \frac{3}{4\pi} F \left(\mu + \frac{2}{3}\right) \qquad (78) \end{aligned}$$

Equation (78) shows that the intensity decreases from the centre of the stellar disk, where $\mu = 1$, to the edge or limb where $\mu = 0$. This phenomenon is known as limb darkening and the ratio $I(0,\mu)/I(0,1)$ gives the relative intensity from the centre to the limb of the disk:

$$\frac{I(0,\mu)}{I(0,1)} = \frac{3\mu + 2}{5}$$

which predicts that the star will have an intensity at its limb of only 40% of its central intensity. Solar observations in the visual part of the spectrum are in good agreement with this prediction.

3.4 The Construction of a Model Stellar Atmosphere

Generally the absorption coefficient depends on frequency and it follows that, for a given geometrical depth, the optical depth also depends on frequency. The problem of constructing a model atmosphere is now more complicated because we have to evaluate the absorption coefficient as a function of frequency before the equation of transfer can be solved.

Here we will look briefly at the equation of hydrostatic equilibrium and then summarise the steps involved for an atmosphere based on the assumptions listed in Section 3.2.

The equation of hydrostatic equilibrium gives the total pressure gradient as:

$$\frac{dP}{dz} = -\,\rho g$$

$$\text{or} \qquad \frac{dP}{d\tau} = \frac{\rho g}{\alpha_{\nu_s}} = \frac{dP_g}{d\tau} + \frac{dP_r}{d\tau}$$

where α_{ν_s} is the absorption coefficient at some standard frequency (or is some mean value) and P_g and P_r are the gas and radiation pressures respectively.

NOTE that $dP_r/d\tau$ is only important for small g, large α_{ν_s} or high T_e and that it is given by:

$$\frac{dP_r}{d\tau} = \frac{1}{c}\int_0^\infty \alpha_\nu\, F_\nu\, \frac{d\nu}{\alpha_{\nu_s}}$$

For $dP_r/d\tau$ negligible we can write:

$$\frac{dP_g}{d\tau} = \frac{\rho g}{\alpha_{\nu_s}}$$

For an ideal gas we have $P_g\, V = RT$ and also $\rho = \mu/V$, where μ is the mass of one mole of gas. Thus, $\rho = \mu P_g/RT$ and the equation of hydrostatic equilibrium can be written:

$$\frac{dP_g}{d\tau} = \frac{\mu P_g g}{RT\alpha_{\nu_s}}$$

$$\text{or} \quad \frac{d(\ln P_g)}{d\tau} = \frac{\mu g}{RT\alpha_{\nu_s}} \tag{79}$$

The steps involved in constructing a model atmosphere are:

1. Adopt an initial temperature distribution $T(\tau)$ where τ is the optical depth at the chosen standard frequency.

2. Integrate the hydrostatic equilibrium equation (28) step by step starting from $\tau \ll 1$ to obtain the run of gas pressure P_g with depth. The electron pressure P_e is then found as a function of depth.

3. Compute other physical variables such as $\rho(\tau)$ and $\alpha_\nu(\tau)$ since, with the assumption of LTE, they depend only on $T(\tau)$ and $P_e(\tau)$.

4. Construct the optical depth scale at each frequency from:

$$\tau_\nu(\tau) = \int_0^\tau \frac{\alpha_\nu(\tau)}{\alpha_{\nu_s}(\tau)}\, d\tau$$

5. The source function $S_\nu(\tau_\nu)$ can now be found as a function of optical depth τ_ν from $\alpha_\nu(\tau_\nu)$ — we will return to this in a later lecture — and so the equation of transfer can be solved.

   $J_\nu(\tau_\nu)$ can be obtained using equations (7), (19) and (20) and $F_\nu(\tau_\nu)$ from equations (6), (19) and (20).

6. With the fluxes at each frequency and depth known, we can check for radiative equilibrium — that the radiative flux is constant with depth.

$$\text{We require:} \qquad \frac{d\left(\int_0^\infty F_\nu\, d\nu\right)}{dz} = \frac{dF}{dz} = 0$$

   If the model is to have a specified effective temperature T_e we can express the condition of radiative equilibrium by requiring that:

$$\int_0^\infty F_\nu\, d\nu = \sigma\, T_e^4$$

   is the same at all points in the atmosphere.

   Generally this will not be true for the temperature distribution assumed initially and it is necessary to invoke an iterative procedure, based on the differences between the desired values of the integrated flux F and its derivative $dF/d\tau$ and the computed values, in order to achieve radiative equilibrium.

The iterations are continued until satisfactory accuracy is obtained. Basically the problem comes down to establishing the correct $T(\tau)$ distribution when LTE is assumed.

3.5 Some Observational Tests of the Predictions of Model Stellar Atmospheres

3.5.1 Flux Distributions

Observations of the flux distribution received from a star may be compared with the predictions of theoretical model atmospheres in basically two ways — relative and absolute.

1. Relative Flux Distributions

If the flux received at the Earth (corrected for extinction by interstellar material and the Earth's atmosphere) is represented by f_λ then photo-electric photometry using narrow spectral bands yields only the relative flux f_λ/f_{λ_o} (or f_ν/f_{ν_o}) where λ_o is some standard wavelength.

Before the observed relative flux distribution can be compared with the predicted relative flux distribution of a model it is necessary to select a suitable model — based on estimates of the star's effective temperature, surface gravity and chemical composition. The problem with this approach is that the effective temperatures are generally known with relatively poor accuracy and the comparison does not provide a very good test of the model.

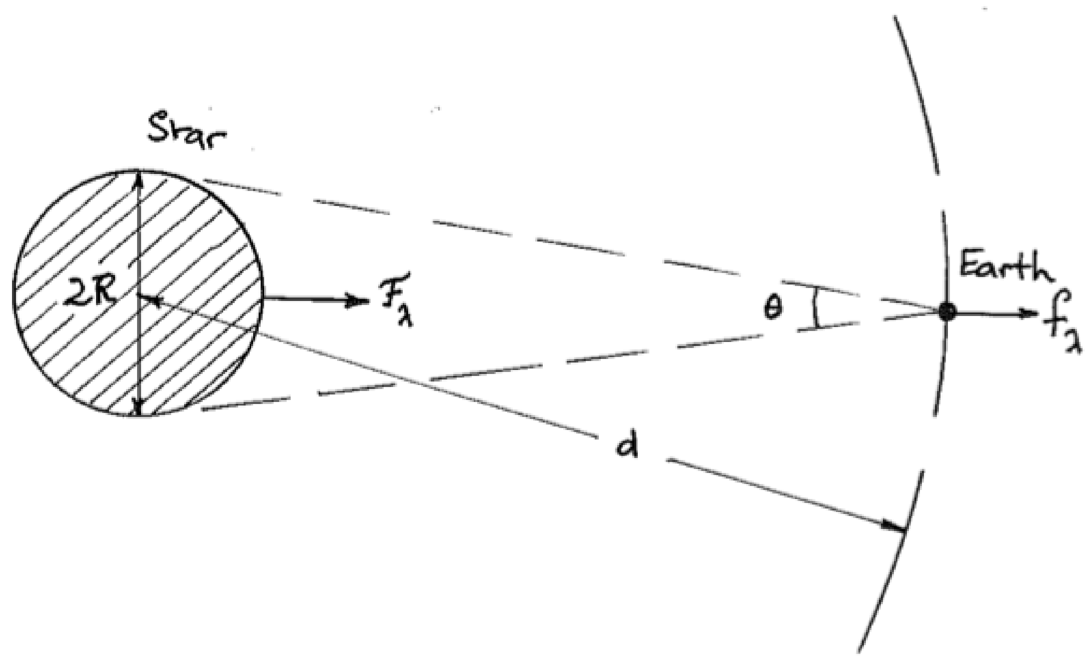
An alternative approach is to compare the observed relative flux distribution with the predicted distributions of a range of models and thereby identify a particular model with the star. This does not test the model but it is a method for obtaining an estimate of the effective temperature of the star. Estimates obtained in this way are only as good as the model.

2. Absolute Flux Distributions

It is necessary to obtain the absolute flux distribution at the surface of a star from observational data if a stringent test is to be applied to the continuum predictions of a model stellar atmosphere. This requires two quite distinct measurements, both of which are technically difficult to make:

- The *absolute* spectral distribution of flux over the entire spectrum, corrected to outside the Earth's atmosphere and for interstellar extinction.
- The angular diameter of the star.

Given these data it is possible to compute the absolute spectral flux distribution at the surface of the star and hence obtain an empirical determination of the effective temperature of the star.



From the diagram we have:

$$4\pi\, R^2\, F_\lambda = 4\pi\, d^2\, f_\lambda$$

where f_λ is the flux at the Earth in W.m^{-2}.m^{-1}.

$$\text{so:} \qquad F_\lambda = \left(\frac{d}{R}\right)^2 f_\lambda = \left(\frac{4}{\theta^2}\right) f_\lambda$$

From the definition of effective temperature T_e:

$$\sigma\, T_e^4 = \int_0^\infty F_\lambda\, d\lambda = \left(\frac{4}{\theta^2}\right) \int_0^\infty f_\lambda\, d\lambda$$

The empirical value of T_e allows a particular model to be identified with the star. The empirical and theoretical distributions of F_λ can then be compared directly and the reasons for any discrepancies sought. The attached paper, even though it is now old, illustrates the results that can come from this type of comparison (model atmospheres have undergone major improvements since this paper was published).

One of the major programs for SUSI is to carry out this type of research for stars of all spectral types and luminosities. This research will also lead to empirical temperature scales.

3.5.2 Limb Darkening

In principle the limb-darkening law can be studied using observations of binary star eclipses and using observations made with a stellar interferometer. They are difficult measurements but again it is a potential SUSI program.

3.6 Model Atmospheres and their Limitations

The limitations of model stellar atmospheres are primarily due to the simplifying assumptions and approximations that are made in their construction. Some of the limitations have already been mentioned but they are summarised here.

<u>Plane Parallel Geometry</u>

This assumption is adequate for thin, high surface gravity cases but not for luminous, low surface gravity stars in which the atmosphere is extended and curvature is not negligible. Models computed for spherical geometry show differences for outer layers and in the predicted spectra.

The assumption that the layers are homogenous obviously is not true for the Sun although it is a reasonable one to make as the convective cells are small. However, for cool stars the convective cells are relatively large and there is evidence for large hot spots.

<u>Steady State and LTE</u>

The assumption of steady state holds for single, stable stars but obviously is not valid for pulsating stars, components of close binary systems, atmospheres with variable magnetic fields etc.

LTE is an adequate assumption for the relatively dense atmospheres of stars with high surface gravity in which collisions dominate. Non-LTE effects become important for extended, low surface gravity atmospheres.

The general approach for constructing non-LTE models is similar to that for LTE models but is more complex. It involves specifying occupation numbers, temperatures and radiation fields so that they satisfy the requirements of steady state and energy equilibrium in a self-consistent way.

Hydrostatic Equilibrium

In the foregoing discussion of model atmospheres it was assumed that gravity was balanced by the gradient of the gas pressure alone. It was noted that radiation pressure was only significant for atmospheres with low surface gravity, high extinction coefficient or high effective temperature. In other words it is important for giant, supergiant and early-type stars (10% at 30 000 K).

Generally, hydrodynamic pressure due to convection can be ignored - for the Sun it is $\sim$ 1% of the gas pressure.

Radiative Equilibrium

Radiative equilibrium prevails for hot stars but convective energy transport in the outer layers becomes increasingly important as we move to cooler stars - it first becomes significant about spectral type F and occurs in increasingly thick and deeper zones for K and M stars. However, the surface layers are always stable to convection, so the biggest influence of the deep ($\tau > 5$ or so) convection on the emergent intensity is to cover the surface of the star in regions of different effective temperature.

4 Radiative Processes

As we have seen, in order to construct a model stellar atmosphere we need to evaluate the opacity as a function of frequency and depth in the atmosphere.

Spectral lines are formed by essentially the same processes as the continuum except that they correspond to bound-bound transitions. The absorption coefficients for spectral lines are larger than for the continuum with the result that spectral lines are formed high in the atmosphere — in regions of lower temperature. In these circumstances they appear as absorption lines.

The simplest possible case of an atomic radiative process is the *two-level atom*, which is represented in Figure 4. This is also representative of transitions between two states in an ion or molecule.

The two levels have energies E_2 and E_1 respectively, degeneracies g_2 and g_1 and the number of atoms in each state is n_2 and n_1 The energy of the transition is $E_{21} = E_2?E_1 = h\nu_{21}$. Transitions between these levels can be achieved by the emission or absorption of a photon with frequency ?21 (taking the atom from the upper state to the lower, or from the lower to the upper respectively), or by collisions with other atoms, ions or molecules or electrons present in the gas. Of course there are in general other transitions, e.g. from level 1 to 3, say, or level 12 to 9. The competition between collisional and radiative processes between all of the energy levels in the atom determines the relative numbers of atoms in each level (the level population), and therefore the macroscopic absorption and emission coefficients of the medium.

