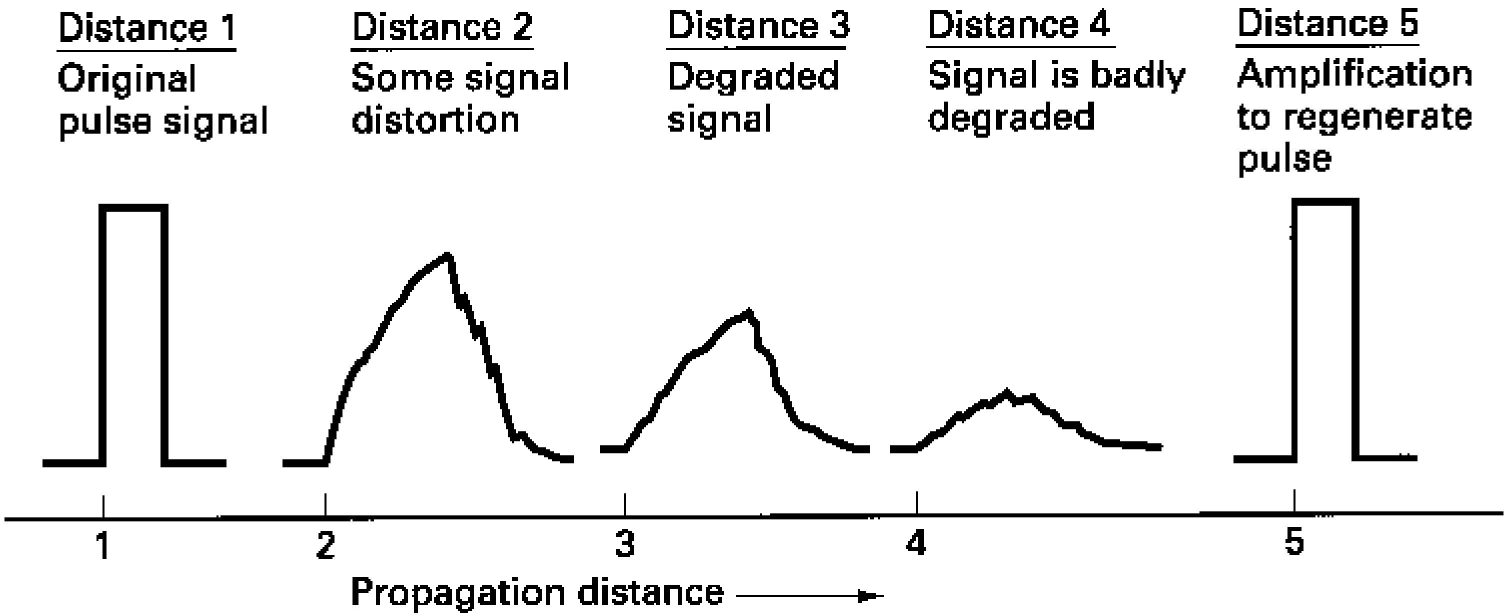
Introduction

In an **analog** communication system, the transmitter sends a waveform from an **infinite** variety of waveform shapes. At the receiver, the objective is to reproduce the transmitted waveform with high fidelity.

In a **digital communication system** (DCS), the transmitter sends a waveform from a **finite** set of possible waveforms. Based on the noisy waveform received, the receiver needs to determine which waveform, from the *finite* set of possible waveforms, has been sent by the transmitter.

Why Digital?



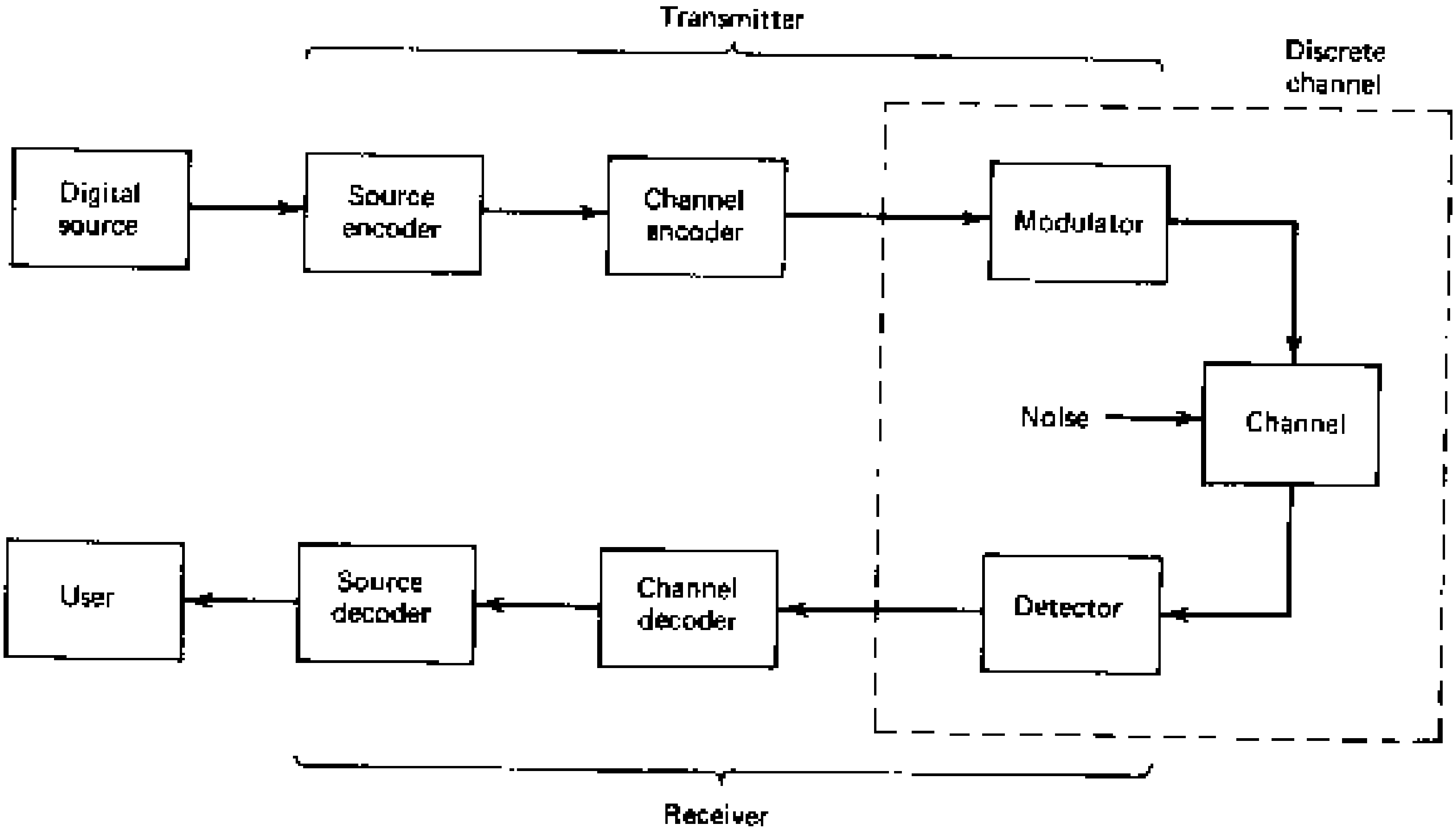
<u>Pulse degradation and regeneration</u>

- Digital communication is more immune to channel noise and channel distortion.
- Regenerative repeaters along the transmission path can detect a digital signal and retransmit a new, clean signal. These repeaters prevent accumulation of noise along the path. This is not possible in analog communications.
- Digital hardware implementation is more flexible and permits the use of microprocessors and large-scale integrated circuits.
- Digital signals can be coded to yield extremely low error rates and high fidelity.
- It is easier and more efficient to multiplex several digital signals.

