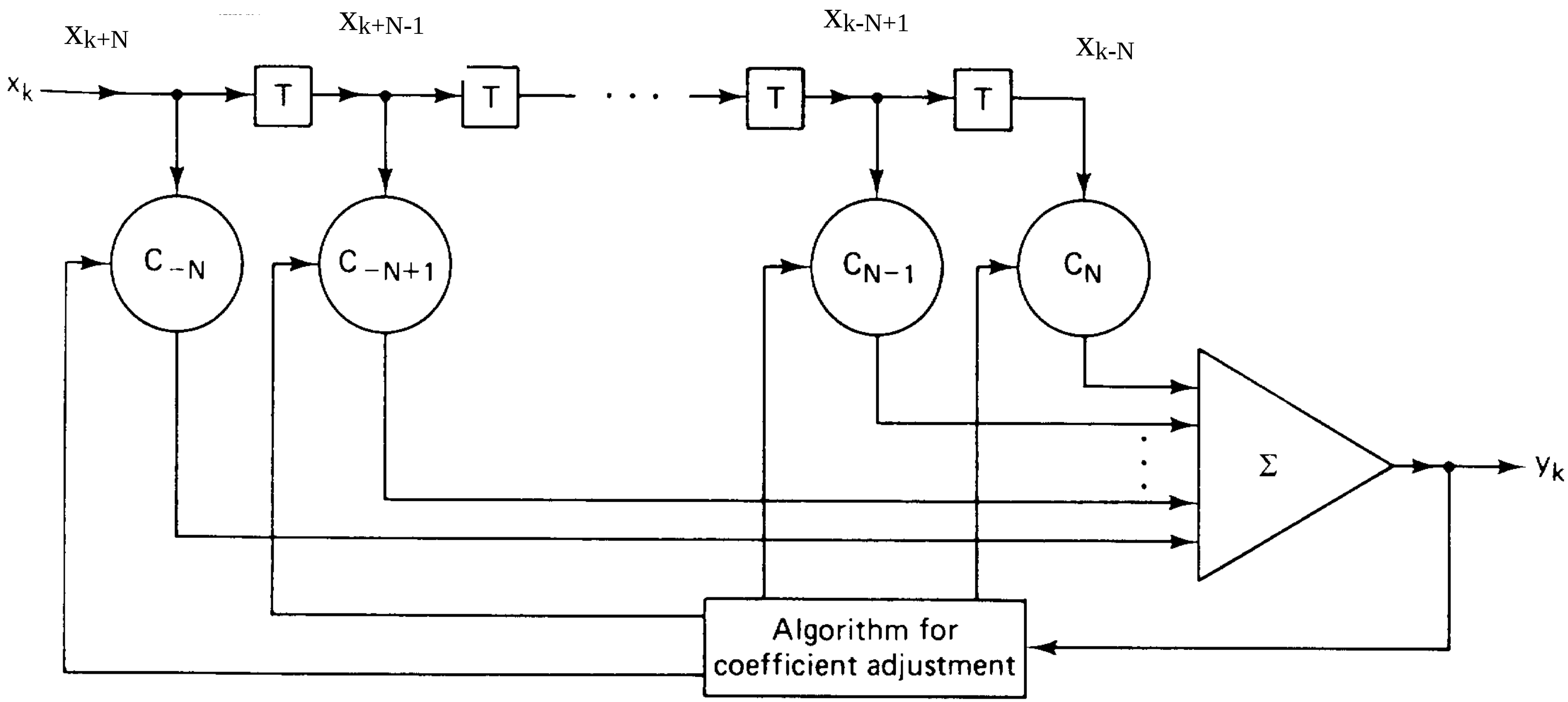
Equalization

From the previous study, the equalizing filter, $H_r(f)$, should be configured to compensate for the distortion caused by the transmitter and the channel, resulting in no ISI. However, the information of the channel transfer function, $H_c(f)$, is often incomplete to allow us to do a compensation for ISI. Hence, in practice, an adjustable equalizing filter is used to handle ISI at the receiver. This process of correcting the channel-induced distortion is called **equalization**.

A **transversal filter** is a common choice for equalization. It has $2N + 1$ taps and a total delay of $2NT$, where T is the symbol duration.



Transversal filter

The output samples, $\{y_k\}$, of the equalizer are expressed in terms of the input samples, $\{x_k\}$, and the $(2N + 1)$ tap coefficients as

$$y_k = \sum_{n=-N}^{N} c_n x_{k-n}$$

The above equation has the mathematical form known as **discrete convolution**.

The criterion for determining the c_n's is typically based on the minimization of either peak distortion or mean-square distortion. Minimizing peak distortion is to set the tap coefficients such that

$$y_k = \begin{cases} 1 & k = 0 \\ 0 & k = \pm 1, \pm 2, \cdots, \pm N \end{cases}$$

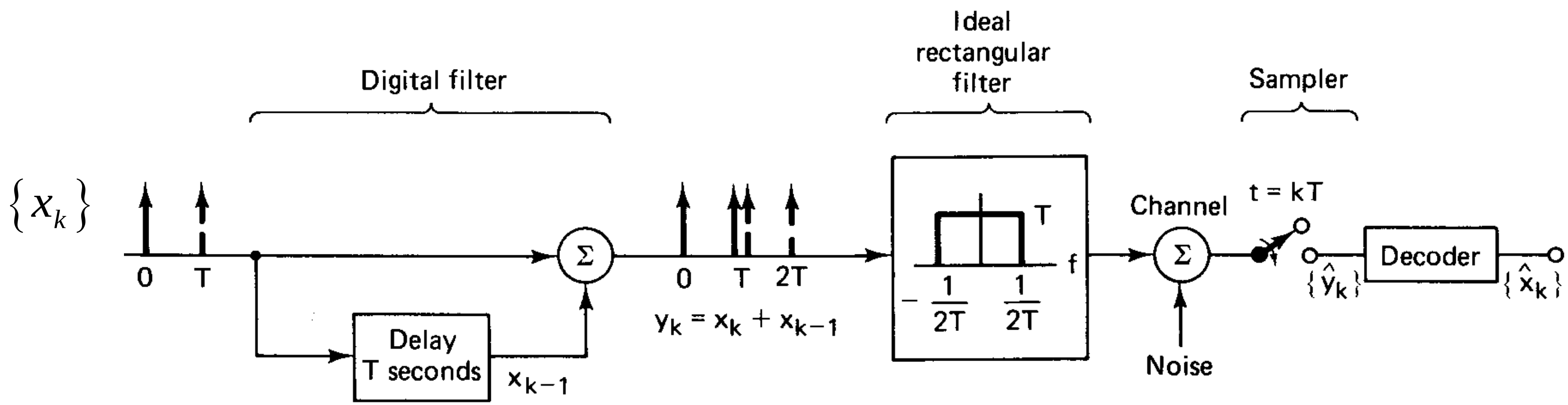
It yields $2N + 1$ simultaneous linear equations that can be solved for the c_n's. Note that the above equation describes a **zero-forcing equalizer** since y_k has N zero values on each side. This strategy is optimum in the sense that it minimizes the peak ISI.

Minimizing the mean-square distortion similarly results in $2N + 1$ simultaneous linear equations. Other strategies are known to be optimum in a different sense.

Duobinary Coding and Decoding

It is well known that a system with bandwidth W Hz can support a maximum symbol rate of $R_s = 2W$ symbols per second without ISI. Also, one way to accomplish this is to use ideal (rectangular) lowpass filters. However, such filters are physically unrealizable.

