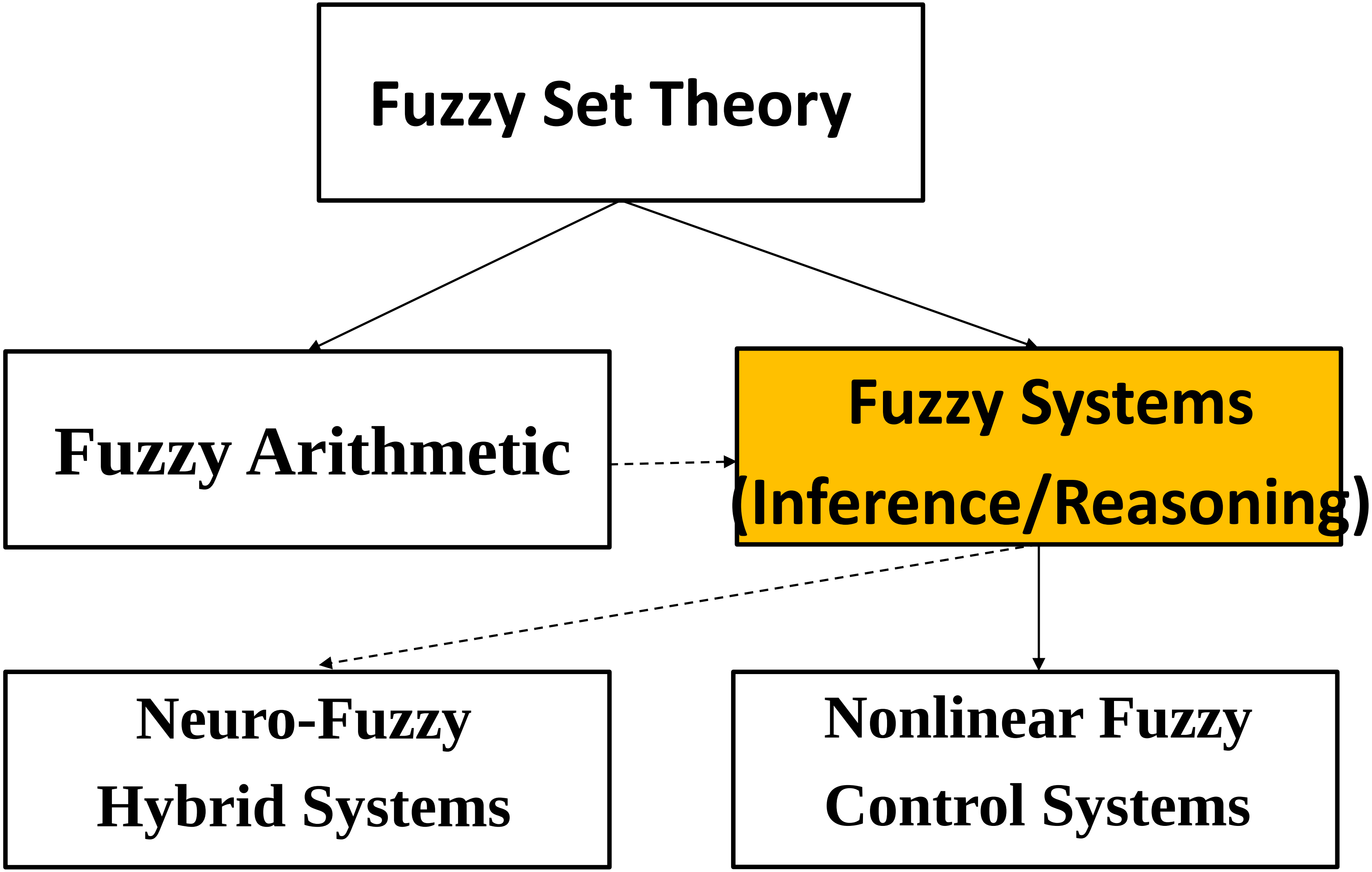
NANYANG TECHNOLOGICAL UNIVERSITY

3. Fuzzy Systems


Fuzzy Set Theory
Fuzzy Arithmetic
Fuzzy Systems
(Inference/Reasoning)
Neuro-Fuzzy
Hybrid Systems
Nonlinear Fuzzy
Control Systems

3.1 Introduction

A **fuzzy system** (**FS**) is defined as a system with operating principles based on ***fuzzy information processing*** and ***decision making.*** Namely the system makes use of fuzzy sets and of the corresponding mathematical framework.

Fuzzy sets can be involved in a system in a number of ways:

1) Systems with fuzzy parameters.

Example

$$y = \tilde{3}x_1 + \tilde{5}x_2$$

where $\tilde{3}$ and $\tilde{5}$ are fuzzy numbers "about three" and "about five", respectively, defined by membership functions.

2) The input, output and state variables of a system may be fuzzy sets.

Example

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = \tilde{2}$$

where $\tilde{2}$ is a fuzzy number

3) Rule-based systems

A fuzzy system in this case is a collection of fuzzy rules that converts inputs to outputs.

The basic configuration of a pure fuzzy system is shown in the Figure. 3.1

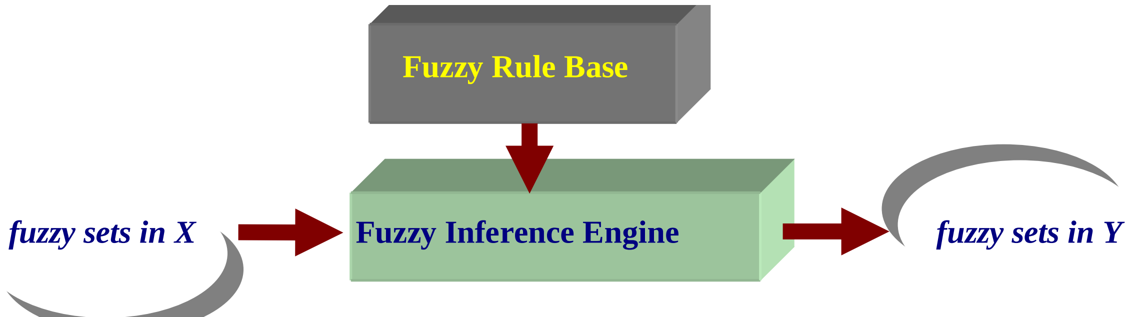


Figure. 3.1

The ***fuzzy inference engine*** (algorithm) combines fuzzy *IF-THEN* rules into a mapping from fuzzy sets in the ***input space*** *X* to fuzzy sets in the ***output space*** *Y* based on fuzzy logic principles.

Rule-based Fuzzy Systems

- Linguistic (Mamdani) fuzzy system

If x **is** A **then** y **is** B

Example
If *the heating power is high*
then *the temperature will increase fast.*

- Takagi-Sugeno fuzzy system

Example
If x **is** A **then** $y = f(x)$

Fuzzy systems are **NONLINEAR MAPPINGS** of inputs (stimuli) to outputs (responses).

The **main characteristics** of fuzzy systems are.

- Fuzzy systems are suitable for uncertain or approximate reasoning, especially for the system with a mathematical model that is difficult to derive;

- Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

3.2 Introduction to Fuzzy Rules

A fuzzy if-then rule (also known as fuzzy rule, fuzzy implication, or fuzzy conditional statement) assumes the form

IF x is A THEN y is B

where x and y are linguistic variables, A and B are linguistic values determined by fuzzy sets on the universe of discourses X and Y, respectively.