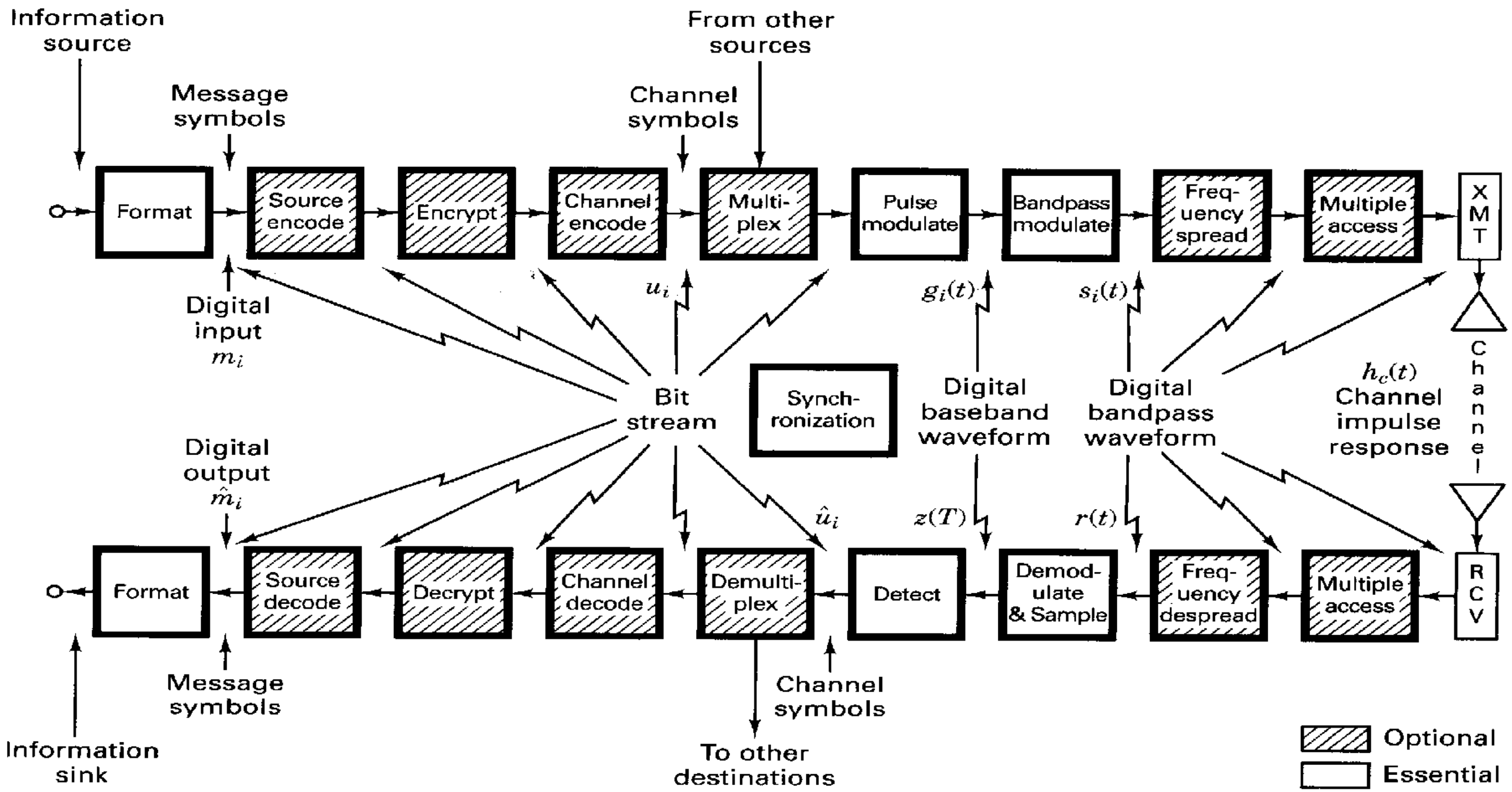
- Digital communication is inherently more efficient than analog in realizing the exchange of signal-to-noise ratio (SNR) for bandwidth.
- Digital communication is more suitable for computer-to-computer communications.

Disadvantages of Digital Transmission

- Digital communication requires more system bandwidth as compared to analog communication.
- Digital detection requires several levels of synchronization (carrier, bit level, symbol level and code level).



Block diagram of a simple digital communication system



Block diagram of a typical DCS

Blocks of a Typical DCS

- The **information source** generates messages that are to be transmitted to the receiver. The messages may be analog or digital.
- **Formatting** transforms the source information into digital symbols. It also makes the information compatible with the signal processing within a DCS.
- The **source encoding** provides an efficient representation that results in little or no redundancy.
- **Encryption** prevents unauthorized users from understanding messages and from injecting false messages into the DCS.

- For a given data rate, **channel coding** can reduce the SNR requirement at the expense of bandwidth (e.g., binary convolutional codes) or decoder complexity (e.g., trellis-coded modulation).
- **Multiplexing** and **multiple access** procedures combine signals that might have different characteristics, or might originate from different sources, so that they can share a portion of the communication resource.
- **Modulation** is the process by which the symbols are converted to waveforms that are compatible with the transmission channel.

- **Frequency spreading** (e.g., spread spectrum) can produce a signal that is less vulnerable to interference (intentional and unintentional) and can be used to enhance the privacy of the communicators.
- The **transmitter** usually consists of a frequency up-conversion stage, a high-power amplifier, and an antenna.
- The **receiver** portion usually consists of an antenna, a low-noise amplifier (LNA), and a down-conversion stage, typically to an intermediate frequency (IF).
- The lower blocks essentially reverse the signal processing steps performed by the upper blocks.

- A signal from the source to the modulator is referred to as a **baseband signal** or a **bit stream**. After modulation, it is called a **digital waveform**. Similarly, in the reverse direction, a received signal appears as a digital waveform until it is demodulated.
- At various points along the signal path, noise corrupts the signals m_i and u_i so that their estimates, $\hat{m}_i$ and $\hat{u}_i$, are used in the lower blocks.
- In digital communications, synchronization involves the estimation of both time and frequency. **Coherent** systems need to synchronize their frequency reference with the carrier in both frequency and phase. For

noncoherent systems, phase synchronization is not needed. The other levels of synchronization include symbol synchronization, frame synchronization and network synchronization.