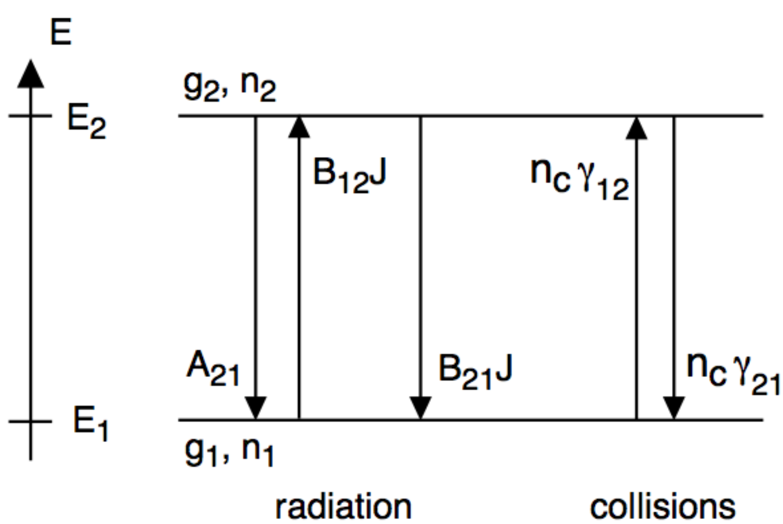


Figure 1: Transitions between two energy levels in an atom occur either by the emission or absorption of a photon or by collisions with other particles.

4.1 Scattering

For many radiative processes, the assumption of local thermodynamic equilibrium is not valid. The simplest non-LTE process is scattering, where the direction of a photon is changed without its energy being changed. Physically, this could relate to scattering off electrons (Thompson scattering), molecules (Rayleigh scattering), particles (dust scattering) or even as an approximation to a two-level atom with negligible collisions.

The extinction coefficient can conveniently be divided into non-scattering (true absorption) (a) and scattering (s) parts:

$$\alpha_\nu = \alpha_{\nu,a} + \alpha_{\nu,s} \tag{80}$$

In true absorption processes the photon energy is converted into kinetic energy of the gas. On the other hand, in scattering processes the photons are not 'destroyed' but are redistributed in angle and, in general, in frequency. Ignoring the redistribution in frequency, we can write the source function as:

$$S_\nu = \frac{\alpha_{\nu,a} B_\nu + \alpha_{\nu,s} J_\nu}{\alpha_\nu} \tag{81}$$

This is clearly not as simple as the LTE assumption. Where multi-level atoms and frequency re-distribution within spectral lines are also taken into account, the situation becomes more complex. We do not need to understand how to model these situations, but do need to understand where the LTE assumption is and is not valid in radiative transfer.

4.2 Einstein Coefficients

[Thanks to Mark Wardle for this and the following few sections]

Kirchoffs law indicates a relation between a bodys emission and absorption properties, suggesting some relationship between the processes at an atomic level. There are three emission and absorption processes:

1. Spontaneous emission: an atom in level 2 has a probability per unit time A_{21} of emitting a photon and dropping to level 1. A21, the Einstein A coefficient, depends only on the properties of the atom.

2. Absorption: an atom in level 1 has a probability per unit time B12J? of absorbing a photon. The probability depends on atomic properties (through B12) and the direction-averaged intensity of the incident radiation averaged over the line profile:

$$\bar{J} = \int J_\nu \phi(\nu) d\nu. \tag{82}$$

   Here $\phi(\nu)$ is the *line profile function* which satisfies $\int_0^\infty \phi(\nu) d\nu$ and accounts for the intrinsic linewidth and Doppler broadening of the line due to thermal motion of the atoms.

3. Stimulated emission: An atom in state 2 is induced to emit a photon by an incident photon at the transition frequency with a probability ? per unit time $B_{21}\bar{J}$. Stimulated emission has an intriguing property – the emitted photon is coherent with the stimulating photon – i.e. it travels in the same direction and has the same phase. B_{12} and B_{21} are the *Einstein B coefficients*.

Spontaneous emission and absorption are fairly intuitive, whereas the introduction of stimulated emission may seem arbitrary. However, we demonstrate that it exists by deriving the relationships between the three Einstein coefficients and showing that B_{21} is nonzero.

To derive the relationships we assume thermodynamic equilibrium. This constrains the level population and radiation field, but does not affect the three Einstein coefficients, which depend only on the internal physics of the atom and not on external factors such as the radiation field, density or temperature unless these are sufficiently extreme to interfere with the internal structure of the atom. We also neglect the effect of collisions. This may seem strange given that we are assuming thermodynamic equilibrium – but as a thought experiment, this could e.g. be a very low density gas in a black box of uniform temperature.

In thermal equilibrium at temperature T the relative numbers of atoms in state 2 and 1 is:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp(-\frac{E_{21}}{kT}) \tag{83}$$

and $J_\nu = B_\nu(T)$. In addition, because the level populations are in equilibrium the rate of transitions from 2 to 1 and vice-versa must be equal:

$$n_1 B_{12}\bar{J} = n_2 A_{21} + n_2 B_{21}\bar{J} \tag{84}$$

and thus

$$\bar{J} = \frac{g_2 A_{21}}{g_1 B_{12}\exp(E_{21}/kT) - g_2 B_{21}} = B_\nu(T). \tag{85}$$

?The second equality above follows because in thermodynamic equilibrium the radiation field is black body. We have neglected the effect of the line profile as it is usually narrow and $B_\nu(T)$ can be regarded as constant over the relevant frequency range. For the second equality to hold true, we have:

$$g_1 B_{12} = g_2 B_{21}, \text{and} \tag{86}$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}. \tag{87}$$

These relationships must hold independently of the assumption of thermodynamic equilibrium as they are properties of an individual atom; they therefore hold in general. Note that $B_{21} \neq 0$, so stimulated emission is necessary. The style of argument we have used here is an example of applying the principle of detailed balance of forward and reverse processes in thermodynamic equilibrium to relate the rates for processes and their inverses more generally.

4.3 Line profiles

Two classes of effect contribute to the line profile $\phi(\nu)$, which is in general a combination of all effects.

First, quantum mechanics implies that any state with a finite lifetime Δt has a corresponding broadening of its energy ΔE determined by Heisenbergs Uncertainty Principle, $\Delta E \Delta t = h/2\pi$. This intrinsic line width is roughly given by $\Delta\nu = \Delta E/h = A_{21}/2\pi$. More precisely, the line profile function for intrinsic broadening is a Lorentzian:

$$\phi_L(\nu) = \frac{\Gamma}{4\pi^2(\nu - \nu_{21})^2 + \frac{1}{4}\Gamma^2}, \tag{88}$$

where in the case of natural broadening only for a 2-level atom, we have $\Gamma = A_{21}$. In general, the Einstein-A coefficients for all transitions from the upper and lower state affect the line profile, as does collisions, leading to a more general form of the Lorentzian width parameter Γ:

$$\Gamma = \Sigma A_{mn} + \gamma_p, \tag{89}$$

where the sum is over all transitions, and γ_p is a *pressure broadening* parameter, which is proportional to the number density of colliding particles. The simplest kind of pressure broadening is impact pressure broadening (e.g. by collisions with fast moving electrons), which reduces the effective lifetime of the upper state.

Second, the motion of atoms in the medium contributes to broadening through the Doppler effect. For speeds $v \ll c$, the Doppler effect produces a change in frequency $\delta\nu/\nu = -v/c$ which is positive (blueshifted) for motion towards the observer and negative (redshifted) for motion away from the observer, where v is the velocity component parallel to the line of sight (positive = moving away from the observer). There are, in principle, three different contributions. Thermal broadening due to the random thermal motion of atoms of mass m at temperature T gives a Gaussian profile

$$\phi_D(\nu) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(\nu - \nu_{21})^2}{2\sigma^2}) \text{ where } \sigma = \frac{\nu_{21}}{c}(\frac{kT}{m_a})^{1/2} \tag{90}$$

with the width determined by the typical thermal speed of an atom with mass m_a. Doppler broadening also occurs due to small scale turbulent or convective motion or other velocity gradients in the medium, called *microturbulence*. Often this produces a Gaussian or near-Gaussian profile, but the width is not determined by T but by the typical size of the random turbulent velocity in the medium. Finally, there is an overall frequency shift because of the average motion of the medium either away or towards the observer.

The total linewidth is a convolution of these profiles ϕ_L and ϕ_D, called a *Voigt profile*. The intrinsic width is usually very small, and a Gaussian profile generally serves as a good approximation except at frequencies several widths away from the line centre, where the $(\nu - \nu_{21})^{-2}$ behaviour of of the Lorentzian dominates the rapid decay of the Gaussian. This becomes important only for very optically thick lines, or for lines that have a great deal of pressure broadening (e.g. in dense atmospheres of brown dwarfs).

4.4 Emission and Absorption Coefficients

Armed with the Einstein coefficients, we can now relate the numbers of atoms in states 1 and 2 per unit volume, n_1 and n_2, to the macroscopic emission and absorption coefficients j_ν and α_ν.

First, consider spontaneous emission. The energy emitted per unit time per unit freuquency interval per unit volume is $n_2 A_{21} h\nu\phi(\nu)$. Assuming that on average the photons are emitted isotropically – usually a very good approximation as we are considering a large number of atoms with random orientations – the spontaneous emission coefficient is:

$$j_\nu = \frac{n_2 A_{21}}{4\pi} h\nu\phi(\nu). \tag{91}$$

Although one might be tempted to attempt to include stimulated emission ?with spontaneous emission, it turns out to be more convenient to combine it with absorption as both processes depend on J. This leads to an absorption coefficient that is the sum of terms for true absorption (the physical absorption of photons) and stimulated emission, the total being referred to as absorption corrected for stimulated emission, or often just as absorption.

The probability that a photon will be absorbed per unit volume per unit time is $n_1 B_{21} \bar{J} = n_1 B_{12} \int J_\nu \phi(\nu) d\nu$. However, we are interested in the absorption of an intensity I? in a particular direction, which contributes $I_\nu d\Omega/4\pi$ to J_ν. So imagine a small volume $dAds$ where dA and ds are perpendicular and parallel to the direction of propagation under consideration. The energy incident on this volume per unit time is $I_\nu dA d\nu d\Omega$, and the energy absorbed per unit time by true absorption within the volume is

$$(n_1 ds dA)\left(B_{12} I_\nu \frac{d\Omega}{4\pi}\phi(\nu) d\nu\right) h\nu. \tag{92}$$

The first term is the number of atoms in the lower state in the volume, the second factor is the probability per atom per unit time of absorbing a photon in frequency interval $d\nu$ propagating into $d\Omega$.

Meanwhile, stimulated emission within the volume returns energy in the form of photons with frequency in $d\nu$ and into $d\Omega$ at a rate

$$(n_2 ds dA)\left(B_{21} I_\nu \frac{d\Omega}{4\pi}\phi(\nu) d\nu\right) h\nu \tag{93}$$

The net energy removal is the difference of these terms, and therefore the absorption coefficient is

$$\alpha_\nu = (n_1 B_{12} - n_2 B_{21}) \frac{h\nu}{4\pi}\phi(\nu). \tag{94}$$

The first and second terms inside the parentheses represent true absorption and the correction for stimulated emission respectively. We can now compute the general source function for this 2-level atom directly from Equations 91 and 94, using relationships 87 and 86 to simplify the equation:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{2h\nu^3}{c^2}\left(\frac{n_1 g_2}{n_2 g_1} - 1\right)^{-1}. \tag{95}$$

This equation depends only on the level populations. If the level populations are populated according to Boltzmann's law, then $S_\nu = B_\nu(T)$ (an exercise for you!) and we obtain a microscopic version of Kirchoff's law. It is quite possible for the population to be Boltzmann even when strict thermodynamic equilibrium does not apply – for example, if collisions determine the level population (see later) the radiation field need not be a black-body. More generally, when LTE does not hold, it is useful to define an excitation temperature $T_{\rm ex}$ for a particular transition via:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left(-\frac{E_{21}}{kT_{\rm ex}}\right), \tag{96}$$

then $S_\nu = B_\nu(T_{\rm ex})$.

4.5 Collisions

So far we have concentrated on the role that radiation plays in an atom making transitions from level 1 to level 2. In this section we discuss collisions, which may also excite or de-excite an atom.

Consider collisions of an atom A with a collision partner B (such as an electron, for example). A collision with centre of mass kinetic energy E_{K1} may take A from state 1 to state 2, in which case the centre-of-mass energy is $E_{K2} = E_{K1}?E_{21}$. (Note that here E_{K1} and E_{K2} do not refer to the energies of states 1 and 2, but E_{21} is the difference between them.)

Typically, the chance of this happening is expressed in the form of a cross section $\sigma_{12}(v)$ which depends on the relative speed of the collision partners. The centre-of-mass energy is $\frac{1}{2}mv2^2$ where $m = m_A m_B/(m_A + m_B)$ is the reduced mass of the partners. The probability per unit time of a collision leading to an excitation is $nB\sigma_{12}(v)v$ averaged over the distribution of relative velocities. For a Maxwellian velocity distribution function in a medium with kinetic temperature T, this leads to the collisional rate coefficient for excitation:

$$\langle\sigma v\rangle_{12} = \left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty \sigma_{12}(v)v4\pi v^2 e^{-mv^2/2kT}v \tag{97}$$

Often this is denoted by γ_{12}. The probability per unit time for an atom in state 1 to be collision ally excited into state 2 is then $n_B\gamma_{12}$.