The signaling scheme, known as **duobinary** or **partial response signaling**, has been developed to overcome some of the difficulties mentioned above. This scheme uses controlled amounts of ISI for transmitting data at a rate of $R_s = 1/T$ symbols per second over a channel with a bandwidth of $W = 1/(2T)$. The shaping filters for the duobinary systems are easier to realize than the ideal rectangular filters. It transmits three possible sampled values (i.e., +2, 0, -2), instead of two (+1, -1), so that it requires more power than the conventional binary signaling scheme for the same performance.



Duobinary signaling

Let us see an example of this scheme:

	Start up digit						
Binary digit sequence $\{x_k\}$:	$\boxed{0}$	0	1	0	1	1	0
Bipolar amplitudes $\{x_k\}$:	-1	-1	$+1$	-1	$+1$	$+1$	-1
Coding rule: $y_k = x_k + x_{k-1}$:		-2	0	0	0	2	0

Decoding decision rule: If $\hat{y}_k = \ \ 2$, decide that $\hat{x}_k = +1$ (or binary one)

If $\hat{y}_k = -2$, decide that $\hat{x}_k = -1$ (or binary zero).

If $\hat{y}_k = \ \ 0$, decide opposite of the previous decision.

Decoded bipolar sequence $\{\hat{x}_k\}$:		-1	$+1$	-1	$+1$	$+1$	-1
Decoded binary sequence $\{\hat{x}_k\}$:		0	1	0	1	1	0

One drawback of this detection technique is that once an error is made, it tends to propagate, causing further errors.

Duobinary Equivalent Transfer Function

Referring to the block diagram of duobinary signaling, the digital filter can be characterized with the transfer function, $H_1(f)$, which is given by

$$H_1(f) = 1 + e^{-j2\pi fT}$$

where the delay of T seconds has a transfer function of $e^{-j2\pi fT}$. The transfer function of the ideal rectangular filter, $H_2(f)$, is

$$H_2(f) = T\,\mathrm{rect}(fT) = \begin{cases} T & |f| < 1/2T \\ 0 & \text{otherwise} \end{cases}$$

The overall transfer function, $H_e(f)$, is given by

$$H_e(f) = H_1(f)H_2(f)$$

$$= \begin{cases} T\left[1 + e^{-j2\pi fT}\right] & |f| < 1/2T \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} T\left[e^{j\pi fT} + e^{-j\pi fT}\right]e^{-j\pi fT} & |f| < 1/2T \\ 0 & \text{otherwise} \end{cases}$$

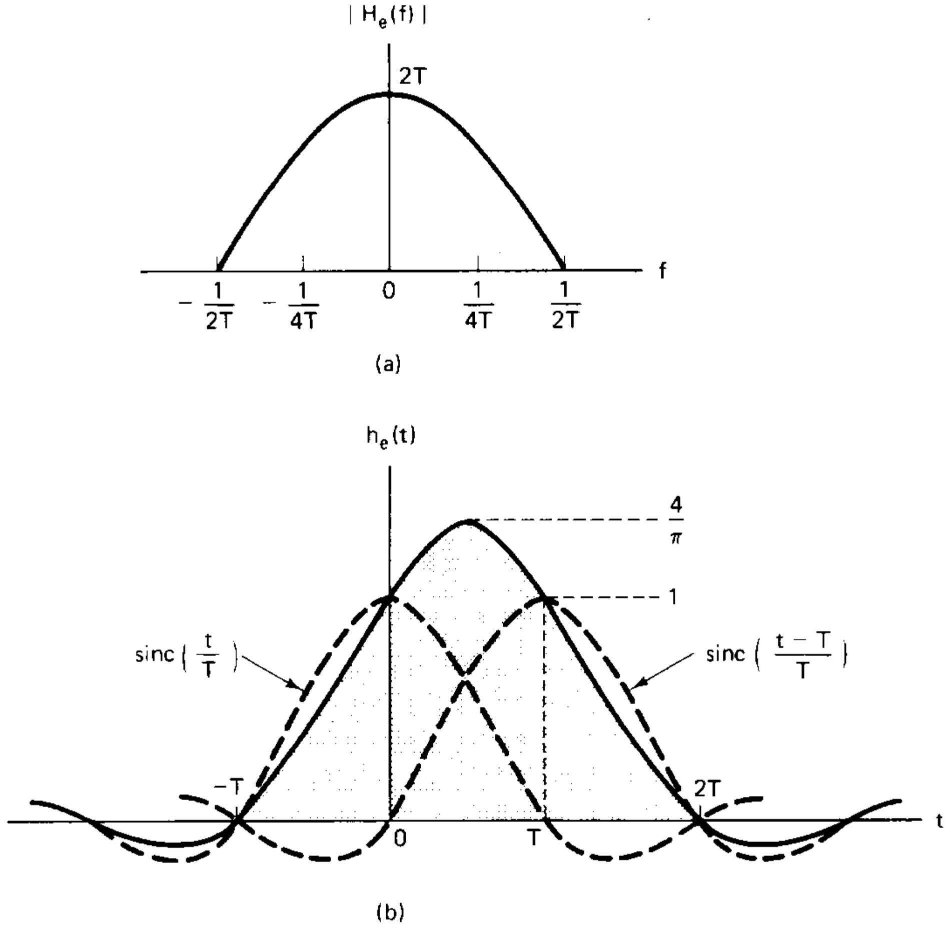
and

$$\left|H_e(f)\right| = \begin{cases} 2T\cos\pi fT & |f| < 1/2T \\ 0 & \text{otherwise} \end{cases}$$

The duobinary equivalent $H_e(f)$ is called a **cosine filter**, which can easily be approximated as it is a smooth cosine function. Although it is noncausal and unrealizable, it can be approximated by truncating it for some negative value and then introducing enough time delay. The corresponding impulse response, $h_e(t)$, is

$$h_e(t) = \operatorname{sinc}\left(\frac{t}{T}\right) + \operatorname{sinc}\left(\frac{t-T}{T}\right)$$

Notice that there are only two nonzero samples, at $t = 0$ and $t = T$, producing controlled ISI from the adjacent bit. The introduced ISI is eliminated by use of the decoding procedure. Error propagation may be eliminated by precoding the input bit stream at the transmitter.



2.40 Duobinary transfer function and pulse shape. (a) Cosine filter. (b) Impulse response of the cosine filter.

Example – Duobinary Precoding

Binary digit sequence $\{x_k\}$:	0	0	1	0	1	1	0
Precoded sequence $w_k = x_k \oplus w_{k-1}$:	0	0	1	1	0	1	1
Bipolar sequence $\{w_k\}$:	-1	-1	$+1$	$+1$	-1	$+1$	$+1$
Coding rule: $y_k = w_k + w_{k-1}$:		-2	0	$+2$	0	0	$+2$

Decoding decision rule: If $\hat{y}_k = \pm 2$, decide that $\hat{x}_k =$ binary zero.

If $\hat{y}_k = 0$, decide that $\hat{x}_k =$ binary one.