- Of all blocks shown in the diagram, only formatting, modulation, and demodulation are essential for a DCS. Other blocks are optional for specific system needs.

Basic Terms of Digital Communications

- **Information source**

It can be analog or digital. Analog information sources can be transformed into digital sources through the use of sampling and quantization.

- **Textual message**

It consists of a sequence of characters.

- **Character**

It is a member of an alphabet or set of symbols. Characters may be mapped into a sequence of binary digits (e.g., ASCII (American Standard Code for Information

Interchange) and EBCDIC (Extended Binary Coded Decimal Interchange Code).

- **Binary digit (bit)**

It is the fundamental unit for all digital systems.

- **Bit stream**

It is a sequence of binary digits.

- **Symbol**

For the transmission of bit stream, groups of k bits are combined to form new symbols. Hence, the size of the symbol set is $M = 2^k$.

- **Digital waveform**

A voltage or current waveform represents a digital symbol.

- **Data rate**

It is normally expressed in bits per second (bps). For the case of symbol transmission with symbol duration T, the data rate in bps is

$$R = 1/T \qquad \text{(in symbols per second)}$$

$$= k/T = \log_2 M/T \qquad \text{(in bps)}$$

Performance Measure

- Analog communication systems draw their waveforms from a continuum (an infinite set). Hence, a figure of merit is a fidelity criterion (e.g., SNR or mean-square error).

- Digital communication systems transmit signals that are from a finite set of waveforms. A suitable performance measure is the probability of error, P_E.

Classification of Signals

- **Deterministic and random signals**

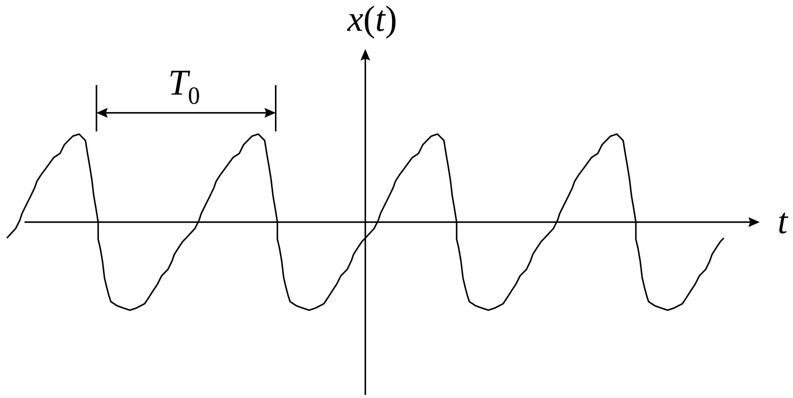
A signal is deterministic if there is no uncertainty with respect to its value at any time. A signal is random if there exists some degree of uncertainty before the signal actually occurs. Unlike the deterministic signals, random signals cannot be modeled by explicit mathematical expressions. However, we may describe the random signals statistically in terms of probabilities and statistical averages.

- **Periodic and nonperiodic signals**

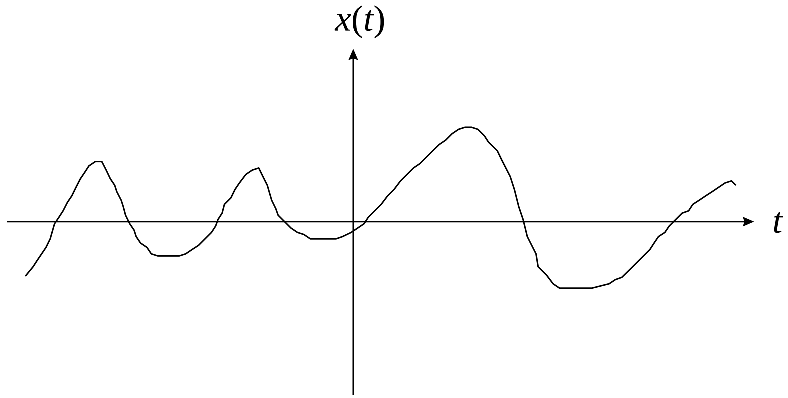
A signal $x(t)$ is periodic if

$$x(t) = x(t + T_0), \quad \forall\, t$$

where $T_0 > 0$. There are many values of T_0 that satisfy the condition. The smallest value of them is called the **period** of $x(t)$. A signal that does not satisfy the above condition is called a **nonperiodic signal** or **aperiodic signal**.



A periodic signal



A nonperiodic signal

- **Analog, discrete and digital signals**

An **analog signal** $x(t)$ is a continuous function of time, with the amplitude being continuous as well.

A **discrete-time signal** $x(kT)$ is defined only at discrete time kT, where k is an integer and T is a fixed time interval. Note that the amplitude of $x(kT)$ may take on continuous values. With each amplitude is quantized and coded, it is called a **digital signal**.

• Energy and power signals

It is well known that the instantaneous power $p(t)$ across a resistor R is given by

$$p(t) = \frac{v^2(t)}{R} = i^2(t)R$$

where $v(t)$ and $i(t)$ are the voltage and the current, respectively. In **communication systems**, power is often normalized by assuming R to be 1 Ω, although R may be another value in the actual circuit.

In this way, the normalized power becomes

$$p(t) = \left| x(t) \right|^2$$

where $x(t)$ is either a voltage or a current. The magnitude square is used to allow the possibility of $x(t)$ being a complex-valued signal.

The energy dissipated during the interval $(-T/2, T/2)$ is given by

$$E_x^T = \int_{-T/2}^{T/2} \left| x(t) \right|^2 \, dt$$

and the corresponding power is

$$P_x^T = \frac{1}{T}\int_{-T/2}^{T/2} \left|x(t)\right|^2 \, dt$$

The **energy** of $x(t)$ is defined by

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

and the **power** of $x(t)$ is defined by

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

A signal $x(t)$ is an **energy signal** if and only if

$$0 < E_x < \infty$$

Similarly, a signal is called a **power signal** if and only if

$$0 < P_x < \infty$$

The energy and power classifications are mutually exclusive. An energy signal has a finite energy but zero average power, whereas a power signal has finite average power but infinite energy.

As a general rule, periodic signals and random signals are classified as power signals, while signals that are both deterministic and nonperiodic are classified as energy signals.

Fourier Series

If a periodic power signal $x(t)$ with period T_0 satisfies the following Dirichlet conditions:

1. $x(t)$ is absolutely integrable over its period, i.e.,

$$\int_0^{T_0} |x(t)| \, dt < \infty$$

2. the number of maxima and minima of $x(t)$ in each period is finite,
3. the number of discontinuities of $x(t)$ in each period is finite,

then $x(t)$ can be represented by a **complex exponential Fourier series** of the form

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

where $f_0 = 1/T_0$ is the **fundamental frequency** and

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi n f_0 t} dt$$

is the **Fourier coefficient** of the nth harmonic. The integration can be performed over any full period, e.g., $(-T_0/2, T_0/2)$ or $(0, T_0)$.

