EE 7207 – Neural & Fuzzy Systems

Part II – Fuzzy Systems

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Outline:

- **Fuzzy Set Theory**
- **Fuzzy Arithmetic**
- **Fuzzy Systems**
- **Neuro-Fuzzy Hybrid Systems**
- **Nonlinear Fuzzy Control Systems**

References

- Guanrong Chen and Trung Tat Pham, ``Introduction to Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems", CRC Press, 2001.

- Jacek M. Zurada, "Artificial Neural Systems", West Publishing Company, 1992.

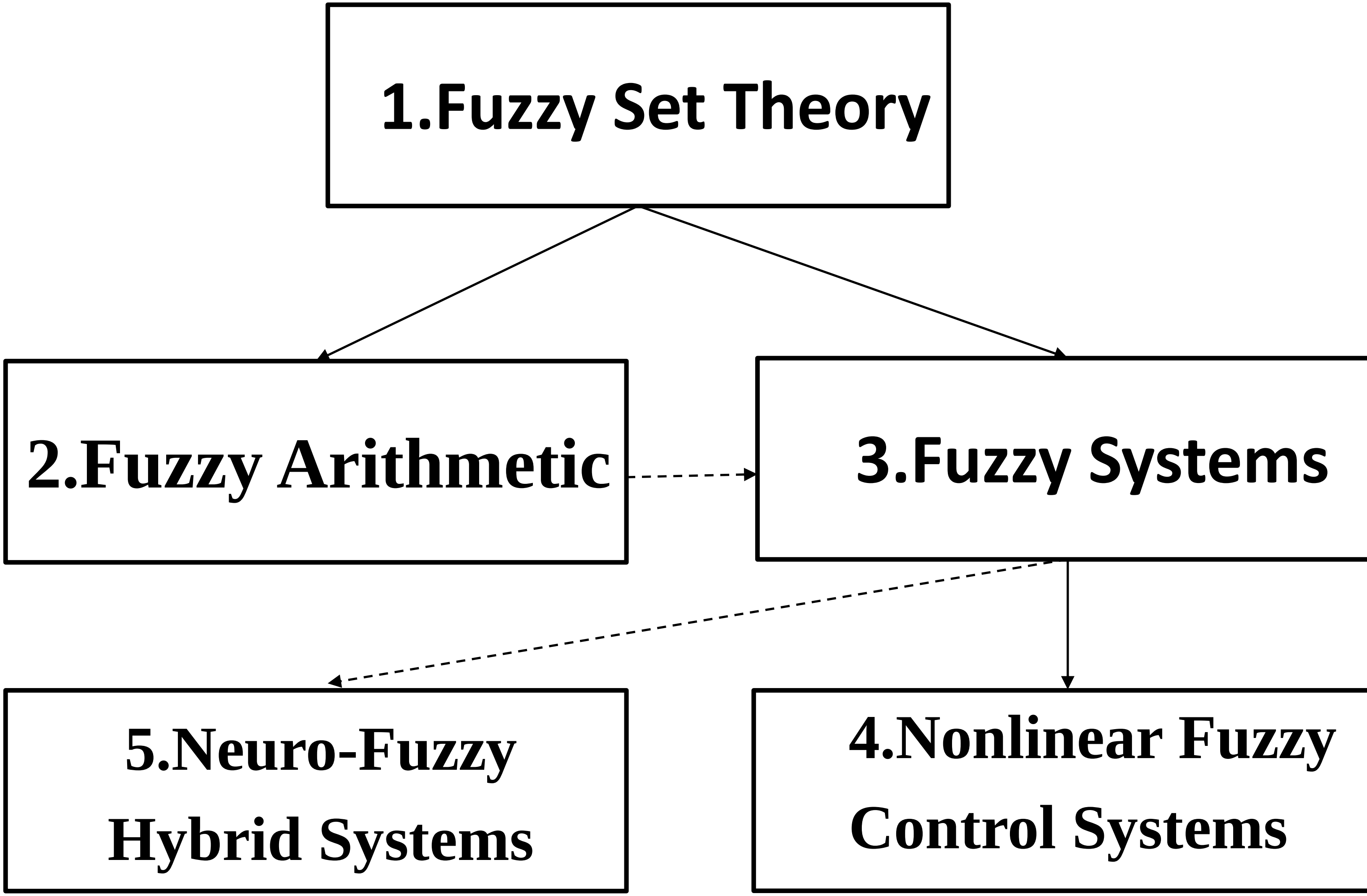
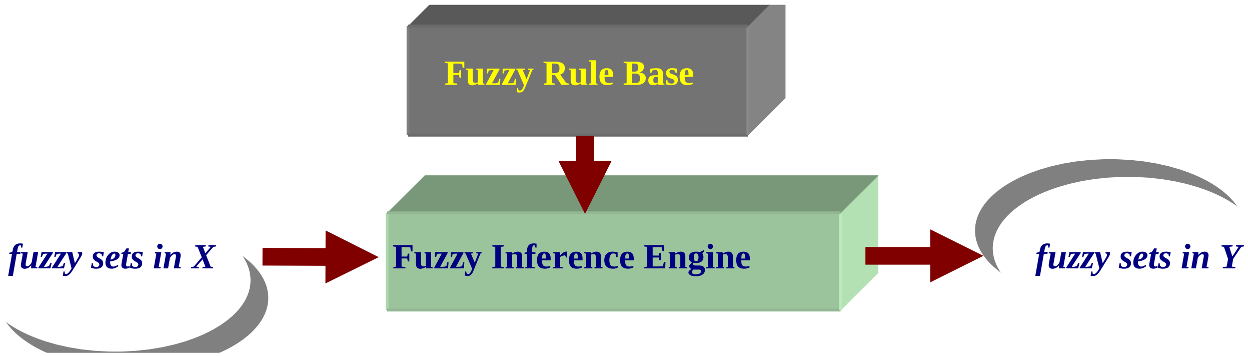