Decoded binary sequence $\{\hat{x}_k\}$:		0	1	0	1	1	0

Polybinary Signaling

Duobinary signaling can be extended to more than three levels (as in duobinary signaling), resulting in greater bandwidth efficiency. Such a system is called **polybinary**. Consider that a binary message is transformed to a signal with j levels, numbered consecutively from 0 to $(j - 1)$. The process consists of two steps:

- The original sequence $\{x_k\}$, consisting of 1's and 0's, is converted into another binary sequence $\{y_k\}$. For example, the present bit y_k is obtained from the modulo-2 sum of the current bit x_k and the $(j - 2) = 3$ immediate preceding bits of $\{y_k\}$. That is,

$$y_k = x_k \oplus y_{k-1} \oplus y_{k-2} \oplus y_{k-3}$$

Note that the value of y_k is either 0 or 1 as modulo-2 addition is used.

- The binary sequence $\{y_k\}$ is transformed into a j-level pulse train $\{z_k\}$ by adding **algebrically**

$$z_k = y_k + y_{k-1} + y_{k-2} + \cdots + y_{k-j+2}$$

Note that the binary 1's and 0's in $\{x_k\}$ are mapped into even and odd values of the sequence $\{z_k\}$.

Bandpass Modulation

Digital modulation is the process by which digital symbols are mapped to waveforms that are compatible with channel characteristics. In the case of baseband modulation, these waveforms are pulses, but in the case of **bandpass modulation**, the desired information signal modulates a sinusoid called a **carrier**. For radio transmission, the carrier is converted to an electromagnetic (EM) waveform for propagation to the desired destination.

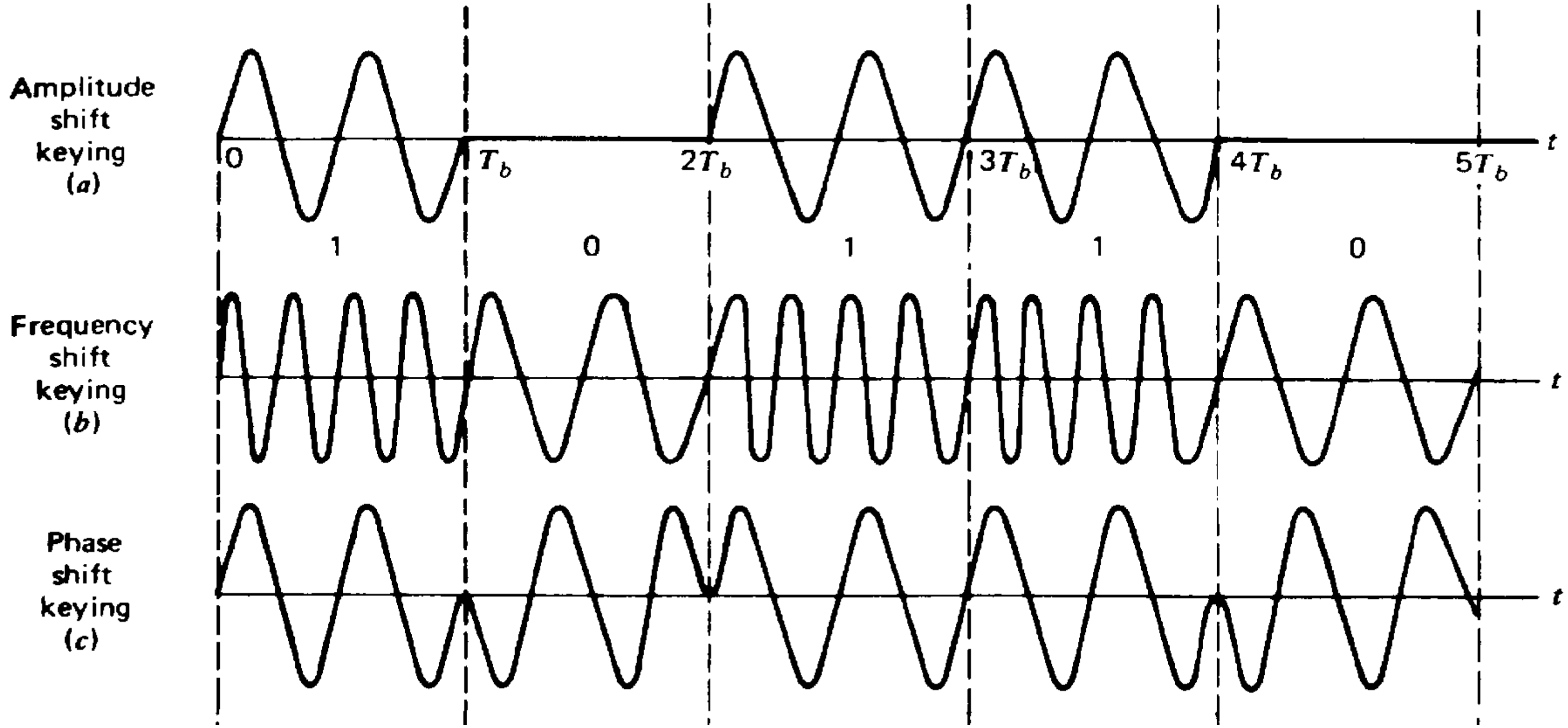
Why Modulate?

- **Antenna Size** -- To efficiently radiate EM energy into space, the size of the antenna should be in the order of the wavelength being used.
- **Frequency-Division Multiplexing (FDM)** -- If many signals share a channel, modulation can be used to assign non-overlapping frequency bands to different signals

Modulation such as spread spectrum modulation can be employed to reduce the effects of interference.

Bandpass Signals

When we transmit digital data over a bandpass channel, it is necessary to modulate the incoming data onto a carrier. The modulation process involves modifying one of the three parameters of the carrier, namely, **amplitude**, **phase** and **frequency**. The resulting waveforms are referred to as **amplitude-shift keying** (ASK), **phase-shift keying** (PSK), and **frequency-shift keying** (FSK), respectively.



(a) ASK, (b) FSK, (c) PSK

It is observed that in PSK and FSK modulation methods, the amplitude of the carrrier remains constant. In addition, all modulated waveforms are continuous in all times.

A digital communication system is called **coherent** if a local reference is available for demodulation which is in phase with the transmitted carrier. Otherwise, it is referred to as **noncoherent**. The advantage of noncoherent over coherent systems is reduced complexity, and the price paid is increased probability of error (P_E).

Likewise, if a periodic signal is available at the receiver (referred to as clock), the system is called **synchronous**. If a signaling technique is employed where such a clock is unnecessary, the system is called **asynchronous**.

Coherent Detection of Binary PSK Signals

The binary PSK (BPSK) signal is represented by

$$s_1(t) = \sqrt{\frac{2E_b}{T}} \cos 2\pi f_c t \qquad 0 \le t \le T \quad \text{binary 1}$$

$$s_2(t) = \sqrt{\frac{2E_b}{T}} \cos(2\pi f_c t + \pi) \qquad 0 \le t \le T \quad \text{binary 0}$$

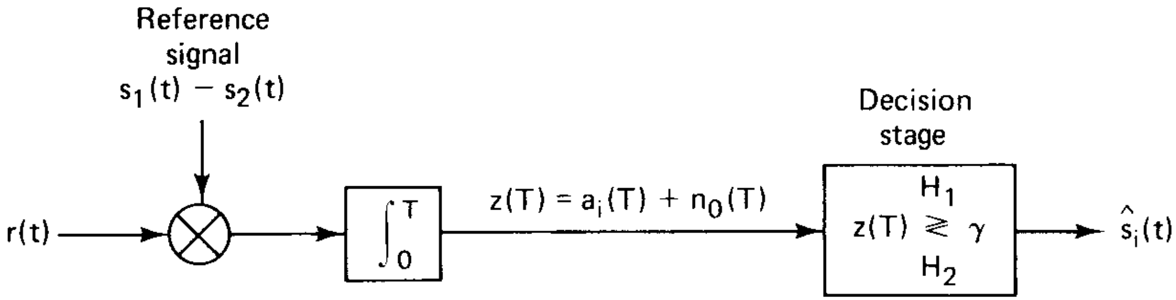
where E_b is the bit energy and T is the bit duration. In BPSK signaling, the modulating signal shifts the phase of the waveform, $s_i(t)$, to 0 or π. Note that $s_1(t) = -s_2(t)$.