Fourier Transform

For nonperiodic signals, we have

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$

where f is frequency measured in Hz. The above expressions are commonly referred to as the **Fourier transform pair**. In particular, $X(f)$ is called the **Fourier transform** of $x(t)$, and $x(t)$ is known as the **inverse**

Fourier transform of $X(f)$. Symbolically, we have the following notation:

$$x(t) \leftrightarrow X(f)$$

$$X(f) = \mathbf{F}\left[x(t)\right]$$

$$x(t) = \mathbf{F}^{-1}\left[X(f)\right]$$

We can obtain $X(f)$ from $x(t)$ and, similarly, reconstruct $x(t)$ from $X(f)$. $X(f)$ is also called the **spectrum** of $x(t)$.

The Unit Impulse Function

The unit impulse $\delta(t)$ is not a function in a strict mathematical sense. It is a member of a special class known as **generalized functions** or **distributions**.

Given any function $x(t)$, the unit impulse $\delta(t)$ is defined by

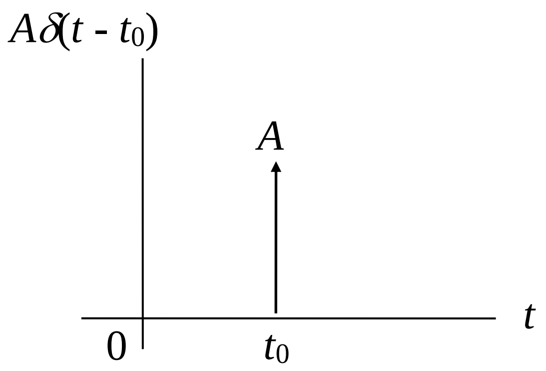
$$\int_{t_1}^{t_2} x(t)\delta(t)dt = \begin{cases} x(0) & t_1 < 0 < t_2 \\ 0 & \text{otherwise} \end{cases}$$

Taking $x(t) = 1$ for all t, it follows that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

It may be interpreted that $\delta(t)$ has **unit area** concentrated at the discrete point $t = 0$ and no net area elsewhere. That is

$$\delta(t) = 0, \quad t \neq 0$$



Properties of $\delta(t)$

By definition, the impulse has no mathematical or physical meaning unless it appears under the operation of integration. Some important properties of the impulse are listed below:

1. Sifting (sampling) operation

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$$

2. Replication operation

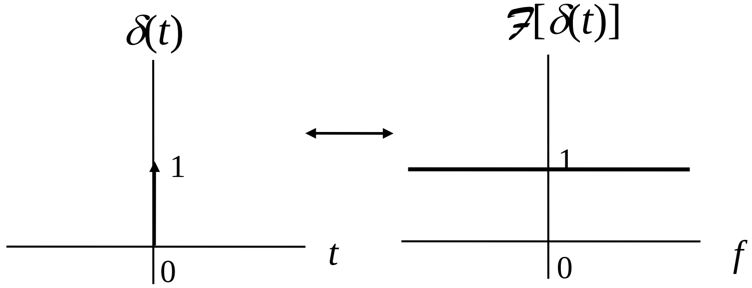
$$x(t)\otimes\delta(t-t_0) = x(t-t_0), \quad \otimes \text{ denotes convolution}$$

3. $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

4. The Fourier transform of $\delta(t)$ is

$$\mathbf{F}\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft}\, dt = 1.$$

Hence, an impulse function has uniform spectrum over the entire frequency range. This type of spectrum is called **white spectrum**.



Spectral Density

The energy (power) spectral density of a signal describes the distribution of signal energy (power) in the frequency domain. The concept is particularly useful when considering filtering in communication systems.

- **Energy spectral density (ESD)**

Using Parseval's theorem, we have

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} \left|X(f)\right|^2 df,$$

where $X(f)$ is the Fourier transform of the nonperiodic energy signal $x(t)$.

Defining the ESD of the signal as

$$\Psi_x(f) = \left|X(f)\right|^2$$

we have

$$E_x = \int_{-\infty}^{\infty} \Psi_x(f)\, df$$

For a real signal $x(t)$, the ESD is even and thus

$$E_x = 2\int_{0}^{\infty} \Psi_x(f)\, df$$

• Power Spectral Density (PSD)

For a periodic signal $x(t)$ with period T_0,

$$\begin{aligned} E_x &= \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt \\ &= \lim_{n\to\infty} \int_{-nT_0/2}^{nT_0/2} |x(t)|^2 dt \\ &= \lim_{n\to\infty} n\int_{-T_0/2}^{T_0/2} |x(t)|^2 dt \\ &= \infty \end{aligned}$$

Therefore, a periodic signal is not typically an energy signal. However, the power of $x(t)$ is

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \\
&= \lim_{n\to\infty} \frac{1}{nT_0} \int_{-nT_0/2}^{nT_0/2} |x(t)|^2 \, dt \\
&= \lim_{n\to\infty} \frac{n}{nT_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt \\
&= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt
\end{aligned}
$$

This means that the power content of a periodic signal is equal to the average power over one period.

Using Parseval's theorem, we have

$$P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

where c_n's are the complex Fourier coefficients of $x(t)$.

Defining the PSD of the signal as

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \, \delta(f - nf_0)$$

we have

$$P_x = \int_{-\infty}^{\infty} G_x(f)\,df$$

For a real signal $x(t)$, the PSD is even and thus

$$P_x = 2\int_{0}^{\infty} G_x(f)\,df$$

If $x(t)$ is a nonperiodic power signal, then we may not find its Fourier series and Fourier transform. However, we may still obtain its PSD in the limiting sense as

$$G_x(f) = \lim_{T \to \infty} \frac{1}{T} \left| X_T(f) \right|^2$$

where $X_T(f)$ is the Fourier transform of $x_T(t)$. Note that $x_T(t)$ is a truncated version of $x(t)$ over $(-T/2, T/2)$.

Example

$$x(t) = A\cos 2\pi f_0 t$$

(a) Time-domain approach

$$P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A^2 \cos^2 2\pi f_0 t \, dt$$

$$= \frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2} \left(1 + \cos 4\pi f_0 t\right) dt$$

$$= A^2 / 2$$

(b) Frequency-domain approach

$$x(t) = A\cos 2\pi f_0 t = \frac{A}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

Hence, the Fourier coefficients are

$$c_n = \begin{cases} A/2 & n = -1 \text{ and } 1 \\ 0 & \text{otherwise} \end{cases}$$

Defining

$$G_x(f) = \left(\frac{A}{2}\right)^2 \left[\delta(f - f_0) + \delta(f + f_0)\right]$$

we have $P_x = \int_{-\infty}^{\infty} G_x(f)\,df = A^2/2$

Autocorrelation

Autocorrelation refers to the similarity between a signal with a delayed version of itself. It has different definitions for energy signals and power signals.

