CHAPTER XI

DEGENERATE STELLAR CONFIGURATIONS AND THE THEORY OF WHITE DWARFS

The white dwarf stars differ from those we have considered so far in two fundamental respects. First, they are what might be called "highly underluminous"; that is, judged with reference to an "average" star of the same mass, the white dwarf is much fainter. Thus, the companion of Sirius, although it has a mass about equal to that of the sun, is yet characterized by a value of L which is only 0.003 times that of the sun. Second, the white dwarfs are characterized by exceedingly high values for the mean density; in fact, we encounter densities of the order of 10^6 and even 10^8 gm cm^{-3}. It is this second characteristic which is generally emphasized, though from a theoretical point of view the fact that $L/L_{\odot}$ is generally very small is of equal importance.

Since the radius of a white dwarf is very much smaller than that of a star on the main series, it follows that for a given effective temperature the white dwarf will be much fainter than the star on the main series. Similarly, for the same luminosity the white dwarf will be characterized by a very much higher effective temperature (i.e., much "whiter") than the main-series star; this, incidentally, explains the origin of the term "white dwarf."

We shall discuss the observational material in somewhat greater detail in § 3, but it should already appear plausible that the white dwarfs differ from other stars in some fundamental way. The clue to the understanding of the structure of these stars was discovered by R. H. Fowler, who pointed out that the electron gas in the interior of the white dwarfs must be highly degenerate in the sense made precise in the last chapter. We shall see that the white dwarfs can, in fact, be idealized to a high degree of approximation as completely degenerate configurations. In this chapter we shall be mainly concerned with the applications of the theory of degeneracy toward the elucidation of the structure of the white dwarfs.

1. *The gaseous fringe of the white dwarfs.*—It is clear that the extreme outer layers of a white dwarf must, in any case, be gaseous, i.e., nondegenerate, with the perfect gas law, $p \propto \rho T$, obeyed. The question then arises as to how far inward we can descend before degeneracy sets in. To answer this question we shall have to consider the criterion for degeneracy which was established in the last chapter (Eq. [211]) and which we shall now write in the form

$$\frac{(\vartheta mc^2)^2}{4\pi^2} \frac{f(x)}{x(1+x^2)^{1/2}} \gg 1 \,, \tag{1}$$

where

$$\vartheta = \frac{1}{kT}\,; \qquad f(x) = x(2x^2-3)(x^2+1)^{1/2} + 3 \sinh^{-1} x \,. \tag{2}$$

Finally, x is related to the mean electron concentration, n, by (Eq. [212], x)

$$n = \frac{8\pi m^3 c^3}{3h^3} x^3 \,. \tag{3}$$

We shall write

$$\rho = \frac{8\pi m^3 c^3}{3h^3} \mu_e H = Bx^3 \,, \tag{4}$$

where

$$B = \frac{8\pi m^3 c^3}{3h^3} \mu_e H = 9.82 \times 10^5 \mu_e \,. \tag{5}$$

Anticipating our result that the region of the white dwarf where the perfect gas law is valid is an outer fringe only, we can use for describing the structure of this gaseous fringe the theory of the stellar envelope given in chapter viii. On account of the very small values of L and R for the white dwarfs, the quantity α as defined in chapter viii (Eq. [54]) is very small indeed ($1-\beta \sim 10^{-4}$), so that we can use the analysis of § 3 of chapter viii. We then have

$$T = \frac{4}{17} \frac{\mu H}{k} \frac{GM}{R} \left(\frac{1}{\xi} - 1\right) \tag{6}$$

and

$$\rho = \frac{1}{30 f(0; w^*)} \bar{\rho} \left(\frac{1}{\xi} - 1\right)^{3.25} \,, \tag{7}$$

where $\bar{\rho}$ is the mean density, ξ is the radius vector expressed in terms of the radius of the star, and $f(0; w^*)$ is defined as in equation (55) of chapter viii. Inserting numerical values and expressing L, M, and R in solar units, we find that

$$T = 5.43 \times 10^6 \mu \frac{M}{R} \left(\frac{1}{\xi} - 1\right) \tag{8}$$

$$\rho = 0.761 \frac{\mu^{3.75} \bar{t}_c^{0.5}}{(1 - X_0^2)^{1/2}} \left(\frac{M^{7.5}}{LR^{6.5}}\right)^{1/2} \left(\frac{1}{\xi} - 1\right)^{3.25} . \tag{9}$$

By (4) and (9), we now find

$$x^3 = 7.75 \times 10^{-7} \frac{\mu^{3.75} \bar{t}_c^{0.5}}{\mu_e (1 - X_0^2)^{1/2}} \left(\frac{M^{7.5}}{LR^{6.5}}\right)^{1/2} \left(\frac{1}{\xi} - 1\right)^{3.25} . \tag{10}$$

By (1) and (8) we find that

$$3.04 \times 10^4 \frac{1}{\mu^2} \frac{R^2}{M^2} \left(\frac{1}{\xi} - 1\right)^{-2} \frac{f(x)}{x(x^2 + 1)^{1/2}} \gg 1 . \tag{11}$$

From (10) and (11) we can determine the point at which the right-hand side of (11) is unity; at this point we may say that "degeneracy sets in."

For most practical purposes it is found that it is sufficient to consider for $f(x)$ the limiting form which it takes for small values of x. By equation (24) of chapter x

$$f(x) \sim \tfrac{8}{5} x^5 \qquad (x \to 0) . \tag{12}$$

The inequality (11) now takes the simpler form,

$$4.86 \times 10^4 \frac{1}{\mu^2} \frac{R^2}{M^2} \left(\frac{1}{\xi} - 1\right)^{-2} x^4 \gg 1 . \tag{13}$$

Eliminating x between (10) and (13), we find that

$$2.54 \times 10^{-3} \frac{\mu^{2.25} \bar{t}_c^{0.5}}{\mu_e (1 - X_0^2)^{1/2}} \left(\frac{M^{4.5}}{LR^{3.5}}\right)^{1/2} \left(\frac{1}{\xi} - 1\right)^{1.75} \gg 1 . \tag{14}$$

Now, since for the white dwarfs L and R are quite small, it follows that for values of ξ appreciably different from unity the right-hand side is, in fact, much greater than unity. Thus, if we consider the case of the companion of Sirius, for which (according to Kuiper) Log $M = -0.01$, Log $L = -2.52$, and Log $R = -1.71$, equation (14) takes the form

$$43\,\frac{\mu^{2.25}\bar{t}_e^{0.5}}{\mu_e(1-X_0^2)^{1/2}}\left(\frac{1}{\xi}-1\right)^{1.75} \gg 1\,. \tag{15}$$

If we assume that $\mu \sim \mu_e = 1.0$, $X_0 = \frac{1}{3}$, $\bar{t}_e = 10$,[1] then the right-hand side of (15) is unity for $\xi = 0.94$. At this point, according to Table 17, the mass traversed from the boundary is only 0.23 per cent of the mass of the star; further, it is found that at this point $x = 0.12$, in agreement with our assumption that x is small. Finally, at $\xi = 0.94$, according to (8) and (9), ρ is found to be 1730 gm cm^{-3}, while T is 1.7×10^7 degrees. For some of the other white dwarfs the situation is even more "favorable," in the sense that the gaseous fringe is of even smaller extent. We thus see that the material of the white dwarf must be almost entirely degenerate; this result is implicitly contained in Fowler's work, but the arguments, essentially in the form we have given them, are due to Strömgren and Sidentopf.

2. *Completely degenerate configurations.* We have seen in § 1 that the gaseous fringe of a white dwarf is of quite negligible extent, and that, further, the radiation is entirely negligible indeed, in the gaseous fringe $1 - \beta \sim 10^{-4}$ or less. It is almost certain (cf. the discussion in § 6) that in the interior $1 - \beta$ does not exceed its value in the gaseous fringe, and we are thus led to consider equilibrium configurations which are completely degenerate and in which the radiation pressure is entirely neglected. The general theory given in this section is due to Chandrasekhar.

The equation of state can be written as (cf. Eqs. [19], [20], and [21] of the last chapter)

$$P = Af(x)\,; \qquad \rho = n\mu_e H = Bx^3\,, \tag{16}$$

[1] According to Strömgren, under the conditions of the gaseous fringe of a white dwarf, the guillotine factor $\bar{t}_e$ must be quite large.

where

$$A = \frac{\pi m^4 c^5}{3h^3} = 6.01 \times 10^{22}\,; \qquad B = \frac{8\pi m^3 c^3 \mu_e H}{3h^3} = 9.82 \times 10^5 \mu_e \quad (17)$$

and

$$f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x\,. \quad (18)$$

The equation of equilibrium is (Eq. [6], iii)

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -\,4\pi G\rho\,. \quad (19)$$

Substituting for P and ρ according to (16), we have

$$\frac{A}{B}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{x^3}\frac{df(x)}{dr}\right) = -4\pi GBx^3\,. \quad (20)$$

From the definition of $f(x)$ we easily verify that

$$\frac{df(x)}{dr} = \frac{8x^4}{(x^2 + 1)^{1/2}}\frac{dx}{dr}\,, \quad (21)$$

or

$$\frac{1}{x^3}\frac{df(x)}{dr} = \frac{8x}{(x^2 + 1)^{1/2}}\frac{dx}{dr} = 8\,\frac{d\sqrt{x^2 + 1}}{dr}\,. \quad (22)$$

Hence, equation (20) can be re-written as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{d\sqrt{x^2 + 1}}{dr}\right) = -\frac{\pi GB^2}{2A}\,x^3\,. \quad (23)$$

Put

$$y^2 = x^2 + 1\,. \quad (24)$$

Then,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dy}{dr}\right) = -\frac{\pi GB^2}{2A}\,(y^2 - 1)^{3/2}\,. \quad (25)$$

Let x take the value x_0 at the center. Further, let y_0 be the corresponding value of y at the center. Introduce the new variables η and ϕ, defined as follows:

$$r = a\eta\,; \qquad y = y_0\phi\,, \quad (26)$$

where

$$\alpha = \left(\frac{2A}{\pi G}\right)^{1/2} \frac{1}{By_0}; \qquad y_0^2 = x_0^2 + 1 . \tag{27}$$

The differential equation finally takes the form

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta}\right) = -\left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2} . \tag{28}$$