"x is A"-Antecedent or Premise
"y is B"-Consequence or conclusion

The expression

If *x* is *A* then *y* is *B*, or *A*→*B*

describes a relation between variables *x* and *y.*

*T*his suggests that a fuzzy rule be defined as a binary fuzzy relation *R* on the product space X × Y (**Mamdani'T-norm Rule)**

$$R = A \to B = A \times B = \int_{X\times Y} \mu_A(x) \wedge \mu_B(y)/(x,y)$$

RECALL : A ***Binary Fuzzy Relation*** is a fuzzy relation between two sets *X* and *Y. It is* a fuzzy set in X ×Y, which map every element in X ×Y into a membership grade between 0 and 1. Hence, ***Binary Fuzzy Relations*** are 2-dimensional MFs

$$\boxed{R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}}$$

Example:

X = {1,2,3,4} - uniqueness of invention

Y = {1, 2, 3, 4, 5, 6} - market sizes

A = medium uniqueness ={0/1+0.6/2 + 1/3 + 0.2/4}

B = medium market size ={0/1+0.4/2 + 1/3 + 0.8/4 + 0.3/5+0.0/6}

$$
\begin{aligned}
&\mu_R(1,2) = (0 \wedge 0.4) = 0 \\
&\mu_R(2,2) = (0.6 \wedge 0.4) = 0.4 \\
&\mu_R(3,2) = (1 \wedge 0.4) = 0.4 \\
&\mu_R(4,2) = (0.2 \wedge 0.4) = 0.2 \\
&\vdots
\end{aligned}
$$

$$A \to B = R = \begin{pmatrix} 0 \\ 0.6 \\ 1 \\ 0.2 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0.4 & 1.0 & 0.8 & 0.3 & 0 \end{pmatrix}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .4 & .6 & .6 & .3 & 0 \\ 0 & .4 & 1 & .8 & .3 & 0 \\ .0 & .2 & .2 & .2 & .2 & 0 \end{bmatrix} \end{array}, \quad \subseteq X \times Y$$

Firing Rules:

In a fuzzy system, all rules fire to some extent, or in other words, some fire fully while others fire partially.

If the **antecedent** is true to **some degree** of membership, then the **consequent** is also true to the **same degree**.

Firing Fuzzy Rules - An Example

IF height is tall THEN weight is heavy

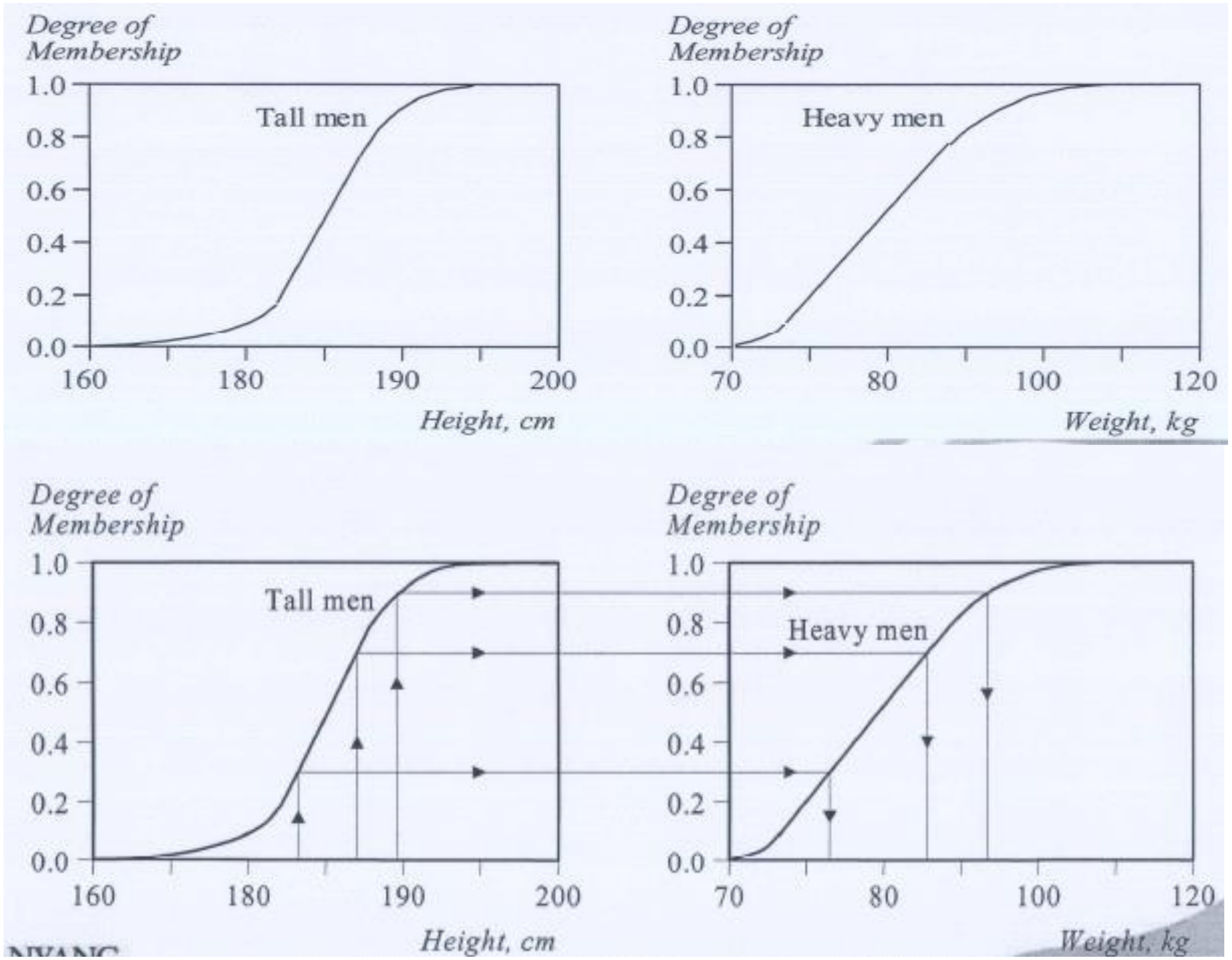


Figure. 3.2

3.3 FUZZY REASONING

- Fuzzy reasoning is an inference procedure that derives conclusions from a set of fuzzy if-then rules and known facts.

- To better understand the reasoning process, we first consider some fundamentals.

Compositional Rule of Inference

- The compositional rule of inference is a generalization of the following familiar notion:
- Suppose y=f(x). If x = a, then from y=f(x) we can infer that y = b = f(a).

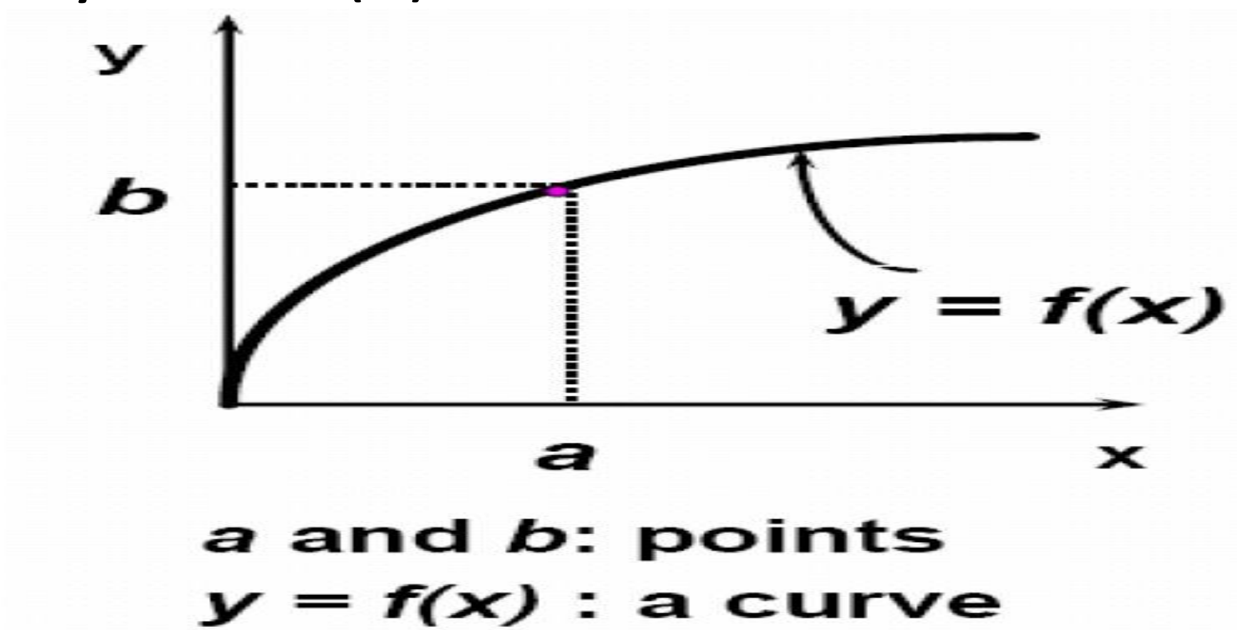


Figure. 3.3

A generalization of the aforementioned process would allow *a* to be an interval and *f(x)* to be an interval valued function.