- **Energy signal**

For a real-valued energy signal $x(t)$, the autocorrelation function is defined as

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\, dt, \quad \forall\ \tau$$

The properties of $R_x(\tau)$ are as follows:

1	$R_x(\tau) = R_x(-\tau)$	even
2	$R_x(\tau) \le R_x(0), \quad \forall \tau$	maximum at $\tau = 0$
3	$R_x(\tau) \leftrightarrow \Psi_x(f)$	Fourier Transform pair
4	$R_x(0) = \int_{-\infty}^{\infty} x^2(t)\, dt$	signal energy

- **Power signal**

The autocorrelation function of a power signal $x(t)$ is defined as

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt, \quad \forall\ \tau$$

For a periodic signal with period T_0,

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x(t+\tau)\, dt, \quad \forall\ \tau$$

The properties of a real-valued periodic signal are similar to those of an energy signal described previously.

Definition of Random Variables (RVs)

Given an experiment E with a sample space Ω and elements $\omega \in \Omega$, we define a function $X(\omega)$ whose **domain** is Ω and whose **range** is a set of numbers on the real line. The function $X(\omega)$ is called a RV.

We shall represent a RV by a capital letter (e.g., X, Y, or Z) and any particular value of the RV by a lower case letter (e.g., x, y, or z).

Example

If we toss a coin, the possible outcome is either a head or a tail. The sample space Ω contains two points labeled H and T. Suppose we define a function

$$X(\omega) = \begin{cases} 1, & \omega = H \\ -1, & \omega = T. \end{cases}$$

Thus we have mapped two possible outcomes of the experiment E into two points on the real line.

Example

An experiment consists of rolling a die. The sample space Ω is

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Let the RV be a function X such that

$$X = X(\omega) = 2\omega$$

where ω is the outcome of the experiment. Here X maps the sample space of 6 elements into 6 even numbers from 2 to 12.

Example

Consider an experiment of measuring the weight of a chicken. Let the possible outcomes be from 0.5 kg to 3 kg. The sample space is

$$\Omega = \{\omega \in R \mid 0.5 < \omega < 3\}$$

We define a RV Y by the function

$$Y = Y(\omega) = 5\omega$$

Points of Y now map onto the real line as the set $\{2.5 < x < 15\}$.

Two Conditions for RVs

1. The set $\{X \leq x\}$ shall be an event for any real number x. That is, $\Pr\{X \leq x\}$ is defined.
2. The probabilities of the events $\{X = \infty\}$ and $\{X = -\infty\}$ are zero. This condition does not prevent X from being either ∞ or $-\infty$ for some values of $\omega \in \Omega$; it only requires that the probabilities of these events be zero.

Distribution function

The distribution function of a RV X is defined as

$$F_X(x) = \Pr\{X \le x\}$$

The distribution function has the following properties:

1. $0 \le F_X(x) \le 1$
2. $F_X(x_1) \le F_X(x_2), \quad x_1 \le x_2$
3. $F_X(-\infty) = 0$
4. $F_X(\infty) = 1$

Probability Density Function (pdf)

The pdf of X is defined in terms of the distribution function by

$$p_X(x) = \frac{dF_X(x)}{dx}$$

Since $F_X(x) = \Pr\{X \le x\}$ and $F_X(-\infty) = 0$, we have

$$F_X(x) = \int_{-\infty}^{x} p_X(\alpha)\, d\alpha$$

When the RV is discrete, the pdf contains delta functions (unit impulses) at the jumps of $F_X(x)$. In such cases, the pdf may be expressed as

$$p_x(x) = \sum_{i=1}^{n} \Pr\{X = x_i\}\,\delta(x - x_i)$$

where x_i's are the possible discrete values of the RV X.

The pdf has the following properties:

1. $$p_X(x) \geq 0$$

2. $$\int_{-\infty}^{\infty} p_X(x)\, dx = 1$$

Statistical Averages

In many cases the distribution function or pdf of a RV contains more information than is really necessary. Thus, we shall often find it easier and more convenient to describe a RV by a few numerical values that roughly characterize the RV.

Consider a single RV X with a pdf $p_X(x)$. The **mean** or **expected value** of X is defined as

$$\mathrm{E}(X) = m_X = \int_{-\infty}^{\infty} x\, p_X(x)\, dx$$

where E(•) denotes expectation. This is also called the **first moment** of *X*. Expectation is a linear operator. That is, if *a* and *b* are constants, then

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

Similarly, the ***n*th moment** of the RV *X* is defined as

$$\mathrm{E}(X^n) = \int_{-\infty}^{\infty} x^n \, p_X(x)\, dx$$

Now, suppose that we denote a RV

$$Y = g(X),$$

where $g(X)$ is some arbitrary function of the RV X. The expected value of Y is

$$\mathrm{E}(Y) = \mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, p_X(x)\, dx$$

The ***nth central moment*** of the RV X is defined as

$$\mathrm{E}\left[(X - m_X)^n\right] = \int_{-\infty}^{\infty} (x - m_X)^n\, p_X(x)\, dx$$

where m_X is the mean value of X. When n = 2, the central moment is called the **variance** of the RV and denoted as σ_X^2. That is,

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - m_X)^2 \, p_X(x) \, dx$$

This parameter provides a measure of the dispersion of the RV X. The positive square root of variance is defined as the **standard deviation** of X and is denoted by σ_X.

By expanding the terms $(x - m_X)^2$ in the integral and noting that the expected value of a constant is equal to the constant, we obtain

$$
\begin{aligned}
\sigma_X^2 &= \mathrm{E}\left[\left(X - m_X\right)^2\right] \\
&= \mathrm{E}\left[X^2 - 2m_X X + m_X^2\right] \\
&= \mathrm{E}\left(X^2\right) - 2m_X \mathrm{E}\left(X\right) + m_X^2 \\
&= \mathrm{E}\left(X^2\right) - 2m_X^2 + m_X^2 \\
&= \mathrm{E}\left(X^2\right) - m_X^2
\end{aligned}
$$

Thus, the variance is equal to the difference between the mean-square value and the square of the mean.

Some Useful Probability Models

There are many probability models, including discrete and continuous. Some models, which will be useful in our future study, are given below.

Binomial Distribution

Consider an experiment having two possible outcomes A and B. Let their probabilities be ε and $1 - \varepsilon$, respectively. If the experiment is repeated m times and we count the number of times that A occurred, then we have created a RV N which takes discrete values $0, 1, \ldots, m$. The corresponding probability is

$$\Pr(N = n) = \binom{m}{n} \varepsilon^n (1-\varepsilon)^{m-n}$$

where

$$\binom{m}{n} \equiv \frac{m!}{n!\,(m-n)!}$$

is the binomial coefficient. The pdf of *N* can be expressed as

$$p_N(y) = \sum_{k=0}^{m} \Pr(N = k)\, \delta(y-k)$$

$$= \sum_{k=0}^{m} \binom{m}{k} \varepsilon^k (1-\varepsilon)^{m-k}\ \delta(y-k)$$

The distribution function is

$$F_N(y) = \Pr(N \leq y)$$
$$= \sum_{k=0}^{[y]} \binom{m}{k} \varepsilon^k (1-\varepsilon)^{m-k}$$

where [y] denotes the largest integer less than or equal to y.