By (26) we have to seek a solution of (28) such that ϕ takes the value unity at the origin. Further, it is clear that the derivative of ϕ must vanish at the origin. The boundary is defined at the point where the density vanishes, and this by (24) means that if η_1 specifies the boundary, then

$$\phi(\eta_1) = \frac{1}{y_0} . \tag{29}$$

From our definitions of the various quantities it is easily seen that

$$\rho = \rho_0 \frac{y_0^3}{(y_0^2 - 1)^{3/2}} \left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2} , \tag{30}$$

where

$$\rho_0 = Bx_0^3 = B(y_0^2 - 1)^{3/2} \tag{31}$$

specifies the central density. Also, we may notice that the scale of length, α, introduced in (27), has, in terms of the natural constants, the form

$$\alpha = \frac{1}{4\pi m \mu_e H y_0} \left(\frac{3h^3}{2cG}\right)^{1/2} , \tag{32}$$

or, inserting numerical values,

$$\alpha = \frac{7.71 \times 10^8}{\mu_e\, y_0} = l_1 y_0^{-1} \text{ cm} . \tag{33}$$

We shall now consider a little more closely the structure of the configurations governed by the differential equation (28).

a) *The potential.*—The function ϕ has a physical meaning. If V is the inner gravitational potential, then from the general theory (chap. iii, § 2)

$$\frac{dV}{dr} = -\frac{1}{\rho} \frac{dP}{dr} . \tag{34}$$

From (16), (18), and (22) we see that

$$\frac{dV}{dr} = -\frac{8A}{B} y_0 \frac{d\phi}{dr} ; \tag{35}$$

or, integrating, we find that

$$V = -\frac{8A}{B} y_0 \phi + \text{constant} . \tag{36}$$

If we choose the zero of the potential at infinity, we have by (29) that the "constant" in (36) is $[(8A/B) - GM/R]$ (cf. Eq. [10], iii). Hence,

$$V = -\frac{8A}{B} y_0 \left(\phi - \frac{1}{y_0}\right) - \frac{GM}{R} \qquad (r \leqslant R) \tag{37}$$

b) The mass relation.—The mass, interior to a specified point η, is given by

$$M(\eta) = 4\pi \int_0^\eta \rho r^2 dr = 4\pi a^3 \int_0^\eta \rho \eta^2 d\eta . \tag{38}$$

By (30),

$$M(\eta) = 4\pi \rho_0 \frac{a^3 y_0^3}{(y_0^2 - 1)^{3/2}} \int_0^\eta \left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2} \eta^2 d\eta ; \tag{39}$$

or, using the differential equation (28),

$$M(\eta) = -4\pi \rho_0 \frac{a^3 y_0^3}{(y_0^2 - 1)^{3/2}} \eta^2 \frac{d\phi}{d\eta} . \tag{40}$$

Substituting for a and ρ_0 according to (27) and (31), we have

$$M(\eta) = -4\pi \left(\frac{2A}{\pi G}\right)^{3/2} \frac{1}{B^2} \eta^2 \frac{d\phi}{d\eta} . \tag{41}$$

The mass of the whole configuration is given by

$$M = -4\pi \left(\frac{2A}{\pi G}\right)^{3/2} \frac{1}{B^2} \left(\eta^2 \frac{d\phi}{d\eta}\right)_{\eta=\eta_1} . \tag{42}$$

We notice that in (41) and (42) y_0 does not occur explicitly. It is, of course, implicitly present inasmuch as y_0 occurs in the differential equation defining ϕ.

c) The relation between the mean and the central density.—Let $\bar{\rho}(\eta)$ be the mean density of the material inside η. Then

$$M(\eta) = \tfrac{4}{3}\pi a^3 \eta^3 \bar{\rho}(\eta) . \tag{43}$$

Comparing (40) and (43), we have

$$\frac{\bar{\rho}(\eta)}{\rho_0} = -3 \frac{y_0^3}{(y_0^2 - 1)^{3/2}} \frac{1}{\eta} \frac{d\phi}{d\eta} . \tag{44}$$

From (44) we deduce that the relation between the mean and the central density of the whole configuration is

$$\rho_0 = -\bar{\rho} \left(1 - \frac{1}{y_0^2}\right)^{3/2} \frac{\eta_1}{3\phi'(\eta_1)} , \tag{45}$$

where ϕ' denotes the derivative of ϕ. It is of interest to notice the similarity between the present relations (42) and (45) and the corresponding relations in the theory of polytropes (Eqs. [69] and [78] of chap. iv).

d) An approximation for configurations with small central densities. —By definition, $y_0^2 = x_0^2 + 1$, and we need a first-order approximation when x_0^2 is small. We shall neglect all quantities of order x_0^4 and higher. Then,

$$y_0 = 1 + \tfrac{1}{2} x_0^2 . \tag{46}$$

Put

$$\phi^2 - \frac{1}{y_0^2} = \theta . \tag{47}$$

In our present approximation we have

$$\phi = 1 - \tfrac{1}{2}(x_0^2 - \theta) . \tag{48}$$

At the origin, ϕ takes the value unity. Hence,

$$\theta(0) = x_0^2 . \tag{49}$$

From (28) we derive the following differential equation for θ.

$$\frac{1}{2} \frac{d^2\theta}{d\eta^2} + \frac{1}{\eta} \frac{d\theta}{d\eta} = -\theta^{3/2} . \tag{50}$$

Finally, introduce the variable ξ, according to

$$\xi = 2^{1/2}\eta \; . \tag{50'}$$

Equation (50) now reduces to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^{3/2} \; , \tag{51}$$

which is the Lane-Emden equation with index $n = 3/2$, but the solution we need is *not* the Lane-Emden function $\theta_{3/2}$. According to (49), we need a solution of (51) which takes the value x_0^2 at $\xi = 0$. Now, according to the homology theorem of chapter iv, § 8, as applied to the case $n = 3/2$, if $\theta(\xi)$ is a solution of (51), then $C^4\theta(C\xi)$ is also a solution, where C is an arbitrary real number. Hence, from $\theta_{3/2}$ we can derive a function satisfying (49) by a homologous transformation of $\theta_{3/2}$:

$$\theta = x_0^2\,\theta_{3/2}(\sqrt{x_0}\,\xi) \; . \tag{52}$$

Hence, by (48), (50), and (52)

$$\phi = 1 - \tfrac{1}{2}x_0^2\{1 - \theta_{3/2}(\sqrt{2x_0}\,\eta)\} + O(x_0^4) \; , \tag{53}$$

which relates ϕ with $\theta_{3/2}$. From (53) we see that for these configurations the boundary η_1 must be such that

$$\theta_{3/2}(\sqrt{2x_0}\,\eta_1) = 0 \; , \tag{54}$$

since, according to (29) and (46), $\phi(\eta_1) = y_0^{-1} = 1 - \frac{1}{2}x_0^2$. If $\xi_1(\theta_{3/2})$ is the boundary of the Lane-Emden function, then from (54) we deduce that

$$\eta_1 = \frac{\xi_1(\theta_{3/2})}{\sqrt{2x_0}} \; . \tag{55}$$

Again, from (53) we have

$$\frac{d\phi}{d\eta} = \tfrac{1}{2}x_0^2\sqrt{2x_0}\,\frac{d\theta_{3/2}(\xi)}{d\xi} \; . \tag{56}$$

Combining (55) and (56), we find that

$$\left(\eta^2\frac{d\phi}{d\eta}\right)_{\eta=\eta_1} = \left(\frac{x_0}{2}\right)^{3/2}\left(\xi^2\frac{d\theta_{3/2}}{d\xi}\right)_{\xi=\xi_1(\theta_{3/2})} \; . \tag{57}$$

Further,

$$\left(\frac{1}{\eta}\frac{d\phi}{d\eta}\right)_{\eta=\eta_1} = x_0^3\left(\frac{1}{\xi}\frac{d\theta_{3/2}}{d\xi}\right)_{\xi=\xi_1(\theta_{3/2})}. \tag{58}$$

From (58) and (45) we have

$$\rho_0 = -\bar{\rho}\,\frac{\xi_1(\theta_{3/2})}{3\theta'_{3/2}(\xi_1)}, \tag{59}$$

which is precisely the relation between the mean and the central density for a Lane-Emden polytrope of index $n = 3/2$. Again, from (42) and (57),

$$M = -4\pi\left(\frac{2A}{\pi G}\right)^{3/2}\frac{1}{B^2}\left(\frac{x_0}{2}\right)^{3/2}\left(\xi^2\frac{d\theta_{3/2}}{d\xi}\right)_{\xi=\xi_1(\theta_{3/2})}. \tag{60}$$

On the other hand, if $x_0 \to 0$, we can write the equation of state (16) in the form

$$P = \frac{8A}{5}x^5; \qquad \rho = Bx^3, \tag{61}$$

or

$$P = K_1\rho^{5/3}, \tag{62}$$

where

$$K_1 = \frac{8A}{5B^{5/3}} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m(\mu_e H)^{5/3}} = \frac{9.91 \times 10^{12}}{\mu_e^{5/3}}. \tag{63}$$

Hence, configurations with small central densities (i.e., x_0 small) are Lane-Emden polytropes of index $n = 3/2$. The results based on (63) and the theory of polytropes, and the approximation derived from the exact differential equation (28) for $x_0 \to 0$ are easily seen to be equivalent. In particular, using (63), the mass relation (60) can be re-written in the form

$$M = -4\pi\left(\frac{5K_1}{8\pi G}\right)^{3/2}\rho_0^{1/2}\left(\xi^2\frac{d\theta_{3/2}}{d\xi}\right)_{\xi=\xi_1(\theta_{3/2})}, \tag{64}$$

which is identical with the mass relation for a polytrope of index $n = 3/2$ based on the law (62) (cf. Eq. [69], iv).

c) The limiting mass.—From the differential equation (28) we see that

$$\phi \to \theta_3 \qquad \text{as} \qquad y_0 \to \infty. \tag{65}$$

But from (33) it follows that at the same time the radius tends to zero. From the mass relation (42), on the other hand, we see that the mass tends to a finite limit:

$$M \to -4\pi \left(\frac{2A}{\pi G}\right)^{3/2} \frac{1}{B^2} \left(\xi^2 \frac{d\theta_3}{d\xi}\right)_{\xi=\xi_1(\theta_3)} . \tag{66}$$

The existence of this limiting mass was first isolated by Chandrasekhar, though its existence had been made apparent from earlier considerations by Anderson and Stoner, who, however, did not consider the problem from the point of view of hydrostatic equilibrium.