Similarly, the reverse process – *collisional de-excitation* of atoms in state 2 to state 1 – may occur, with a corresponding cross section σ_{21} and rate coefficient $\langle\sigma v\rangle_{21}$. In an ionised medium, electrons are usually the most effective collision partners as they are abundant and have a high thermal speed. The rate coefficient for electron collisions is often parameterised in terms of a downward *collision strength* Ω_{21}, via

$$\langle\sigma v\rangle_{21} = \frac{h^2}{(2\pi m_e)^{3/2}(kT)^{1/2}}\frac{\Omega_{21}}{g_2}. \tag{98}$$

The reason for this parameterisation is that it turns out that the downward collision strength is roughly independent of T and often is of order unity.

As usual, the cross sections and rate coefficients for excitation and the reverse process, deexcitation, are related by simple expressions that can be obtained using the principle of detailed balance. Thus one need only calculate or measure the rate coefficient for one of the processes. Here, we just quote the result:

$$\langle \sigma v \rangle_{12} = \frac{g_2}{g_1} \langle \sigma v \rangle_{21} \exp\left(-\frac{E_{21}}{kT} \right). \tag{99}$$

4.5.1 Example: A two-level atom with no radiation field.

To illustrate the role collisional processes play in determining the level popu- lations we consider a model atom containing only two levels, 1 and 2. In this section we assume that the radiation field is negligible, so that transitions oc- cur via collisional excitation and deexcitation or spontaneous emission. The total number of atoms, $n_1 + n_2$ is constant, so we can restrict our attention to the atoms in state 2, whose population varies according to:

$$\frac{dn_2}{dt} = n_c \gamma_{12} n_1 - n_c \gamma_{21} n_2 - A_{21} n_2. \tag{100}$$

Steady state is usually achieved rapidly compared to the time scales on which conditions in the medium change. In this case, $dn_2/dt = 0$ and the levels are populated according to

$$\frac{n_2}{n_1} = \frac{\gamma_{12} n_c}{\gamma_{21} n_c + A_{21}} \tag{101}$$

In the limit $n_c \rightarrow 0$, $n_2/n_1 \rightarrow 0$. In this case collisions are rare and atoms in state 2 spontaneously decay by emitting a photon.

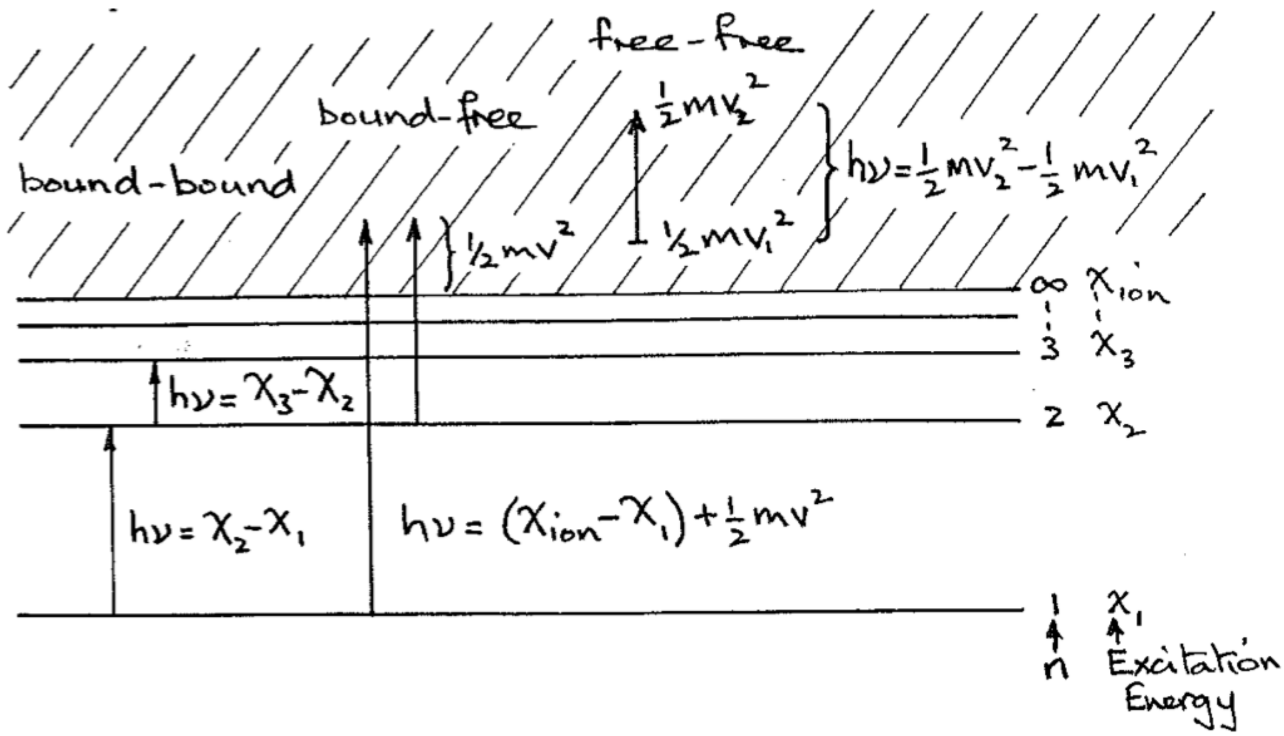
In the opposite limit, $n_c \rightarrow \infty$, $n_2/n_1 \rightarrow \gamma_{12}/\gamma_{21} = g_2/g_1 \exp(-E_{21}/kT)$ and the levels are populated according to the Boltzmann distribution. Physically, this occurs because the Maxwellian velocity distribution of the colliding species brings the level populations into thermal equilibrium if the collisions are sufficiently rapid – i.e. at high densities, where $n_c \gg A_{21}/\gamma_{21}$. This motivates the definition of the critical density for the transition,

$$n_{\rm crit} = \frac{A_{21}}{\gamma_{21}} \tag{102}$$

at which collisional de-excitation and spontaneous decay of the upper level occur at the same rate.

4.6 The Continuous Absorption Coefficient

4.6.1 Absorption Processes



- Bound-free transitions

$$h\nu = (\chi_{ion} - \chi_n) + \frac{1}{2}mv^2$$

Continuous absorption for frequencies given by $h\nu > (\chi_{ion} - \chi_n)$.

- Free-free transitions

$$h\nu = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Continuous absorption at all frequencies.

The details of the determination of the bound-free and free-free absorption coefficients are outside the scope of this course but we will outline what is involved.

To compute the total bound-free absorption at frequency ν we have to:

Multiply the absorption cross-section (a_n) at frequency ν by the number of atoms in the state of quantum number n and then sum over all the states that contribute at the given frequency ν.

For the hydrogen atom a quantum mechanical analysis leads to the following expression for the cross-section for continuous absorption from level n:

$$a_n = \frac{64\,\pi^4}{3\sqrt{3}}\,\frac{m\,e^{10}}{c\,h^6}\,\frac{1}{n^5\,\nu^3}\,G$$

where G is known as the Gaunt factor and is close to unity. This version of the formula is only valid in cgs units, where $4\pi\epsilon_0 = 1$ by definition. A more modern way to write the formula includes the dimensionless fine structure constant α, the absorbed photon frequency $\nu_n = 2\pi^2 me^2/h^3n^2$ and the Bohr radius a_0:

$$a_n = (\frac{64\,\pi n}{3\sqrt{3}Z^2})\,\alpha a_0^2(\frac{\nu_0}{\nu})^3\,G$$

Thus, the bound-free absorption from level n commences abruptly at frequency

$$\nu_n = \frac{(\chi_{ion} - \chi_n)}{h}$$

and falls off as ν^{-3} at higher frequencies.

For the helium ion He^+ the cross-section is larger by Z^2 (=4) but the excitation energies and the ionisation energy are also larger by a factor of four. The discontinuities in the continuous absorption coefficient are shifted to higher frequencies by a factor of four (see Figure 5.2).

In order to find the number of atoms in different quantum states - the *occupation numbers* - we must use the Boltzmann and Saha equations.

<u>The Boltzmann Equation</u>

In thermodynamic equilibrium, at temperature T, the relative populations of the ground level (N_1) and the nth level (N_n) is given by

$$\frac{N_n}{N_1} = \frac{g_n}{g_1}\,e^{-(\chi_n-\chi_1)/\,kT} = \frac{g_n}{g_1}\,e^{-h\nu/\,kT} \tag{103}$$

where g_n and g_1 are the statistical weights for the levels.

<u>The Saha Equation</u>

The relative number of ions to neutral atoms is given by

$$\frac{N_i\,n_e}{N} = \frac{U_i}{U}\,2\left[\frac{2\pi mkT}{h^2}\right]^{\frac{3}{2}}\,e^{-(\chi_{ion}-\chi_1)/\,kT} \tag{104}$$

where	N_i	is the number of singly ionised atoms (summed over all excitation levels)/ unit volume
	N	is the number of atoms (summed over all excitation levels)/ unit volume
	n_e	is the number of free electrons/ unit volume
	U_i and U	are known as the *partition functions* for the ionisation states and are given by:

$$U = g_1 + \sum_{n=2}^{\infty} g_n \, e^{-(\chi_n - \chi_1)/\,kt}$$

Equation (31) can be re-written to represent successive states of ionisation

$$\frac{N_{i+1}\, n_e}{N_i} = \frac{U_{i+1}}{U_i} \, 2 \left[\frac{2\pi mkT}{h^2}\right]^{\frac{3}{2}} \, e^{-\chi^*/\,kT} \tag{105}$$

where χ^* is the ionisation energy of the ith ion.

<u>Example</u>

Consider hydrogen atoms in the solar photosphere with $T \approx 6000$ K and electron pressure $p_e \approx 3$ Pa.

Calculate the fraction of the atoms in the $n = 2$ state and the fraction that is ionised.

Note that the ionisation potential (χ_{ion}) is 13.6 eV; the excitation potential for the $n = 2$ level is 10.2 eV; and, since $g_n = 2n^2$, we have $g_1 = 2$ and $g_2 = 8$.

Substitution in the Boltzmann equation gives:

$$\frac{N_2}{N_1} \simeq 1.1 \times 10^{-8}$$

i.e. Only ~ 1 atom in 10^8 is in the $n = 2$ state at 6000 K.

The Saha equation can be re-written using $p_e = n_e kT$ as:

$$\frac{N_i}{N} = \frac{kT}{p_e} \frac{U_i}{U} \, 2 \left[\frac{2\pi mkT}{h^2}\right]^{\frac{3}{2}} \, e^{-(\chi_{ion} - \chi_1)/\,kT}$$

With $U(H) = 2$ and $U(H^+) = 1$ (since the proton can only exist in one form $g_p = 1$ and $U_p = 1$ and only g_1 is significant in evaluating $U(H)$):

$$\frac{N_{H^+}}{N_H} \simeq 1.2 \times 10^{-4}$$

i.e. Only ~ 1 hydrogen atom in 10^4 is ionised.

4.6.2 The Rosseland Mean Opacity

In practice, taking into account all these processes in detail is very complex, and can only be done "properly" in surface layers. What we really want is to reduce the monochromatic transfer equation, when integrated over frequency, to the same form as the gray equation. There are several ways of defining a mean opacity but Rosseland mean opacities are used in stellar interiors and in the deeper layers of stellar atmospheres.

If we take the first moments of the transfer equation we have

$$\text{for the non-gray case:} \qquad -\frac{1}{\alpha_\nu}\frac{dK_\nu}{dz} = H_\nu$$

$$\text{and, for the gray case:} \qquad -\frac{1}{\alpha}\frac{dK}{dz} = H$$

If we now wish to ensure that $\int_0^\infty H_\nu \, d\nu = H$ then we must have

$$-\int_0^\infty \frac{1}{\alpha_\nu}\frac{dK_\nu}{dz}\, d\nu = \int_0^\infty H_\nu \, d\nu = H = -\frac{1}{\overline{\alpha}}\frac{dK}{dz}$$

$$\text{or} \qquad \frac{1}{\overline{\alpha}} = \frac{\displaystyle\int_0^\infty \frac{1}{\alpha_\nu}\frac{dK_\nu}{dz}\, d\nu}{\displaystyle\int_0^\infty \frac{dK_\nu}{dz}\, d\nu}$$

K_ν is not known so we have to introduce an approximation. We use the Eddington factor ($K_\nu/J_\nu = 1/3$) and $J_\nu = B_\nu$ both of which hold deep in the atmosphere (and in the interior of the star) and so

$$\frac{dK_\nu}{dz} \approx \frac{1}{3}\frac{dB_\nu}{dT}\frac{dT}{dz}$$

We define the *Rosseland mean opacity* $\overline{\alpha_R}$ by

$$\frac{1}{\overline{\alpha_R}} = \frac{\displaystyle\frac{1}{3}\frac{dT}{dz}\int_0^\infty \frac{1}{\alpha_\nu}\frac{dB_\nu}{dT}\, d\nu}{\displaystyle\frac{1}{3}\frac{dT}{dz}\int_0^\infty \frac{dB_\nu}{dT}\, d\nu}$$

$$\text{or} \qquad \frac{1}{\overline{\alpha_R}} = \frac{\displaystyle\int_0^\infty \frac{1}{\alpha_\nu}\frac{dB_\nu}{dT}\, d\nu}{\displaystyle\frac{dB}{dT}} = \frac{\pi}{4\sigma T^3}\int_0^\infty \frac{1}{\alpha_\nu}\frac{dB_\nu}{dT}\, d\nu$$

Note that the weighting process gives the highest weight to the regions with lowest opacity (smallest α_ν) and the flux is largest in these regions.

