6101 assignment 1

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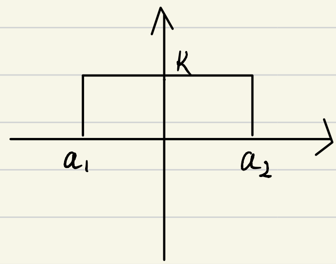
1、(a) $f_Z(z) = \begin{cases} k & a_1 \leqslant z \leqslant a_2 \\ 0 & \text{otherwise} \end{cases}$



(i) $\because$ uniform probability density

$\therefore\ k = \frac{1}{a_2 - a_1}$

(ii) $a_1 = -1$; $a_2 = 2$

$\therefore\ k = \frac{1}{3}$

$P(|Z| \leqslant \frac{1}{2}) = P_r\{-\frac{1}{2} \leqslant Z \leqslant \frac{1}{2}\}$

$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_z(z)\,dz = \frac{1}{3} z \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{3}$

$\therefore\ P(|Z| \leqslant \frac{1}{2}) = \frac{1}{3}$

(iii) $E(Z) = \int_{-\infty}^{+\infty} z \cdot f_z(z) = \int_{a_1}^{a_2} z \cdot k = \frac{1}{2} k (a_2^2 - a_1^2) = \frac{1}{2} \frac{a_2^2 - a_1^2}{a_2 - a_1} = \frac{a_2 + a_1}{2}$

if $a_1 = -1$ $a_2 = 2$

$E(z) = \frac{1}{6}(4 - 1) = \frac{1}{2}$

$E[z^2] = \int_{-\infty}^{+\infty} z^2 f(z) = k \int_{a_1}^{a_2} z^2\,dz = \frac{1}{3} k\, z^3 \Big|_{a_1}^{a_2} = \frac{1}{3} \frac{a_2^3 - a_1^3}{a_2 - a_1}$

$= \frac{1}{3}\, a_2^2 + a_1 a_2 + a_1^2$

$Var = E[z^2] - E[z]^2 = \frac{a_2^2 + a_1^2 + 2ab}{4} - \frac{a_1^2 + a_2^2 + a_1 a_2}{3} = \frac{(a_2 - a_1)^2}{12} = \frac{3}{4}$

b. i. $R_Y(\tau) = E\left[\frac{dX(t)}{dt} \cdot \frac{dX(t+\tau)}{dt}\right]$

$$= -\frac{d^2 R_X(\tau)}{d\tau^2}$$

$$\updownarrow$$

$$S_Y(f) = \mathcal{F}[R_Y(\tau)] = -(j2\pi f)^2 S_X(f)$$

$$= 4\pi^2 f^2 S_X(f)$$

ii $R_X(\tau) = 16\ \mathrm{sinc}(4\tau)$

$$S_X(f) = 4\ \mathrm{Rect}\left(\frac{f}{4}\right)$$

$$S_Y(f) = 16\pi^2 f^2\ \mathrm{Rect}\left(\frac{f}{4}\right)$$

$$P_x = \int_{-\infty}^{+\infty} S_Y(f)\,df = \int_{-2}^{2} 16\pi^2 f^2\,df$$

$$= 16\pi^2 \frac{1}{3} f^3 \Big|_{-2}^{2}$$

$$= \frac{32}{3}\pi^2 2^3$$

$$= \frac{256}{3}\pi^2$$

for higher frequency, the power spectral will be larger, contribute more to the average power but the components higher than $f=2$ have no contribution.

2.(a) $X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

$$\int_{-\infty}^{+\infty} |x(t)|^2 = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} X(f) e^{j\omega t} df \right] x(t)\, dt$$

$$= \int_{-\infty}^{+\infty} X(f) \left[\int_{-\infty}^{+\infty} e^{j\omega t} \cdot x(t)\, dt \right] df$$

$$= \int_{-\infty}^{+\infty} X(f) \cdot X(-f)\, df$$

$\because X(-f) = X^*(f)$

$\therefore \int_{-\infty}^{+\infty} |x(t)|^2 = \int_{-\infty}^{+\infty} |X(f)|^2$

meaning: the intergration of spetrum is equal to the energy of a signal

(b) $x(t) = \left[\frac{2}{T_b} \mathrm{sinc}\left(\frac{2t}{T_b}\right) \right] * \left[\frac{1}{T_b} \mathrm{sinc}\left(\frac{t-T_b}{T_b}\right) \right]$

$$X(f) = \mathrm{rect}\left(\frac{T_b f}{2}\right) \cdot \mathrm{rect}(T_b f) \cdot e^{-2\pi f T_b} = \mathrm{rect}(T_b f) e^{-2\pi f T_b}$$

$$E_x = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

$$= \int_{-\infty}^{+\infty} X(f) \cdot X^*(f)\, df$$

$$= \frac{1}{T_b}$$

2. c. $F(t) = x(t) \cdot x_\delta(t) = x(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$

$x_\delta(t) = \sum_{-\infty}^{+\infty} \delta(n - nT_s)$ is periodic.