For $x_0 \to \infty$ we can write (16) in the form

$$P = 2Ax^4 ; \qquad \rho = Bx^3 , \tag{67}$$

or

$$P = K_2 \rho^{4/3} , \tag{68}$$

where

$$K_2 = \frac{2A}{B^{4/3}} = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8(\mu_e H)^{4/3}} = \frac{1.231 \times 10^{15}}{\mu_e^{4/3}} . \tag{69}$$

By equation (70) of chapter iv the mass of a Lane-Emden configuration based on (68) is given by

$$M = -4\pi \left(\frac{K_2}{\pi G}\right)^{3/2} \left(\xi^2 \frac{d\theta_3}{d\xi}\right)_{\xi=\xi_1} , \tag{70}$$

which is seen to be equivalent to (66) on substituting for K_2 according to (69).

We shall denote by M_3 the limiting mass (66).[2] The mass relation (42) can then be written in the form

$$M(y_0) = M_3 \frac{\Omega(y_0)}{{}_0\omega_3} , \tag{71}$$

where

$${}_0\omega_3 = -\left(\xi^2 \frac{d\theta_3}{d\xi}\right)_{\xi=\xi_1(\theta_3)} ; \qquad \Omega(y_0) = -\left(\eta^2 \frac{d\phi}{d\eta}\right)_{\eta=\eta_1} \tag{72}$$

[2] We denote the limiting mass by M_3 since, as $x \to \infty$, $\phi \to \theta_3$, the Lane-Emden function of index 3.

As the mass of the configuration increases monotonically with increasing y_0, we have the useful inequality

$$\Omega(y_0) < {}_0\omega_3 \qquad (y_0 \text{ finite}) . \tag{73}$$

Finally, we may note that the insertion of numerical values in the formula for M_3 yields

$$M_3 = 5.75\mu_e^{-2} \times \odot . \tag{74}$$

f) The internal energy.—By equation (23) of chapter x, the internal energy U of the configuration is given by

$$U = A\int_0^R \{8x^3[(1 + x^2)^{1/2} - 1] - f(x)\}dV ; \tag{75}$$

or, using equations (16) and (17) which express the equation of state, we can re-write the foregoing in the form

$$U = \frac{8A}{B}\int_0^R \rho[(1 + x^2)^{1/2} - 1]dV - \int_0^R PdV . \tag{75'}$$

But by equation (32) of chapter iii the second term on the right-hand side of (75′) is $-\Omega/3$ where Ω is the potential energy. Hence,

$$U = \frac{8A}{B}\int_0^R [(1 + x^2)^{1/2} - 1]dM(r) + \tfrac{1}{3}\Omega ; \tag{76}$$

or, expressing x in terms of ϕ (cf. Eqs. [24] and [27]), we have

$$U = \frac{8Ay_0}{B}\int_0^R \left(\phi - \frac{1}{y_0}\right) dM(r) + \tfrac{1}{3}\Omega . \tag{76'}$$

Using (37) for expressing ϕ in terms of the potential V, we obtain

$$U = -\int_0^R \left(V + \frac{GM}{R}\right) dM(r) + \tfrac{1}{3}\Omega . \tag{77}$$

Finally, using equation (16) of chapter iii, we find

$$U = -\tfrac{2}{3}\Omega - \frac{GM^2}{R} . \tag{78}$$

For the case under consideration the internal energy is due entirely to the kinetic energies of the motions of the electrons; we can, therefore, write

$$T = U = -\tfrac{5}{3}\Omega - \frac{GM^2}{R} . \tag{79}$$

The total energy, E, of the configuration is

$$E = U + \Omega = -\tfrac{2}{3}\Omega - \frac{GM^2}{R} . \tag{79'}$$

For stars of small mass the configurations are (as we have shown in section d, above) polytropes of index $n = 3/2$, and by equation (90) of chapter iv,

$$\Omega = -\frac{6}{7}\frac{GM^2}{R} \qquad (M \ll M_3) . \tag{80}$$

By (79) and (80) we have

$$T = -\tfrac{1}{2}\Omega , \tag{80'}$$

which is the statement of the virial theorem (chap. ii, § 10) derived on the basis of Newtonian mechanics. On the other hand, if $M \to M_3'$, then (again by Eq. [90], iv),

$$\Omega = -\frac{3}{2}\frac{GM^2}{R} \qquad (M \to M_3) . \tag{81}$$

By (79) and (81) we now have

$$T = -\Omega , \tag{81'}$$

which must be the statement of the virial theorem for material particles moving with very nearly the velocity of light.

g) General results.—In sections d and e we have considered certain limiting cases. However, the exact treatment on the basis of the differential equation (28) will provide much more quantitative information. The boundary conditions,

$$\phi = 1 , \qquad \frac{d\phi}{d\eta} = 0 \qquad \text{at} \qquad \eta = 0 ,$$

combined with a particular value for y_0 will determine ϕ completely and therefore the mass of the configuration as well. Equation (28) does not admit of a homology constant, and hence *each mass has a density distribution characteristic of itself which cannot be inferred from the density distribution in a configuration of a different mass.* This is the most fundamental difference between our present configurations and the polytropes. We thus see that each specified value for y_0 determines uniquely the mass M, the radius R, the ratio of the mean to the central density, and the march of the density distribution. We have (collecting our results):

$$\left.\begin{aligned} \frac{M}{M_3} &= \frac{\Omega(y_0)}{{}_0\omega_3}\,, \\ \frac{R}{l_1} &= \frac{\eta_1}{y_0}\,, \\ \frac{\rho_0}{B} &= (y_0^2 - 1)^{3/2}\,, \\ \frac{\bar{\rho}}{\rho_0} &= -\frac{1}{\left(1 - \frac{1}{y_0^2}\right)^{3/2}} \frac{3}{\eta_1}\left(\frac{d\phi}{d\eta}\right)_{\eta=\eta_1}. \end{aligned}\right\} \tag{82}$$

In (82) we have introduced the unit of length ($l_1 = ay_0$),

$$l_1 = \frac{1}{4\pi m \mu_e H}\left(\frac{3h^3}{2cG}\right)^{1/2} = 7.71\mu_e^{-1} \times 10^8 \text{ cm}\,, \tag{82'}$$

which, therefore, does not involve the factor in y_0. Further, the physical variables determining the structure of the configurations are:

$$\left.\begin{aligned} \rho &= \rho_0 \frac{1}{\left(1 - \frac{1}{y_0^2}\right)^{3/2}} \left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2}\,, \\ \bar{\rho}(\eta) &= -\rho_0 \frac{1}{\left(1 - \frac{1}{y_0^2}\right)^{3/2}} \frac{3}{\eta}\frac{d\phi}{d\eta}\,, \\ \frac{M(\eta)}{M_3} &= \frac{\left(\eta^2 \frac{d\phi}{d\eta}\right)}{\left(\xi^2 \frac{d\theta_3}{d\xi}\right)_{\xi=\xi_1(\theta_3)}}\,. \end{aligned}\right\} \tag{83}$$

h) *Numerical results.*—In section *g* we reduced the problem of the structure of degenerate gas spheres to a study of the function ϕ for different initially prescribed values of the parameter y_0. The integration has been numerically effected by Chandrasekhar for ten different values of the parameter:

$$\frac{1}{y_0^2} = 0.8, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.02, 0.01 .$$

The integration is started at the origin by a series expansion and then continued by standard numerical methods. The following expansion for ϕ near the origin may be noted here:

$$\left.\begin{aligned}\phi = 1 - \frac{q^3}{6}\eta^2 + \frac{q^4}{40}\eta^4 - \frac{q^5(5q^2+14)}{7!}\eta^6 + \frac{q^6(339q^2+280)}{3\times 9!}\eta^8 \\ - \frac{q^7(1425q^4+11436q^2+4256)}{5\times 11!}\eta^{10} + \dots ,\end{aligned}\right\} \quad (84)$$

where $q^2 = (y_0^2 - 1)/y_0^2$. The important quantities of interest are the boundary quantities occurring in equation (82). These are tabulated in Table 25. From the figures in Table 25 it is easy to calculate the

TABLE 25

THE CONSTANTS OF THE WHITE-DWARF FUNCTIONS

$1/y_0^2$	η_1	$-\eta_1^2\phi'(\eta_1)$	$\rho_0/\bar{\rho}$
0	6.8968	2.0182	54.182
0.01	5.3571	1.9321	26.203
0.02	4.9857	1.8652	21.486
0.05	4.4601	1.7096	16.018
0.1	4.0690	1.5186	12.626
0.2	3.7271	1.2430	9.9348
0.3	3.5803	1.0337	8.6673
0.4	3.5245	0.8598	7.8886
0.5	3.5330	0.7070	7.3505
0.6	3.6038	0.5679	6.9504
0.8	4.0446	0.3091	6.3814
1.0	∞	0	5.9907

mass in units of M_3, the radius in units of l_1, and the central density in units of B ($= 9.82 \times 10^5 \mu_e$ gm cm^{-3}). These express the chief physical characteristics in the "natural system" of units occurring in the theory of these configurations (see Table 26). In Table 27 they are converted into the more conventional system of units which express the radius and the density in c.g.s. units and the mass in units of the

TABLE 26

THE PHYSICAL CHARACTERISTICS OF COMPLETELY DEGENERATE CONFIGURATION IN THE "NATURAL" UNITS

$1/y_0^2$	M/M_3	R/l_1	ρ_0/B
0	1	0	∞
0.01	0.95733	0.53571	985.038
0.02	0.92419	0.70508	343.
0.05	0.84709	0.99732	82.8191
0.1	0.75243	1.28674	27.
0.2	0.61589	1.66682	8.
0.3	0.51218	1.96102	3.56423
0.4	0.42600	2.22908	1.83711
0.5	0.35033	2.49818	1.[illegible]
0.6	0.28137	2.79148	0.54433
0.8	0.15316	3.61760	0.125
1.0	0	∞	0

TABLE 27*

THE PHYSICAL CHARACTERISTICS OF COMPLETELY DEGENERATE CONFIGURATIONS

$1/y_0^2$	$M/\odot$	ρ_0 in Grams per Cubic Centimeter	ρ_{mean} in Grams per Cubic Centimeter	Radius in Centimeters
0	5.75	∞	∞	0
0.01	5.51	9.85×10^8	3.70×10^7	4.13×10^8
0.02	5.32	3.37×10^8	1.57×10^7	5.44×10^8
0.05	4.87	8.13×10^7	5.08×10^6	7.69×10^8
0.1	4.33	2.65×10^7	2.10×10^6	9.92×10^8
0.2	3.54	7.85×10^6	7.9×10^5	1.29×10^9
0.3	2.95	3.50×10^6	4.04×10^5	1.51×10^9
0.4	2.45	1.80×10^6	2.29×10^5	1.72×10^9
0.5	2.02	9.82×10^5	1.34×10^5	1.93×10^9
0.6	1.62	5.34×10^5	7.7×10^4	2.15×10^9
0.8	0.88	1.23×10^5	1.02×10^4	2.79×10^9
1.0	0	0	0	∞

* The values given in this table differ slightly from the published values (S. Chandrasekhar *M.N.*, **95**, 208, 1935, Table III). The difference is due to the change in the accepted values of the fundamental physical constants.

