The problem

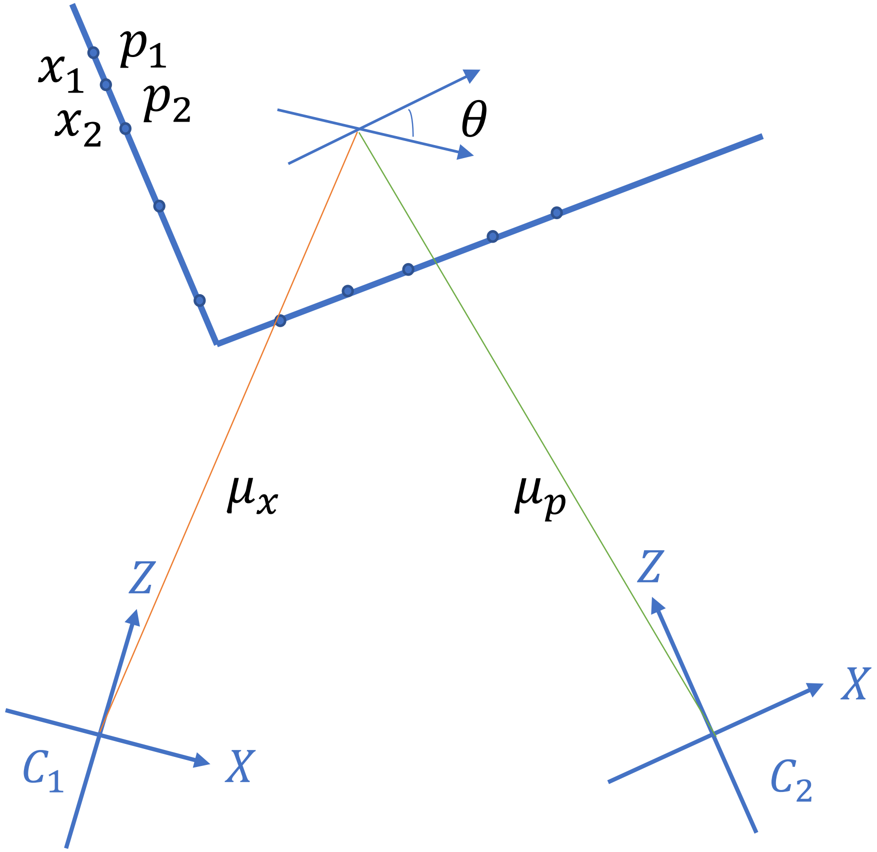
- Given two corresponding point sets:

$$X = \{x_1, \ldots, x_{N_x}\}$$
$$P = \{p_1, \ldots, p_{N_p}\}$$

- Wanted: Translation t and rotation R that minimize the sum of the squared error:

$$E(R,t) = \frac{1}{N_p} \sum_{i=1}^{N_p} ||x_i - Rp_i - t||^2$$

where x_i and p_i are corresponding points.



$$X_1 = \begin{pmatrix} x_{1x} \\ x_{1y} \\ x_{1z} \end{pmatrix} \qquad P_1 = \begin{pmatrix} p_{1x} \\ p_{1y} \\ p_{1z} \end{pmatrix}$$

$$X_2 = \begin{pmatrix} x_{2x} \\ x_{2y} \\ x_{2z} \end{pmatrix} \qquad P_2 = \begin{pmatrix} p_{2x} \\ p_{2y} \\ p_{2z} \end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$X_N \qquad\qquad P_N$$

Center of Mass

$$\mu_x = \frac{1}{N_x}\sum_{i=1}^{N_x} x_i \quad \text{and} \quad \mu_p = \frac{1}{N_p}\sum_{i=1}^{N_p} p_i$$

are the centers of mass of the two point sets.

Ideas:

- Subtract the corresponding center of mass from every point in the two point sets before calculating the transformation.
- The resulting point sets are:

$$X' = \{x_i - \mu_x\} = \{x_i'\}$$

and

$$P' = \{p_i - \mu_p\} = \{p_i'\}$$

Singular Value Decomposition

Let $W = \sum_{i=1}^{N_p} x_i' p_i'^T$

donate the singular value decomposition (SVD) of W by:

$$W = U \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} V^T$$

where $U, V \in R^{3\times 3}$ are unitary, and $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the singular values of W.

SVD

Theorem (without proof):

If rank(W)=3, the optimal solution of $\mathrm{E}(R,t)$ is unique and is given by:

$$R = UV^T$$

$$t = \mu_x - R\mu_p$$

The minimal value of error function at (R,t) is :

$$E(R,t) = \sum_{i=1}^{N_p} (||x_i'||^2 + ||y_i'||^2) - 2(\sigma_1 + \sigma_2 + \sigma_3)$$

Proof: Assume $N_x = N_p = N$

$$E(R,t) = \frac{1}{N}\sum_{i=1}^{N} \left|\left|x_i - (Rp_i + t)\right|\right|^2$$

To find t that minimize the square error: $\frac{\partial \mathrm{E}}{\partial \mathrm{t}} = 0$