$\Downarrow$

$= \sum_{-\infty}^{+\infty} C_n \cdot e^{-j2\pi f_s n t}$

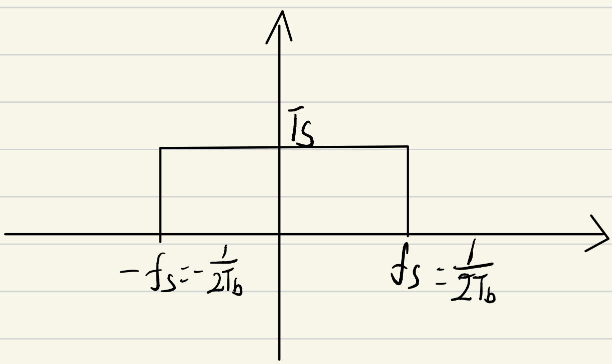
$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_\delta(t) e^{-j2\pi f_s n t} dt = \frac{1}{T_s}$

$\therefore F(t) = \frac{1}{T_s} \sum_{-\infty}^{+\infty} x(t) e^{-j2\pi f_s n t}$

$\Downarrow$

$F(f) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(f - nf_s)$

The bandwidth of LPF is $f_s = \frac{1}{T_s} = \frac{1}{2T_b}$
gain of LPF is T_s.



3.(a) i: $P_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$

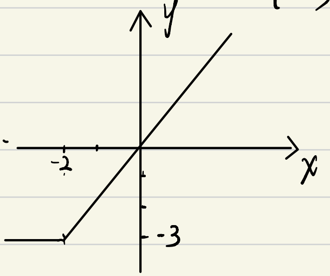
$\because$ Normal Distribution

$\therefore \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\therefore E[X] = 1$

$Var[X] = 1$

ii: $Y = \begin{cases} \frac{3x}{2} & x > -2 \\ -3 & x \le -2 \end{cases}$



$$P_Y(y=-3) = P_r\{x \le -2\} = \int_{-\infty}^{-2} P_X(x)\,dx = Q\left(\frac{4-1}{1}\right) = Q(3)$$

$y > -3$: $x = \frac{2y}{3} \Longrightarrow dx = \frac{2}{3}dy$

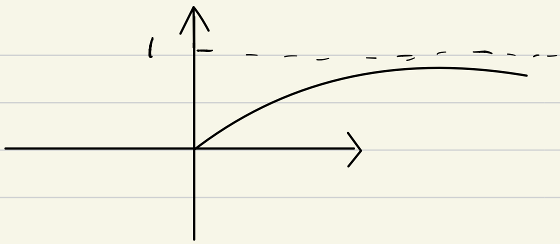
$\therefore P_Y(y)dy = P_X(x)dx$

$$P_Y(y) = P_X(x) \cdot \frac{2}{3} = \frac{2}{3} P_X\left(\frac{2}{3}y\right)$$

$$= \frac{2}{3}\frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{2}{3}y-1)^2}{2}} = \frac{2}{3}\frac{1}{\sqrt{2\pi}} e^{-\frac{4(y-\frac{3}{2})^2}{18}}$$

$$P_Y(y) = \begin{cases} 0 & y < -3 \\ Q(3) & y = -3 \\ \frac{2}{3\sqrt{2\pi}} e^{-\frac{4(y-\frac{3}{2})^2}{18}} & y > -3 \end{cases}$$

3.(b)



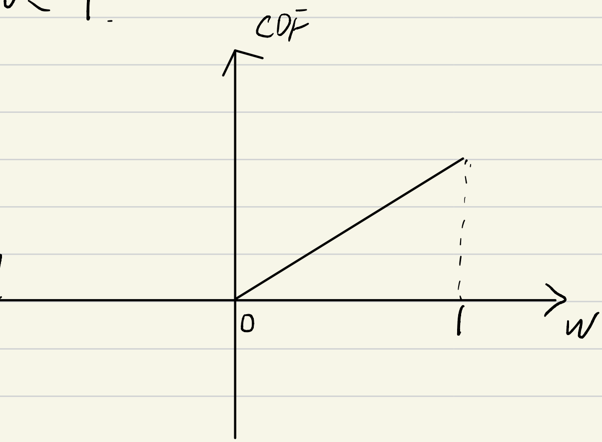
$$P_V(v) = F_V'(v) = \begin{cases} 0 & v < 0 \\ 2e^{-2v} & v \geq 0 \end{cases}$$

$$W = F_V(V) = \Pr\{V \leq v\}$$

$$\begin{aligned} F_W(w) &= \Pr\{W \leq w\} \\ &= \Pr\{F_V(V) \leq w\} \\ &= \Pr\{V \leq F_V^{-1}(w)\} \\ &= F_V[F_V^{-1}(w)] \\ &= w \end{aligned}$$