The first two moments and variance of N are

$$\mathrm{E}(N) = m\varepsilon$$
$$\mathrm{E}(N^2) = m\varepsilon(1-\varepsilon) + m^2\varepsilon^2$$
$$\sigma^2 = m\varepsilon(1-\varepsilon)$$

Uniform Distribution

When a continuous RV X is equally likely to take any value in the range [a, b], it is said to be uniformly distributed. The pdf is

$$p_X(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

The first two moments and variance are

$$\mathrm{E}\left(X\right) = (b+a)/2$$

$$\mathrm{E}(X^2) = \left(b^2 + ab + a^2\right)/3$$

$$\sigma_X^2 = \left(b-a\right)^2/12$$

Gaussian Distribution

Gaussian distribution is also called **normal distribution**. It is the most widely used probability model in communications, due to the following two reasons:

- **Central-limit theorem**

  If $X_1, X_2, \ldots, X_n$ are n independent RVs and Y is the sum of them, then as n becomes very large, the distribution of Y approaches a Gaussian distribution. This result is independent of the distributions of the individual components as long as the contribution of each is small compared to the total.

- **Noise** considerations in system analysis are extremely difficult unless the underlying noise statistics are Gaussian. Thus, even if the actual noise statistics are not Gaussian, it is often necessary to approximate noise as Gaussian in order to obtain some useful results.

The pdf of a Gaussian RV *X* is the familiar bell-shape curve

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-(x-m_X)^2/2\sigma_X^2}$$

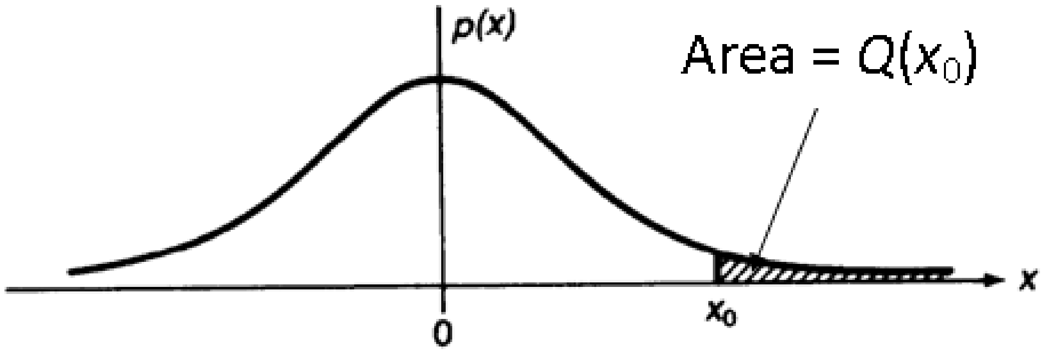
where m_X and σ_X^2 are the mean and variance of the RV.

The distribution function is

$$F_X(x) = \int_{-\infty}^{x} p_X(u)\,du = 1 - Q\left(\frac{x - m_X}{\sigma_X}\right)$$

where

$$Q(x_0) = \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$



Standard normal pdf with mean 0 and variance 1

We note that $Q(-x) = 1 - Q(x)$, $Q(0) = 0.5$, $Q(-\infty) = 1$, and $Q(\infty) = 0$.

The nth central moments of X are

$$\mathrm{E}\left[(X - m_X)^n\right] = \mu_n$$

$$= \begin{cases} 1 \cdot 3 \cdots (n-1)\, \sigma_X^n & n \quad \text{even} \\ 0 & n \quad \text{odd} \end{cases}$$

and the nth moments are

$$\mathrm{E}\left[X^n\right] = \sum_{i=0}^{n} \binom{n}{i} m_X^i \, \mu_{n-i}$$

Random Processes

In our study of digital communications, we often encounter **random processes**, or **stochastic processes**, in the description of

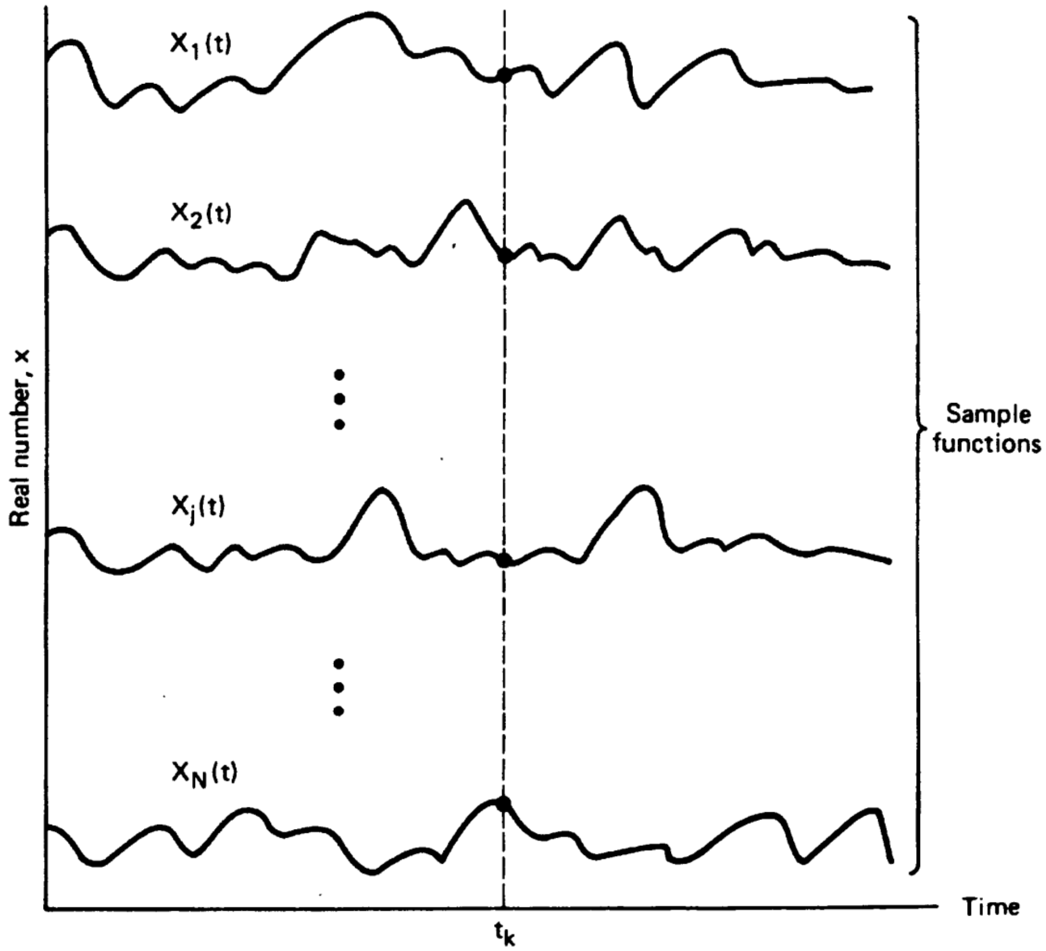
- information sources
- communication channels
- noise generated at a receiver
- the optimum receiver

A random process, $X(\omega, t)$ or $X_\omega(t)$, can be viewed as a function of two variables, where ω is a member of a sample space Ω and t is the time variable.