The calculations are for $\mu_e = 1$. For the other values of μ_e, M should be multiplied by μ_e^{-2}, R by μ_e^{-1}, and ρ_0 by μ_e.

sun. To see the order of magnitude of the quantities involved, it is of interest to point out that the mass $4.87\,\odot\mu_e^{-2}$ has a radius only slightly larger than the radius of the earth, while the mass $0.957M_3$ has a radius considerably less than the radius of the earth. In Fig-

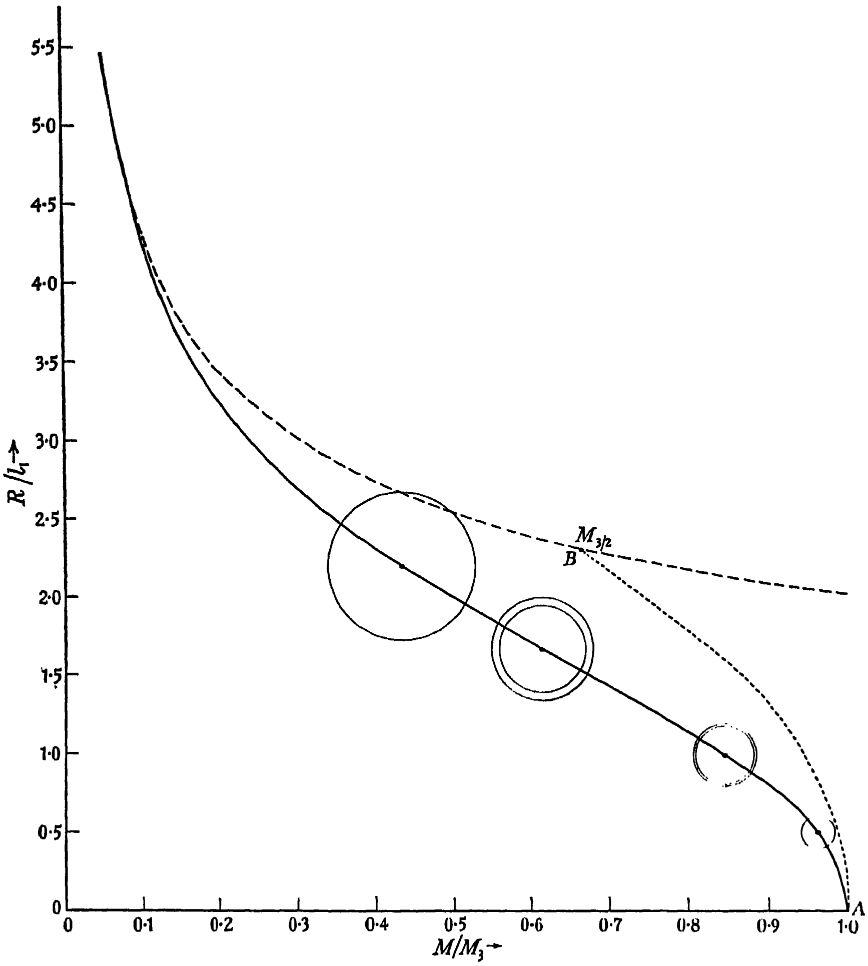

FIG. 31.—The solid-line curve represents the exact (mass, radius) relation for the completely degenerate configurations. This curve tends asymptotically to the dotted curve as $M \to 0$.

ures 31 and 32 we have illustrated the mass-radius and the mass-central density relationships. The dotted curves in the two cases are the corresponding relations based on the Lane-Emden polytrope of index $n = 3/2$ (the approximation considered in section d, above), and the exact curves tend toward these asymptotically for $M \to 0$. We notice from Figures 31 and 32 how marked the deviations of the dotted curves from the exact curves become even for quite small masses. Thus, for $M = 0.15 M_3$ the central density predicted by the exact treatment is about 25 per cent greater and the radius about 5 per cent smaller. The relativistic effects on the equation of state

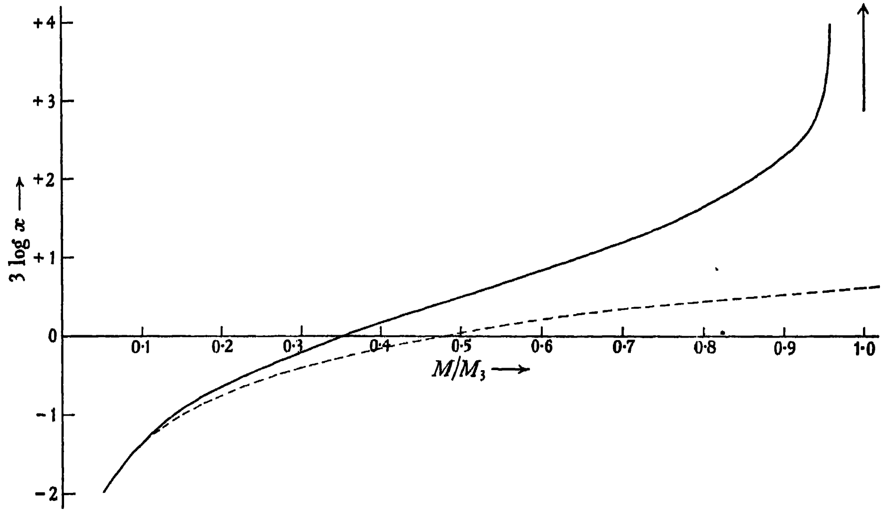

FIG. 32.—The solid-line curve represents the exact (mass, Log ρ_0) relation for the completely degenerate configurations. This curve tends asymptotically to the dotted curve as $M \to 0$.

are therefore quite significant even for small masses. They certainly cannot be ignored for masses greater than $0.2 M_3$. Of course, the extrapolation of the $n = 3/2$ configurations for masses approaching M_3 is quite misleading. The completely degenerate configurations have a natural limit, and our discussion based on the differential equation shows how this limit is reached.

i) The relative density distributions in the different configurations.— Our main diagram (Fig. 33) now illustrates the relative density distributions in the configurations studied. Here we have plotted ρ/ρ_0 against η/η_1 for the different masses for which we have numerical results. The two limiting density-distributions specified by the Lane-

Emden functions $\theta_{3/2}$ and θ_3 are also shown (dotted) in the same diagram. The density distributions specified by the differential equa-

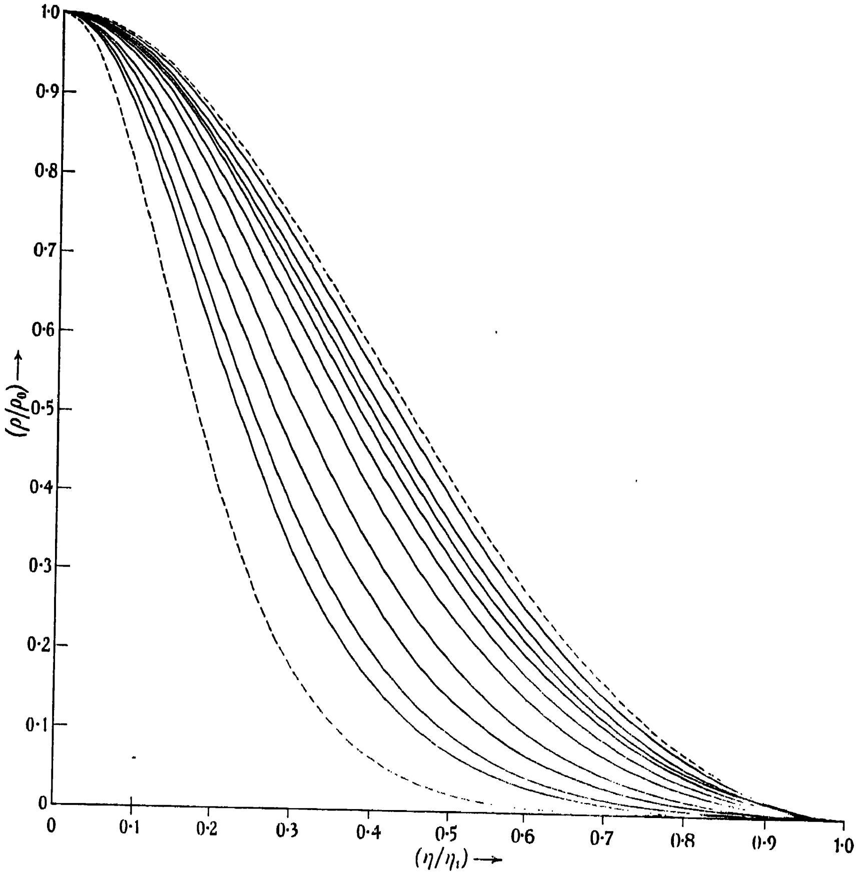

FIG. 33.—The relative density distributions in the completely degenerate configurations. The upper dotted curve corresponds to the polytropic distribution $n=\frac{3}{2}$, and the lower dotted curve to the polytropic distribution $n=3$. The inner curves represent the density distributions for $1/y_0^2=0.8$, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.02, and 0.01, respectively.

tion (28) thus form a continuous family which covers the range specified by the polytropic distributions of indices 3/2 and 3.