Mean opacities are essential for modelling the interiors of stars, and are useful in providing a first estimate of the temperature distribution in a stellar photospheres.

5 Opacity Sources and Applications

5.1 The Major Sources of Continuous Absorption

The sources of continuous absorption in stellar photospheres differ with spectral type and the following listing gives the major sources in the visible part of the spectrum for the different spectral classes.

Spectral Type	Major Sources of Continuous Absorption
O	He, He^+ and Thomson scattering by free electrons
B & early A	H and He
A & F	H
G & K	H^-
M	H^-, molecular absorption (TiO in visible, H_2O and CO in infrared) and Rayleigh scattering

In addition to the sources listed above the effects of atomic spectral line absorption can be significant. In the lecture, I also discussed the effects of further molecular and atomic absorption in L and T dwarf atmospheres.

5.2 Scattering Processes

Thomson Scattering

Thomson scattering is scattering of photons by free electrons with scattering cross-section σ_T given by

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_o mc^2} \right)^2$$

Thomson scattering is independent of frequency and is only important for hot stars (O stars and B supergiants) where it becomes the principal source of opacity. In the absence of nuclei, conservation of energy and momentum means that sub-relativistic electrons can not significantly change the energy of photons that they interact with. In the rest-frame of the electron, the momentum given to the electron by the photon is $h\nu/c$. This is insignificant whenever the photon energy hf is much smaller than the rest-mass energy of the electron $m_e c^2$. Equally importantly, in a scattering process where a photon is thought of as recoiling off an electron, the typical blue-shift or red-shift is $\sqrt{2kT/m}/c$, which is $\sim 3 \times 10^{-3}$ for the hottest (T$\sim$30 000 K) stars. Treating this process as separate from LTE absorption is the simplest kind of non-LTE treatment in stellar atmospheres.

Rayleigh Scattering

Rayleigh scattering is scattering of photons by atoms and molecules at frequencies much lower than their characteristic transition frequencies. The classical scattering cross-section σ_R is given by

$$\sigma_R \approx \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_o mc^2}\right)^2 \left(\frac{\omega}{\omega_o}\right)^4 \propto \frac{1}{\lambda^4}$$

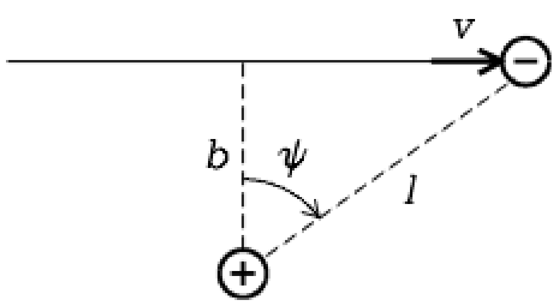
where ω_o represents a resonant frequency.

Rayleigh scattering can be important for stars of moderate temperature — spectral types G and K. It also becomes increasingly important for cool stars at high frequencies (i.e in the ultraviolet).

Thomson and Rayleigh scattering are not strictly isotropic and there will also be some frequency redistribution due to Doppler shifts resulting from the motions of the scattering particles. In practice the processes can generally be taken to be isotropic and, in the continuum, the effects of frequency redistribution can generally be neglected and the scattering taken to be coherent.

5.3 Free-free Opacity

Free-free opacity (thermal Bremsstrahlung) describes the interaction between photons and electrons in the presence of nuclei. In the presence of nuclei, electrons can change there energy state. This is well-known for bound states of atoms, and it applies equally well to unbound states. A schematic of the low energy limit of these interaction is as follows:



This interaction emits a spectrum that is roughly flat up to a frequency of v/b, where v is the electron velocity and b the impact parameter. The sum of these contributions ends up having an opacity that is proportional to ν^{-2}. Although this is very important in the radio, it it is not important at wavelengths most relevant to energy transport in stellar atmospheres.

5.4 Bound-free Opacity

The peak in the Rosseland mean opacity plot versus temperature (previous lecture) is dominated by bound-free opacity. In a very qualitative sense, a free electron can only couple to another free electron state when it is in the vicinity of a nucleus. However, a bound electron is *always* in the vicinity of a nucleus, so bound-free opacity is clearly more important than free-free opacity when photons have enough energy to ionize atoms. For stars at least as metal-rich as the sun, bound-free opacity can dominate in stellar interiors temperatures of 10^4 to 10^6 K as atoms become successively ionized. In stellar photospheres, bound-free opacity is most important for effective temperatures in the range ~8000-25 000 K.

Bound-free opacity goes as ν^{-3}, cutting off when photons no longer have enough energy to ionize an atom. A schematic diagram of bound-free opacity for a mix of elements is given below (source: Cox and Guili).

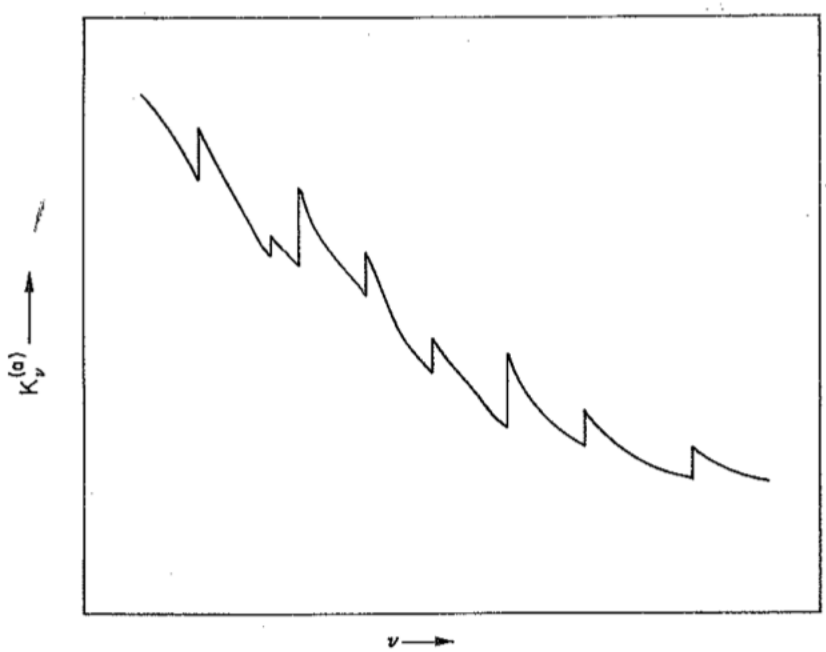


Fig. 16.2 Schematic dependence of the monochromatic mass absorption coefficient (for a mixture of elements) on frequency.

5.5 The Negative Hydrogen Ion

The negative hydrogen ion (H^-) is a hydrogen atom with an additional electron. The binding energy is only 0.7 eV and there are no excited states between its ground and ionisation levels. The two electrons are both in the $n = 1$ level — one with spin-up and the other with spin-down. Hence $g_{H^-} = 1$ and $U_{H^-} = 1$.

In applying the Saha equation to this case we must note that the neutral H atom now has the role of the ion and the negative hydrogen ion has the role of the neutral atom.
Substitution gives:

$$\frac{N_H}{N_{H^-}} \simeq 3.2 \times 10^7$$

$$\frac{N_{H^-}}{N_H} \approx 3 \times 10^{-8}$$

i.e. Less than 1 hydrogen atom in 10^7 is in the form of H^- at 6000 K and with $p_e = 3$ Pa.

HOWEVER, the H^- absorption coefficient plays a major role in the solar photosphere. In the visual region of the spectrum only hydrogen atoms in the $n = 3$ state contribute to the continuum absorption. Substitution in the Boltzmann equation gives

$$\frac{N_3}{N_1} \simeq 6 \times 10^{-10} \quad \text{at 6000 K}$$

$$\text{Therefore} \quad \frac{N_3}{N_{H^-}} = 6 \times 10^{-10} \ \times \ 3.2 \times 10^7 \simeq 2 \times 10^{-2}$$

The absorption coefficients per absorbing atom are generally of the same order of magnitude so we would expect H^- absorption to be ~ 50 times more important than neutral hydrogen absorption from the $n = 3$ level.

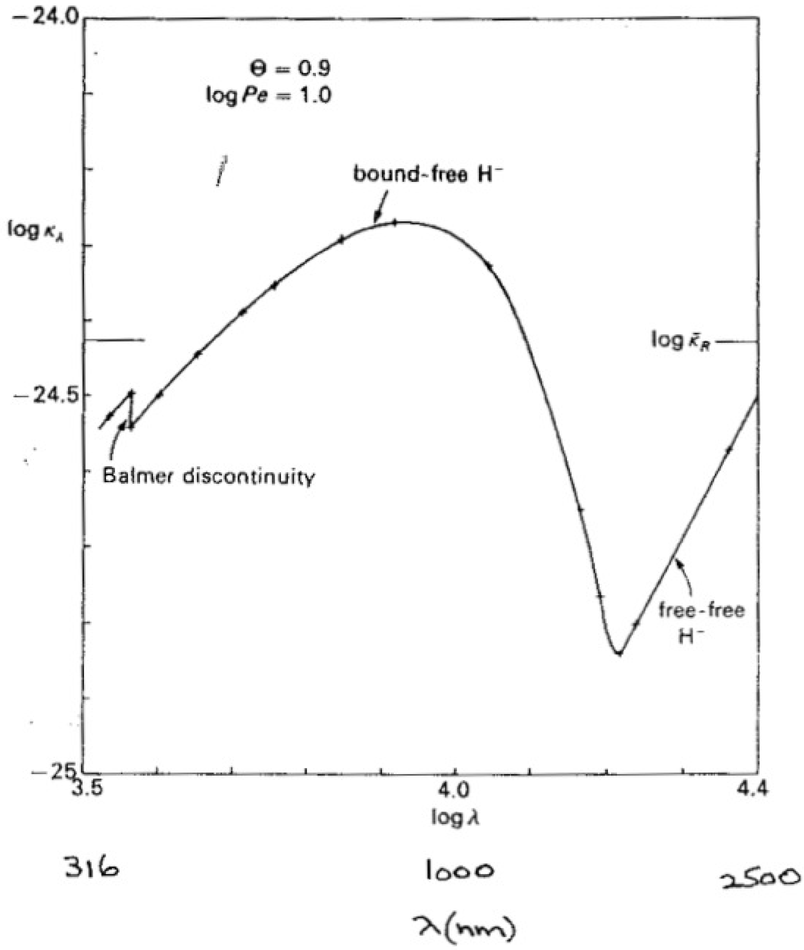
$$\text{Similarly} \quad \frac{N_2}{N_{H^-}} \simeq 3.5 \times 10^{-1}$$

Hence the Balmer continuum absorption can be expected to contribute significantly to the total absorption and we will see an edge (the *Balmer discontinuity*) at the frequency at which bound-free transitions from the $n = 3$ level start.

The absorption coefficient for free-free transitions is derived in an analogous manner — the cross-section for absorption, as a function of velocity and transition energy, is derived and then summed assuming a Maxwellian velocity distribution in order to obtain the absorption coefficient as a function of frequency.

The figure below shows the continuous absorption coefficient per particle for a temperature of 5600 K and electron pressure of 1 Pa. The contributions from the bound-free and free-free transitions of H^- dominate. The Balmer discontinuity can be seen at a wavelength of 365 nm.

This figure also shows that the absorption coefficient varies by $< 2\times$ in the wavelength range $\lambda\lambda 400 - 1300$ nm. This explains why gray models work as well as they do in representing stars like the Sun.



Helium Absorption in the Solar Photosphere

Substitution in the Boltzmann equation for helium gives:

$$\frac{N_2}{N_1} \simeq 10^{-16}$$

and, since the abundance of helium is only $\sim 10\%$ of that of hydrogen by number, only about 1 in 10^{17} atoms are helium atoms in the $n = 2$ state.

For helium in the $n = 1$ state, continuous absorption is for wavelengths less than about 50 nm and there is negligible solar radiation in this spectral range.

Metallic Absorption in the Solar Photosphere

Metallic absorption is not important in the visual region of the solar spectrum. However, in spite of the very much lower abundances (iron has an abundance $\sim 10^{-4}$ that of hydrogen), the metals become much more important than H^- or H in the ultraviolet. For example, iron atoms in the ground state absorb radiation of wavelength shorter than 157 nm and silicon atoms at wavelengths shorter than 152 nm.

5.6 Line Blanketing

A single spectral line has little effect on the structure of a stellar atmosphere or on the transmitted radiation field. However, there are generally a very large number of lines (thousands, or millions in cool stars) and their cumulative effect is known as *line blanketing*.

It is not necessary to know the opacity at every frequency in order to establish the structure of a stellar atmosphere since it is determined by integrals over the radiation field and depends on the average properties of the spectrum rather than the details. Thus some form of averaging of the line opacities is generally used.

In the simplest treatment the absorption lines are assumed to have square profiles and at any given wavelength a relative probability of being in a line or between lines is adopted. This approach is known as the *picket fence* representation for obvious reasons.