If we fix the random element $\omega = j$, then $X_j(t)$ is a time function or a **sample function**. The totality of all sample functions is called an **ensemble**. For a specific value of $t = t_k$, $X_\omega(t_k)$ is a RV, depending on the choice of ω. Finally, if $\omega = j$ and $t = t_k$, then $X_j(t_k)$ is simply a real number.

For notational convenience, we drop the subscript ω and use $X(t)$ to designate the random process.

Example - Random noise process



Description of a Random Process

A general description of a random process $X(t)$ requires a very complicated joint pdf, which may not be known in most situations. Fortunately, a partial description consisting of the mean and autocorrelation function is often found to be adequate for the needs of communication systems.

The mean of the RV $X(t_k)$ is defined as

$$\mathrm{E}\left[X(t_k)\right] = \int_{-\infty}^{\infty} x\, p_k(x)\, dx = m_X(t_k)$$

where $X(t_k)$ is the RV at time t_k, and $p_k(x)$ is the corresponding pdf.

Next we consider two random variables $X(t_1)$ and $X(t_2)$ that are generated from the random process $X(t)$ at times t_1 and t_2, respectively. The autocorrelation function between $X(t_1)$ and $X(t_2)$ is given by

$$R_X(t_1, t_2) = \mathrm{E}\left[X(t_1)X(t_2)\right]$$

which measures the degree to which the two time samples of $X(t)$ are related.

Stationary

A random process $X(t)$ is **stationary in the strict sense** if its statistics are not affected by a shift in the time origin. This means that the two random processes $X(t)$ and $X(t+\tau)$ have the same statistics for any τ.

We say that a random process $X(t)$ is **stationary in the wide sense**, if its mean value is a constant and its autocorrelation function depends on the time difference. That is,

$$\mathrm{E}\left[X(t)\right] = m_X = \text{constant}$$

and

$$R_X(t_1, t_2) = R_X(t_1 - t_2)$$

Clearly, if $X(t)$ is stationary in the strict sense, then it is **wide-sense stationary** (WSS); however, the converse is not true. Fortunately, most of the useful results in communication theory are predicated on random information signals and noise being WSS.

Autocorrelation of a WSS Process

For a real-valued WSS process,

$$R_X(\tau) = \mathrm{E}\left[X(t+\tau)X(t)\right]$$

If $R_X(\tau)$ changes slowly as τ increases from 0 to some value, it means that, on the average, $X(t)$ also changes slowly. It is expected that $X(t)$ contains mainly low-frequency components.

If $R_X(\tau)$ decreases rapidly as τ is increased, we would expect that $X(t)$ changes rapidly with time and contains mostly high frequencies.

Properties of $R_X(\tau)$ for a real-valued WSS random process $X(t)$ are

- $R_X(\tau) = R_X(-\tau)$
- $R_X(\tau) \leq R_X(0)$
- $R_X(\tau) \leftrightarrow G_X(f)$ PSD
- $R_X(0) = \mathrm{E}\left[X^2(t) \right]$

Time Averaging and Ergodicity

- Computing m_X and $R_X(\tau)$ by ensemble averaging is normally very difficult and tedious.
- When a random process belongs to a special class, known as **ergodic process**, its time and ensemble averages are the same. Hence, it is possible to compute time averages from a single sample function.
- For a random process to be ergodic, it must be stationary in the strict sense.

In particular, a random process is **ergodic in the mean** if

$$m_X = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\, dt$$

and it is **ergodic in the autocorrelation function** if

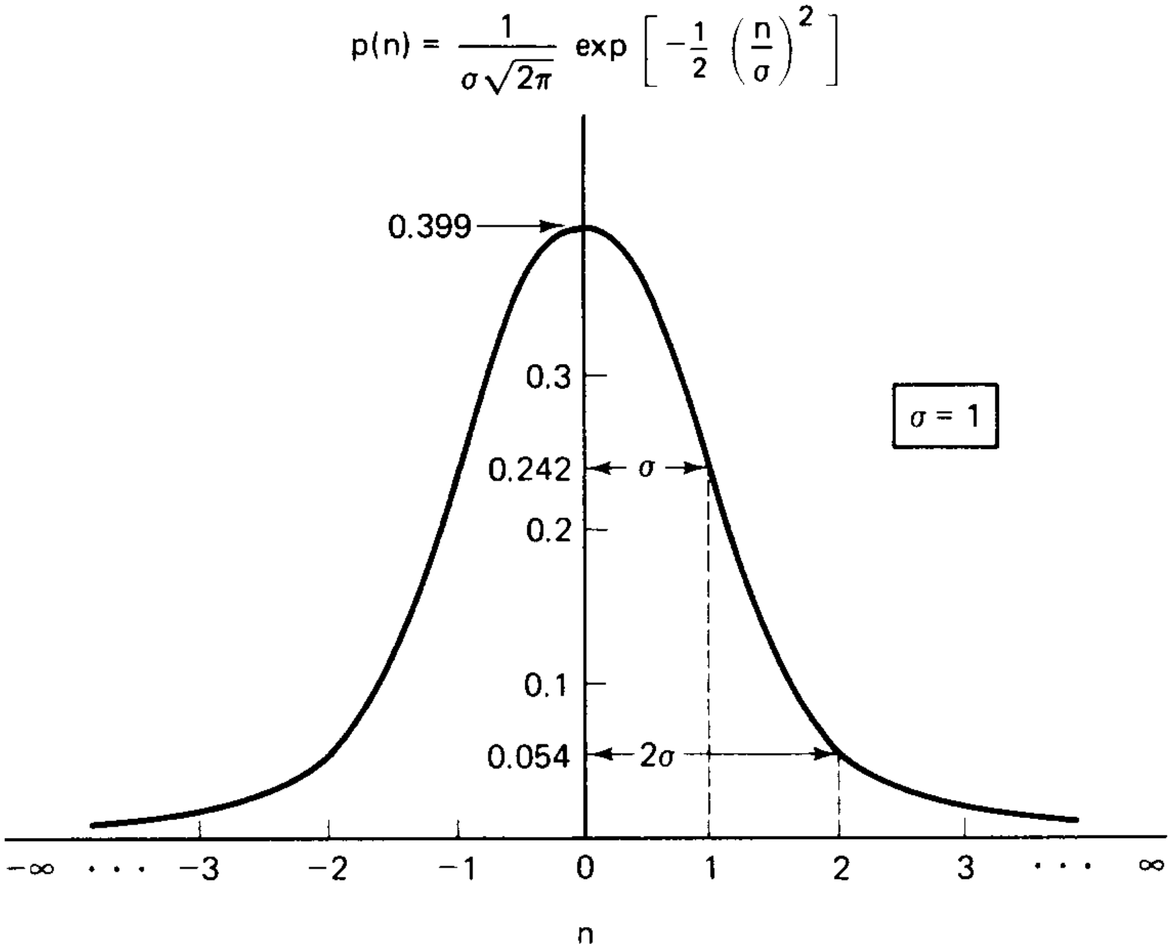
$$R_X(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t+\tau)\, X(t)\, dt$$

Thermal Noise

- Noise is normally referred to as unwanted electrical signals that are always present in electrical systems.
- Good engineering design can eliminate much of the noise or its undesirable effects through filtering, shielding, the choice of modulation, and receiver design.
- Thermal noise is caused by the random motion of electrons in all dissipative components. Thermal noise cannot be eliminated.
- Thermal noise can be described as a zero-mean Gaussian random process $n(t)$ with pdf

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-n^2/2\sigma^2}$$

where σ^2 is the variance of the thermal noise. Note that the pdf is independent of time. This noise model is commonly used in the analysis of communication systems.