To find the resulting interval *y = b* corresponding to the interval *x = a*, we first construct a cylindrical extension of *a.*

Then find its intersection *I* with the interval-valued curve. The projection of *I* onto the y-axis yields the interval *y-b*.

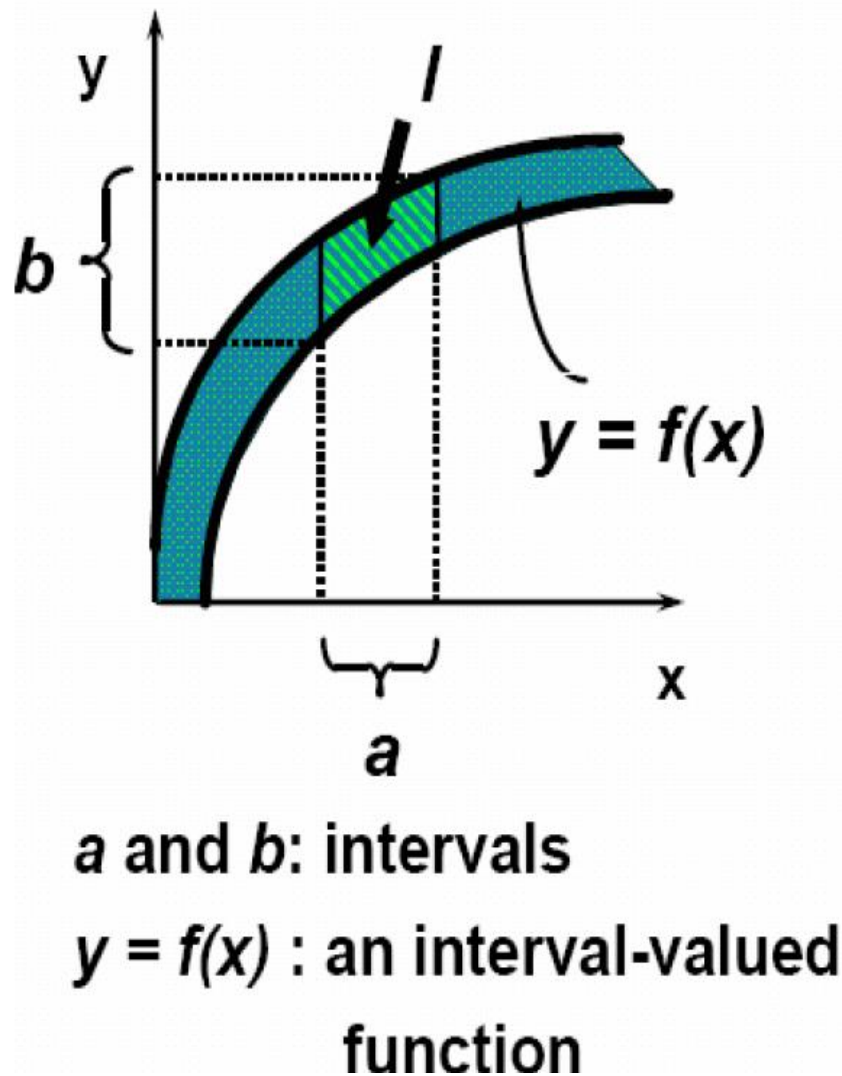
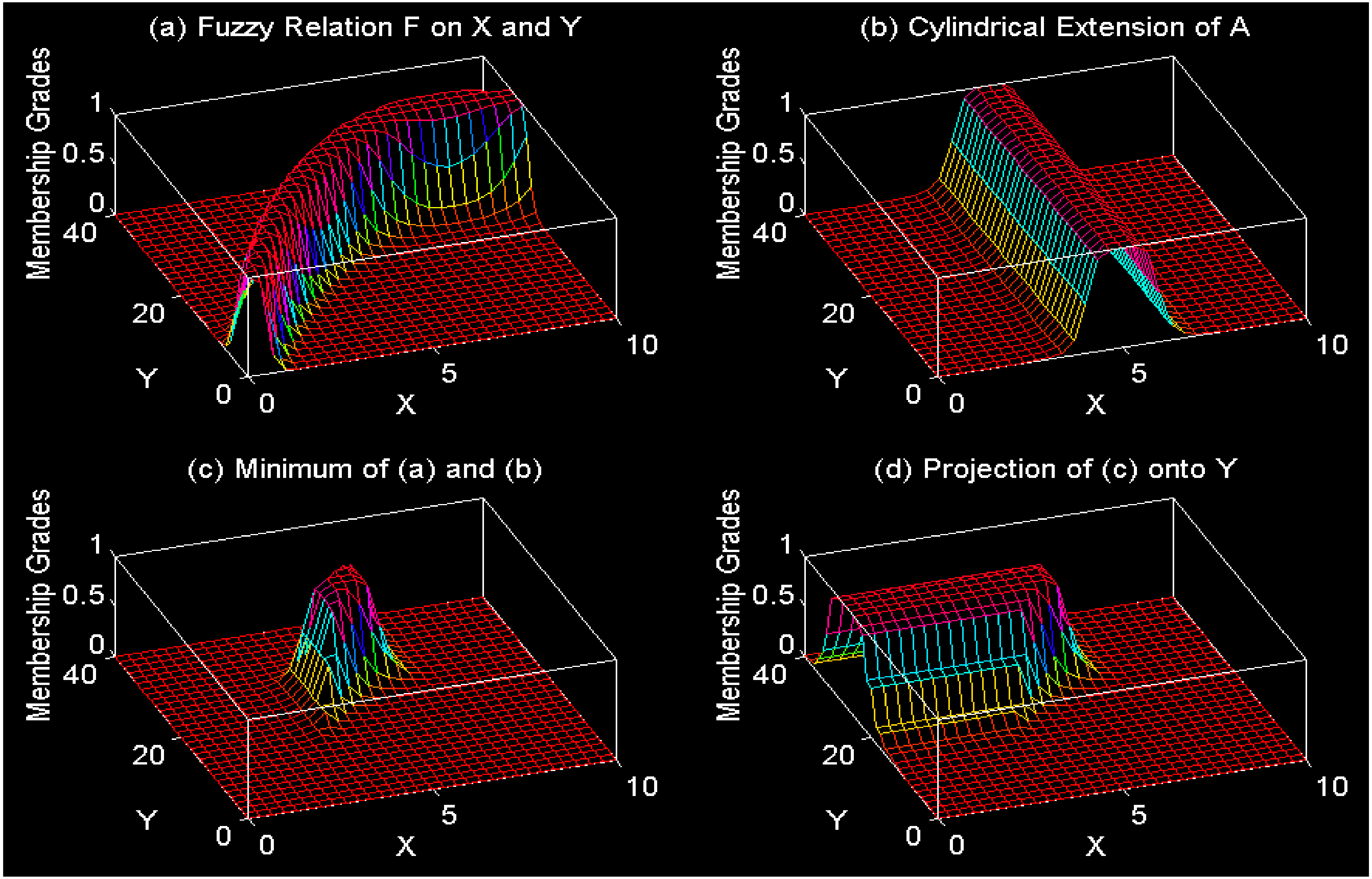


Figure. 3.4

Going one step further in our generalization, we may assume that *R* is a fuzzy relation on X ×Y and *A* is a fuzzy set of *X*, as shown in the Figure 3.5.

To find the resulting fuzzy set *B*, again we construct a cylindrical extension *c(A)*with base *A*. The intersection of *c(A)* and *R* forms the analog of the region of intersection *I*. By projecting $c(A) \cap R$ on to the *y*-axis, we infer *y* as a fuzzy set *B* on the *y*-axis .