1.Fuzzy Set Theory
2.Fuzzy Arithmetic
3.Fuzzy Systems
5.Neuro-Fuzzy Hybrid Systems
4.Nonlinear Fuzzy Control Systems

Figure 1.1

The Core of Fuzzy Systems



One equation approach (Zadeh, L.A., 1965)

max-min composition

1. Fuzzy Set Theory

1.1 Introduction

The height of a group of men

Name	Height, cm
Chris	208
Mark	205
John	198
Tom	181
David	179
Mike	172
Bob	167
Steven	158
Bill	155
Peter	152

In your opinion, who are tall men?

Classically, we may define a set to denote all tall men as follows:

$$A=\{height \geq 180 \text{ cm}\};$$

Based on the above David is not in the set A and cannot be classified as a tall man.

The above logic, called Boolean logic, uses sharp distinctions. It forces us to draw lines between members of a class and non-members.

David is a short man, but Tom who is 181 cm is a tall man.

Is David really a short man?

The decision is a bit vague or *fuzzy*.

Our perception of the real world is pervaded by concepts which do not have sharply defined boundaries – for examples, ***many***, ***tall***, ***much larger than***, ***young***, etc. are true only to some degree and they are false to some degree as well.

Many decision-making and problem solving tasks are too complex to be understood quantitatively. However, people succeed by using knowledge that is imprecise rather precise.

Most of the phenomena we encounter everyday are *imprecise*.

These concepts with imprecision or uncertainty can be called **fuzzy** concepts

To mathematically represent uncertainty and to provide formalized tools for dealing with the imprecision intrinsic to many problems, a **fuzzy set** is specifically designed.

Fuzzy theory is established.