Another approach, which is more realistic, is to divide the spectrum up into a number of intervals and to compute a 'distribution function' for the opacity in each interval. As many lines as possible are included in the computations and the distribution function is a histogram of line opacities across the interval. The distribution function is interpolated across the interval to give the opacity for a selected number of points for use in the stellar model computations. The weakness in this approach is that the wavelengths that contain most the opacity (i.e. at the peak of the distribution function) can chance as a function of depth in the atmosphere.

The most common modern approach to line opacity is called *opacity sampling*, where all opacity sources at individual wavelength sampling points are taken into account, and these wavelength points are spread evenly throughout the spectrum. Models take into account opacities at typically 10^3 to 10^5 sampling points. Note that to fully sample the Doppler broadening of $\sim$2 km/s over 3 orders of magnitude required to cover the energetically important regions of the spectrum, 10^6 or more wavelengths would be required. Modern codes typically spend more effort on non-LTE effects than such large numbers of opacity sampling points.

Absorption lines are more common at shorter wavelengths and the result of line blanketing is to distort the spectral distribution of the light emitted by the star. By suppressing the radiation at short wavelengths it causes more radiation to be emitted at longer wavelengths than would be the case in the absence of line blanketing.

Blanketing is sensitive to the composition of the atmosphere and is mainly caused by lines due to heavy elements. Care must be taken to ensure that compositions do not differ significantly in inter-comparing the colours of stars and in comparing them with model predictions.

5.7 Further Applications

This section not written yet, but we will discuss e.g. radiative transfer in disks around young stars in the lectures.

6 Advanced Stellar Structure

In 2nd year, you've already come across some of the basics of stellar structure. Now we'll get into the physics in a little more detail, enabling us to talk about models of the sun, other stars, compact stellar remnants (white dwarfs, neutron stars) and planets.

For the major part of their life, stars evolve so slowly that dynamical effects can be ignored. Under these conditions the structure of a star is determined by equations describing pressure balance, conservation of energy, and energy transport, together with equations for the various properties of the gas and appropriate boundary conditions. Stars evolve (on a long time scale) because the internal composition changes as a result of nuclear reactions.

There is a great deal of similarity between the study of stellar interiors and stellar atmospheres. Obvious differences are that one has to deal with spherical shells rather than plane-parallel layers, the gravitational force is varying radially and energy is being generated in some regions.

As in the case of stellar atmospheres some simplifying assumptions are made:

- Spherical symmetry.
- Hydrostatic equilibrium.
- Thermodynamic equilibrium - for most purposes this is a very reasonable assumption - LTE is a very good approximation for stellar interiors.

6.1 The Equations of Stellar Structure

There are four differential equations resulting from the equilibrium conditions that characterise the general conditions in the interior of a star. They are the equations of:

- Mass continuity or mass conservation
- Hydrostatic equilibrium
- Energy transport
- Energy generation

6.1.1 Mass Continuity

Equation (38) is known as the equation of mass continuity or mass conservation and can be written as

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho_r \tag{106}$$

6.1.2 Hydrostatic Equilibrium

Hydrostatic equilibrium requires the inward gravitational force on a spherical shell of stellar material to be balanced by the outward pressure force:

$$-4\pi r^2\,dP = \frac{M(r)G}{r^2}\,4\pi r^2\,\rho_r\,dr$$

where P is the total pressure supporting the shell (gas plus radiation), $M(r)$ is the mass inside a sphere of radius r, and ρ_r is the density at radius r. $M(r)$ is given by:

$$M(r) = \int_0^r 4\pi r'^2\,\rho_{r'}\,dr' \tag{107}$$

Thus the equation of hydrostatic equilibrium can be written:

$$\frac{dP}{dr} = -\rho_r\,\frac{M(r)G}{r^2} \tag{108}$$

$$\text{or}\quad \frac{dP}{dm} = -\frac{M(r)G}{4\pi r^4} \tag{109}$$

The equation of hydrostatic equlibrium can be used to obtain a lower limit estimate for the central pressure in a star given its mass and radius. Assuming the star to behave as an ideal gas — a reasonable assumption since the internal temperatures are so high that stars are gaseous throughout and indistinguishable from an ideal gas — we can also get an estimate for the central temperature. These estimates are made without reference to energy sources with the implication that the basic structure of the star does not depend on them.

<u>The Virial Theorem</u>

If we multiply Equation 169 through by $4\pi r^3$ we have

$$4\pi r^3\,\frac{dP}{dr} = -\,\rho_r\,4\pi r^2\,\frac{M(r)\,G}{r}$$

If we now integrate by parts from $r = 0$ to $r = R$ where R is the radius of the star we get

$$[4\pi r^3 P]_0^R - \int_0^R 4\pi r^2\,3P\,dr = -\int_0^R \rho_r\,4\pi r^2\,\frac{M(r)\,G}{r}\,dr$$

The first term on the left hand side is zero since $P = 0$ at $r = R$ and $r^3 = 0$ at $r = 0$. The second term is equal to $-2U$ where U is the total thermal energy of the star. We get this from the ideal has law from a monatomic gas $P = nk_BT$ together with with the specific internal kinetic energy per unit volume being $nk_BT/2$ per degree of freedom for particle densith n, with 3 spatial degrees of freedom. The right hand side is equal to Ω, the total gravitational binding energy of the star. We get this by realizing that as we construct a model star shell by shell, the gravitational potential energy of each shell of mass dM is $-GMdM/r$, with $dM = 4\pi r^2\rho dr$.

$$\text{Thus:} \quad 2U + \Omega = 0$$

Also, if E is the total energy of the star we have:

$$E = U + \Omega$$

$$\text{and so:} \quad E + U = 2U + \Omega = 0$$

These equations are forms of the *virial theorem* which is essentially a statement of how the energy of a body is distributed among its various forms as a function of the body's relevant properties.

The virial theorem holds exactly for an ideal gas in hydrostatic equilibrium — no other assumptions have been made other than to assume that the thermal energy of the star is equal to the kinetic energy of its constituents.

Consider the formation of a star by contraction of a gas cloud. During the contraction gravitational potential energy will be released making Ω more negative. From the virial theorem we know that the thermal energy will increase by an amount equal to $|\Delta\Omega|/2$ and an equal amount of energy must be radiated away. As a result of the contraction the star gets hotter, radiates energy and becomes more tightly bound.

It is possible to make a rough estimate of the time taken for the Sun to contract to the main sequence by estimating the total gravitational energy released (E_G) and assuming that it radiated half this energy away at a constant rate equal to its current luminosity.

$$E_G = -\int_0^R \frac{M(r)G}{r}\, \rho_r \, 4\pi r^2 \, dr$$

If we assume ρ_r is constant and equal to the mean density $\overline{\rho}$ and note that the mass interior to a radius r can then be written as $4\pi r^3 \overline{\rho}/3$, we can show that

$$E_G = -\frac{3}{5}\frac{GM^2}{R}$$

Substitution of the parameters for the Sun shows that it would take of the order of 10^7 years to radiate this energy away. It also follows that in the hypothetical case of the Sun continuing to radiate at the same rate after reaching the main sequence it would exhaust the thermal energy produced by gravitational contraction in a further 10^7 years. The 'contraction time' estimated in this simple manner is generally known as the *Kelvin-Helmholtz time.*

Gas and Radiation Pressure

The pressure gradient appearing in the equation of hydrostatic equilibrium is the total gradient — gas plus radiation. The radiation pressure is almost isotropic and, as noted before, is only important for hot stars. The gas pressure in principle also includes the effects of electron degeneracy - important for very low mass stars, evolved stars and white dwarfs. This is something we will consider in later lectures, but for now, lets consider the ideal gas law:

$$P_g = NkT = \frac{\rho kT}{\overline{\mu} m_H}$$

where $\overline{\mu}$, the mean molecular weight, depends on the composition of the star.

It is usual to express the composition in terms of X, Y and Z — the fractional abundances of hydrogen (X), helium (Y) and heavier elements (Z) by mass. Values for middle of the road stars like the Sun are $X = 0.73$, $Y = 0.25$ and $Z = 0.02$ giving $\overline{\mu} = 0.6$. Abundances by volume are $H \sim 92.5\%$, $He \sim 7.3\%$ and heavier elements $\leq 0.2\%$.

For a fully ionised gas we have

$$\overline{\mu} = \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}$$

$$\text{Thus:} \quad P_g = (2X + \frac{3}{4}Y + \frac{1}{2}Z)\,\frac{\rho kT}{m_H}$$

6.1.3 Energy Transport

Energy is transported through a star mainly by radiation and convection. Transfer of energy by radiation is described by the equation of radiative transfer (16). However, the high opacity of matter inside stars allows a simpler description of radiative energy transport — the diffusion approximation.

The Diffusion Approximation

A simpler derivation of this will be given in lectures. The derivation below is more complete, and also helps to implicitly demonstrate where the approximation will break down.

Consider depths far from the boundary of the atmosphere where LTE holds and the source function S_ν is given by the Planck function B_ν. We can represent the source function at some optical depth t_ν near τ_ν by a power series expansion:

$$S_\nu(t_\nu) = \sum_{n=0}^{\infty} \frac{(t_\nu - \tau_\nu)^n}{n!}\,\frac{d^n B_\nu}{d\tau_\nu^n}$$

Substitution in Equation 68 gives, for $0 \leq \mu \leq 1$:

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu\,\frac{dB_\nu}{d\tau_\nu} + \mu^2\,\frac{d^2 B_\nu}{d\tau_\nu^2} + \cdots \tag{110}$$

Substitution in Equation 69 gives, for $-1 \leq \mu \leq 0$, a similar series differing only by terms of order $\exp(-\tau/\mu)$. Thus, for great depths $\exp(-\tau/\mu) \approx 0$, and Equation 110 represents the total radiation field.

Substitution in Equations 55, 56 and 57 leads to

$$J_\nu(\tau_\nu) = B_\nu(\tau_\nu) + \frac{1}{3}\frac{d^2B_\nu}{d\tau_\nu^2} + \cdots \tag{111}$$

$$H_\nu(\tau_\nu) = \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} + \frac{1}{5}\frac{d^3B_\nu}{d\tau^3} + \cdots \tag{112}$$

$$K_\nu(\tau_\nu) = \frac{1}{3}B_\nu(\tau_\nu) + \frac{1}{5}\frac{d^2B_\nu}{d\tau_\nu^2} + \cdots \tag{113}$$

For large optical depth we can write equations 111-113) as

$$I_\nu(\tau_\nu,\mu) \approx B_\nu(\tau_\nu) + \mu\frac{dB_\nu}{d\tau_\nu} \tag{114}$$

$$J_\nu(\tau_\nu) \approx B_\nu(\tau_\nu) \tag{115}$$

$$H_\nu(\tau_\nu) \approx \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} \tag{116}$$

$$K_\nu(\tau_\nu) \approx \frac{1}{3}B_\nu(\tau_\nu) \tag{117}$$

Note that the Eddington factor $K_\nu/J_\nu = 1/3$ holds for large optical depths.

Recalling the equation of transfer 65 and taking the first order moment gives

$$\frac{dK_\nu}{d\tau_\nu} = H_\nu$$

Substitution from Equation 117 for K_ν gives

$$H_\nu = \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} = -\frac{1}{3}\frac{1}{k_\nu}\frac{dB_\nu}{dr} = -\frac{1}{3}\left(\frac{1}{k_\nu}\frac{dB_\nu}{dT}\right)\frac{dT}{dr}$$

$$\text{so that}\quad \mathcal{F}_\nu = \pi F_\nu = 4\pi\,H_\nu = -\frac{4\pi}{3}\left(\frac{1}{k_\nu}\frac{dB_\nu}{dT}\right)\frac{dT}{dr} \tag{118}$$

Integrating equation 118 over frequency gives a *diffusion equation* relating the radiative energy flux to the temperature gradient.

$$\mathcal{F} = \int_0^\infty \mathcal{F}_\nu\,d\nu = -\frac{4\pi}{3}\int_0^\infty \frac{1}{k_\nu}\frac{dB_\nu}{dT}\,d\nu\,\frac{dT}{dr}$$

Recalling the definition of the Rosseland mean opacity $\overline{k}$ (Section 5.1.3) and noting that $\int_0^\infty (dB_\nu/dT)d\nu = 4\sigma T^3/\pi$ we get

$$\mathcal{F} = -\frac{4\pi}{3}\frac{4\sigma T^3}{\pi}\frac{1}{\overline{\overline{k}}}\frac{dT}{dr} \tag{119}$$

$$L_r = -4\pi r^2\frac{4\pi}{3}\frac{4\sigma T^3}{\pi}\frac{1}{\overline{\overline{k}}}\frac{dT}{dr} = -\frac{16}{3}\frac{\sigma T^3}{\overline{k}}4\pi r^2\frac{dT}{dr} \tag{120}$$

These have the form of a diffusion equation because the flux is proportional to the temperature gradient. It is analogous to thermal conductivity with the conduction coefficient being determined by the properties of the gas.

Convection

If the Rosseland opacity becomes too large for a given L and T, the temperature gradient becomes too steep for stability and *convection* sets in.