(a) Fuzzy Relation F on X and Y
(b) Cylindrical Extension of A
(c) Minimum of (a) and (b)
(d) Projection of (c) onto Y
Membership Grades
1
0.5
0
40
20
Y
0
0
5
10
X

Figure. 3.5

Specifically, let $\mu_{A,}$ $\mu_{c(A),}$ μ_B and μ_R be MFs of *A, c(A), B* and *R*, where $\mu_{c(A)}$ is related to μ_A through $\mu_{c(A)}(x,y)= \mu_A(x)$.

Then

$$\mu_{c(A)\cap R}=\min\{ \mu_{c(A)}(x,y), \mu_R(x,y)\}$$

$$=\min\{ \mu_A(x), \mu_R(x,y)\}$$

By projecting *c(A)∩R* on to the y-axis, we have

$$\mu_B(y)=\max_x \min\{ \mu_A(x,y), \mu_R(x,y)\}$$

$$=\vee_x [\mu_A (x)\wedge\mu_R(x,y)]$$

This formula reduces to the max-min composition and is represented as:

$$B=A \circ R$$

RECALL : 1.2.13 Composition of Fuzzy Relation with Fuzzy set

The *composition* is defined as follows:

Suppose there exists a fuzzy relation *R* in X × Y and *A* is a fuzzy set in X. Then, fuzzy subset *B* of Y can be induced by A through the composition of A and R:

$$B = A \circ R$$

i.e. the composition is defined by:

$$B = \mathrm{proj}_Y \left(R \cap \mathrm{ext}_{X \times Y}(A) \right).$$

The composition can be regarded in two phases: ***combination*** (intersection) and ***projection***.

This is actually the *sup-min* composition. Assume that *A* is a fuzzy set with membership function $\mu_A(x)$ and *R* is a fuzzy relation with membership function $\mu_R(x, y)$. Then

$$\mu_B(y) = \sup_x \min\left(\mu_A(x), \mu_R(x, y)\right),$$

The above is called f**uzzy or approximate reasoning (**also called generalized modus ponens(GMP)) which is formally summarized as follows**:**

Premise 1 (fact):	x is A',
Premise 2 (rule):	if x is A then y is B
consequence (conclusion)	y is B'

where *A, A'*, and *B* are fuzzy sets of *X, X*, and *Y*.

Note : $p \to q \equiv \overline{p} \vee q$ and $\left(p \wedge q\right) \vee \overline{q}$ which can be verified by truth table.

Example:

X = {20 25 30 35 40 45 50}; Y = {10 30 50 70 80 100}

A = 1/20+0.8/25+0.4/30 , B = 0.6/70+0.8/80+1.0/100

If x is A Then y is B

If x is A' what is y ?,

where A' = 0.4/30+0.6/35+1.0/40+0.2/45+0/50

$$\mu_R(x,y) = \mu_A(x) \wedge \mu_B(y)$$

$$= \begin{pmatrix} 1 \\ .8 \\ .4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 & .6 & .8 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & .6 & .8 & 1 \\ 0 & 0 & 0 & .6 & .8 & .8 \\ 0 & 0 & 0 & .4 & .4 & .4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example: Continued

$$B' = A' \circ R = \begin{pmatrix} 0 & 0 & .4 & .6 & 1 & .2 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & 0 & .6 & .8 & 1 \\ 0 & 0 & 0 & .6 & .8 & .8 \\ 0 & 0 & 0 & .4 & .4 & .4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .4 & .4 & .4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\max_x = \begin{bmatrix} 0 & 0 & 0 & .4 & .4 & .4 \end{bmatrix}$$

Further simplification of the rule gives:

$$\mu_{B'}(y) = [\vee_x(\mu_{A'}(x) \wedge \mu_A(x))] \wedge \mu_B(y)$$
$$= w \wedge \mu_B(y)$$

In other words, first we find the degree of match *w* as the maximum of $\mu_{A'}(x) \wedge \mu_A(x)$ then the MF of the resulting B' is equal to the MF of B clipped by w.

Graphic Representation:

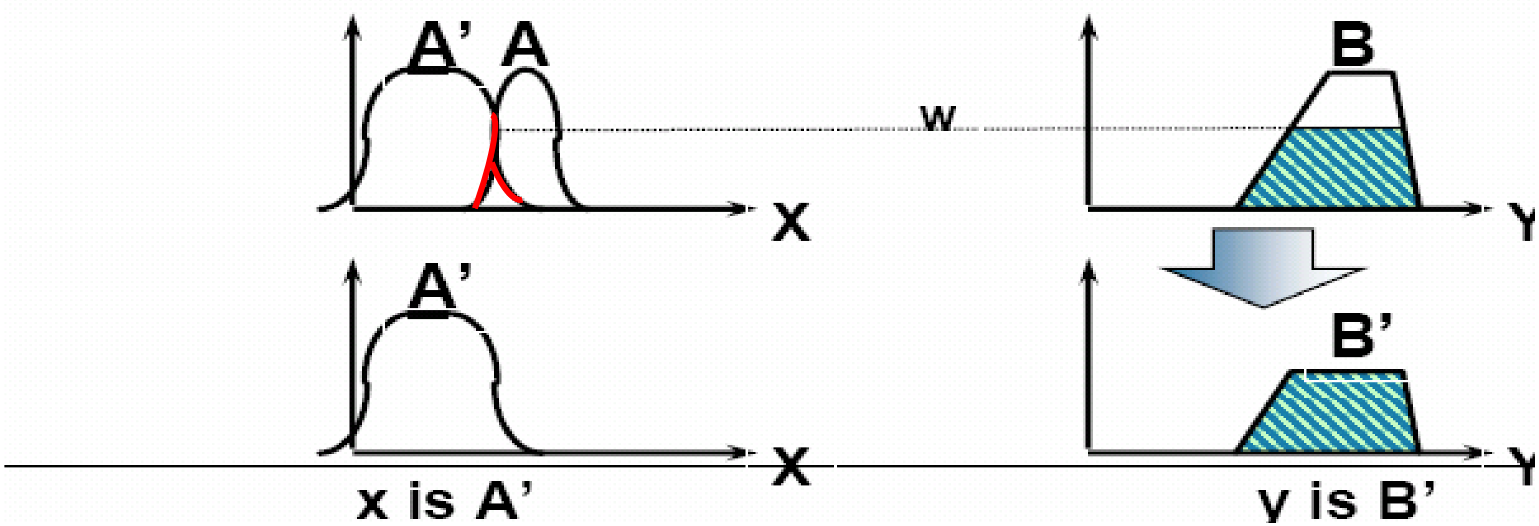


Figure. 3.6

More on Clipping will be discussed later.

Single rule with multiple antecedent

A fuzzy if-then rule with two antecedents is usually written as

"if *x* is A and *y* is B then *z* is C".

premise 1 (fact)	x is A' and y is B'
premise 2(rule)	if x is A and y is B then z is C

____

consequence (conclusion) z is C'

The fuzzy rule in premise 2 can be put into a simpler form "A x B -> C".

Based on Mamdani's fuzzy implication function,

$$C' = (A' \times B') \circ (A \times B \to C)$$

$$\begin{aligned}
\mu_{C'}(z) &= \vee_{x,y}[\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)] \\
&= \vee_{x,y}\{[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)]\} \wedge \mu_C(z) \\
&= \underbrace{\{\vee_x[\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\vee_y[\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z) \\
&= \underbrace{(w_1 \wedge w_2)}_{\textit{firing strength}} \wedge \mu_C(z)
\end{aligned}$$

w_1 and w_2 are maxima of the MFs of $A \cap A'$ and $B \cap B'$

w_1 denote the <u>degree of compatibility</u> between A and A' and similarly for w_2. $w_1 \wedge w_2$ is called the <u>firing strength</u> or <u>degree of fulfillment</u> of the fuzzy rule.