From the statistical decision theory and the description of matched filter (or correlator), we may implement the binary detection in two ways. The first configuation uses a single product integrator, with the reference signal being the difference between the two binary signals,

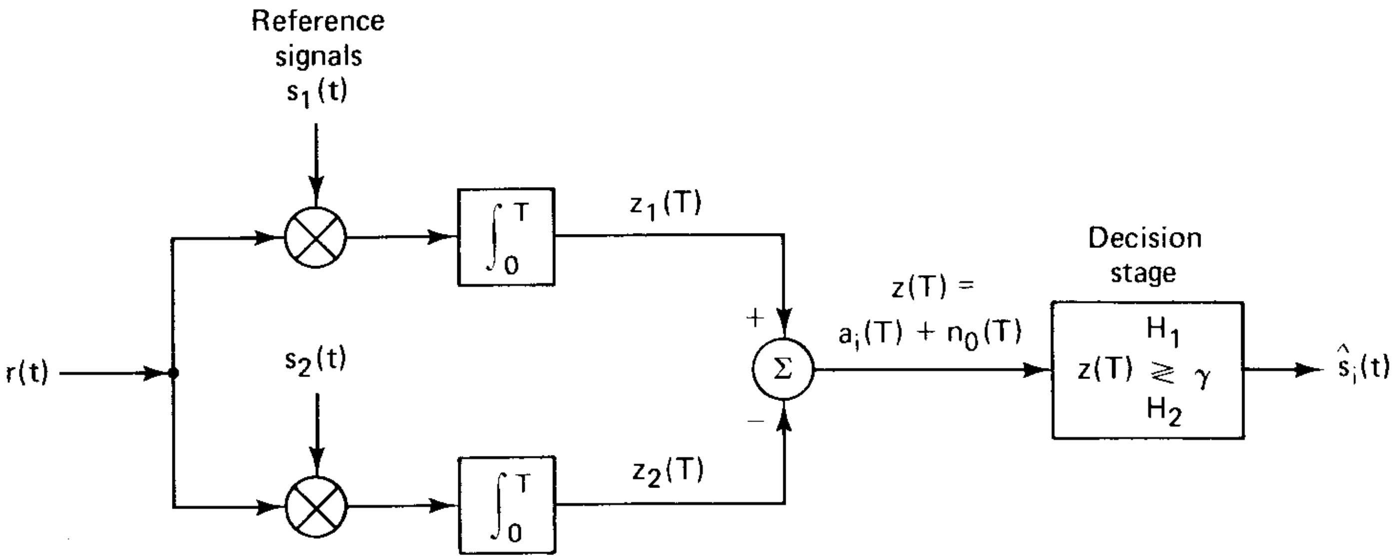
$$d(t) = s_1(t) - s_2(t).$$

The output of the correlator, $z(T)$, is compared with the threshold, γ.

Another implementation is to use two product integrators, each of which is matched to one of the two possible signals, $s_1(t)$ or $s_2(t)$. It is observed that these two implementations are equivalent.



(a)



(b)

Binary correlator receiver. (a) Using a single correlator. (b) Using two correlators.

Suppose $s_1(t)$ has been sent at the transmitter. Then

$$a_1 = \int_0^T s_1(t)\left[s_1(t) - s_2(t)\right] dt = 2E_b$$

Similarly,

$$a_2 = \int_0^T s_2(t)\left[s_1(t) - s_2(t)\right] dt = -2E_b$$

For maximum-likelihood (ML) detection, the threshold is

$$\gamma = (a_1 + a_2)/2 = 0$$

and

$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} 0$$

The test statistic $z(T)$ is

$$z(T) = a_i(T) + n_0(T)$$

where $n_0 = n_0(T)$ is the noise component due to AWGN at the input of the receiver. It is a zero-mean Gaussian RV, and thus $z(T)$ is a conditional Gaussian RV with a mean of either $a_1 = a_1(T)$ and $a_2 = a_2(T)$, depending on whether a binary one or binary zero was sent.

To derive the probability of error, we first compute the energy of $d(t) = s_1(t) - s_2(t)$ as

$$E_d = \int_0^T d^2(t)\, dt = \int_0^T \left[s_1(t) - s_2(t)\right]^2 dt = a_1 - a_2 = 4E_b$$

Note that

$$\sigma_0^2 = \frac{N_0}{2}\int_0^T \left[s_1(t) - s_2(t)\right]^2 dt = \frac{N_0}{2} E_d$$

Hence, the BER is

$$P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0^2}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

Coherent Detection of Binary FSK Signals

FSK modualtion is characterized by the information being contained in the frequency of the carrier. The binary FSK (BFSK) signal is represented by

$$s_1(t) = \sqrt{\frac{2E_b}{T}} \cos\left(2\pi f_1 t + \phi\right) \qquad 0 \le t \le T \qquad \text{binary 1}$$

$$s_2(t) = \sqrt{\frac{2E_b}{T}} \cos\left(2\pi f_2 t + \phi\right) \qquad 0 \le t \le T \qquad \text{binary 0}$$

where the frequency spacing $f_1 - f_2$ is assumed to be greater than $1/2T$ so that the two possible signals are orthogonal.

We may use the receiver shown for coherent BPSK to detect the incoming signal. Note that

$$a_1 = \int_0^T s_1(t)\left[s_1(t) - s_2(t)\right] dt = E_b$$

and

$$a_2 = \int_0^T s_2(t)\left[s_1(t) - s_2(t)\right] dt = -E_b$$

This gives

$$\gamma = (a_1 + a_2)/2 = 0$$

The energy of $d(t) = s_1(t) - s_2(t)$ is

$$
\begin{aligned}
E_d &= \int_0^T d^2(t)\, dt \\
&= \int_0^T \left[s_1(t) - s_2(t)\right]^2\, dt \\
&= 2E_b
\end{aligned}
$$

Therefore, the BER is

$$
P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right).
$$

Coherent Detection of Binary ASK Signals

The binary ASK (BASK) signal is represented by

$$s_1(t) = \sqrt{\frac{2E_1}{T}} \cos\left(2\pi f_1 t + \phi\right) \qquad 0 \le t \le T \qquad \text{binary 1}$$

$$s_2(t) = 0 \qquad 0 \le t \le T \qquad \text{binary 0}$$

For coherent detection, we may use the receiver shown on pp. 202. The mean value at the output of the correlator at $t = T$ is either

$$a_1 = \int_0^T s_1(t)\left[s_1(t) - s_2(t)\right] dt = E_1 = 2E_b$$

or

$$a_2 = \int_0^T s_2(t)\left[s_1(t) - s_2(t)\right] dt = 0$$

where $E_b = E_1/2$ is average energy per bit. Hence, the threshold is

$$\gamma = (a_1 + a_2)/2 = E_b$$

The energy in the different signal $d(t) = s_1(t) - s_2(t)$ at the receiver input is