3. *The discussion of the observational material and of the theoretical mass-radius relation.*—We have already seen in § 1 that the gaseous

fringe of the known white dwarfs can be neglected (in the first approximation) and that we can regard them (in the first approximation) as completely degenerate configurations. The theory developed in § 2 can therefore be applied, as it stands, to the known white dwarfs. A glance at Table 27 shows that the mean density, the mass, and the radius of these degenerate configurations are all of the right order of magnitude to provide the basis for the theoretical discussion of the structure of the white dwarfs. However, a really satisfactory test of the theory will consist in providing an observational basis for the existence of a mass such that as we approach it the mean density increases several times, even for a slight increase in the mass. At the present time there is just one case which seems to support this aspect of the theoretical prediction.

The case in question is Kuiper's white dwarf (AC 70°8247), which is, from several points of view, a most remarkable star; for instance—and this is very unfortunate—in this star no spectral lines have been detected so far and only a pure continuous spectrum has been observed. According to Kuiper, the most probable values of L and R are

$$\operatorname{Log} L = -1.76\,, \qquad \operatorname{Log} R = -2.38\,, \tag{85}$$

L and R being expressed in solar units. From (85) we derive that

$$\bar{\rho} = 19{,}600{,}000 \left(\frac{M}{\odot}\right) \text{gm cm}^{-3}\,. \tag{86}$$

It is seen that we have here an unusually dense star. If we assume that $\mu_e = 1.48$, then the mass-radius relation established in § 2 leads to a mass of $2.5\odot$, which would correspond to an actual mean density of 49,000,000 gm cm^{-3}. On the other hand, if we use the approximation $P = K\rho^{5/3}$ (Eqs. [62] and [63]), then from the mass-radius relation for the polytropes (Eq. [74], iv) we easily derive that

$$\operatorname{Log} R = -\tfrac{1}{3} \operatorname{Log} M - \tfrac{5}{3} \operatorname{Log} \mu_e - 1.397\,, \tag{87}$$

where R and M are expressed in solar units. Assuming $\mu_e = 2.0$ (which is the maximum we can permit), we find that (87) leads to a mass of $28\odot$ for Kuiper's white dwarf; it should be noticed that this is the minimum mass predicted on the basis of (87). (If we assume for μ_e the more probable value of 1.5, then (87) leads to $M = 118\odot$.) Since the mass predicted on the model $P \propto \rho^{5/3}$ comes out unusually high, it seems likely that Kuiper's white dwarf does, in fact, provide a confirmation of the theory. In any case, it is clear that if spectral lines could be detected and identified in this star and the red shift measured, we might have a most valuable astronomical confirmation of the physical theory of degeneracy.[3]

However, since the theory is such a straightforward consequence of the quantum mechanics and, further, uses Dirac's theory of the electron only in that phase of its application which has been confirmed by laboratory experiments (Klein-Nishina formula, production of cosmic ray showers, etc.), there can be little doubt that it is essentially correct.

We have seen that the theory provides a unique mass-radius relation if the radius is measured in units of l_1 (Eq. [82]) and the mass in units of M_3. But these units involve the "molecular weight," μ_e, so that we can apply the theory to determine μ_e for white dwarfs for which both M and R are known, or to determine M for a white dwarf for which only the radius is known (assuming, however, a value for μ_e). It should be noticed that μ_e is not the same as the mean molecular weight μ used in the theory of gaseous stars. For, as the definition of μ_e we have used

$$\rho = n\mu_e H \,, \tag{88}$$

where n is the number of electrons per unit volume. For a mixture of elements which are all completely ionized we can write, in the notation of § 3 of chapter vii,

$$n = \frac{\rho}{H}\sum \frac{x_Z Z}{A_Z} \,, \tag{89}$$

where the element of atomic number Z and atomic weight A_Z is assumed to occur with an abundance x_Z by weight. The summa-

[3] There is a possibility that Wolf 219, another white dwarf discovered by Kuiper, for which Humason found recently a continuous spectrum, may be comparable to AC 70°8247. If confirmed, this star would be even more extraordinary than AC 70°8247, since it is of lower luminosity.

tion in (89) is extended over the elements present. Comparing (88) and (89), we derive

$$\mu_e = \frac{1}{\sum \left(x_Z \frac{Z}{A_Z} \right)} . \tag{90}$$

If X_0 is the abundance of hydrogen, we can re-write (90) as

$$\frac{1}{\mu_e} = X_0 + \sum_{Z \neq 1} \frac{x_Z Z}{A_Z} . \tag{91}$$

As a first approximation we can write $Z/A_Z = 1/2$ for all the metals and obtain

$$\mu_e = \frac{2}{1 + X_0} . \tag{92}$$

For the Russell mixture considered in chapter vii we find that

$$\mu_e = \frac{1}{0.492 + 0.508 X_0} . \tag{93}$$

We shall now consider briefly the other white dwarfs for which we have data.

a) *Sirius B*.—We have already considered this star in § 1. Using the data given there and using the theoretical mass radius relation, it is found that $\mu_e = 1.32$, $X_0 = 0.52$.

b) *o_2 Eridani B*.—According to Kuiper,

$$\operatorname{Log} L = -2.26 , \qquad \operatorname{Log} M = -0.35 , \qquad \operatorname{Log} R = -1.74 . \tag{94}$$

The mean density is 91,000 gm cm^{-3}. The theoretical mass-radius relation leads to $X_0 = 0.15$.

c) *Van Maanen No. 2*.—From the reliably known parallax and spectral type, Kuiper derives for this star

$$\operatorname{Log} L = -3.85 , \qquad \operatorname{Log} R = -2.05 . \tag{95}$$

The radial velocity of this star has been determined and found to be +238 km/sec. According to Oort, most of this must be due to the Einstein gravitational red shift. Assuming that the full amount is due to the red shift (which will give the right order of magnitude), it is found, with the value of R given (Eq. [95]), that

$$\operatorname{Log} M = 0.53\,, \qquad \bar{\rho} = 6{,}800{,}000 \text{ gm cm}^{-3}\,. \tag{96}$$

The mass-radius relation now leads to $\mu_e = 1.206$, $X_0 = 0.66$.

4. *A stellar criterion for degeneracy.*—In the last chapter we showed that the criterion for the applicability of the degeneracy formulae is (Eq. [211], x),

$$\frac{4\pi^2}{(\vartheta mc^2)^2}\,\frac{x(1+x^2)^{1/2}}{f(x)} \ll 1\,. \tag{97}$$

However, for applications to stellar problems it is more convenient to state the criterion for degeneracy in a rather different form.

Consider an assembly of N electrons contained in a volume V at temperature T. Then, on the basis of the perfect gas law, the electron pressure p_e would be given by

$$p_e = \left(\frac{N}{V}\right) kT\,. \tag{98}$$

At temperature T we also have radiation pressure of amount given by the Stefan-Boltzmann law

$$p_r = \tfrac{1}{3} a T^4\,. \tag{98'}$$

Let us denote by P the total pressure $(=p_r + p_e)$ and introduce a parameter β_e, defined as follows:

$$P = p_r + p_e = \frac{1}{\beta_e}\, p_e = \frac{1}{1-\beta_e}\, p_r\,. \tag{99}$$

Eliminating T between the relations (99), we find

$$p_e = \left[k^4\,\frac{3}{a}\,\frac{1-\beta_e}{\beta_e}\right]^{1/3} n^{4/3}\,, \tag{100}$$

where we have used n for (N/V). Let

$$n = \frac{8\pi m^3 c^3}{3h^3}\, x^3\,, \tag{101}$$

as in equation (3). Then (100) can be transformed into

$$p_e = \frac{\pi m^4 c^5}{3h^3}\left(\frac{512\pi k^4}{h^3c^3a}\,\frac{1-\beta_e}{\beta_e}\right)^{1/3} 2x^4\,. \tag{102}$$

Since the radiation constant a can be expressed in terms of the other natural constants as (Eq. [107], v)

$$a = \frac{8}{15}\,\frac{\pi^5 k^4}{h^3c^3}\,, \tag{103}$$

equation (102) can be simplified to

$$p_e = A\left(\frac{960}{\pi^4}\,\frac{1-\beta_e}{\beta_e}\right)^{1/3} 2x^4\,, \tag{104}$$

where A is defined as in (17). It must, of course, be understood that (104) is simply another form of (98).

Now for an assembly having the same number N of electrons in the volume V, we can formally calculate the electron pressure that would be given by the degenerate formula, namely,

$$p_{\deg} = Af(x)\,. \tag{105}$$

We have already shown (Eq. [26], x) that for all finite values of x

$$\frac{f(x)}{2x^4} < 1 \qquad (x < \infty)\,. \tag{106}$$

Hence, comparing (104) and (105), we have the result that if for a prescribed N and T, the value of β_e, calculated on the basis of the perfect gas equation (98), be such that

$$\frac{960}{\pi^4}\,\frac{1-\beta_e}{\beta_e} \geqslant 1\,, \tag{107}$$

then the pressure given by the perfect gas formula is greater than that given by the degenerate formula—not only for the prescribed N and T, but for all values of N and T which specify the same β_e. Let β_ω be such that

$$\frac{960}{\pi^4}\,\frac{1-\beta_\omega}{\beta_\omega} = 1\,, \tag{108}$$

or

$$1 - \beta_\omega = 0.09212 \ldots ; \qquad \beta_\omega = 0.90788 \ldots . \qquad (109)$$

We can state the result just obtained in the following alternative form. *If for material at density* ρ *and temperature* T *the fraction* $(1 - \beta_e)$, *calculated according to* (98), (98′), *and* (99), *is greater than* $(1 - \beta_\omega)$, *then the system is definitely not degenerate.*

On the other hand, if

$$\frac{960}{\pi^4} \frac{1 - \beta_e}{\beta_e} < 1 , \qquad (110)$$

or

$$1 - \beta_e < 1 - \beta_\omega ; \qquad \beta_e > \beta_\omega , \qquad (111)$$

then for the specified β_e the electron assembly becomes degenerate for sufficiently high electron concentrations. The criterion for degeneracy under these circumstances would then be the following.

For the specified N and T, calculate β_e on the perfect gas law (i.e., $p_e = n_e kT$) and solve the equation

$$\left(\frac{960}{\pi^4} \frac{1 - \beta_e}{\beta_e}\right)^{1/3} = \frac{f(x)}{2x^4} . \qquad (112)$$

(A solution exists, since [110] holds.) Denote the solution by x'. If x for the prescribed N (according to Eq. [101]) is much less than x', then the system is far removed from degeneracy, while if x is much greater than x' the system will be more or less completely degenerate.