Graphic Representation:

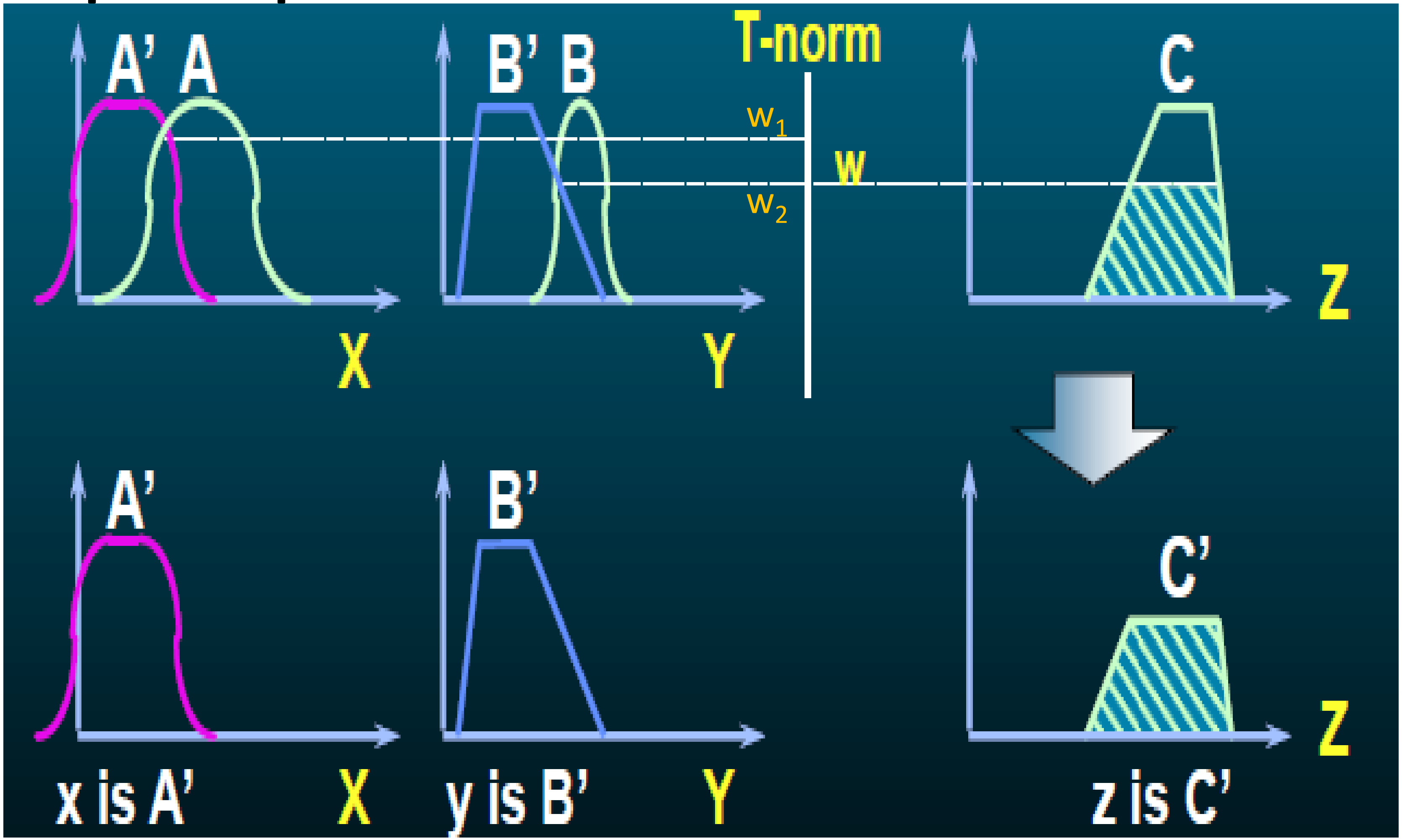


Figure. 3.7

3.4 Fuzzy Inference Systems

As mentioned, a **FUZZY SYSTEM** is a collection of fuzzy rules that converts inputs to outputs.

Usually we start with **crisp inputs and find** crisp outputs.

With such considerations, we have the following four steps of fuzzy inference process:

- Fuzzification of the input variables
- Evaluation of output for each rule
- Aggregation of the rules' outputs
- Defuzzification

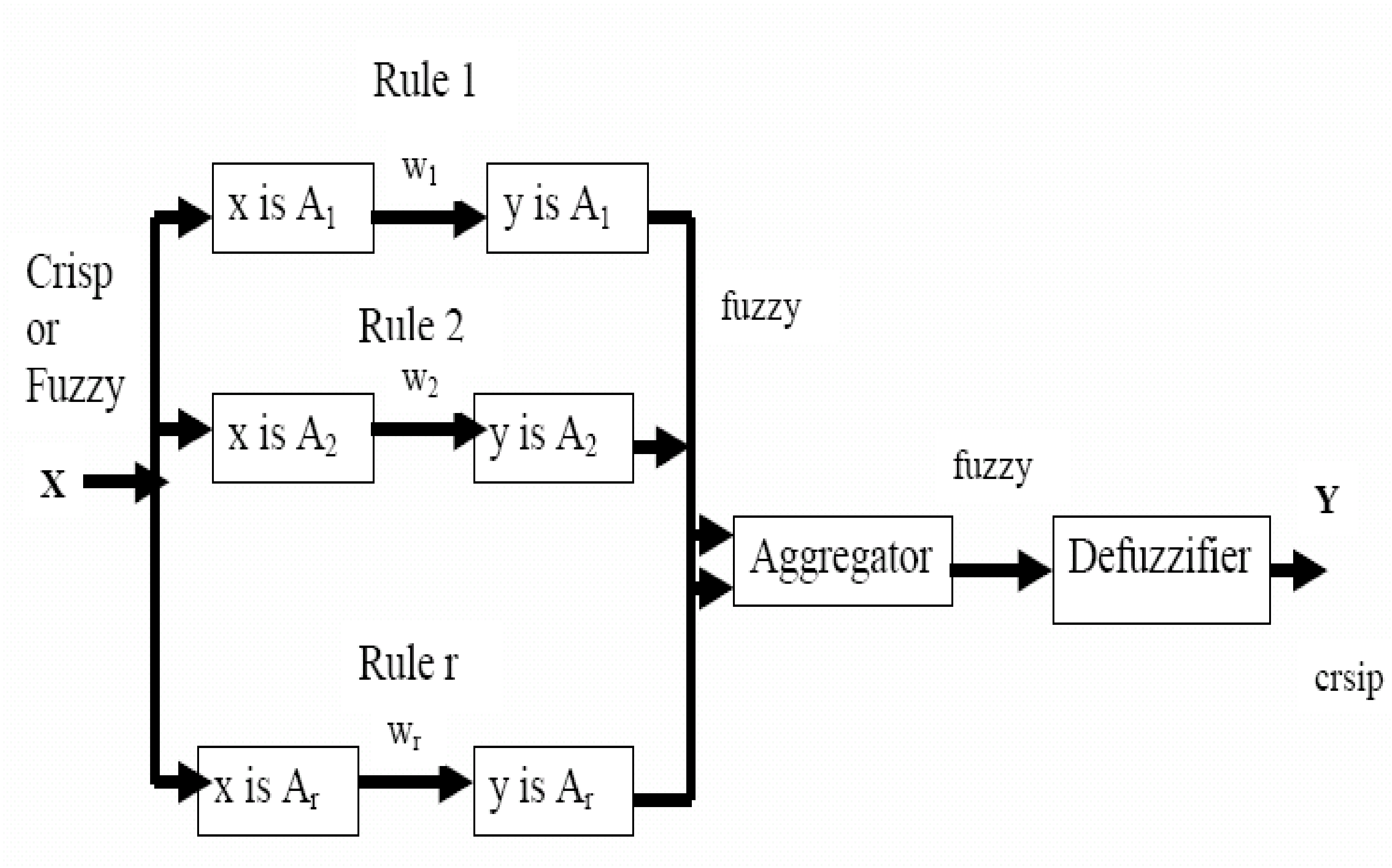


Figure. 3.8

We illustrate the above steps with ***examples.***

Step 1: Fuzzification

The first step is to take the crisp inputs x_1 and y_1, and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.