Instability will occur if the actual temperature gradient is larger than the adiabatic temperature gradient experienced by a parcel of gas rising adiabatically. Instability will occur if:

$$\left|\frac{dT}{dr}\right| > \left|\frac{dT}{dr}\right|_{ad}$$

If we divide through by dP/dr and invert both sides we obtain:

$$\left|\frac{dP}{dT}\right| < \left|\frac{dP}{dT}\right|_{ad}$$

$$\text{or} \qquad \left|\frac{d\,\ln P}{d\,\ln T}\right| < \left|\frac{d\,\ln P}{d\,\ln T}\right|_{ad}$$

Now, for adiabatic changes $PT^{\frac{\gamma}{1-\gamma}} = constant$ so:

$$\left|\frac{d\,\ln P}{d\,\ln T}\right|_{ad} = \frac{\gamma}{\gamma - 1}$$

Thus, instability will occur for:

$$\left|\frac{d\,\ln P}{d\,\ln T}\right| < \frac{\gamma}{\gamma - 1}$$

If this condition is met, convection will occur, and convection will cause the temperature gradient to be equal to the adiabatic temperature gradient

$$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dP}{dr} \tag{121}$$

In general, if convection occurs, the temperature gradient given by Equation 121 is the appropriate one to use. Otherwise the temperature gradient from the radiative equilibrium condition should be used (Equation 120). In practice, this is complicated by mixing-length theory, which we won't go in to.

6.1.4 Energy Generation

NOT COVERED. SEE MIKE IF YOU WANT NOTES.

6.1.5 Summary of Equations of Stellar Structure

$$\begin{aligned}
\frac{dP}{dr} &= -\rho_r \frac{M(r)G}{r^2} && \text{Hydrostatic equilibrium} \\
\frac{dM(r)}{dr} &= 4\pi r^2 \rho_r && \text{Mass continuity} \\
\frac{dT}{dr} &= -\frac{3}{16}\frac{\overline{k}}{\sigma T^3}\frac{L_r}{4\pi r^2} && \text{Radiative transport} \\
\text{or} \quad \frac{dT}{dr} &= \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dr} && \text{Convective transport} \\
\frac{dL_r}{dr} &= 4\pi r^2 \rho_r \epsilon_r && \text{Energy generation}
\end{aligned}$$

Plus:

$$P = \frac{\rho k T}{\overline{\mu} m_H} \qquad \text{Equation of state (Ideal gas)}$$

Plus "constitutive" relations:

$$\begin{aligned}
\overline{k} &= \overline{k}\ (\rho, T, \text{composition}) && \text{Opacity} \\
\epsilon &= \epsilon\,(\rho, T, \text{composition}) && \text{Energy generation rate} \\
\overline{\mu} &= \overline{\mu}\,(\rho, T, \text{composition}) && \text{Mean molecular weight}
\end{aligned}$$

6.2 Polytropic Stars

The equations for stellar structure in general require complex numerical solutions, but in certain cases we can neglect the equations of energy transport and energy generation, turning a 4th order system of differential equations to a 2nd order system. If we further make the approximation that the equation of state is of the form:

$$P = \alpha\rho^{1+1/n}, \tag{122}$$

then we call the star a polytrope of index n. n is not necessarily an integer, but analytic solutions only exist for n=0, 1 or 5. Examples of *physical situations* where polytropes are justified include fully convective stars, where if we assume the ideal monatomic gas law, $n = 1.5$, and white dwarfs, where $n = 1.5$ in the non-relativistic limit, $n = 3$ in the relativistic limit and in-between a polytrope isn't appropriate. The $n = 0$ solution is non-physical, where ρ is constant and independent of pressure – but is is a pretty good approximation for low-mass well mixed solid planets. A convective star consisting of a monatomic ideal gas has $n = 1.5$, and as we will see in following lectures, a fully relativistic degenerate electron gas has $n = 3$, approximating the structure of a white dwarf approaching the mass where it becomes a supernova Ia. Real stars have structures that are roughly bounded by these two polytropes.

Polytropic models can be constructed from a dimensionless form of the mass continuity and hydrostatic equilibrium equations called the Lane-Emden equation. The first step in its derivation is to introduce a new variable θ:

$$\rho = \rho_c \theta^n \tag{123}$$

Here ρ_c is the central density. This conveniently means that the starting point for the solution is $\theta = 1$ and $d\theta/dr = 0$. Next, we substitute this into the hydrostatic equilibrium equation, and obtain:

$$\frac{dP}{dr} = (n+1)\alpha\rho_c^{1+1/n}\theta^n\frac{d\theta}{dr} = -\frac{GM\rho_c}{r^2}\theta^n. \tag{124}$$

The θ^n terms cancel out, we write the central pressure $P_c = \alpha\rho_c^{1+1/n}$, and have a simple equation relating $d\theta/dr$ to M. Substitution into the mass conservation equation then gives the Lane-Emden equation:

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}(\zeta^2\frac{d\theta}{d\zeta}) + \theta^n = 0, \tag{125}$$

where we have introduced a dimensionless radius ζ:

$$\zeta = r\sqrt{\frac{4\pi G\rho_c^2}{(n+1)P_c}} \tag{126}$$

A function providing numerical solutions to this equation is given as a MATLAB program on the course website. Importantly, this program provides the output $\theta^n(\zeta)$, where:

$$\begin{aligned}
\rho &= \rho_c\theta^n && (127)\\
r &= \zeta(\frac{(n+1)P_c}{4\pi G\rho_C^2})^{1/2} && (128)\\
&= \zeta(\frac{\alpha(n+1)\rho_c^{1/n-1}}{4\pi G})^{1/2}. && (129)
\end{aligned}$$

The mass of the polytropic star is then given by:

$$\begin{aligned}
M_{\rm tot} &= \int 4\pi r^2\rho dr && (130)\\
&= 4\pi\rho_c(\frac{\alpha(n+1)\rho_c^{1/n-1}}{4\pi G})^{3/2}\int_0^{\zeta_{\rm max}}\zeta^2\theta^n d\zeta. && (131)\\
&= \frac{1}{2\sqrt{\pi}}\rho^{\frac{1}{2}(\frac{3}{n}-1)}(\frac{\alpha(n+1)}{G})^{3/2}I_n, \text{where} && (132)\\
I_n &= \int_0^{\zeta_{\rm max}}\zeta^2\theta^n d\zeta. && (133)
\end{aligned}$$

Values of I_n and $\zeta_{\rm max}$ are tabulated below for various useful values of n. From Equation 133, we can see that for $n < 3$, increasing ρ_c increases the total mass, but for $n > 3$, increasing ρ_c decreases the total mass. For $n = 3$, the total mass is independent of ρ_c, and depends on the equation of state (i.e. α) only.

An equivalent consideration about the radius of the star as a function of central density results in R being independent of ρ_c for polytropes with $n = 1$. Most stars that can be approximated by a polytrope have $1.5 < n < 3$.

6.3 The Chandrasekhar Limit

In the extremely relativistic limit, we can use Equation 45 and see that a relativistic electron gas can be described by a polytrope with $n = 3$. We can therefore derive the mass:

$$M_{\rm lim} = \frac{1}{2\sqrt{\pi}}(\frac{4}{24\pi^{1/3}})^{3/2}(\frac{ch}{G})^{3/2}(\frac{3\rho}{m_{\rm H}\mu_e})^2 I_3 \tag{134}$$

$$= \frac{\sqrt{6}}{8\pi}(\frac{ch}{G})^{3/2}(m_{\rm H}\mu_e)^{-2} I_3. \tag{135}$$

This gives $M_{\rm lim} = 5.82/\mu_e^2 M_\odot$, or $M_{\rm lim} = 1.46 M_\odot$ for $\mu_e = 2$. In practice, this means that as the mass of a white dwarf becomes closer and closer to $M_{\rm lim}$, it becomes more and more extremely relativistic. This also corresponds to $\rho_c \to \infty$, and $R \to 0$. A white dwarf or indeed the degenerate core of a star can not have a mass larger than $M_{\rm lim}$ and be supported by electron degeneracy pressure.

This is a rather incredible result: ultimately we have derived the maximum mass of a white dwarf or stellar core directly from quantum mechanics. The final result includes Planck's constant (quantum mechanics), the speed of light (special relativity), the gravitational constant (Newtonian gravity) as well as the mass of the proton. Interestingly, this formula does not include the mass of the electron: in the extreme relativistic limit this is unimportant.

7 Stellar Models

7.1 Constructing a Model Stellar Interior

The problem of constructing a model of a stellar interior is to solve the equations for the structure given an initial mass for the star, a chemical composition, and appropriate boundary conditions. The equations have to be satisfied in every layer of the star.

Four boundary conditions are required and we can readily provide two for the outer edge of the star and two at the centre. At the outer edge we require that $T = T_e$ at $\overline{\tau} = 2/3$. Generally the pressure and temperature decrease rapidly near the surface and we can take $T(R) = 0$ where R is the radius of the star. We also have $P_g(R) = 0$. At the centre of the star we must have $m(r) = 0$ and $L_r = 0$. Thus the four boundary conditions are:

At Outer Edge of Star	At Centre of Star
$T(R) = 0$	$m(0) = 0$
$P_g(R) = 0$	$L_r(0) = 0$

The radius of the star (R) is determined by $m(R) = M$.

The fact that we only have two boundary conditions at each end complicates the solution of the differential equations. If the integration is started from the outer boundary, values for R and L must be assumed and, similarly, if the integration is started from the centre, values for $T(0)$ and $P(0)$ must be adopted. Unfortunately, in both cases, the solutions diverge at the opposite end. One approach to this problem is to start the integrations from both ends and adjust the assumed values for R, L, $P(0)$ and $T(0)$ until the solutions 'match' where they meet in the stellar interior at, say, $r = R/2$. Inside the star, T_r, P_r, L_r and $m(r)$ and their derivatives must be continuous and these are the conditions required for a 'match'.

A theoretical model of a stellar interior is a tabulation of ρ_r, T_r, P_r, L_r, $m(r)$ etc. as a function of radius r plus predictions of the observables R and L for the given mass M. The effective temperature T_e is also predicted since $L = 4\pi R^2 \sigma T_e^4$.

7.2 Linear Stellar Models

An initial understanding of stellar structure can be obtained using a simple model in which the density is a linear function of radius

$$\rho_r = \rho_c \left(1 - \frac{r}{R}\right)$$

The equation of mass continuity gives

$$m(r) = M \left[\frac{4r^3}{R^3} - \frac{3r^4}{R^4}\right] \qquad \text{and} \qquad \rho_c = \frac{3M}{\pi R^3}$$

The equation of hydrostatic equilibrium gives

$$P_r = P_c - \pi G \rho_c^2 \left[\frac{2r^2}{3} - \frac{7r^3}{9R} + \frac{r^4}{4R^2} \right] \qquad \text{where} \qquad P_c = \frac{5}{36} \pi G \rho_c^2 R^2$$

This implies

$$T_r = T_c \left[1 + \frac{r}{R} - \frac{19r^2}{5R^2} + \frac{9r^3}{5R^3} \right] \qquad \text{where} \qquad T_c = \frac{5\pi}{36} \frac{G\mu m_H}{k} \rho_c R^2$$

Exercise Show that these equations follow from the assumption that $\rho_r = \rho_c(1 - r/R)$.

The early homogeneous structure of stars of the lower main sequence can be approximated by the linear model and, assuming the sole energy source is the pp chain, it can be shown that

$$\begin{aligned}
\frac{R}{R_\odot} &= 0.312\mu^{-0.358} \left(\frac{M}{M_\odot} \right)^{0.0769} \\
\frac{L}{L_\odot} &= 49.1\mu^{7.77} \left(\frac{M}{M_\odot} \right)^{5.46} \\
T_e &= 27300\mu^{2.21} \left(\frac{M}{M_\odot} \right)^{1.327} \\
T_c &= 3 \times 10^7 \mu^{1.54} \left(\frac{M}{M_\odot} \right)^{0.923} \\
\rho_c &= 186\mu^{-1.615} \left(\frac{M}{M_\odot} \right)^{0.769}
\end{aligned}$$

A series of such models for different masses represents the zero-age main sequence (ZAMS).

7.3 Mixing Length Theory

Last lecture, we spoke about how a star becomes unstable to convection if the radiative temperature gradient became steeper than the adiabatic temperature gradient. Assuming that the adiabatic temperature gradient holds is assuming that all energy can easily be carried by convection. Lets examine if this is true. A rough estimate of the maximum energy that could be carried by convection is the product of sound speed with internal energy per unit volume:

$$\begin{aligned}
F_{\max} &= c_s \times (3/2) n k_B T && (136) \\
&= \sqrt{\frac{\gamma P}{\rho}} \frac{3\rho k_B}{2u\bar{\mu}} T && (137) \\
&= \frac{3\gamma^{1/2}\rho}{2} \left(\frac{k_B T}{u\bar{\mu}}\right)^{3/2}. && (138)
\end{aligned}$$

Here all symbols have there usual meaning, including the atomic mass unit u, roughly the mass of the Hydrogen atom. For the sun, $F = 6.6 \times 10^{10}\,\mathrm{erg/s/cm^2}$, and $F_{\max} = 2 \times 10^{10}\,\mathrm{erg/s/cm^2}$ at the top of the photosphere ($\tau = 0.1$) or 2×10^{11} at the base of the photosphere ($\tau = 5$). So this means that the layers at the base of the photosphere have $F \sim F_{\max}$, and convection can not carry all the sun's luminosity. However, deep within stars, $F_{\max} \gg F$, so whenever the star is convectively unstable, it is a good approximation that convection carries all the flux and the temperature gradient is the adiabatic temperature gradient.