The key idea of fuzziness :
Everything is a matter of degree

Major Research Fields in Fuzzy Theory

By fuzzy theory we mean all the theories that use the basic concept of fuzzy set or membership function. Fuzzy theory can be roughly classified according to Fig 1.1

Fuzzy set theory was invented by the father of fuzzy systems – Zadeh (1965). However, like most of theoretical systems, the best approach is for linear systems only, in particular, the fuzzy arithmetic in section 2, for example., which can be applied only to certain applications like neuro-fuzzy classification in Chapter 4.

From a practical point of view, the majority of applications of fuzzy theory has concentrated on fuzzy systems, especially fuzzy control in Chapter 5, for example.

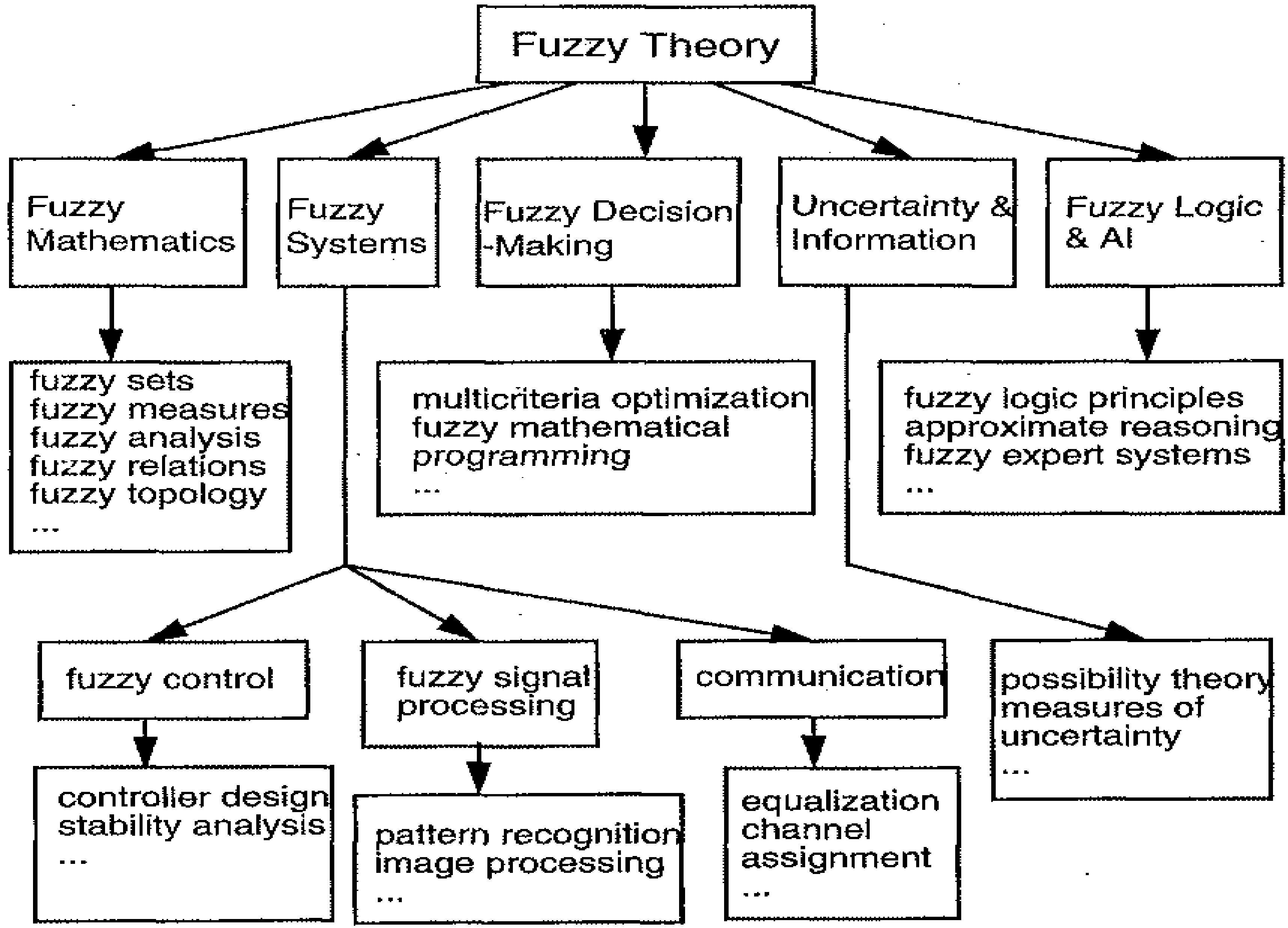

Fuzzy Theory
Fuzzy Mathematics
Fuzzy Systems
Fuzzy Decision -Making
Uncertainty & Information
Fuzzy Logic & AI
fuzzy sets
fuzzy measures
fuzzy analysis
fuzzy relations
fuzzy topology
...
multicriteria optimization
fuzzy mathematical programming
...
fuzzy logic principles
approximate reasoning
fuzzy expert systems
...
fuzzy control
fuzzy signal processing
communication
possibility theory
measures of uncertainty
...
controller design
stability analysis
...
pattern recognition
image processing
...
equalization
channel assignment
...