Table 28 provides solutions of (112) for different values of $1 - \beta_e$.

If (110) holds, we can use the following approximation for the real equation of state:

$$\left.\begin{aligned} p_e &= Af(x) && (x \geqslant x') \\ \text{and} \quad p_e &= 2A\left(\frac{960}{\pi^4} \frac{1 - \beta_e}{\beta_e}\right)^{1/3} x^4 && (x \leqslant x') , \end{aligned}\right\} \qquad (113)$$

x' being such that

$$\left(\frac{960}{\pi^4} \frac{1 - \beta_e}{\beta_e}\right)^{1/3} = \frac{f(x')}{2x'^4} . \qquad (114)$$

5. *The effect of radiation pressure. The mass* $\mathfrak{M} = M_3\beta_\omega^{-3/2}$.—In § 2 we considered the equilibrium of completely degenerate configurations, neglecting the radiation pressure entirely. This was justified in § 1, where it was shown that for the known white dwarfs these assumptions (of complete degeneracy and zero radiation pressure) were entirely justified and our object in the study of the completely degenerate configurations is primarily one of obtaining a satisfactory theory for the white dwarfs. It is, however, of some theoretical interest to consider the effect of "introducing" radiation pressure in these configurations.

Let us, in the first instance, consider a degenerate configuration which is built on the standard model. Then the total pressure, P, will be given by

$$P = \beta_e^{-1} p_e , \tag{115}$$

where p_e is the electron pressure and β_e is a constant. Then, according to equation (16),

$$P = \beta_e^{-1} A f(x) \; ; \qquad \rho = B x^3 . \tag{116}$$

It is clear that the analysis of § 2 applies to our present models if we replace A (wherever it occurs) by $\beta_e^{-1}A$. In particular, the mass relation (42) now takes the form

$$M(\beta_e; y_0) = -4\pi \left(\frac{2A}{\pi\beta_e G}\right)^{3/2} \frac{1}{B^2} \left(\eta^2 \frac{d\phi}{d\eta}\right)_{\eta=\eta_1} , \tag{117}$$

where ϕ is, as before, a solution of (28). We can also write (117) in the form

$$M(\beta_e; y_0) = M(1; y_0)\beta_e^{-3/2} , \tag{118}$$

in an obvious notation. In particular,

$$M(\beta_e; \infty) = M_3 \beta_e^{-3/2} . \tag{119}$$

From (118) and (119) it would at first sight appear that by allowing $\beta_e \to 0$ we can obtain degenerate configurations for any mass. This is, however, incorrect. For, according to the criterion of degeneracy established in § 4, β_e has to be greater than β_ω if the matter is to be regarded as degenerate, and we see that the maximum mass of the configurations which can be regarded as degenerate is therefore given by

$$\mathfrak{M} = M_3 \beta_\omega^{-3/2} . \tag{120}$$

The result just stated is extremely general and can be proved as follows: Consider a completely degenerate configuration of mass M, slightly less than M_3. The density will everywhere be so great that we can increase the radiation pressure from zero to a value only slightly less than $(1 - \beta_\omega)$ at each point of the configuration and still regard the matter as degenerate. According to (118), the mass of the new configuration so obtained will be approximately $M\beta_\omega^{-3/2}$. When $M \to M_3$, the result becomes exact. We have thus proved that the *maximum mass of a stellar configuration which, consistent with the physics of degenerate matter, can be regarded as wholly degenerate, is* $\mathfrak{M} = \mathrm{M}_3\beta_\omega^{-3/2}$.

We may notice that

$$\mathfrak{M} = 1.156 M_3 = 6.65 \odot \mu_e^{-2} . \tag{121}$$

6. *Composite configurations.*—We shall now give some elementary considerations concerning stellar configurations with degenerate cores, a subject initiated by Milne. Milne, however, considered degenerate cores at such densities that the approximation $P = K\rho^{5/3}$ could be made. Since the exact treatment based on the differential equation (28) leads to the existence of the two masses M_3 and $\mathfrak{M}$, and since, further, there are no analogues to these on the approximate considerations, it is clear that very considerable care should be exercised in interpreting the results derived on the basis of the approximate considerations. In particular, the formal results which are derived for masses greater than $\mathfrak{M}$ have no physical meaning. On the other hand, it is possible to indicate the general characteristics of these composite configurations by allowing the degenerate core to be described by ϕ without any elaborate machinery.

First of all, it is important to bear in mind that, while in the degenerate regions the electrons contribute toward the pressure almost entirely, the situation is different in the gaseous region: depending on the abundance of hydrogen, the atomic nuclei would also contribute appreciably toward the gas pressure. The consideration of the composite configurations which allow for these factors is elementary but complicated. However, the essential features of the situation can be understood by considering the case where we can put $\mu_e = \mu$; this implies that $1 - \beta_e = 1 - \beta$.

According to (104), we have (for the case under consideration) in the gaseous region

$$P = \frac{1}{\beta} p_e = 2A \left(\frac{960}{\pi^4} \frac{1-\beta}{\beta^4} \right)^{1/3} x^4 \tag{122}$$

and

$$\rho = Bx^3 . \tag{123}$$

Eliminating x between (122) and (123), we have

$$P = 2A \left(\frac{960}{\pi^4} \frac{1-\beta}{\beta^4} \right)^{1/3} \frac{1}{B^{4/3}} \rho^{4/3} . \tag{124}$$

We shall assume that the gaseous region is governed by the standard-model equations, i.e., β is constant in (124). The gaseous region must then be governed by a solution $\theta(\xi)$ of the Lane-Emden equation of index 3—not necessarily θ_3. The mass relation (Eq. [70], iv) is now

$$M = -4\pi \left(\frac{2A}{\pi G} \right)^{3/2} \frac{1}{B^2} \left(\frac{960}{\pi^4} \frac{1-\beta}{\beta^4} \right)^{1/2} \left(\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_1(\theta)} , \tag{125}$$

which by (66) can be written as

$$M = M_3 \left(\frac{960}{\pi^4} \frac{1-\beta}{\beta^4} \right)^{1/2} \frac{\omega_3}{{}_0\omega_3} , \tag{126}$$

where, in the notation of chapter iv,

$${}_0\omega_3 = -\left(\xi^2 \frac{d\theta_3}{d\xi} \right)_{\xi=\xi_1(\theta_3)} ; \qquad \omega_3 = -\left(\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_1(\theta)} . \tag{127}$$

If the configuration is wholly gaseous, we have

$$M = M_3 \left(\frac{960}{\pi^4} \frac{1-\beta}{\beta^4} \right)^{1/2} , \tag{128}$$

which is Eddington's quartic equation in a different form.

Now for a given mass M, equation (128) determines a $\beta = \beta(M)$. Start with this mass having an infinite radius and imagine it being slowly contracted. At first the configuration will be so rarefied that it will be wholly gaseous and the path of the "representative point" in the $(R, 1 - \beta)$ plane will be along the line parallel to the R-axis through $\beta = \beta(M)$. How far is this process of contraction possible? From our criterion of degeneracy we can now conclude that if $1 - \beta(M) > 1 - \beta_\omega$, then the process of contraction is theoretically possible to an unlimited extent. Since β_ω, according to definition, is given by

$$\frac{960}{\pi^4}\frac{1 - \beta_\omega}{\beta_\omega} = 1 , \tag{129}$$

it follows that a configuration for which $\beta(M) = \beta_\omega$ is, according to (128),

$$M_3 \beta_\omega^{-3/2} = \mathfrak{M} . \tag{130}$$

a) The domain of degeneracy.—For configurations of mass greater than $\mathfrak{M}$, the appropriate $1 - \beta(M)$ is greater than $(1 - \beta_\omega)$ and the representative point will travel down the straight line $\beta = \beta(M)$, however far the contraction may proceed. But the situation is different when the mass of the configuration is less than $\mathfrak{M}$. For such masses, $1 - \beta(M) < 1 - \beta_\omega$, and hence a stage must be reached when the configuration should begin to develop central regions of degeneracy. On the scheme of approximation (113) and (114), we can now easily see how far the process of contraction is possible before degeneracy sets in.

Let the central density be ρ_0. Then

$$\rho_0 = B x_0^3 . \tag{131}$$

Degeneracy would just begin to develop at the center for a value of $x = x_0$ such that

$$\frac{f(x_0)}{2x_0^4} = \left(\frac{960}{\pi^4}\frac{1 - \beta}{\beta}\right)^{1/3} . \tag{132}$$

For this configuration the mean density $\bar{\rho}$ is (according to Eq. [78], iv, which gives the ratio of the mean to the central density for a polytrope)

$$\bar{\rho} = -3\left(\frac{1}{\xi}\frac{d\theta_3}{d\xi}\right)_{\xi=\xi_1(\theta_3)} B x_0^3 . \tag{133}$$

The radius R_0 of the configuration is, therefore, given by

$$\tfrac{4}{3}\pi R_0^3 = \frac{\text{Mass}}{\text{Mean density}} . \tag{134}$$

Substituting (125) and (133) in the foregoing expression, we obtain

$$R_0 = \left(\frac{2A}{\pi G}\right)^{1/2} \left(\frac{960}{\pi^4} \frac{1-\beta}{\beta^4}\right)^{1/6} \frac{1}{Bx_0} \xi_1(\theta_3) . \tag{135}$$

Define a unit of length by (cf. Eqs. [27] and [33])

$$l = \left(\frac{2A}{\pi G}\right)^{1/2} \frac{\xi_1(\theta_3)}{B} = \frac{7.71 \times 10^8 \times 6.897}{\mu} , \tag{136}$$

or, numerically,

$$l = 5.32 \times 10^9 \mu^{-1} \text{ cm} . \tag{137}$$

From (135), then,

$$\frac{R_0}{l} = \left(\frac{960}{\pi^4} \frac{1-\beta}{\beta^4}\right)^{1/6} \frac{1}{x_0} , \tag{138}$$

where x_0 is again determined from (132). By using (132), we can write (138) more conveniently as

$$\frac{R_0}{l} = \left(\frac{f(x_0)}{2x_0^4} \frac{1}{\beta}\right)^{1/2} \frac{1}{x_0} . \tag{139}$$

It is a fairly simple matter to calculate from (132) and (139) corresponding pairs of values for (R_0/l) and β. These are tabulated in Table 28. This $(R_0, 1-\beta)$ curve can therefore be drawn in the $(R, 1-\beta)$ plane (see Fig. 34). The region bounded by this curve and the two axes then defines the domain of degeneracy meaning that it is only in this region that the curves of constant mass are distorted from straight lines parallel to the R-axis.