Hence $\frac{1}{N}\sum_{i=1}^{N} 2\,(X_i - Rp_i - t) = 0$

$$\underbrace{\frac{1}{N}\sum_{i=1}^{N} X_i}_{\mu_x} - \frac{1}{N}\sum_{i=1}^{N} (Rp_i) - \underbrace{\frac{1}{N}\sum_{i=1}^{N} t}_{t} = 0$$

$$\mu_x - R\underbrace{\left(\frac{1}{N}\sum_{i=1}^{N} p_i\right)}_{\mu_p} - t = 0$$

$$t = \mu_x - R\mu_p$$

$$
\begin{aligned}
E(R,t) &= \frac{1}{N}\sum_{i=1}^{N} ||x_i - Rp_i - t||^2 \\
&= \frac{1}{N}\sum_{i=1}^{N} ||x_i - Rp_i - \left(\mu_x - R\mu_p\right)||^2 \\
&= \frac{1}{N}\sum_{i=1}^{N} ||\underbrace{\left(x_i - \mu_x\right)}_{x_i'} - R\underbrace{\left(p_i - \mu_p\right)}_{p_i'}||^2 \\
&= \frac{1}{N}\sum_{i=1}^{N} ||x_i' - Rp_i'||^2
\end{aligned}
$$

$$\boxed{(ab)^T = b^T a^T}$$

$$\begin{aligned} ||x_i' - Rp_i'||^2 &= (x_i' - Rp_i')^T (x_i' - Rp_i') \\ &= \left(x_i'^T - p_i'^T R^T\right)(x_i' - Rp_i') \\ &= x_i'^T x_i' - 2p_i'^T R^T x_i' + p_i'^T p_i' \end{aligned}$$

To find R, we need to minimize the squared error.

$$argmin_R ||x_i' - Rp_i'||^2 = argmin_R \left(\underbrace{x_i'^T x_i'}_{independent\ of\ R} - 2p_i'^T R^T x_i' + \underbrace{p_i'^T p_i'}_{independent\ of\ R} \right)$$

$$= argmin_R \left(-2p_i'^T R^T x_i'\right)$$

$$= argmax_R \left(p_i'^T R^T x_i'\right)$$

$$= argmax_R \left(x_i'^T R p_i'\right)$$

$$\sum \left(x_i'^T R p_i'\right) = trace(X'^T R P')$$

where $\quad X' = \{x_i'\}, \quad P' = \{p_i'\}$

$$trace \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} + a_{22} + a_{33}$$

Let

$$X' = \{x_i'\} = \begin{pmatrix} x_1 & x_1' & x_1'' & \dots \\ x_2 & x_2' & x_2'' & \dots \\ x_3 & x_3' & x_3'' & \dots \end{pmatrix}$$

$$P' = \{p_i'\} = \underbrace{\begin{pmatrix} p_1 & p_1' & p_1'' & \dots \\ p_2 & p_2' & p_2'' & \dots \\ p_3 & p_3' & p_3'' & \dots \end{pmatrix}}_{N}$$

$$\sum\left(x_i'^T R p_i'\right) = trace[\begin{pmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2' & x_3'' \\ \vdots & \vdots & \vdots \end{pmatrix} R \begin{pmatrix} p_1 & p_1' & p_1'' & \dots \\ p_2 & p_2' & p_2'' & \dots \\ p_3 & p_3' & p_3'' & \dots \end{pmatrix}]$$

$$= \begin{pmatrix} x_1^T R p_1 \\ x_2^T R p_2 \\ \vdots \end{pmatrix} = trace(X'^T R P')$$

$$\boxed{\begin{gathered}(abc)^T = c^T b^T a^T \\ trace(a \cdot b) = trace(b \cdot a) \\ trace(a \cdot b) = trace(a \cdot b)^T\end{gathered}}$$

$$trace\left(\underbrace{X'^T}_{a}\underbrace{RP'}_{b}\right) = trace\left[R\begin{pmatrix} p_1 & p_1' & p_1'' & \dots \\ p_2 & p_2' & p_2'' & \dots \\ p_3 & p_3' & p_3'' & \dots \end{pmatrix}\begin{pmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2' & x_3'' \\ \vdots & \vdots & \vdots \end{pmatrix}\right]$$

$$= trace(RP'X'^T)$$

$$trace(X'^T RP') = trace(RP'X'^T)$$

$$= trace(RP'X'^T)^T$$

$$= trace(X'P'^T R^T)$$

Let $W = X'P'^T \in R^{3\times 3}$

SVD of W

$$W = U\Sigma V^T$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

singular values $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0, \text{and } R = UV^T$

$$trace(X'P'^T R^T) = trace(WR^T)$$

$$= trace\left(\underbrace{U}_{a}\underbrace{\Sigma V^T R^T}_{b}\right)$$

$$= trace\left(\Sigma\underbrace{V^T R^T U}_{I}\right)$$

$$= trace(\Sigma) = \sigma_1 + \sigma_2 + \sigma_3$$

maximum!

$U\ and\ V$ are orthogonal.

$$V^T R^T U = I$$

$$V(V^T R^T U) = V \cdot I$$

$$R^T U = V$$

$$R^T = V U^T$$

$$R = (V U^T)^T = U V^T$$