$$E_d = \int_0^T d^2(t)\, dt = E_1 = 2E_b$$

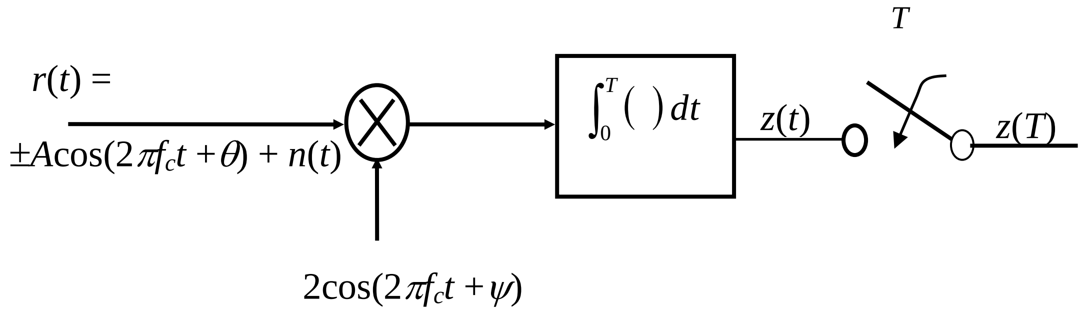
Therefore, the BER is

$$P_B = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

It is observed that the performance of BASK is the same as that of BFSK and 3 dB inferior to BPSK signaling.

BPSK with Imperfect Reference

The results obtained earlier for BPSK are for the case of a perfect reference $d(t) = s_1(t) - s_2(t)$ at the receiver. Consider the correlation detection of BPSK in the following diagram:



The input is one of

$$s_1(t) = A\cos\left(2\pi f_1 t + \theta\right) \qquad 0 \le t \le T \qquad \text{binary 1}$$

$$s_2(t) = -A\cos\left(2\pi f_1 t + \theta\right) \qquad 0 \le t \le T \qquad \text{binary 0}$$

where θ is an unknown carrier phase. The reference is $v(t) = 2\cos(2\pi f_c t + \psi)$, where ψ is the phase estimate at the receiver. The mean value of the output $z(T)$ due to $s_1(t)$ is

$$\begin{aligned} a_1 &= \int_0^T s_1(t) v(t) \; dt \\ &= \int_0^T A\cos(2\pi f_c t + \theta) \times 2\cos(2\pi f_c t + \psi) \; dt \\ &\cong AT\cos\phi \end{aligned}$$

where $\phi = \theta - \psi$ and the integration value due to the higher frequency component is ignored. Similarly,

$$a_2 = \int_0^T s_2(t)v(t) \ dt \cong -AT\cos\phi$$

Hence, the threshold is

$$\gamma = (a_1 + a_2)/2 = 0$$

Now, we consider the noise component at the correlator output.

$$n_0 = \int_0^T n(t)v(t)\ dt$$

Its variance is

$$
\begin{aligned}
\sigma_0^2 &= E\left[\int_0^T n(t_1)v(t_1)\,dt_1 \int_0^T n(t_2)v(t_2)\,dt_2\right] \\
&= \int_0^T\int_0^T E\left[n(t_1)n(t_2)\right] v(t_1)v(t_2)\,dt_1 dt_2 \\
&= \int_0^T\int_0^T \frac{N_0}{2}\delta(t_1 - t_2)\,v(t_1)v(t_2)\,dt_1 dt_2 \\
&= \frac{N_0}{2}\int_0^T v(t_1)v(t_1)\,dt_1 \\
&= \frac{N_0}{2} \times 2^2 \times \frac{1}{2}T \\
&= N_0 T
\end{aligned}
$$

where $N_0/2$ is the two-sided PSD of thermal noise.

The probability of bit error is given by

$$P_B(\phi) = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right) = Q\left(\frac{2AT\cos\phi}{2\sqrt{N_0 T}}\right)$$

$$= Q\left(\sqrt{\frac{A^2 T\cos^2\phi}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b\cos^2\phi}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\cos\phi\right)$$

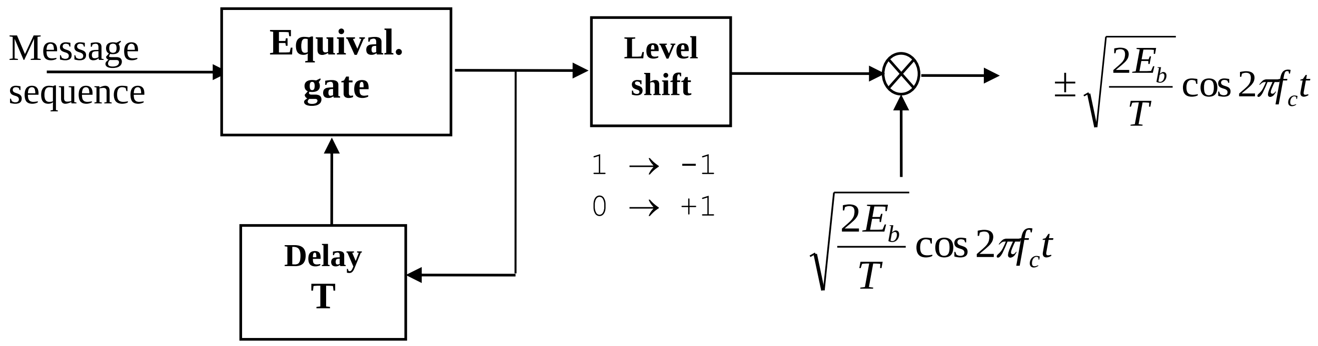
The performance is degraded by 20 $\log_{10}\cos\phi$ in dB as compared with the perfect reference case. The unconditional P_B can be obtained by taking the expectation over $P_B(\phi)$.

Differential Phase-Shift Keying (DPSK)

We may view DPSK as the noncoherent version of PSK. It obtains a phase reference for the demodulation process by using the carrier phase of the previous signaling interval. The implementation of DPSK assumes that the unknown phase contained in the received signal varies slowly, i.e., slow enough for it to be considered essentally unchanged over a two-bit interval.

Message seq.		1	0	0	1	0	0	1	1
Encoded seq.	1	1	0	1	1	0	1	1	1
Ref. bit									
Phase	π	π	0	π	π	0	π	π	π

Differential encoding of a message is illustrated in the above table. A 1 has been chosen as an initial reference bit. For each bit of the encoded sequence, the previous encoded bit is used as a reference bit. If the present message bit and the reference bit are the same, a 1 is encoded. Otherwise, a zero is encoded. The encoded message sequence then BPSK modulates a carrier with phase 0 or π as shown in the table.



DPSK modulator

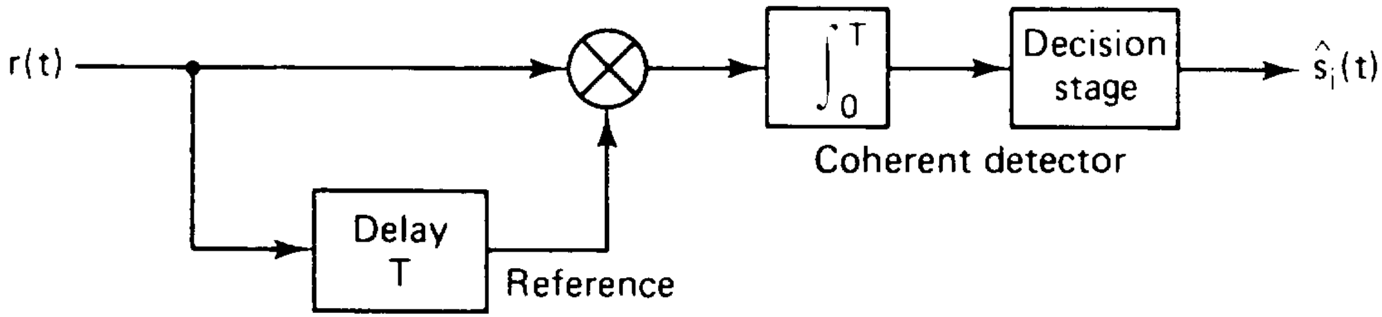
The block diagram above illustrates the generation of DPSK signaling. The truth table of the equivalence gate, which is the negation of an XOR, is shown as follows.