$\because F_V(v) \geq 0 \quad \therefore w \geq 0$

$\because F_V(v) \leq 1 \quad \therefore w \leq 1$

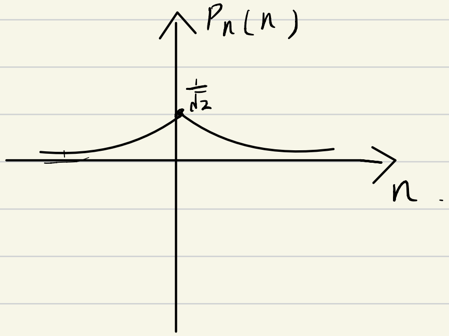


$$P_W(w) = \begin{cases} 1 & 0 \leq w \leq 1 \\ 0 & \end{cases}$$

$\therefore$ w is uniform distributed between 0 and 1

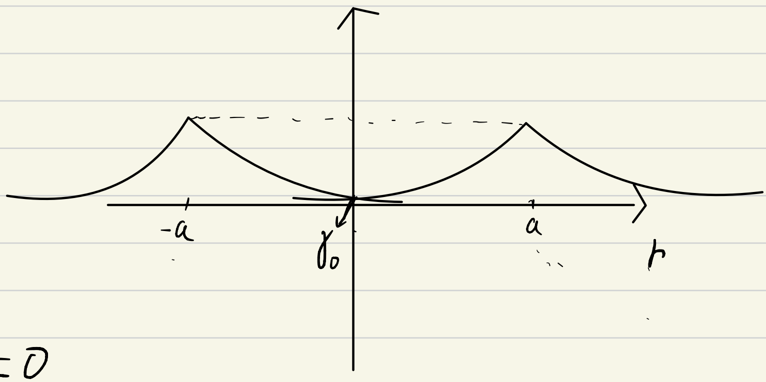
4. (a)

$P_n(n)$:



$$P_n(n) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|n|}$$

$\therefore$



$$\gamma_0 = \frac{a-a}{2} = 0$$

(b) $+a$ and $-a$ with equal probability

$$\therefore P_B = P(S_a|-a)P(-a) + P(S_{-a}|a)P(a)$$
$$= \frac{1}{2}[P(S_a|-a) + P(S_{-a}|a)]$$

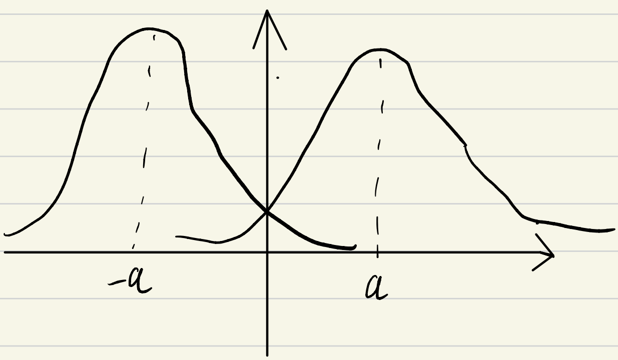
$$\because P(S_a|-a) = P(S_{-a}|a)$$
$$\therefore P_B = P(S_a|-a) = P(r > \gamma_0|-a) = P_r(\text{error}|-a)$$

c. $P_{error} = P(a)P(-a|a) + P(-a)P(a|-a)$

$= \frac{1}{2}P_r\{r < \gamma_0 | a) + \frac{1}{2}P(r > \gamma_0 | -a)$

$= \frac{1}{2}e^{-\sqrt{2}a}$

d. replace by Gaussian PDF



$$P_n(n) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(n-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{2\sqrt{\pi}}e^{-\frac{n^2}{4}}$$

$m_n = \int_{-\infty}^{+\infty} n \cdot \frac{1}{\sqrt{2}}e^{-\sqrt{2}|n|}dn = 0$

$\sigma^2 = \int_{-\infty}^{+\infty} n^2 \cdot \frac{1}{\sqrt{2}}e^{-\sqrt{2}|n|}dn = \sqrt{2}$

bit error probability will be smaller.

Gaussian distribution will accumulate more in the center and decrease faster

$\therefore$ $P_r(r > \gamma_0 | -a)$ will be smaller.