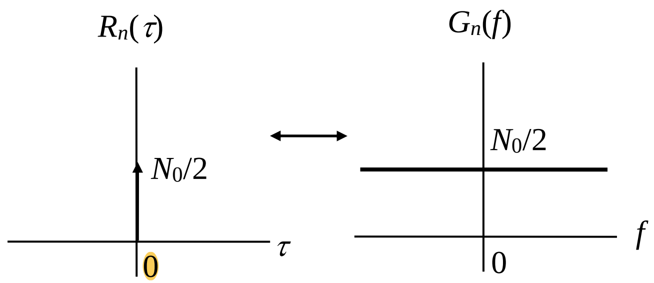


Normalized Gaussian PDF

White Noise

The PSD of thermal noise is flat (up to 10^{12} Hz). Therefore,

$$G_n(f) = \frac{N_0}{2} \quad \text{(watts/Hz)}$$



The autocorrelation function is

$$R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

Note that the power $P_n = R_n(0) = \infty$.

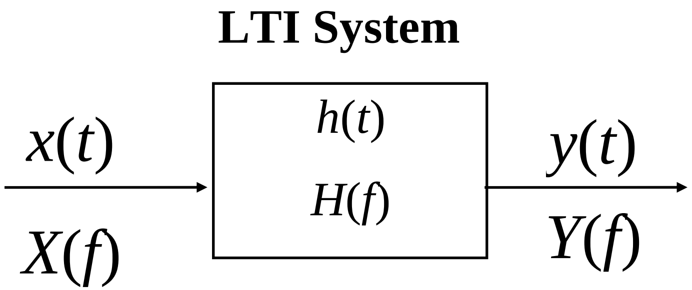
In reality no noise process can be truly white. However, as long as the bandwidth of noise is much larger than that of real systems, this approximation greatly simplifies our analysis.

Since the autocorrelation function is a delta function, the noise samples in different time instants are **statistically uncorrelated**. Moreover, thermal noise is a Gaussian process. It implies that the noise samples are **independent** and they affect each transmitted symbol independently. We refer to this type of noise as **additive white Gaussian noise** (AWGN). The adjective “additive” means that the noise is simply added to the signal. Such a channel is a **memoryless channel**.

In the subsequent analysis, we shall assume that the system is corrupted by AWGN even though this is sometimes an oversimplification.

Signal Transmission through LTI Systems

For a **linear time-invariant** (LTI) system, the output can be obtained in two ways.



In the time domain,

$$y(t) = x(t) \otimes h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

where $h(t)$ is the impulse response of the LTI system.

In the frequency domain,

$$Y(f) = X(f)H(f)$$

where $H(f)$ is the transfer function of the LTI system.
Note that $h(t) \leftrightarrow H(f)$.

If a WSS random process is transmitted through an LTI system, the output PSD, $G_Y(f)$, is related to the input PSD, $G_X(f)$, in the form

$$G_Y(f) = G_X(f)\left|H(f)\right|^2$$

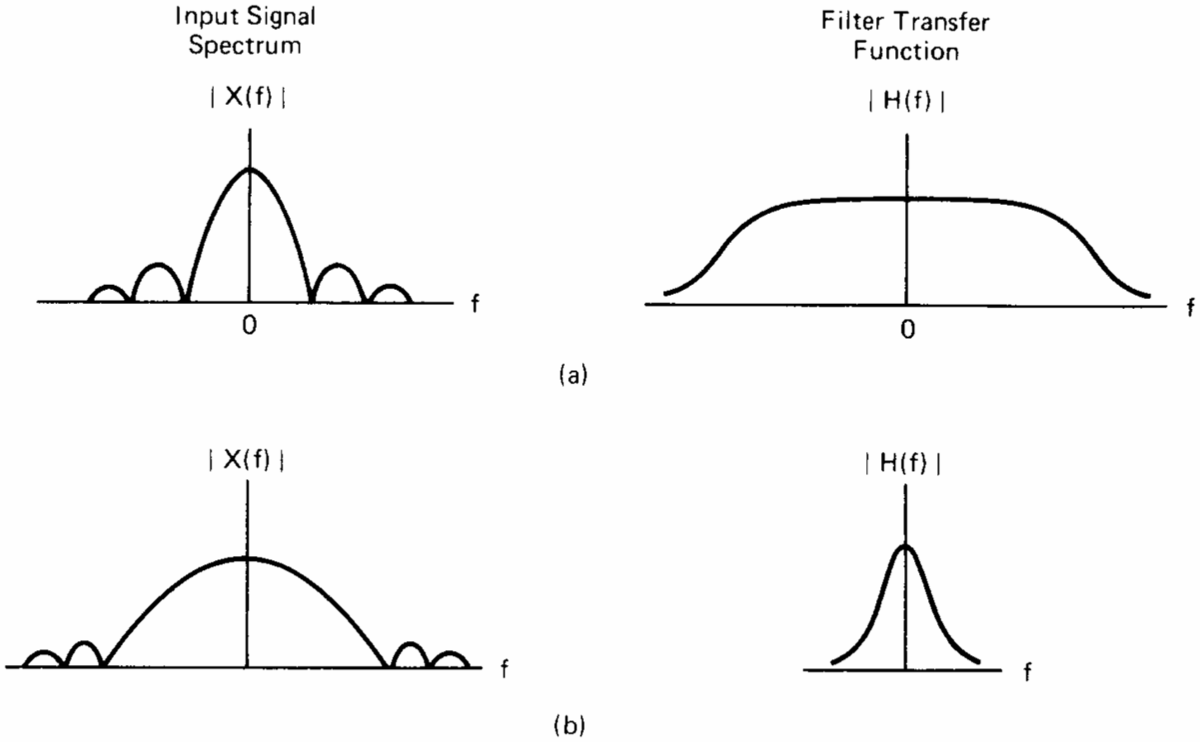
It can be shown that if a Gaussian process $x(t)$ is applied to an LTI system, the output $y(t)$ is also Gaussian.

Signal, Circuits, and Spectra

- Both signals and circuits can be described in the frequency domain in terms of $X(f)$ and $H(f)$.
- The output spectrum is

$$Y(f) = X(f)H(f)$$

Hence, the output bandwidth is constrained by the smaller of the two bandwidths (when one of the two spectral functions goes to zero, the multiplication yields zero).



Spectral characteristics of the input signal and the circuit contribute to the spectral characteristics of the output signal. (a) Case 1: Output bandwidth is constrained by input signal bandwidth. (b) Case 2: Output bandwidth is constrained by filter bandwidth.