Truth table for A ≡ B

A	***B***	***A ≡ B***
0	0	1
0	1	0
1	0	0
1	1	1

By performing a simple level shift at the output of the logic circuit, so that the encoded message is bipolar, the DPSK signal is produced by modulating a sinusoidal carrier.

A method of demodulating the DPSK signal is as follows.



The received signal is correlated bit by bit with a one-bit delayed version of the received signal. The output of the correlator is then compared with a threshold set at zero, a decision being made in favor of a 1 or 0 depending on whether the correlator output is positive or negative, respectively.

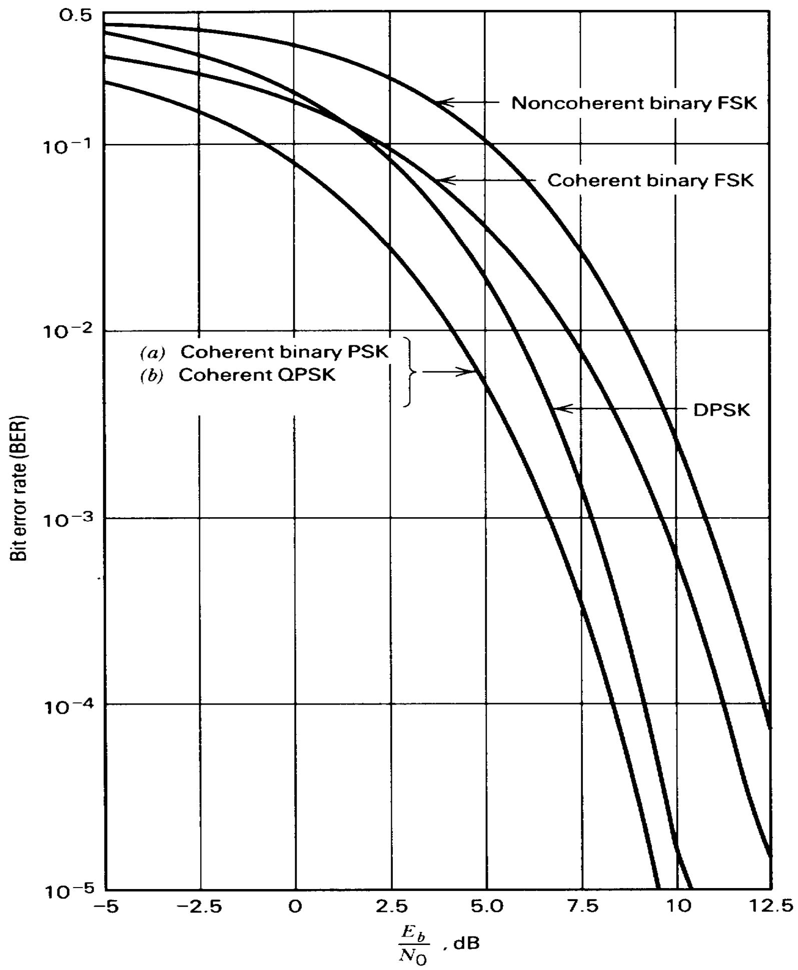
The result for the error probability, although complicated to derive, is surprisingly simple in form. The result is

$$P_B = \frac{1}{2} e^{-E_b / N_0}$$

For $P_B \leq 10^{-4}$, the E_b/N_0 values between DPSK and BPSK is less than 1 dB, which makes DPSK extremely attractive. However, the errors occurred in DPSK tend to be in groups of two because of the correlation imposed between successive bits by the encoding process.

Probabilities of Error for Selected Binary Modulation Schemes

Modulation	P_B
Coherent PSK	$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$
Coherent FSK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
Coherent ASK	$Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
DPSK	$\frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$
Noncoherent FSK	$\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$



Comparison of the noise performances of different PSK and FSK schemes.

M-ary Signaling Techniques

With the binary digital communication systems that we have considered so far, only two possible signals are used during each T-second signaling interval. It is perhaps the most important mode of communications today. In the M-ary case, we may send any one of $M = 2^k$ possible signals, $s_0(t), s_1(t), \cdots, s_{M-1}(t),$ during each signaling (or symbol) interval of T_s seconds. The symbol duration is thus equal to $T_s = kT$, where T is the bit duration. These signals are generated by changing the amplitude, phase, or frequency of a carrier in M discrete steps. Thus, we have M-ary ASK, M-ary PSK, and M-ary FSK digital modulation schemes.

Coherent *M*-ary PSK

For *M*-ary PSK (MPSK) systems, the signal can be expressed as

$$s_i(t) = \sqrt{\frac{2E_s}{T_s}}\cos\left(2\pi f_c t + \frac{2\pi i}{M}\right)$$

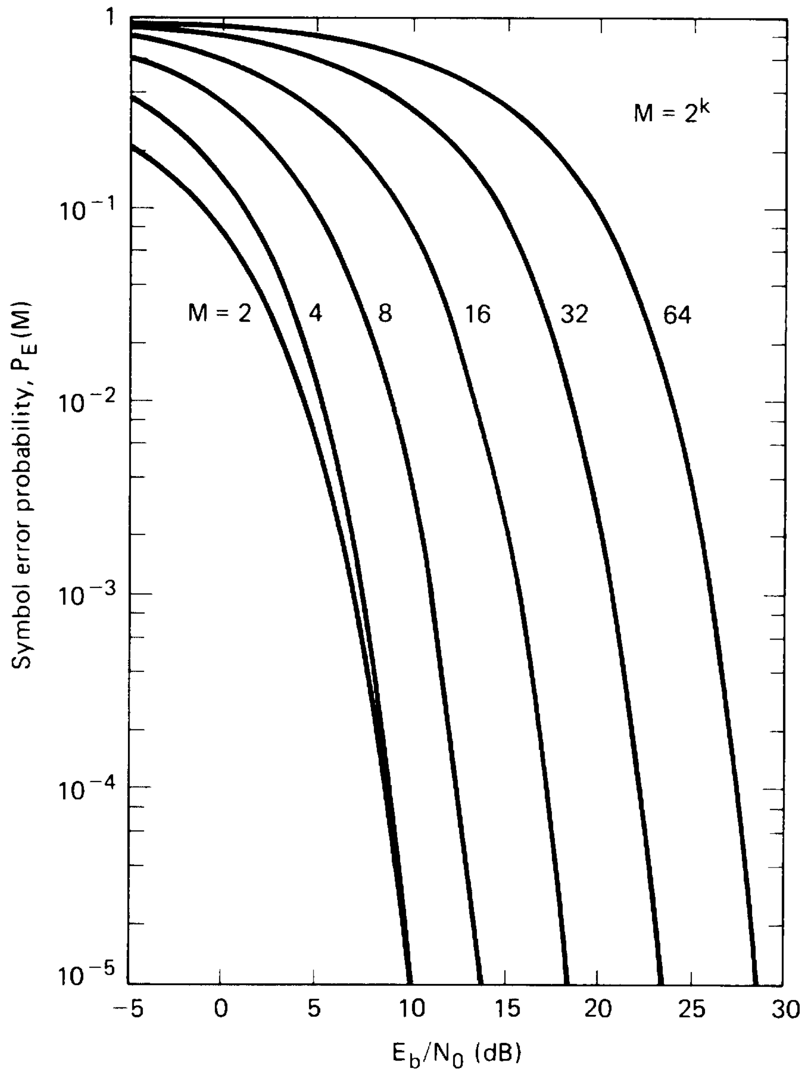
for $i = 0, 1, .., M\text{-}1$ and $0 \le t \le T_s$, where E_s is the signal energy per symbol.