From (132) and (139) we see that, as $\beta \to \beta_\omega$,

$$x_0 \to \infty , \quad R_0 \to 0 . \tag{140}$$

Hence, as we should expect, the $(R_0, 1-\beta)$ curve intersects the $(1-\beta)$ axis at a point where $\beta = \beta_\omega$. It can be proved easily that the $(R_0, 1-\beta)$ curve intersects the $(1-\beta)$ axis vertically.

b) The nature of the curves of constant mass for $\mathfrak{M} \leqslant \mathfrak{M}_3$ *in the* *domain of degeneracy*.—In (*a*), above, we have shown at what stage a configuration of mass less than $\mathfrak{M}$ (contracting from infinite ex-

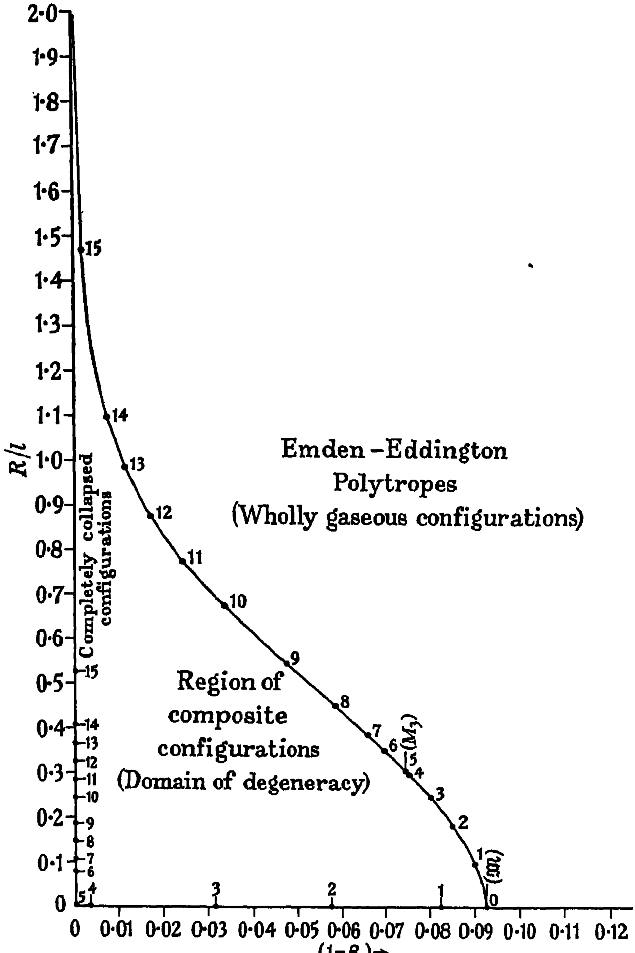

FIG. 34.—The curve running from $1-\beta=0.092\ldots$ to infinity along the R-axis is the $(R_0, 1-\beta)$ curve (see Eq. [139]). The points marked $(5, \ldots, 15)$ on the $(R_0, 1-\beta)$ curve and the R-axis are the end-points (in the domain of degeneracy) of the curves of constant mass for the values of M tabulated in Table 29. The points marked $(1, \ldots, 4)$ on the $(R_0, 1-\beta)$ curve and on the $(1-\beta)$ axis are the corresponding end-points for some curves of constant mass $(M_3 \leqslant M \leqslant \mathfrak{M})$ on the standard model (see Table 30).

tension) begins to develop degeneracy at the center. This happens when the appropriate line $(1-\beta) = 1-\beta(M)$ intersects the

(R_0, $1 - \beta$) curve. If the contraction continues further, the configuration will begin to develop finite degenerate cores, and our problem now is to examine how the curves of constant mass run inside the domain of degeneracy.

TABLE 28

THE STELLAR CRITERION FOR DEGENERACY AND THE (R_0, $1 - \beta$) CURVE

x	$1 - \beta$	R_0/l	x	$1 - \beta$	R_0/l
0..........	0	∞	2.8..........	0.06919	0.3515
0.2..........	0.00040	1.9868	3.0..........	.07149	.3304
0.4..........	.00282	1.3787	3.5..........	.07598	.2870
0.6..........	.00793	1.0956	4.0..........	.07920	.2535
0.8..........	.01505	0.9187	4.5..........	.08158	.2268
1.0..........	.02305	0.7934	5.0..........	.08337	.2051
1.2..........	.03101	0.6985	6.0..........	.08583	.1721
1.4..........	.03839	0.6235	7.0..........	.08739	.1481
1.6..........	.04495	0.5627	8.0..........	.08844	.1299
1.8..........	.05068	0.5123	9.0..........	.08918	.1157
2.0..........	.05561	0.4699	10.0..........	.08972	.1043
2.2..........	.05983	0.4337	20.0..........	.09150	.0524
2.4..........	.06344	0.4025	30.0..........	.09185	.0350
2.6..........	0.06653	0.3753	∞..........	0.09212	0

In § 2 we made an analysis of completely degenerate configurations. Each mass (less than M_3) has a certain uniquely determined radius. Thus, if the mass under consideration has a central density corresponding to $y = y_0$, then the radius, R, is given by

$$R = \alpha\eta_1 = \left(\frac{2A}{\pi G}\right)^{1/2} \frac{\eta_1}{By_0}, \tag{141}$$

where η_1 is the boundary of the corresponding function $\phi(y_0)$. In terms of the unit of length, l (Eq. [136]),

$$\frac{R}{l} = \frac{1}{y_0} \frac{\eta_1[\phi(y_0)]}{\xi_1(\theta_3)}. \tag{142}$$

These completely degenerate configurations correspond to $\beta = 1$. Hence, we know from (142) the point at which the curves of constant mass for $M < M_3$ must intersect the R-axis. Also, for any mass M we can calculate the value of β in the wholly gaseous state. Let $\beta\dagger$ be the value of β for a wholly gaseous configuration which in its completely degenerate state has a central density corresponding to $y = y_0$. Then, according to equations (71) and (128), we have the relation

$$\left(\frac{960}{\pi^4}\frac{1-\beta\dagger}{\beta\dagger^4}\right)^{1/2} = \frac{M}{M_3} = \frac{\Omega(y_0)}{{}_0\omega_3}\,, \tag{143}$$

where, as in equation (72),

$$\Omega(y_0) = -\eta_1^2\left(\frac{d\phi}{d\eta}\right)_{\eta=\eta_1(\phi(y_0))}. \tag{144}$$

Now the line through $\beta\dagger$ parallel to the R-axis will intersect the $(R_0, 1-\beta)$ curve at $[R_0(M(y_0)), 1-\beta\dagger]$. In the domain of degeneracy the continuation of the curve must in some way connect the point $[R_0(M(y_0)), 1-\beta\dagger]$ and the point R on the R-axis, where

$$\frac{R}{l} = \frac{1}{y_0(M)}\,\frac{\eta_1[\phi(y_0(M))]}{\xi_1(\theta_3)}. \tag{145}$$

From the numerical values for η_1, Ω, etc., for the ten different values of y_0 given in Table 25, the corresponding values of R/l (according to [145]) and $\beta\dagger$ can be evaluated. The results are given

TABLE 29

$1/y_0^2$	M/M_3	$1-\beta\dagger$	R/l
0	1.	0.07446	0
0.01	0.95733	.06966	0.07767
0.02	0.92419	.06596	0.10223
0.05	0.84709	.05746	0.14460
0.1	0.75243	.04732	0.18657
0.2	0.61589	.03358	0.24168
0.3	0.51218	.02414	0.28434
0.4	0.42600	.01718	0.32320
0.5	0.35033	.01187	0.36222
0.6	0.28137	.00779	0.40475
0.8	0.15316	.00236	0.52453
1.0	0	0	∞

in Table 29. We have thus fixed the "end-points" for the curves of constant mass for $M \leqslant M_3$ in the domain of degeneracy. The corresponding pairs of points on the $(R_0, 1-\beta)$ curve and the R-axis are shown in Figure 34.

It is clear that the curve for M_3 must pass through the origin of our system of co-ordinates. Further, if β_0 is the value of β for M_3 in the wholly gaseous state, then, according to (143),

$$\frac{960}{\pi^4}\frac{1-\beta_0}{\beta_0^4}=1\,, \tag{146}$$

or

$$1-\beta_0=0.07446\,;\qquad \beta_0=0.92554\,. \tag{147}$$

c) The nature of the curves of constant mass for M > M$_3$ *in the domain of degeneracy.*—In (*b*), above, the end-points for the curves of constant mass (for configurations with mass less than, or equal to, M_3) have been fixed. We further saw that the curve for M_3 must pass through the origin. The question now arises: What happens for configurations with $\mathfrak{M} \geqslant M > M_3$? The answer to this question can be given quite simply if $(1-\beta)$ has the same value in the degenerate core as in the gaseous envelope. We have already shown (Eq. [119]) that the completely relativistic configuration has a mass

$$M=M_3\beta^{-3/2}\qquad (1\geqslant\beta\geqslant\beta_\omega) \tag{148}$$

and is of zero radius. Hence, the curves of constant mass for $M>M_3$ must cross the $(1-\beta)$ axis at a point $(1-\beta^*)$, say, such that

$$M=M_3\beta^{*-3/2}\,. \tag{149}$$

Let us denote by $\beta\dagger$ the value of β in the wholly gaseous state. There is a simple relation between β^* and $\beta\dagger$. Comparing (143) and (149), we derive that

$$\beta^*=\left(\frac{\pi^4}{960}\frac{\beta\dagger^4}{1-\beta\dagger}\right)^{1/3}. \tag{150}$$

From (150) we see that $\beta^*=1$, $\beta\dagger=\beta_0$ (Eq. [146]), is a solution; in other words, the appropriate curve for M_3 must pass through the origin which in fact it does. Again, $\beta^*=\beta\dagger=\beta_\omega$ is also a solution of (150); the appropriate curve for $\mathfrak{M}$ is therefore the full line through $(1-\beta_\omega)$ parallel to the R-axis, as we should have expected.