Fig 1.1

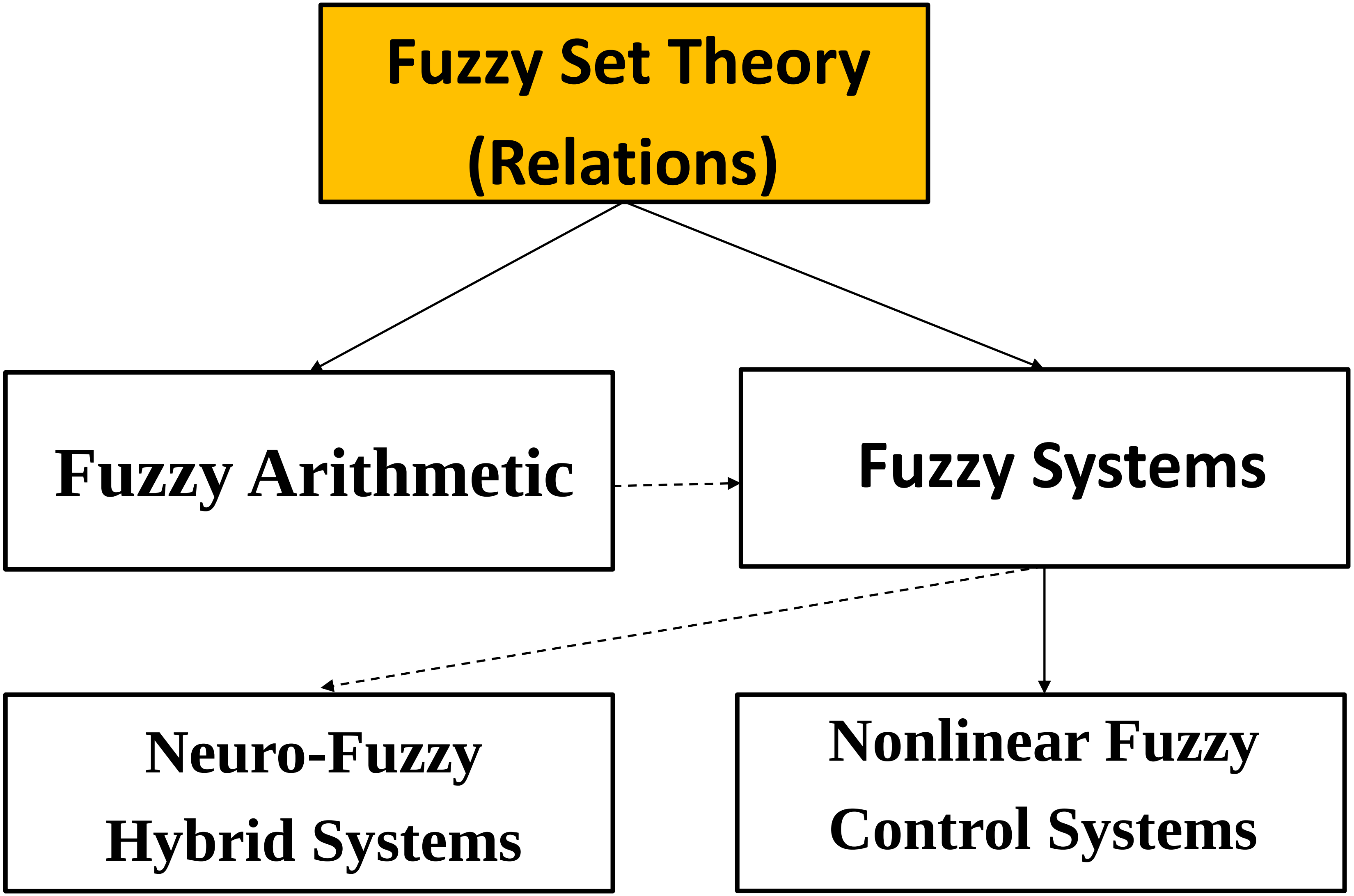

Fuzzy Set Theory
(Relations)
Fuzzy Arithmetic
Fuzzy Systems
Neuro-Fuzzy
Hybrid Systems
Nonlinear Fuzzy
Control Systems

Figure 1.1

1.2 Fuzzy Set Theory

- Fuzzy sets were introduced by Zadeh in 1965 to represent/manipulate data and information possessing non-statistical uncertainties.
- L.A.Zadeh, Fuzzy Sets, *Information and Control*, Vol. 8(1965) 338-353.

The concept of a fuzzy set contrasts with a **classical** concept of a **crisp set**.

- A Fuzzy Set is a set with a **smooth boundaries**
- Fuzzy Set Theory generalizes classical set theory to allow ***partial membership***

1.2.1 Review of Classical Set Theory

A set is a collection of objects with a common property.

A **classical** (**crisp**) set **A** in the universe of discourse $\mathbf{X}$ can be defined in three ways:

- by **enumerating** (listing) elements (often called list or extensional definition)

- by specifying the common **properties** of elements (rule definition).

The notation **A = {x | P(x)}** means that set **A** is composed of elements x such that *every* **x** has the *property* **P(x)**

- by introducing a **zero-one membership function** (characteristic or indicator definition)

Characteristic function: $\mu_A(x) : X \rightarrow \{0,1\}$

$$\mu_A(x) = \begin{cases} 1 & x \text{ is member of } A \\ 0 & x \text{ is not member of } A \end{cases}$$

Example:

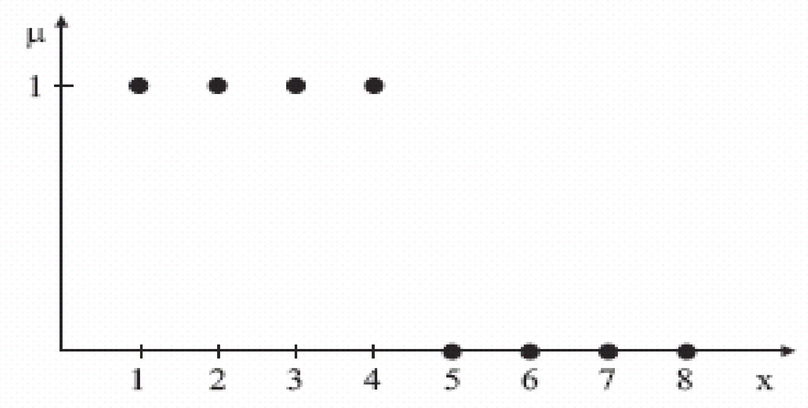
$U = \Im$ **(set of integer numbers),**

$\mathbf{A = \{x \in U \mid x \text{ is divisible by } 3\}}$

$$\mu_A(x) = \begin{cases} \mathbf{1}, & \textbf{if } \mathbf{x \in U}\textbf{, and x is divisible by 3} \\ \mathbf{0}, & \textbf{if } \mathbf{x \in U}\textbf{, and x is not divisible by 3} \end{cases}$$