If we take a block of *k* data bits and send the block using an MPSK scheme, then the required transmission bandwidth is roughly $1/(kT) = 1/T_s$. Comparing with the BPSK scheme, the use of MPSK reduces the transmission bandwidth by a factor of k.

The M-ary PSK receiver consists of a phase discriminator whose output is directly proportional to the phase of the incoming waveform (i.e., desired signal plus noise) as measured over a symbol duration of T_s seconds. The decision on the phase of the signal component at the receiver output is correct if the noise-induced phase error is within $\pm(\pi/M)$. Otherwise, the receiver makes an error decision.

M	**BW gain**	**Power Loss at 10^{-4}**
4	0.500	0.34 dB
8	0.333	3.91 dB
16	0.250	8.52 dB
32	0.200	13.52 dB

This table shows that the QPSK (M = 4) offers the best trade-off between BW requirements and power. It is for this reason that QPSK is widely used in practice. For M > 8, power requirements become excessive so those MPSK schemes are seldom used in practice.



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
Symbol error probability, $P_E(M)$
$M = 2^k$
M = 2
4
8
16
32
64
−5
0
5
10
15
20
25
30
E_b/N_0 (dB)

QPSK

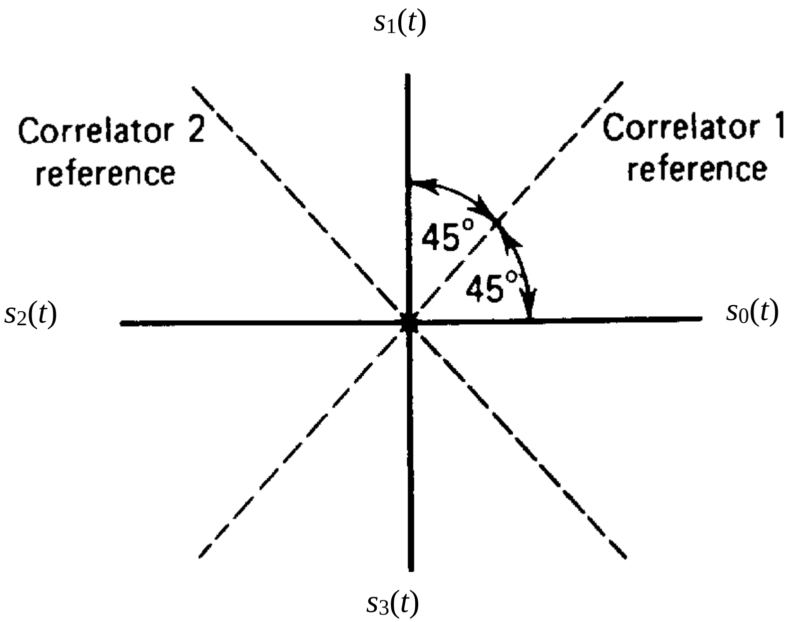
In QPSK, one of four possible waveforms is transmitted during each signaling interval T_s. These waveforms are:

$$s_0(t) = \sqrt{\frac{2E_s}{T_s}}\cos(2\pi f_c t)$$

$$s_1(t) = -\sqrt{\frac{2E_s}{T_s}}\sin(2\pi f_c t)$$

$$s_2(t) = -\sqrt{\frac{2E_s}{T_s}}\cos(2\pi f_c t)$$

$$s_3(t) = \sqrt{\frac{2E_s}{T_s}}\sin(2\pi f_c t)$$



These waveforms correspond to phase shifts of 0°, 90°, 180° and 270°, as shown in the phase diagram.

The receiver for the system requires two local reference waveforms

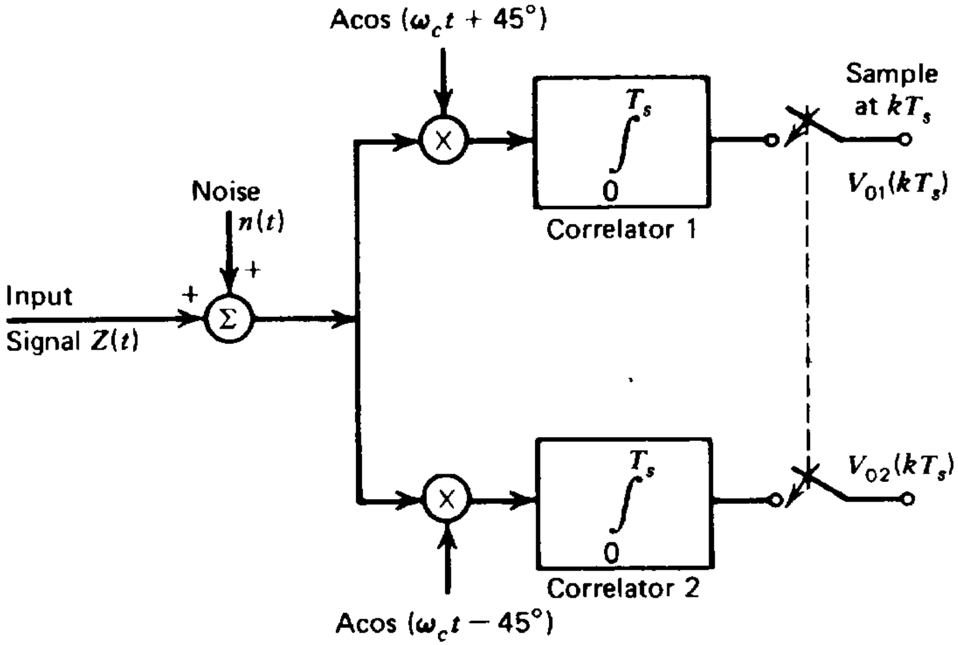
$$\sqrt{2E_s/T_s}\cos\left(2\pi f_c t+\pi/4\right)$$

and

$$\sqrt{2E_s/T_s}\cos\left(2\pi f_c t-\pi/4\right)$$

that are derived from a coherent local carrier reference

$$\sqrt{2E_s/T_s}\cos\left(2\pi f_c t\right).$$



For purposes of analysis, let us consider the operation of the receiver during the signaling interval $(0, T_s)$. Let us denote the signal component at the output of the correlators by s_{01} and s_{02}, respectively, and the noise component by $n_0(t)$. If we assume that $s_0(t)$ was the transmitted signal during the signaling interval $(0, T_s)$, then we have

$$
\begin{aligned}
s_{01}(T_s) &= \int_0^{T_s} \sqrt{\frac{2E_s}{T_s}} \cos\left(2\pi f_c t + \frac{\pi}{4}\right) s_0(t)\, dt \\
&= E_s \cos\left(\frac{\pi}{4}\right)
\end{aligned}
$$

and

$$s_{02}(T_s) = \int_0^{T_s} \sqrt{\frac{2E_s}{T_s}} \cos\left(2\pi f_c t - \frac{\pi}{4}\right) s_0(t)\, dt$$
$$= E_s \cos\left(\frac{\pi}{4}\right)$$