Table 30 gives a set of corresponding pairs of values for β^* and

$\beta\dagger$ (see also Fig. 34, where the corresponding pairs of points are marked [1, 2, 3, 4] on the [R_0, $1 - \beta$] curve and the [$1 - \beta$] axis).

The results described above (in [*b*]) are true for the usual standard model. If we consider as an another limiting case configurations in which $\beta = 1$ in the degenerate core and $\beta \leqslant 1$ in the gaseous envelope, then the discussion is similar but somewhat more complicated (cf. Chandrasekhar's papers quoted in the Bibliographical Notes).

TABLE 30

$1 - \beta\dagger$	$1 - \beta^*$	$M/\mathfrak{M}$
0.09212	0.09212	1.
.090	.08220	0.9838
.085	.05768	0.9457
.080	.03143	0.9075
.075	.00319	0.8692
0.07446	0	0.8651

A more detailed discussion of composite configurations would consist in describing the mathematical methods for handling them precisely, i.e., by a consideration of the methods of fitting a solution of the Lane-Emden equation of index 3 to a solution of the differential equation for ϕ. Such discussions, however, are beyond the scope of the monograph. Reference may be made to the literature quoted in the Bibliographical Notes.

7. *Partially degenerate configurations.*—So far we have considered completely degenerate configurations and also stellar configurations with degenerate cores. For describing the degenerate state we have used the exact equation of state (allowing for relativistic effects) which should be valid if the degeneracy criterion is satisfied. In considering the composite configurations in § 6, we changed over from the perfect gas equation to the degenerate equation of state at a definite interface, the interface being defined in such a way that both the equations of state give the same numerical value for the pressure for the density and the temperature at the interface. We have seen that this approximation is quite good so long as we deal with configurations of not too small masses (in units of M_3). However, for stars of small mass ($\sim 0.1\ M_3$) the central density, even in the completely degenerate state, is not unduly high. Under these cir-

cumstances, we may expect that in actual stars (e.g., Krüger 60) the "transition zone" between the perfect gas region and the region of more or less complete degeneracy will be quite extensive. It is therefore a matter of some importance to allow for these incipiently degenerate regions in a satisfactory way.

We shall illustrate a method of approach to the problem just stated in one case, namely, that in which the configuration is so poor in hydrogen that we can put $\mu = \mu_e = 2$. Further, we shall assume that the star is of such small mass that relativistic effects can be neglected. Under these circumstances, the equation of state can be parametrically expressed as follows (cf. Eqs. [262] and [263], x):

$$p_{\text{gas}} = \frac{2}{h^3}(2\pi m)^{3/2}(kT)^{5/2}U_{3/2}\,, \tag{151}$$

$$\rho = \frac{2}{h^3}(2\pi m)^{3/2}(kT)^{3/2}\mu H\, U_{1/2}\,, \tag{152}$$

where U_ν stands for the integral

$$U_\nu = \frac{1}{\Gamma(\nu+1)}\int_0^\infty \frac{u^\nu du}{\frac{1}{A}e^u + 1}\,. \tag{153}$$

We shall assume that $U_{1/2}$ and $U_{3/2}$ are known functions of A, so that the parametric representation of the equation of state is in terms of A.

We shall consider two classes of equilibrium configurations built on the equation of state (151) and (152) which allows for the transition between $p \propto \rho T$ to $p \propto \rho^{5/3}$ quite accurately.

a) The isothermal gas sphere.—In this case, T is assumed constant, and the equation of equilibrium,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho\,, \tag{154}$$

on inserting for p and ρ according to (151) and (152), becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{U_{1/2}}\frac{dU_{3/2}}{dr}\right) = -4\pi G\left(\frac{2}{h^3}(2\pi m)^{3/2}\right)(kT)^{1/2}(\mu H)^2U_{1/2}\,. \tag{155}$$

Let

$$r = a\zeta \, , \tag{156}$$

where

$$a = \left(\frac{h^3}{8\pi G(2\pi m)^{3/2}(kT)^{1/2}(\mu H)^2} \right)^{1/2} . \tag{157}$$

Equation (154) now reduces to

$$\frac{1}{\zeta^2} \frac{d}{d\zeta} \left(\frac{\zeta^2}{U_{1/2}} \frac{d}{d\zeta} U_{3/2} \right) = -U_{1/2} \, . \tag{158}$$

Now it is easily seen that

$$\frac{d}{d\Lambda} U_v = \frac{1}{\Gamma(v+1)\Lambda^2} \int_0^\infty \frac{u^v e^u du}{\left(\frac{1}{\Lambda} e^u + 1 \right)^2} = \frac{1}{\Lambda} U_{v-1} \, . \tag{159}$$

Equation (158) can therefore be simplified to the form

$$\frac{1}{\zeta^2} \frac{d}{d\zeta} \left(\zeta^2 \frac{d \log \Lambda}{d\zeta} \right) = -U_{1/2}(\Lambda) \, . \tag{160}$$

If $\Lambda \ll 1$, we have (cf. Eq. [270], x)

$$U_v = \Lambda \, . \tag{161}$$

Hence, if we write

$$\Lambda = e^{-\psi} \, ; \qquad \zeta = \xi \, , \tag{162}$$

equation (160) transforms to

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\psi}{d\xi} \right) = e^{-\psi} \, , \tag{163}$$

which is the isothermal equation of a classical perfect gas sphere (cf. § 22, iv).

On the other hand, if $\Lambda \gg 1$, then (Eq. [266], x)

$$U_{1/2} = \frac{4}{3\sqrt{\pi}} (\log \Lambda)^{3/2} \, . \tag{164}$$

Hence, if we make the substitutions

$$\log A = \theta \; ; \qquad \zeta = \left(\frac{3\sqrt{\pi}}{4}\right)^{1/2} \xi \, , \tag{165}$$

equation (160) reduces to

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^{3/2} \, , \tag{166}$$

which is the Lane-Emden equation of index $n = 3/2$. We thus see that, depending upon T, we obtain from (160) either the classical isothermal case ($T \rightarrow \infty$) or the polytrope, $n = 3/2$ ($T \rightarrow 0$). A closer study of the differential equation (160) than has yet been made will make it possible to study how the change from the classical isothermal gas sphere to the polytrope $n = 3/2$ takes place as A increases from very near 0 to ∞. The discussion of (160) may lead to results of cosmological importance.

b) The standard model.—We shall next consider the standard model built on the equation of state, (151) and (152). Quite generally, on the standard model, we have

$$P = \frac{1}{\beta} p_{\text{gas}} = \frac{1}{1 - \beta} \frac{a}{3} T^4 \, . \tag{167}$$

Let

$$Q_1 = \frac{2}{h^3} (2\pi m)^{3/2} \; ; \qquad Q_2 = k^4 \frac{3}{a} \frac{1 - \beta}{\beta} \, . \tag{168}$$

Equations (151), (152), and (167) can now be written as

$$p_{\text{gas}} = Q_1 (kT)^{5/2} U_{3/2} \, , \tag{169}$$

$$\rho = Q_1 (kT)^{3/2} \mu H \, U_{1/2} \, , \tag{170}$$

and

$$(kT)^4 = Q_2 p_{\text{gas}} \, . \tag{171}$$

From (169) and (171), we obtain

$$(kT)^{3/2} = Q_1 Q_2 U_{3/2} \, . \tag{172}$$

Substituting for kT according to (172) in (169) and (170), we obtain

$$\beta P = p_{\text{gas}} = Q_1^{8/3} Q_2^{5/3} U_{3/2}^{8/3} \tag{173}$$

and

$$\rho = Q_1^2 Q_2 \mu H \, U_{1/2} U_{3/2} \, . \tag{174}$$

Substituting (173) and (174) in the equation of equilibrium (154), we find

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{U_{1/2} U_{3/2}} \frac{d}{dr} U_{3/2}^{8/3} \right) = -4\pi G \beta Q_1^{4/3} Q_2^{1/3} (\mu H)^2 U_{3/2} U_{1/2} \, . \tag{175}$$

By (159) we can simplify (175) somewhat into the form

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 U_{3/2}^{2/3} \frac{d \log \Lambda}{dr} \right) = -\frac{3\pi G \beta Q_1^{4/3} Q_2^{1/3} (\mu H)^2}{2} U_{3/2} U_{1/2} \, . \tag{176}$$

Let

$$r = \alpha \zeta = \left(\frac{2}{3\pi G \beta Q_1^{4/3} Q_2^{1/3} (\mu H)^2} \right)^{1/2} \zeta \, . \tag{177}$$

Equation (176) now reduces to

$$\frac{1}{\zeta^2} \frac{d}{d\zeta} \left(\zeta^2 U_{3/2}^{2/3} \frac{d \log \Lambda}{d\zeta} \right) = -U_{3/2} U_{1/2} \, . \tag{178}$$

If $\Lambda \ll 1$, we have $U_\nu = \Lambda$, and (178) can be written as

$$\frac{1}{\zeta^2} \frac{d}{d\zeta} \left(\zeta^2 \Lambda^{-1/3} \frac{d\Lambda}{d\zeta} \right) = -\Lambda^2 \, ; \tag{179}$$

or, if

$$\theta = \Lambda^{2/3} \, ; \qquad \zeta = \sqrt{\tfrac{3}{2}} \, \xi \, , \tag{180}$$

equation (179) reduces to the Lane-Emden equation of index $n = 3$, as would be expected.

On the other hand, if $\Lambda \gg 1$, then, according to equations (266) and (267) of chapter x,

$$U_{1/2} = \frac{4}{3\sqrt{\pi}} (\log \Lambda)^{3/2} \, ; \qquad U_{3/2} = \frac{15}{8\sqrt{\pi}} (\log \Lambda)^{5/2} \, . \tag{181}$$

If we now put

$$(\log A)^{8/3} = \theta\ ; \qquad \zeta = \sqrt{\frac{3\pi}{20}\left(\frac{15}{8\sqrt{\pi}}\right)^{2/3}}\,\xi\ , \tag{182}$$

equation (178) reduces to the Lane-Emden equation of index $n = 3/2$. We thus see that a detailed study of (178), which has not as yet been made, should give insight into the structure of partially degenerate stars. The numerical discussion of the models (Eqs. [160] and [178]) cannot be very difficult; a one-parametric series of integrations would be sufficient.

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