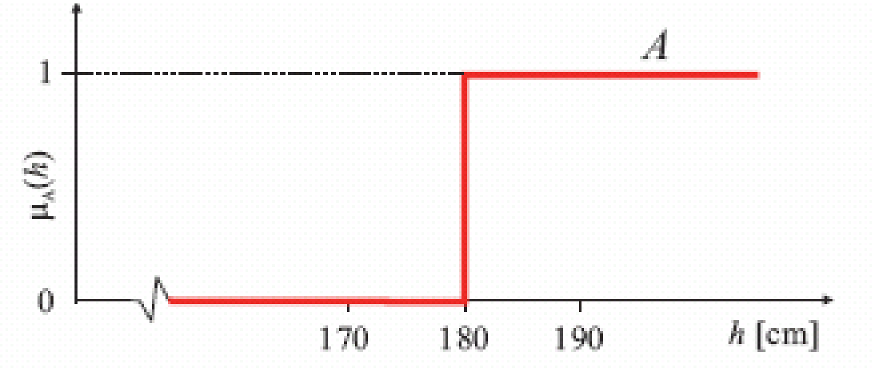
Set of natural numbers smaller than 5:

A = {1*;* 2*;* 3*;* 4}



Set of tall men

A={men with height≥ 1.80 cm}



Convex subset

A subset *A* in R^n is said to be *convex* if

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in A \qquad \text{and} \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in A$$

implies

$$\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in A \quad \text{for any } \lambda \in [0,1].$$

Properties of Classical Set Operations

Involutive law	$\overline{\overline{A}} = A$
Commutative law	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associative law	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cup A = A$$
$$A \cap A = A$$
$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$
$$A \cup (\overline{A} \cap B) = A \cup B$$
$$A \cap (\overline{A} \cup B) = A \cap B$$
$$A \cup S = S$$
$$A \cap \varnothing = \varnothing$$
$$A \cup \varnothing = A$$
$$A \cap S = A$$
$$A \cap \overline{A} = \varnothing$$
$$A \cup \overline{A} = S$$

DeMorgan's law

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

1.2.2 Fuzzy Set

Definition *(Fuzzy Set)*

*Let **X** be a nonempty set. A fuzzy set A in **X*** is completely determined by the set of tuples

$$\mathbf{A = \{ (x,}\ \mu_A(x)\mathbf{)\ |\ x \in X\}}$$

where $\mu_A(x) : \boldsymbol{X} \longrightarrow [0, 1]$ *denotes the degree of membership of element x in fuzzy set A for each* $\boldsymbol{x} \in \boldsymbol{X}$ and thus is called *the membership function* .

Remarks:

- Set ***X*** is called Universal set (or Universe of Discourse and sometimes denoted as **U**). It contains all possible elements of concern for a particular application.

- Fuzzy Set A in an universal set ***X*** is determined by a **membership function** $\mu_A(x)$ that assigns to each element $\boldsymbol{x} \in \boldsymbol{X}$ a value in the unit interval [0,1]. It means that a fuzzy set **A** contains an object **x** to degree $\boldsymbol{\mu_A(x)}$, i.e.