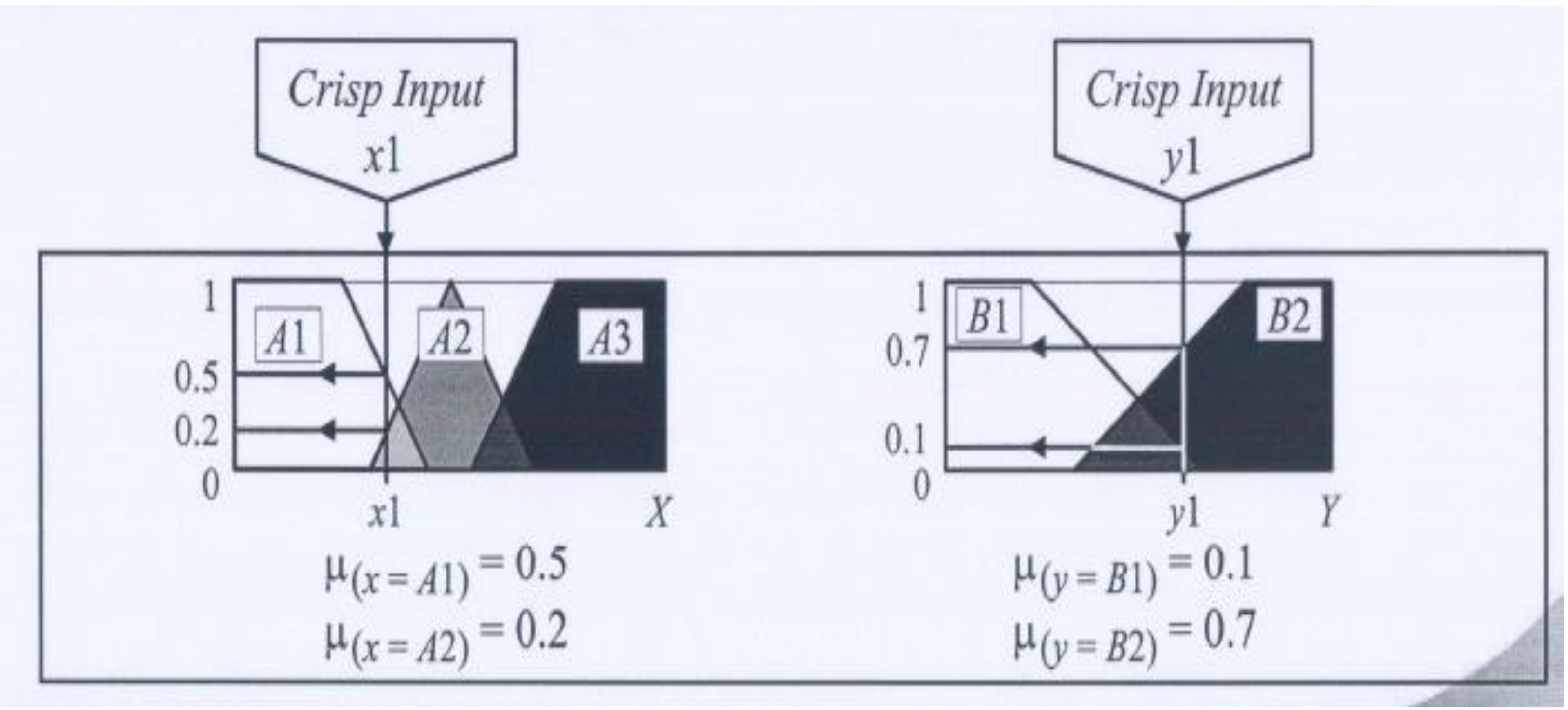


Figure. 3.9

Step 2: Rule Evaluation

- The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$, $\mu_{(x=A2)} = 0.2$, $\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$, and apply them to the antecedents of the fuzzy rules.

- If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.

- This number (the truth value) is then applied to the consequent membership function.

Example

R1: If *x* is A_1 AND *y* is B_1, THEN *z* is C_1
R2: If *x* is A_2 AND *y* is B_2, THEN *z* is C_2

Mamdani's minimum fuzzy implication rule:

The firing levels of the rules, denoted by α_i, *i* = 1, 2, are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),$$
$$\alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

The individual rule outputs are obtained by

$$\mu_{c_1'} = \alpha_1 \wedge \mu_{c_1}(Z)$$
$$\mu_{c_2'} = \alpha_2 \wedge \mu_{c_2}(Z)$$

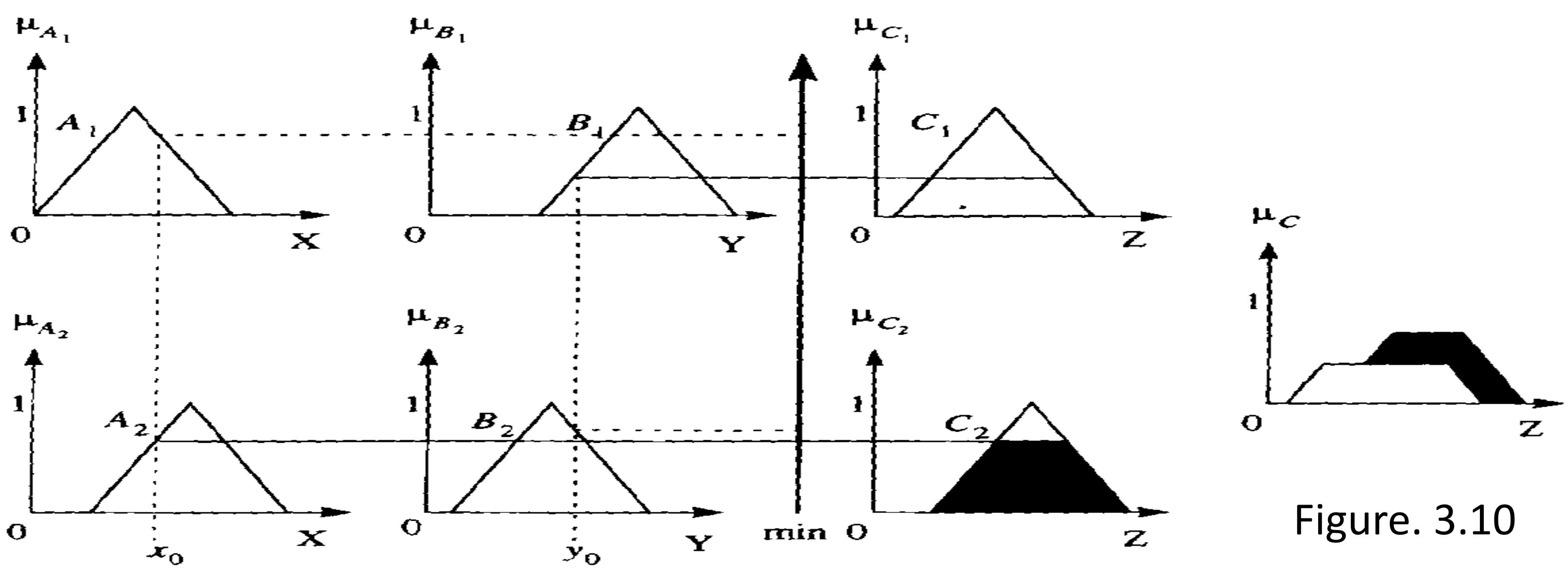


Figure. 3.10

Larsen's product fuzzy implication rule:

The firing levels of the rules, denoted by α_i, i = 1, 2, are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),$$
$$\alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