The detailed 1-dimensional theory of inefficient convection is called *mixing length* theory. This is quite complex, and it isn't worthwhile going in to the details because the newest dynamical models of the solar surface include the dynamical motions of 3D convection properly, so do not need mixing length theory. It is important, however, to recognise that the theory has one important free parameter α_M, the *mixing length*, usually expressed as a constant multiple of pressure scale heights (the distance where pressure decreases by a factor of e). Roughly speaking, α_M is the length a convective element travels before losing its identity and transfering its thermodynamic properties to the surrounding material. The larger α_M is, the more efficient convection is and the closer the temperature gradient is to the adiabatic temperature gradient. Typically, a series of models fixes α_M so that the model radius matches a particular star, e.g. the sun.

7.4 Stellar Models in MATLAB

On the course website `http://www.physics.usyd.edu.au/~mireland/astrop/` there a a bunch of MATLAB programs that can model the interior of stars. We will go through them in the lecture, and use them in assignment 2. They are relatively simple, as all the tedious physics has been done by others:

- The thermonuclear reaction rates come from Caughlan, G. & Fowler, W., 1998, Atomic Data and Nuclea Data Tables, Vol 40, p. 283. This data originates from Oak Ridge National Laboratory (`http://www.phy.ornl.gov/`).

- The chemical composition of the sun comes from Grevesse and Sauval, 1998, Space Science Reviews, v. 85, Issue 1/2, p. 161-174. Everything more modern than this remains contraversial due to the complexity of 3D solar photosphere models (look up Martin Asplund on Google).

- The Rosseland mean opacities come from the OPAL project. Iglesias C. & Rogers F., 1996, ApJ, 464, 943 (`http://adg.llnl.gov/Research/OPAL/opal.html`). Although some work on these opacities is ongoing, most of this work finished in the mid-1990s. These updated opacities were an essential ingredient in precise modelling of old stars in globular clusters, resolving the contraversy about the Hubble constant and the age of the Universe.

The MATLAB code integrates outwards only at a fixed core density ρ_c, modifying the core temperature T_c so that the surface condition $L = 4\pi r^2 \sigma T^4_{\tau=2/3}$ is met. For a radiative outer atmosphere, even double-precision arithmetic is inadequate to meet this condition within e.g. 10%, because of the divergent nature of the equations of stellar structure. So as the code runs out of precision, it instead fixes the interior structure out to some radius r_{fix} and modifies the temperature from this point. This code will be demonstrated in the lecture in detail.

7.5 Normalized and log Equations

For the purposes of solving the equations of stellar structure, it is best to use normalized units, so that numerical factors are of order unity. In addition, the hydrostatic equilibrium and convective energy transport equations are approximately of the form $dy/dr \propto y$, so the equations become easier to solve if the variables $\log(P)$ and $\mathrm{lot}(T)$ are chosen. The equations then become:

$$\begin{aligned}
\frac{dM'}{dx} &= \frac{4\pi r_\odot^3 x^2}{M_\odot}\rho && (139)\\
\frac{dL'}{dx} &= \frac{4\pi r_\odot^3 x^2}{L_\odot}\epsilon\rho && (140)\\
\frac{d\ln P}{dx} &= -\frac{GM'M_\odot}{r_\odot x^2 P}\rho && (141)\\
\frac{d\ln T}{dx}|_r &= -\frac{3L'L_\odot}{64\pi\sigma T^4 r_\odot x^2}\kappa\rho \text{ , or} && (142)\\
\frac{d\ln T}{dx}|_c &= \frac{\gamma-1}{\gamma}\frac{d\ln P}{dx} && (143)
\end{aligned}$$

where we have put $x = r/r_\odot$, $L' = L/L_\odot$ and $M' = M/M_\odot$. A further simplification sometimes used is to put dM' as the independent variable. This is particularly useful if following evolutionary models where composition is a function of mass fraction, or 1-dimensional (i.e. spherically symmetric) dynamical models the Lagrangian formalism. It is convenient because the factors of ρ on the right hand side of the above equation cancel out:

$$\begin{aligned}
\frac{dx}{dM'} &= \frac{M_\odot}{4\pi r_\odot x^2\rho} && (144)\\
\frac{dL'}{dM'} &= \frac{M_\odot}{L_\odot}\epsilon && (145)\\
\frac{d\ln P}{dM'} &= -\frac{GM'M_\odot^2}{4\pi r_\odot^2 x^4 P} && (146)\\
\frac{d\ln T}{dM'}|_r &= -\frac{3}{256\pi^2\sigma T^4 r_\odot^2 x^4}\kappa \text{ , or} && (147)\\
\frac{d\ln T}{dM'}|_c &= \frac{\gamma-1}{\gamma}\frac{d\ln P}{dM'} && (148)
\end{aligned}$$

We will not be using the equations of stellar structure in this form, but you may encounter them in future courses.

8 Compact Stellar Remnants

8.1 The Chandrasekhar Limit

In the extremely relativistic limit, we can use Equation 45 to find:

$$
\begin{aligned}
M_{\mathrm{lim}} &= \frac{1}{2\sqrt{\pi}}(\frac{4}{24\pi^{1/3}})^{3/2}(\frac{ch}{G})^{3/2}(\frac{3\rho}{m_{\mathrm{H}}\mu_e})^2 I_e && (149)\\
&= \frac{\sqrt{6}}{8\pi}(\frac{ch}{G})^{3/2}(m_{\mathrm{H}}\mu_e)^{-2} I_3. && (150)
\end{aligned}
$$

This gives $M_{\mathrm{lim}} = 5.82/\mu_e^2 M_\odot$, or $M_{\mathrm{lim}} = 1.46 M_\odot$ for $\mu_e = 2$. In practice, this means that as the mass of a white dwarf becomes closer and closer to M_{lim}, it becomes more and more extremely relativistic. This also corresponds to $\rho_c \to \infty$, and $R \to 0$. A white dwarf or indeed the degenerate core of a star can not have a mass larger than M_{lim} and be supported by electron degeneracy pressure.

This is a rather incredible result: ultimately we have derived the maximum mass of a white dwarf or stellar core directly from quantum mechanics. The final result includes Planck's constant (quantum mechanics), the speed of light (special relativity), the gravitational constant (Newtonian gravity) as well as the mass of the proton. Interestingly, this formula does not include the mass of the electron: in the extreme relativistic limit this is unimportant.

8.2 Supernovae Ia

[Some pictures for this section are in the power-point slides] A white dwarf in a binary star system will accrete mass until it approaches the Chandrasehkar limit. At this point, there are many things that could happen in principle: e.g. the protons could undergo electron capture, the neutrinos could escape and the star could collapse to form a neutron star... or maybe the protons could undergo electron capture and the neutrinos could be absorbed by the outer layers driving an explosion.

What actually occurs in a carbon/oxygen white dwarf has been a major subject of research for several decades. For reasonable accretion rates the system is not in equilibrium, and the adiabatic compression of the nuclear gas in the white dwarf causes an increase in temperature until carbon and oxygen are burnt. This process is a runaway process, as the white dwarf will not expand and cool until the nuclear temperature exceeds the Fermi temperature of the electrons. The total energy release in burning carbon and oxygen to form "iron-peak" elements exceeds the gravitational potential energy of the white dwarf, and the entire star explodes. These explosions have very repeatable light curves, and are the "standard candles" used to measure the distance to the most distant galaxies, and have enabled the cosmological acceleration term "dark energy" to be directly measured. It has only recently ($\sim$2007) become generally accepted that supernovae Ia are essentially all formed this way, rather than e.g. merging white dwarfs.

8.3 Core-Collapse Supernovae

The cores of the most massive stars contain "iron-peak" elements, that is, the cores are nearly in nuclear equilibrium where the most probable configuration of nuclei is a Fe nucleus. The difference in mass-energy between ^{56}Fe and ^{12}C is $\sim$1 MeV per nucleon, which can be though of as a characteristic energy where nuclear reactions become very likely. The actual energies in the nuclear burning regions must be significantly less than this, i.e. less than the mass-energy of the electron (0.512 MeV). This means that as the iron core of a massive star will be supported by electron degeneracy pressure as the electrons become relativistic. In turn, this means that the centers of these stars are described with a polytrope with $n = 3$.

Assuming that Fe has 26 protons and 30 neutrons (applicable to the most common isotope), $\mu_e = 2.15$, and the limiting mass for the core of a massive star is $1.26\,\mathrm{M}_\odot$. As the core approaches this mass, it collapses to form a neutron star or black hole, and in general (or so it is thought) a supernova occurs. The details of this process are extremely complex and the subject of active research. For example, models where the neutrinos produced by electron-capture (p + e$^-$ $\leftrightarrow$ n + ν_e) escape the star have resulted in solutions where the entire star collapses into a black hole leaving no explosion.

9 Asteroseismology

In this lecture, we will only consider acoustic (sound) waves in what is called the Cowling approximation – where the influence of a wave on the gravitational potential of a star is neglected. This is such a terrible over-simplification that you will not find it in any textbook I know of on asteroseismology. However, as we'll see, all the main results of asteroseismology are reproduced, and it enables us to consider how interior models of stars can be tested using observations.

9.1 General Stellar Dynamics

In earlier lectures, we have calculated and talked about the Kelvin-Helmholtz timescale (∼1 Myr for the sun) and the nuclear burning timescale (∼10 GYr for the sun). A very much shorter timescale is the dynamical timescale for the sun. In the case of the Kelvin-Helmholtz timescale, we divided the internal energy of the star by its luminosity to get time: and because of the virial theorem, it didn't matter (within a factor of 2) whether we chose the gravitational potential energy or the total internal energy of the star. For similar reasons (again due to the virial theorem - an exercise for the reader!) we could use the virial theorem to derive two equivalent dynamical timescales for a star, which would be roughly equal. The first of these is typically called the *dynamical timescale*:

$$t_{\mathrm{dyn}} = \sqrt{\frac{R_*^3}{GM_*}} \tag{151}$$

$$= \sqrt{1/G\rho_m} \tag{152}$$

This is simply the square root of the stellar radius divided by the surface gravity. ρ_m is the mean density of the star. A second timescale is the *sound-crossing time*:

$$t_{\mathrm{sound}} = \int_0^{R_*} c_s^{-1} dr. \tag{153}$$

To understand how sound propagates in a star, we will first derive the acoustic wave equation from general considerations of dynamics in a star.

9.2 Acoustic Wave Equation

We begin with assuming that the pressure, density and displacement in a star are given by their means, plus some small perturbation:

$$P = P_0 + p \tag{154}$$

$$\rho = \rho_0 + \rho' \tag{155}$$

$$\mathbf{r} = \mathbf{r_0} + \zeta. \tag{156}$$

We are considering the star here in a Lagrangian formalism, because $\mathbf{r}$ is a variable describing the position of each unit of mass and not a coordinate. By drawing a picture, it is rather easy to show that:

$$\frac{\rho'}{\rho_0} = -\nabla \cdot \zeta. \tag{157}$$

If we consider that the wave travels adiabatically, then we can relate the variation in density directly to a variation in pressure:

$$\frac{p}{P_0} = \Gamma_1 \frac{\rho'}{\rho_0}. \tag{158}$$

Here Γ_1 is a more generalized version of the adiabatic exponent γ for non-ideal gases. Putting the two above equations together gives:

$$p = -\Gamma_1 P_0 \nabla \cdot \zeta. \tag{159}$$

To obtain our wave equation for p, all we have to do is consider Newton's laws in the same way as the force balance in the hydrostatic equilibrium equation:

$$\begin{aligned} \frac{d^2\zeta}{dt^2} &= -\frac{1}{\rho}(\nabla p + [\nabla P_0 + \frac{GM}{|\mathbf{r_0} + \zeta|^2}]) && (160) \\ &\approx -\frac{1}{\rho}\nabla p. && (161) \end{aligned}$$

We consider that the terms in the square brackets above balance, because the unperturbed star satisfies hydrostatic equilibrium. This really isn't correct, because a radial displacement of the gas in the star changes the effective gravity, and this change is a vector change. But for high frequency waves (i.e. acoustic waves), this will be a reasonable approximation and enables us to end up with a scalar wave equation. There is a further approximation in Equation 160, which is the writing of GM as the numerator of the gravitational acceleration. Actually, a wave will change the density in the star, which changes the gravitational potential. Using GM in the potential is a rather common approximation however, called the Cowling approximation after a 1941 paper. It is also valid for high-frequency waves. We now take the second derivative of Equation 159 and arrive at:

$$\begin{aligned} \frac{d^2 p}{dt^2} &= -\Gamma_1 P_0 \nabla \cdot \frac{d^2\zeta}{dt^2} && (162) \\ &= \frac{\Gamma_1 P_0}{\rho_0}\nabla^2 p && (163) \end{aligned}$$

This is the wave equation for acoustic waves with sound speed $c_s = \sqrt{\Gamma_1 P_0/\rho_0}$.