$$\boldsymbol{\mu_A(x)} = \textbf{\textit{Degree}}(\mathbf{x \in A}).$$

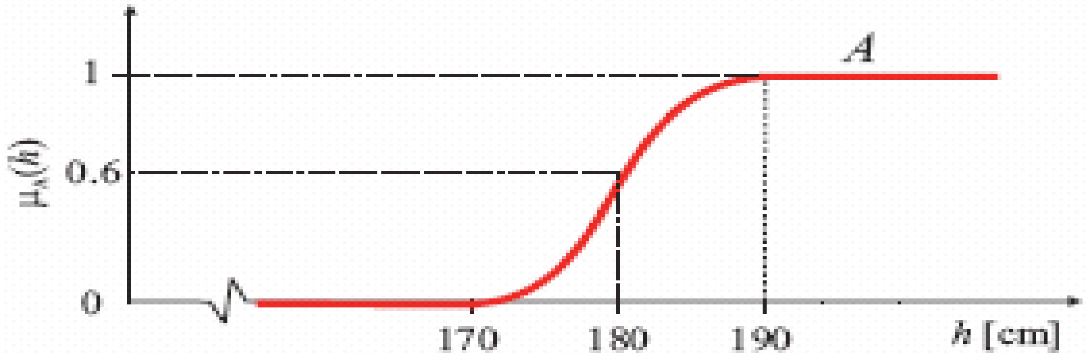
- Fuzzy Set Theory generalizes classical set theory to allow ***partial membership***.

- A membership function can be either discrete or continuous.

- The membership functions themselves are NOT fuzzy - they are **precise mathematical functions**; once a fuzzy property is represented by a membership function, nothing is fuzzy anymore.

Example:

For the Tall Men example, we may have



$$\mu_A(h) = \begin{cases} 1 & h \text{ is full member of } A \quad (h \geq 190) \\ (0,1) & h \text{ is partial member of } A \ (170 < h < 190) \\ 0 & h \text{ is not member of } A \quad (h \leq 170) \end{cases}$$

In fuzzy logic proposition, someone is tall with certain degree of truth. *For examples*,

Chris is tall with membership function 1;

Tom is tall with membership function 0.62.

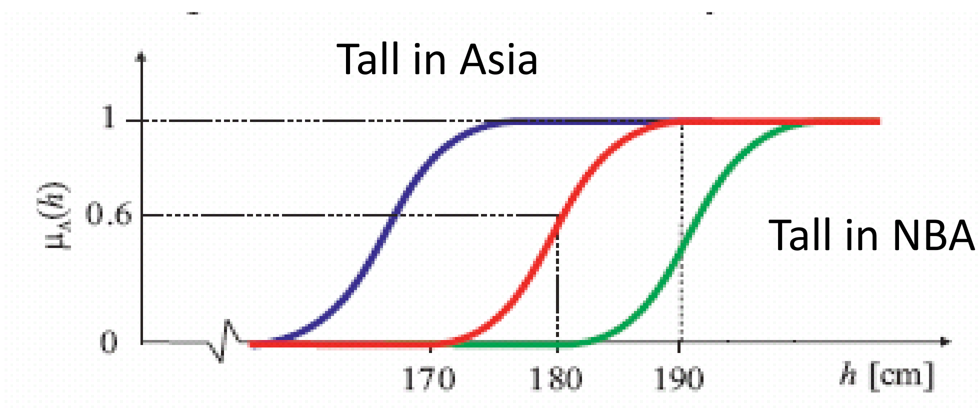
David is tall with membership function 0.58.

Peter is tall with membership function 0.

In each application of fuzzy set theory, we must construct appropriate fuzzy sets (i.e. their membership functions) which adequately capture the intended meanings of relevant linguistic terms.

The membership is subjective and context dependent.

Example:



Tall in Europe

Membership functions can be represented
(a)graphically, **(b)** analytically **(c)** in a tabular or list form.

1.2.3 Point-wise Representation of Fuzzy Sets

In a discrete set

$$X = \{x_i \mid i = 1, 2, \ldots, n\}$$

a fuzzy set **A** may be defined by a list of ordered pairs in the form of membership degree/set element:

$$A = \{\mu_A(x_1)/x_1,\ \mu_A(x_2)/x_2,\ \ldots, \mu_A(x_n)/x_n\} = \{\mu_A(x)/x \mid x \in X\}$$

Normally, only elements $x \in X$ with non-zero membership degrees are listed.

