Stellar Structure — Polytrope models for White Dwarf density profiles and masses

We assume that the star is spherically symmetric. Then we want to calculate the mass density $\rho(r)$ (units of $\mathrm{kg/m^3}$) as a function of distance r from its centre. The mass density will be highest at its centre and as the distance from the centre increases the density will decrease until it becomes very low in the outer atmosphere of the star. At some point $\rho(r)$ will reach some low cutoff value and then we will say that that value of r is the radius of the star.

The mass dnesity profile is calculated from two coupled differential equations. The first comes from conservation of mass. It is

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r), \tag{1}$$

where $m(r)$ is the total mass within a sphere of radius r. Note that $m(r \to \infty) = M$, the total mass of the star is that within a sphere large enough of include the whole star. In practice the density profile $\rho(r)$ drops off rapidly in the outer atmosphere of the star and so can obtain M from $m(r)$ at some value of r large enough that the density $\rho(r)$ has become small. Also, of course if $r = 0$, then $m = 0$: the mass inside a sphere of 0 radius is 0. Thus the one boundary condition we need for this first-order ODE is $m(r = 0) = 0$.

The second equation comes from a balance of forces on a spherical shell at a radius r. The pull of gravity on the shell must equal the outward pressure at equilibrium; see Ref. [1], Chapter 1. It is

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}, \tag{2}$$

where $p(r)$ is the pressure at a radius r. This equation is sometimes known as the equation of hydrostatic equilibrium.

The pressure is a maximum at the centre and decrease as r increases, because the pressure at a radius r must be enough, at equilibrium to support the weight of all the mass of the star which is further than r from the centre. The density has a maximum at $r = 0$ and then at large r, $\rho(r \to \infty) = 0$.

Our two differential equations have three unknown functions: $\rho(r)$, $m(r)$ and $p(r)$. As we have one less equation than unknown, we need a relation between two of these three functions, otherwise we don't have enough equations and we cannot solve them.

White Dwarf model

One simple type of relation is a polytropic relation between the pressure and the density. This simply says that the pressure is a power law of the density with an exponent α,

$$p(r) = K\rho(r)^{\alpha}. \tag{3}$$

White dwarfs are quite simple stars as their pressure is dominated by that of degenerate electrons, which means that simple polytropic pressure functions are good approximations for this type of star, see Ref. [1].

For example, for a highly degenerate but non-relativistic electron gas with a density of n_e electrons per unit volume

$$p = K_{NR} n_e^{5/3} \qquad K_{NR} = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3}$$

This can be substituted into our differential equation for the pressure above, and then we have two differential equations for two unknown functions, which we can solve. Note that to do this, we need to convert from number density of electrons to mass density. We can do this if we, for example, assume that the white dwarf is made entirely of protons and electrons [1], see Ref. [1]. Then $n_e = \rho/m_p$, and the pressure is given by

$$p = \frac{K_{NR}}{m_p^{5/3}}\rho^{5/3} \qquad K_{NR} = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} \tag{4}$$

If we sub this in Eq. (2) we get (using chain rule for differentiation)

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = \frac{\mathrm{d}\rho(r)}{\mathrm{d}r} \times \frac{\mathrm{d}p(r)}{\mathrm{d}\rho} = \frac{\mathrm{d}\rho(r)}{\mathrm{d}r} \times \frac{5}{3}\frac{K_{NR}}{m_p^{5/3}}\rho^{2/3} = -\frac{Gm(r)\rho(r)}{r^2}, \tag{5}$$

which simplifies to

$$\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} = -\frac{3}{5}\frac{Gm_p^{5/3}m(r)[\rho(r)]^{1/3}}{K_{NR}r^2}, \tag{6}$$

The two coupled first order ODEs, Eqs. (1) and (6) can be solved, given two boundary conditions, and so the density and mass profiles inside a white dwarf obtained. The two boundary conditions are taken to be almost at $r = 0$ (not quite at $r = 0$ as RHS of Eq. (6) diverges there). Take this small value of r to be $r = \delta$. Then the mass boundary condition is trivial, $m(\delta) = 0$. This leaves only one non-trivial boundary condition, $\rho(\delta)$, i.e., the mass density at the centre of the white dwarf. As a first guess you can try $\rho(\delta) = 10^7 \mathrm{kg/m}^3$, but in fact it is by varying this boundary condition that you can vary the size and the mass of the white dwarf. You will want to do this to understand how the mass and size of a white dwarf are related — a major, testable, prediction of this model.

From the numericall calculated functions $\rho(r)$ and $m(r)$, we can estimate the size of the white dwarf, which is the value of the radius, call it r_{WD}, beyond which the density is so low we can neglect it. Then the mass of the white dwarf is just the mass inside this radius, $m(r_{WD})$.

[1] Note that this is typically not correct, composition can be closer to that of say Carbon-12. This changes the ratio n_e/m_p and is something you can explore.

Pressure when density is so high the electrons are relativistic

Equation (4) for the pressure applies at densities that are not too high. At even higher densities, the electrons are so confined that they become relativistic. The expression for a highly degenerate *relativistic* quantum gas with a density n_e is also polytropic. It is

$$p = K_R n_e^{4/3} \qquad K_R = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3}. \tag{7}$$

In general, once the pressure p has been written in terms of the density ρ using a polytrope equation as above, the two coupled differential equation, can be solved if both $m(\delta)$ and $\rho(\delta)$ are specified.

You can solve for the density function and mass of a white dwarf predicted by each model.

But it is good to understand when (at what densities) you expect each model to be a reasonable model of the pressure. To do this you need to know if the electrons are going to be relativistic, i.e., is their speed close to the speed of light. This can be done by estimating the magnitude of the momentum, p_e, from Heisenberg's Uncertainty Principle, and noting that when electrons are non-relativistic the speed of the electron $v_e = p_e/m_e$.

Heisenberg's Uncertainty Principle states that the product of the uncertainty in the momentum, Δp, times the uncertainty in the position, Δx, is at least equal to Planck's constant (over 2π)

$$\Delta p_e \times \Delta x_e \gtrsim \hbar \tag{8}$$

which we rewrite to get an estimate for the size of p_e

$$p_e \approx \frac{\hbar}{n_e^{-1/3}} \tag{9}$$

because the uncertainty in the position of the electrons $\Delta x_e \approx n_e^{-1/3}$, because at a density of electrons n_e the electrons are on average about $n_e^{-1/3}$ apart. For a white dwarf made of protons and neutrons $n_e = \rho/m_p$, so we have

$$v_e \approx \frac{\hbar \rho^{1/3}}{m_e m_p^{1/3}} \qquad \text{for} \qquad v_e \lesssim c \tag{10}$$

when the v_e from this expression is much less than c then Eq. (4) should be valid (except perhaps a very low densities). However, at densities where this gives a $v_e \approx c$ and above, we need to use the expression for the pressure of relativistic electrons, Eq. (7).

Useful data:
Mass of the proton $m_p = 1.67 \times 10^{-27}$kg
Mass of the electron $m_e = 9.11 \times 10^{-31}$kg
Speed of light $c = 3.00 \times 10^8$m/s
Planck's constant $h = 6.63 \times 10^{-34}$Js

Gravitational constant $G = 6.67 \times 10^{-11}$kg
Mass of the sun $= 1.99 \times 10^{30}$kg
Radius of the sun $= 6.96 \times 10^{8}$m

References

[1] A. C. Phillips, *The Physics of Stars* (Wiley).