9.3 Oscillations in Spherical Coordinates

You should all be familiar with the Laplacian ∇^2 in spherical coordinates from quantum mechanics. We search for periodic solutions of the form $p = p_r p_A \exp(i\omega t)$, where we employ separation of variables for a radial (p_r), angular (p_A) and time-dependent ($\exp(i\omega t)$) component of p. Equation 163 then becomes in spherical coordinates:

$$\begin{aligned} c_s^2(r)(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{dp_r}{dr}) - \frac{l(l+1)p_r}{r^2}) &= \omega^2 p_r & (164)\\ \hat{Y}_l^m p_A &= l(l+1)p_a. & (165) \end{aligned}$$

We have written the angular component of Laplace's equation in spherical coordinates in short-hand using a Legendre operator. The angular wave-functions have the form of Legendre functions $p_A = Y_l^m(\theta, \phi)$ analogous to Schrödinger's equation or any other place where a Laplacian is used in spherical coordinates. The angular quantum number l refers to the number of nodes that a mode has around the surface of a star. The radial wave-function equation is an eigenvector equation that appears at first sight to be rather complicated. However, if we make the substitution $u = rp_r$, then the radial equation becomes:

$$c_s^2(r)[\frac{\partial}{\partial r^2} - \frac{l(l+1)}{r^2}]u = \omega^2 u. \qquad (166)$$

In the case of $l = 0$, this is just a standard 1-dimensional wave equation, however with a sound speed that varies with our coordinate r. For any value of l, this equation is a well-behaved Sturm–Liouville equation, which has an ordered set of eigenvalues ω^2 and orthonormal eigenfunctions. The boundary condition at $r = 0$ is $u = 0$, because otherwise the mode amplitude $p_r = u/r$ would diverge. The boundary condition at $r = R_*$ is not so simple, but we will make the approximation that we have a free boundary with $u = 0$ at $r = R_*$. This is certainly very close to correct, as the pressure P_0 becomes vanishingly small and the amplitude p_r must also be small.

On the course website, I have provided code that takes as input the sound speed as a function of depth in a stellar model, and solves this eigenvalue problem. In the lecture, we looked at eigenfunctions and Echelle diagrams of the eigenvalues.

9.4 Measuring Age and Mass with Asteroseismology

Solar-like oscillations in principle come in 3 varieties: p-modes which are described approximately by the equations above and are relatively easily observed, g-modes (gravity modes) which primarily occur deep within the sun and have not yet been observed, and surface f-modes.

The modes are described by 3 "quantum" numbers n, l and m. In the slides and the code demonstration in the lecture, pictorial examples of these modes were given. n describes how many nodes there are inside the star, l describes the number of nodes around the surface of the star and m describes how these nodes are distributed over both angular coordinates.

The frequencies of these modes at high n are roughly proportional to $n + 2l$. The *large spacing* is the separation between two modes of the same l and with n separated by 1. The *small spacing* is the separation between modes (n, l) and $(n - 1, l + 2)$

As the dynamical timescale of a star is proportional to $\rho_m^{-1/2}$ (Equation 152), as a star evolves and its radius grows, the mean density becomes smaller and all frequencies decrease. However, for a given star it is not possible to use the large spacing alone to measure both mass and age. In order to achieve this, it is necessary to measure the *small spacing*. Modes of higher l penetrate less deeply into the star, meaning that the measurement of the *small spacing* can distinguish differing sound speeds in the core or a star due to the conversion of hydrogen to helium.

10 Fluid Dynamics and Shocks

As a general rule, astrophysical fluid dynamics consists of conserving mass, momentum and energy. As this is just an introduction to fluid dynamics, we will (mostly) ignore energy for now, because it can be complex due to the interaction of matter with radiation. It can also be simple if we make approximations such as constant temperature, which can be valid where there is rapid radiative cooling and an external radiation field.

10.1 Hydrostatics

If gas is not moving, then we have a somewhat trivial case of fluid dynamics. There are nonetheless some key concepts even for static gas.

10.1.1 Mass Continuity

This may seem trivial but will be important in stellar structure. The total mass contained within a volume is the volume integral of density.

10.1.2 Hydrostatic Equilibrium

In general, hydrostatic equilibrium requires that the force due to a pressure gradient is balanced by any external forces.:

$$\nabla P(\boldsymbol{r}) = f_{\rm ext}(\boldsymbol{r}), \tag{167}$$

where $f_{\rm ext}$ is the external force per unit mass, e.g. the acceleration due to gravity. In spherical coordinates, with gravity as our external force, equilibrium requires the inward gravitational force on a spherical shell of stellar material to be balanced by the outward pressure force:

$$-4\pi r^2\, dP = \frac{M(r)G}{r^2}\, 4\pi r^2\, \rho_r\, dr$$

where P is the total pressure supporting the shell (gas plus radiation), $M(r)$ is the mass inside a sphere of radius r, and ρ_r is the density at radius r. $M(r)$ is given by:

$$M(r) = \int_0^r 4\pi r'^2\, \rho_{r'}\, dr' \tag{168}$$

Thus the equation of hydrostatic equilibrium can be written:

$$\frac{dP}{dr} = -\rho_r\, \frac{M(r)G}{r^2} \tag{169}$$

$$\text{or} \quad \frac{dP}{dm} = -\frac{M(r)G}{4\pi r^4} \tag{170}$$

10.1.3 Gas and Radiation Pressure

The pressure gradient appearing in the equation of hydrostatic equilibrium is the total gradient — gas plus radiation. The radiation pressure is almost isotropic and is only important for hot stars. The gas pressure in principle also includes the effects of electron degeneracy - important for very low mass stars, evolved stars and white dwarfs. This is something we will consider in later lectures, but for now, lets consider the ideal gas law:

$$P_g = NkT = \frac{\rho kT}{\overline{\mu} m_H}$$

where $\overline{\mu}$, the mean molecular weight, depends on the composition of the star.

It is usual to express the composition in terms of X, Y and Z — the fractional abundances of hydrogen (X), helium (Y) and heavier elements (Z) by mass. Values for middle of the road stars like the Sun are $X = 0.73$, $Y = 0.25$ and $Z = 0.02$ giving $\overline{\mu} = 0.6$. Abundances by volume are $H \sim 92.5\%$, $He \sim 7.3\%$ and heavier elements $\leq 0.2\%$.

For a fully ionised gas we have

$$\overline{\mu} = \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}$$

$$\text{Thus:} \quad P_g = (2X + \frac{3}{4}Y + \frac{1}{2}Z)\,\frac{\rho kT}{m_H}$$

10.2 Hydrodynamics

Hydrodynamics can be considered in two main ways:

- Lagrangian flow, where individual fluid parcels or mass elements are considered. The co-ordinates of each parcel changes with time, and forces on the parcel (which can be thought of as an infinitesimal volume dV) are easy to compute by considering Newton's laws and thermodynamic laws on this parcel.
- Eulerian flow, where the coordinate system is fixed, and the fluid flows past the fixed coordinates. This is the natural coordinate system where fluid is flowing by is in steady state (the same amount of fluid flows in the same direction at all time).

These two co-ordinate systems are related by the *Lagrangian derivative*:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \tag{171}$$

This and the following equations in an Eulerian flow have derivations throughout the literature, and if time permits, will be derived in the lectures.

10.2.1 Equation of Continuity

The equation of continuity states that the rate of change of density balances the divergence of fluid flow, i.e. if mass is diverging from a point, then the density will go down:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{172}$$

10.2.2 Momentum Equation (Euler's Equation)

Momentum is a little more complicated, as there is the possibility of external forces per unit volume, and a gradient in pressure exerts a force on a gas particle. Eulers' equation is:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho}\nabla P = f_{\text{ext}}. \tag{173}$$

10.2.3 Example: The solar wind

In this example we will follow the groundbreaking work of Parker (1958, ApJ, 128, 644). We can use the two equations above to derive the structure of the solar wind after making the (tiny) approximation of a constant temperature and therefore a constant isothermal sound speed:

$$c_s = \sqrt{kT/\mu}, \tag{174}$$

where μ is the mean molecular weight (about half a proton mass for a fully ionised gas). We also make the approximation that the solar wind is in a steady, spherically symmetric state. This means we are searching for a function $v(r)$ which is not an explicit function of t. In the continuity equation, $\partial\rho/\partial t = 0$, and we compute the divergence in spherical coordinates:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{175}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2 \rho v) = 0 \tag{176}$$

$$r^2 \rho v = \text{constant} = \dot{(M)}/4\pi. \tag{177}$$

Despite the apparent complexity of having to use the divergence operator in spherical co-ordinates, the end result is very simple: the mass dM lost in a time $dt$is just equal to the volume of a shell of thickness $dr = vdt$ () multiplied by the shell density, i.e.:

$$dM = 4\pi r^2 v dt \rho. \tag{178}$$

The momentum equation is similar: the ∂t derivative is zero, and we need to put the gravitational force on the right-hand side:

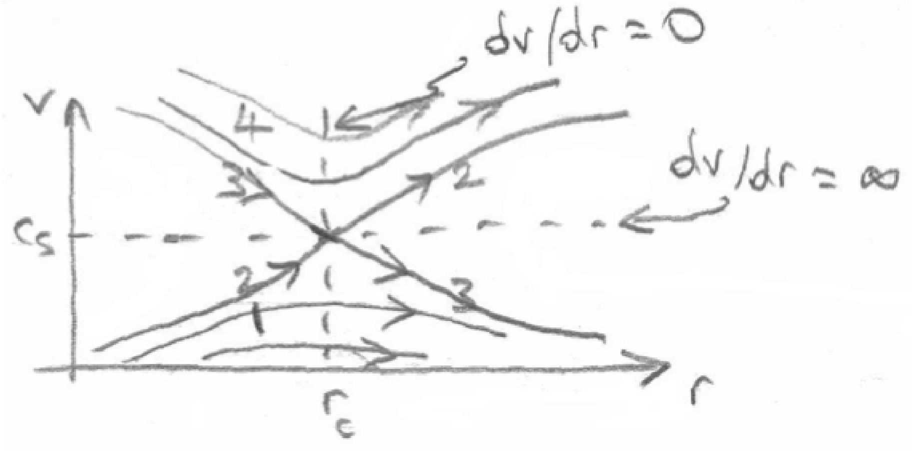


Figure 2: A sketch of the solar wind solutions, courtesy of Mark Wardle.

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho}\nabla P = f_{\text{ext}} \tag{179}$$

$$v\frac{dv}{dr} + \frac{c_s^2}{\rho}\frac{d\rho}{dr} = -\frac{GM}{r^2}, \tag{180}$$

where we have used the ideal gas equation along with the definition of the isothermal sound speed given above. Putting equations 177 and **??** together (exercise for the reader or done on the whiteboard in class) we arrive at:

$$\frac{dv}{dr} = \frac{2v}{r}\frac{1 - r_c/r}{v^2/c_s^2 - 1} \tag{181}$$

$$\text{where } r_c = \frac{GM}{2c_s^2} \tag{182}$$

The possible solutions to this equation are shown in Figure 2. Note that there is a discontinuity at $v = c_s$ unless $r = r_c$. Only the class 1 and 2 solutions are physical, due to low velocity at the photosphere and no discontinuity. It turns out that the class 2 solution most closely matches reality. This involves a very high velocity wind ($\sim$500 km/s) and low densities. This *solar wind* eventual interacts with the interstellar medium at the *termination shock*.

10.3 Shocks

Whenever two media are travelling towards each other at more than the speed of sound, a shock front is eventually formed. This is a discontinuity, where we still have to conserve mass, momentum and energy. This is best achieved with the use of the *Rankine-Hugoniot conditions*, written for a planar shock in Eulerian coordinates at rest with respect to the shock front. They equate conditions in front of (variables with subscript 0) and behind (variables with subscript 1) the shock front. They are normalised per unit area of the shock front. The first of these equations is the mass conservation equation:

$$\rho_0 v_0 = \rho_1 v_1. \tag{183}$$

This equation can be used to find the shock front velocity u_s' in some other reference frame. For example, if velocities in some primed (') coordinate frame are known, then:

$$\begin{aligned} \rho_0(v_0' - v_s') &= \rho_1(v_1' - v_s') && (184)\\ v_s' &= \frac{\rho_1 v_1' - \rho_0 v_0'}{\rho_1 - \rho_0}. && (185)\\ & && (186) \end{aligned}$$

The second equation is a momentum conservation equation:

$$P_0 + \rho_0 v_0^2 = P_1 + \rho_1 v_1^2. \tag{187}$$

The third equation is an energy conservation equation. The difference between the rate of energy entering and leaving the shock is equated to the work done by the pressure difference:

$$\dot{E}_0 - \dot{E}_1 = P_1 v_1 - P_0 v_0. \tag{188}$$

The energy flow rates on the left hand side consist of kinetic energy plus gas internal energy. This equation is written with a sign convention that velocities are negative in the shock front's reference frame. The right hand side of this equation is easily justified by considering the shock front to be within a closed system where Fdx work is done on the gas by membranes on either side of the shock front.

There are two easy-to-understand classes of shocks in astrophysics. The first is the *adiabatic* shock, where radiation is neglected and the work done by the shock completely goes into an increase in gas internal energy. The second kind of shock is the *isothermal* shock, where radiation is approximated as being very efficient, losing all its energy in a very narrow region after the shock. In this case, where where the pressure difference does not produce a phase change and we can therefore assume the ideal gas equation, the right hand side of Equation 188 is equal to zero (from Equation 183, with P and ρ proportional to each other for constant T). The energy is radiated out of the shocked region, and the shock front luminosity is given by:

$$L_s = \frac{1}{2}\rho_0 u_0^3 - \frac{1}{2}\rho_1 u_1^3. \tag{189}$$

We will discuss examples of this in class.