The individual rule outputs are obtained by

$$\mu_{c_1'} = \alpha_1 \mu_{c_1}(Z)$$
$$\mu_{c_2'} = \alpha_2 \mu_{c_2}(Z)$$

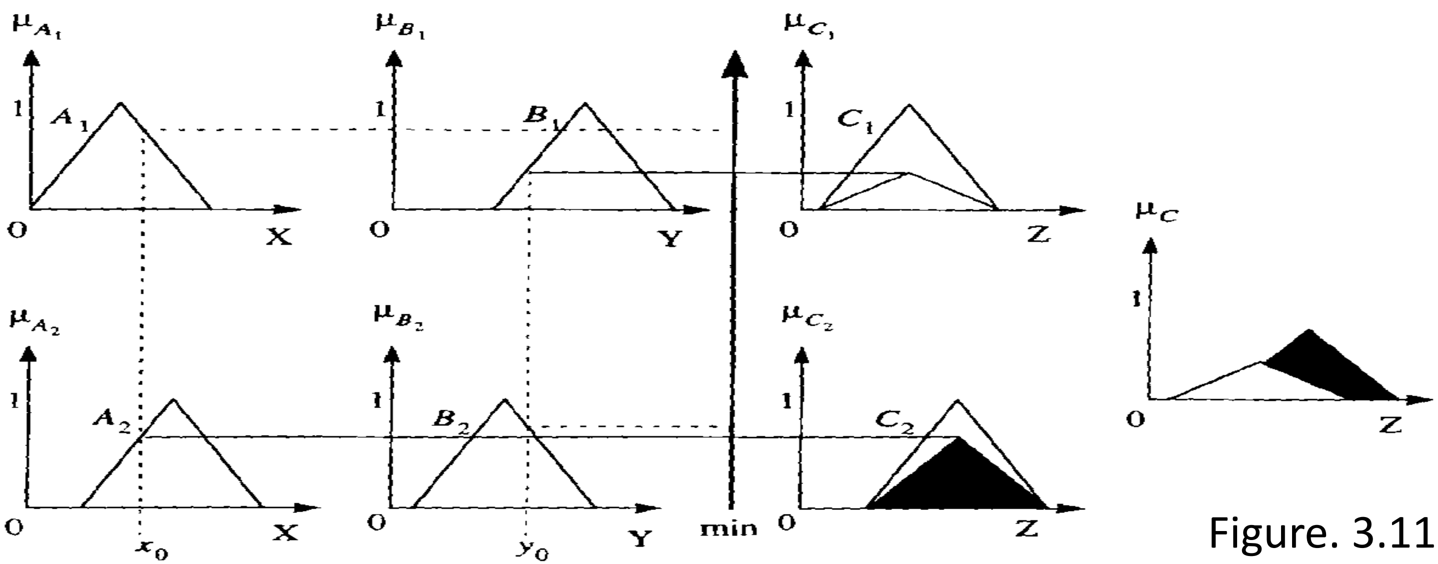


Figure. 3.11

From the above, there are two main methods for applying the result of the antecedent evaluation to the membership function of the consequent

- Clipping
- Scaling

The most common method of correlating the rule consequent with the truth value of the rule antecedent is to cut the consequent membership function at the level of the antecedent truth. This method is called clipping.

Since the top of the membership function is sliced, the clipped fuzzy set loses some information.

However, clipping is still often preferred because it involves less complex and faster mathematics, and generates an aggregated output surface that is easier to defuzzify.

While clipping is a frequently used method, scaling offers a better approach for preserving the original shape of the fuzzy set.

The original membership function of the rule consequent is adjusted by multiplying all its membership degrees by the truth value of the rule antecedent.

This method, which generally loses less information, can be very useful in fuzzy expert systems.

Step 3: Aggregation of Rule Outputs

Aggregation is the process of unification of the outputs of all rules.

We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.

$$C(w) = C_1'(w) \vee C_2'(w)$$

Example

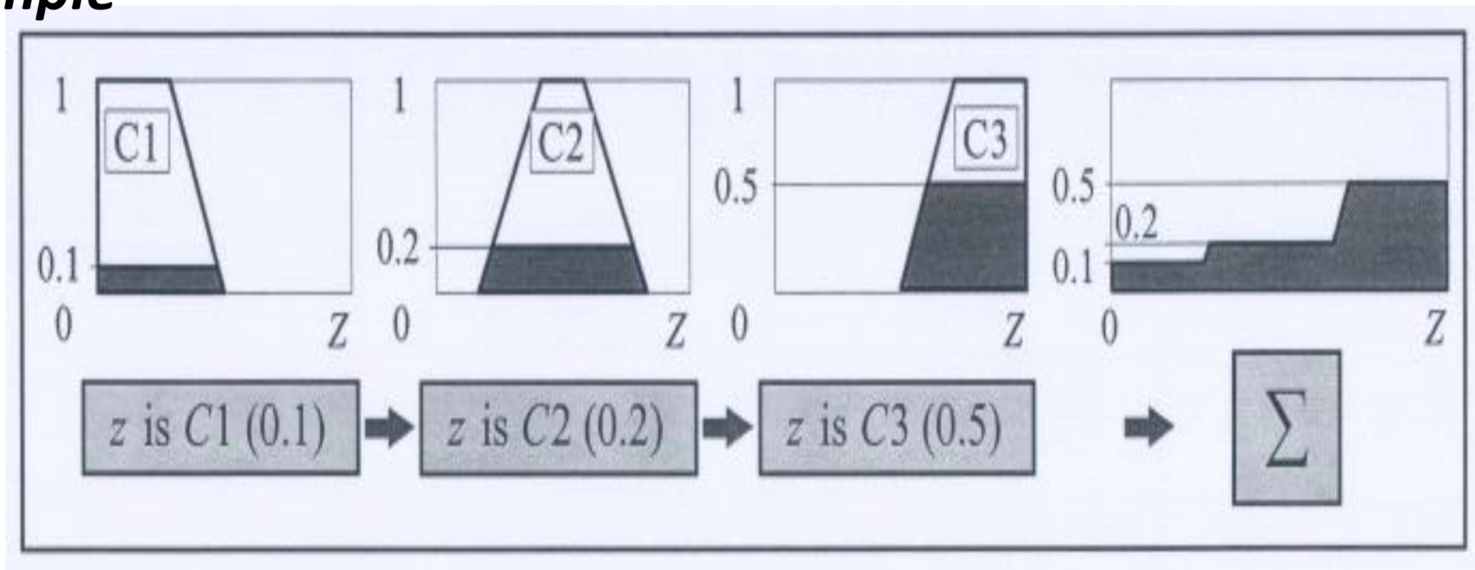


Figure. 3.12

Step 4: Defuzzification

Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.

The input for the defuzzification process is the aggregate output fuzzy set *C* and the output is a single crisp number.

There are several defuzzification methods, but probably the most popular one is the centroid technique.

Centroid defuzzification method finds a point representing the centre of gravity of the fuzzy set, *A,* on the interval, [*a b*].

A reasonable estimate can be obtained by calculating it over a sample of points.

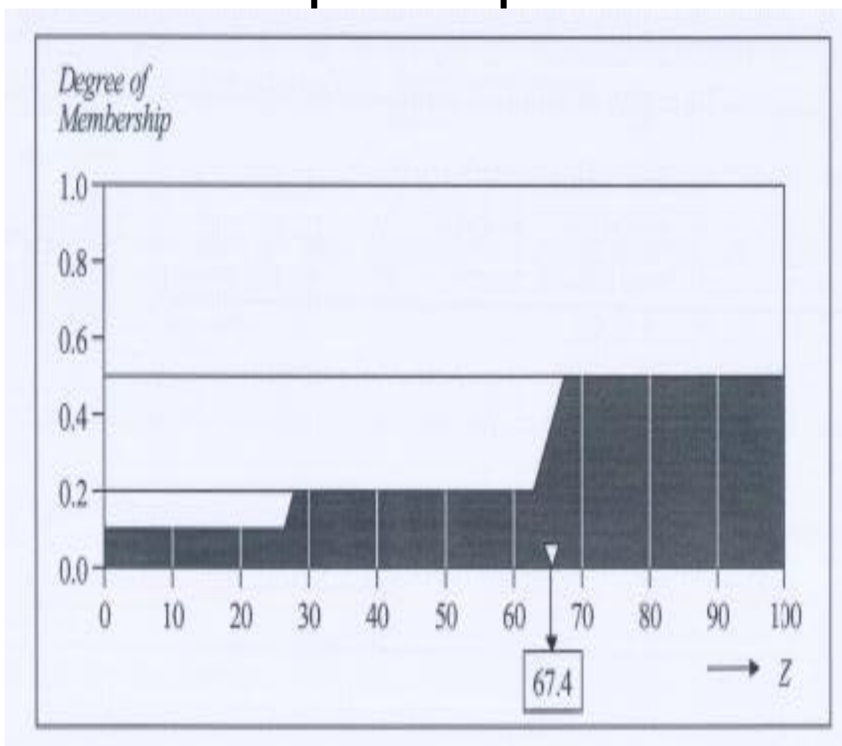


Figure. 3.14

Mathematically this Centre Of Gravity (COG) can be expressed as:

$$z_{COG} = \frac{\int\limits_{Z} z \cdot \mu_C(z)\,dz}{\int\limits_{Z} \mu_C(z)\,dz}$$

Approximating the integral by discretization & summation

$$z_{COG} = \frac{\sum\limits_{i} z_i \mu_C(z_i)\,dz_i}{\sum\limits_{i} \mu_C(z_i)\,dz_i}$$

The advantage of the COG defuzzifier lies in its intuitive plausibility. The disadvantage is that it is computationally intensive. In fact, the *membership function is usually irregular and therefore the integrations in general are difficult to compute*.