In the presence of noise, there will be some probability that an errror will be made by one or both correlators. The outputs of the correlators at time $t = T_s$ are

$$V_{01}(T_s) = s_{01}(T_s) + n_{01}(T_s)$$
$$V_{02}(T_s) = s_{02}(T_s) + n_{02}(T_s)$$

where

$$n_{01}(T_s) = \int_0^{T_s} n(t)\sqrt{\frac{2E_s}{T_s}}\cos\left(2\pi f_c t + \frac{\pi}{4}\right)dt$$

$$n_{02}(T_s) = \int_0^{T_s} n(t)\sqrt{\frac{2E_s}{T_s}}\cos\left(2\pi f_c t - \frac{\pi}{4}\right)dt$$

and $n(t)$ is assumed to be AWGN with two-sided power spectral density $N_0/2$. We can show that $n_{01}(T_s)$ and $n_{02}(T_s)$ are independent Gaussian random variables with mean zero and equal variance

$$\operatorname{var}\left[n_{01}(T_s)\right] = \operatorname{var}\left[n_{02}(T_s)\right] = \frac{N_0 E_s}{2}$$

Let us now compute the probability of error by assuming that $s_0(t)$ was the transmitted signal. If we denote the probability that correlator i makes an error by P_{Ei}, i = 1, 2, then

$$
\begin{aligned}
P_{E1} &= \Pr\left[V_{01} < 0\right] \\
&= \Pr\left[s_{01}(T_s) + n_{01}(T_s) < 0\right] \\
&= \Pr\left[n_{01}(T_s) < -E_s \cos\frac{\pi}{4}\right] \\
&= \Pr\left[n_{01}(T_s) > E_s \cos\frac{\pi}{4}\right] \\
&= Q\left(\frac{E_s \cos\left(\pi/4\right)}{\sqrt{N_0 E_s / 2}}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)
\end{aligned}
$$

and

$$P_{E2} = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

The probability P_C that the transmitted signal is received correctly is

$$P_C = (1 - P_{E1})\ (1 - P_{E2}) = 1 - 2P_{E1} + P_{E1}^2$$

Now, the probability of error P_E for the system is

$$P_E = 1 - P_C = 2P_{E1} - P_{E1}^2$$

$$\cong 2P_{E1}$$

$$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Coherent *M*-ary FSK

For *M*-ary FSK (MFSK) systems, the signal can be described as

$$s_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left(2\pi f_i t + \varphi\right) \quad i = 0,1,\ldots,M-1 \quad 0 \le t \le T_s$$

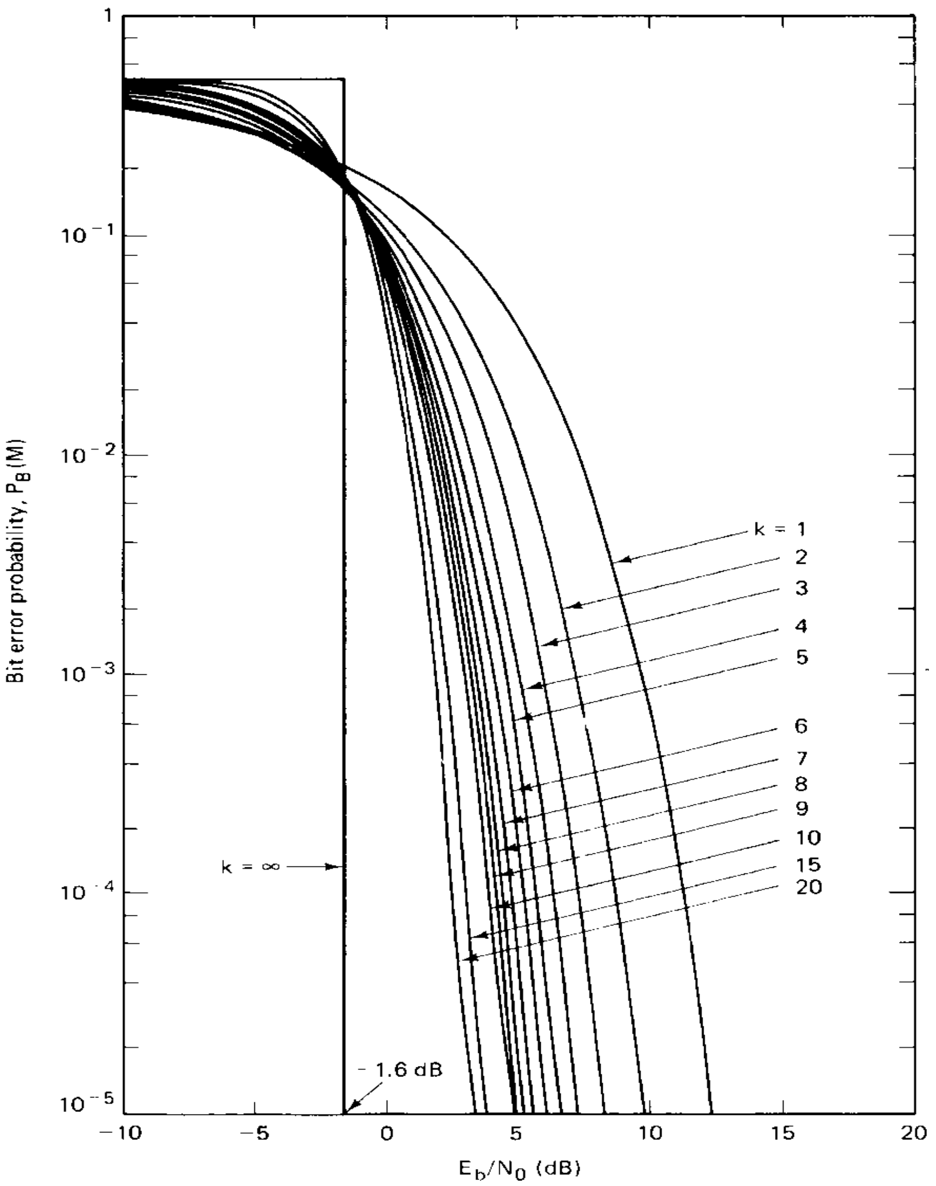
where E_s is the signal energy per symbol. The frequency separation between two adjacent frequencies is assumed to be $1/(2T_s)$ to ensure that the signals are orthogonal.

For fixed values of bit error $P_B(M)$, an increase in M results in a reduction in E_b/N_o. However, this is achieved at the cost of increased bandwidth.

In the limiting case as $M \to \infty$, the average probability of bit error $P_B(M)$ satisfies the condition

$$P_B = \begin{cases} 1 & E_b / N_0 < -1.6 \text{ dB} \\ 0 & E_b / N_0 > -1.6 \text{ dB} \end{cases}$$

Note that the bandwidth required for such a transmission is infinite, since the bandwidth of the signal set approaches infinity as M approaches infinity.



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
Bit error probability, $P_B(M)$
k = 1
2
3
4
5
6
7
8
9
10
15
20
k = ∞
– 1.6 dB
−10
−5
0
5
10
15
20
E_b/N_0 (dB)