Example: Given a fuzzy control system:

Rule 1: IF x is A_1 AND y is B_1, THEN z is C_1.

Rule 2: IF x is A_2 AND y is B_2, THEN z is C_2.

uppose x_0 and y_0 are the sensor readings for linguistic input variables x and y and the follow-
g membership functions for fuzzy predicates A_1, A_2, B_1, B_2, C_1, and C_2 are given:

$$\mu_{A_1}(x) = \begin{cases} \dfrac{x-2}{3} & 2 \le x \le 5 \\ \dfrac{8-x}{3} & 5 < x \le 8, \end{cases} \qquad \mu_{A_2}(x) = \begin{cases} \dfrac{x-3}{3} & 3 \le x \le 6 \\ \dfrac{9-x}{3} & 6 < x \le 9, \end{cases}$$

$$\mu_{B_1}(y) = \begin{cases} \dfrac{y-5}{3} & 5 \le y \le 8 \\ \dfrac{11-y}{3} & 8 < y \le 11, \end{cases} \qquad \mu_{B_2}(y) = \begin{cases} \dfrac{y-4}{3} & 4 \le y \le 7 \\ \dfrac{10-y}{3} & 7 < y < 10, \end{cases}$$

$$\mu_{C_1}(z) = \begin{cases} \dfrac{z-1}{3} & 1 \le z \le 4 \\ \dfrac{7-z}{3} & 4 < z \le 7, \end{cases} \qquad \mu_{C_2}(z) = \begin{cases} \dfrac{z-3}{3} & 3 \le z \le 6 \\ \dfrac{9-z}{3} & 6 < z \le 9. \end{cases}$$

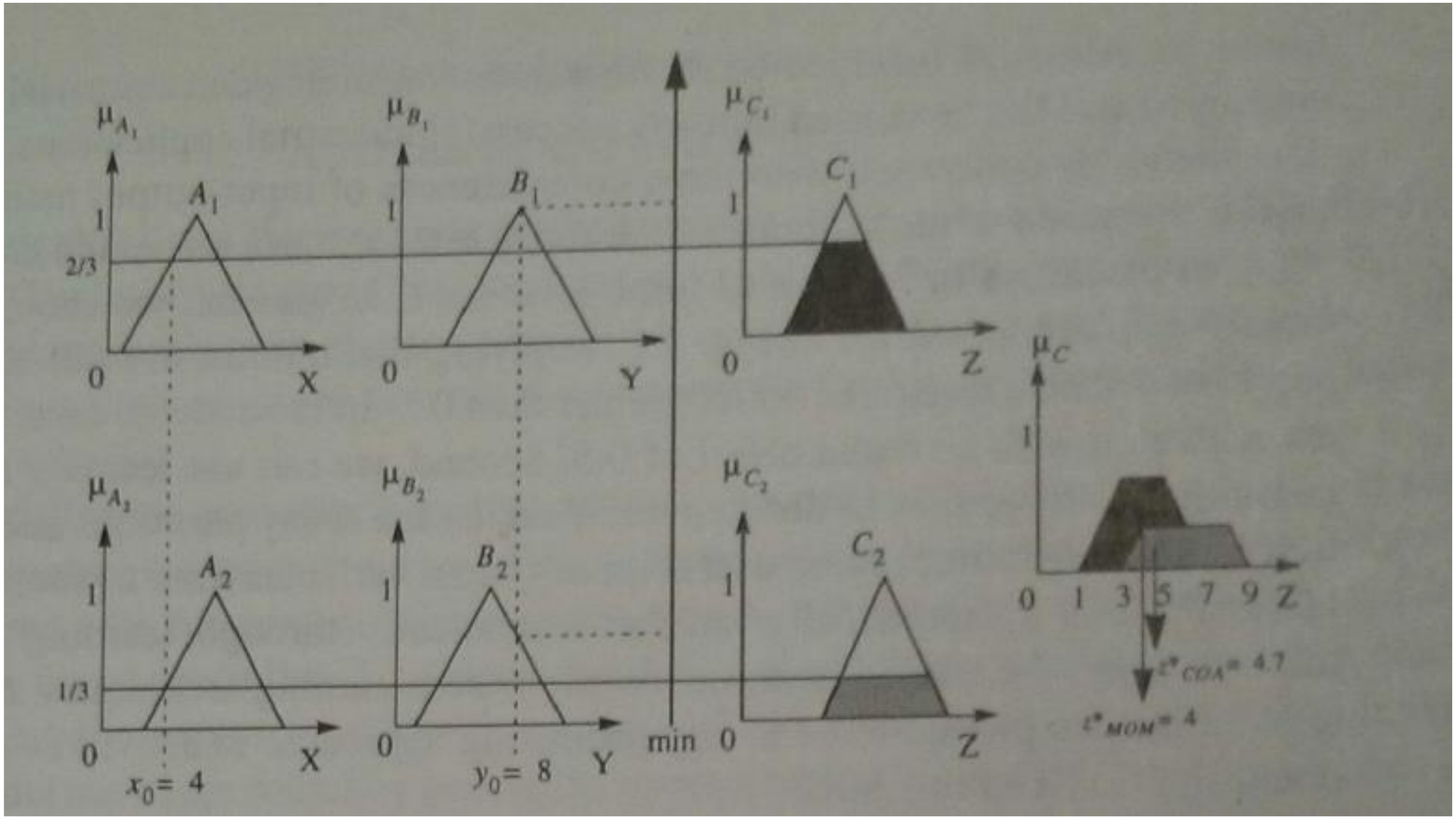
Further assume that at time t_1 we are reading the sensor values at $x_0(t_1) = 4$ and $y_0(t_1) = 8$. Let us illustrate how the final control output is computed.

First, the sensor readings $x_0(t_1)$ and $y_0(t_1)$ have to be matched against the preconditions A_1 and B_1 of rule 1, respectively. This will produce $\mu_{A_1}(x_0 = 4) = \frac{2}{3}$ and $\mu_{B_1}(y_0 = 8) =$ Similarly, for rule 2, we have $\mu_{A_2}(x_0) = \frac{1}{3}$ and $\mu_{B_2}(y_0) = \frac{2}{3}$. The firing strength of rule 1 calculated by

$$\alpha_1 = \min\left(\mu_{A_1}(x_0), \mu_{B_1}(y_0)\right) = \min\left(\frac{2}{3}, 1\right) = \frac{2}{3},$$

and similarly for rule 2, the firing strength is

$$\alpha_2 = \min\left(\mu_{A_2}(x_0), \mu_{B_2}(y_0)\right) = \min\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}.$$



Center of Gravity (COG) or also called Center of Area (COA) using Discrete Approximation

$$z^*_{\text{COA}} = \frac{2\left(\frac{1}{3}\right) + 3\left(\frac{2}{3}\right) + 4\left(\frac{2}{3}\right) + 5\left(\frac{2}{3}\right) + 6\left(\frac{1}{3}\right) + 7\left(\frac{1}{3}\right) + 8\left(\frac{1}{3}\right)}{\frac{1}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 4.7.$$