The following alternatives to the above notation can be encountered:

$$A = \mu_A(x_1)/x_1 + \cdots + \mu_A(x_n)/x_n = \sum_{i=1}^{n} \mu_A(x_i)/x_i$$

For continuous domains,

$$A = \int_X \mu_A(x)/x$$

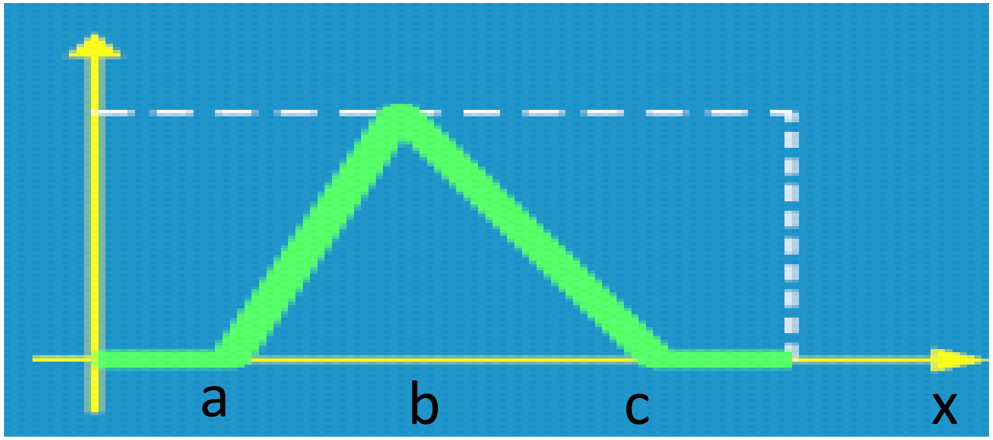
Note that rather than summation and integration in this context, the

$\Sigma, +$ and $\int$ symbols represent a collection (union) of elements

1.2.4 Some Common Membership Functions:

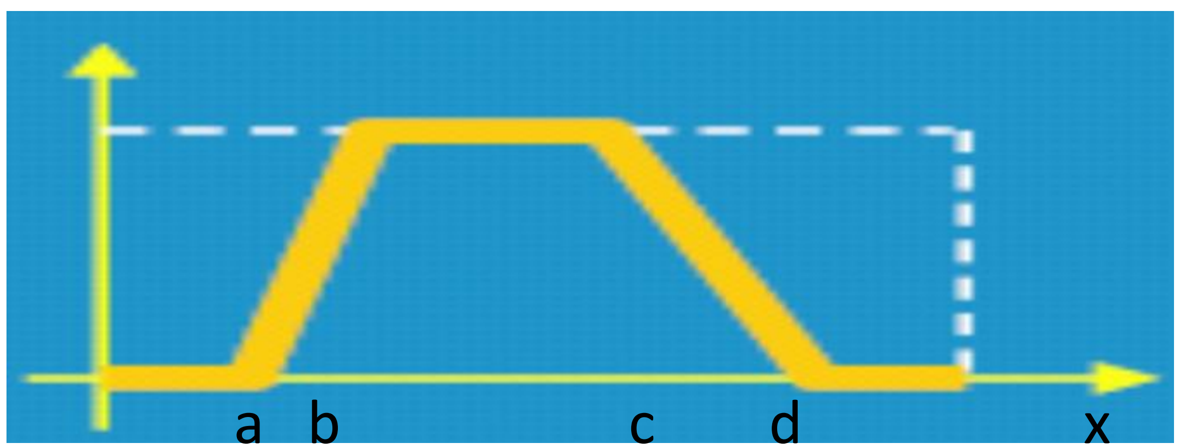
(a) Triangular MF is specified by 3 parameters {**a**,**b**,**c**}:

$$trn(x:a,b,c) = \begin{cases} 0, & \text{if } x < a \\ (x-a)/(b-a), & \text{if } a \le x \le b \\ (c-x)/(c-b), & \text{if } b \le x \le c \\ 0, & \text{if } x > c \end{cases}$$



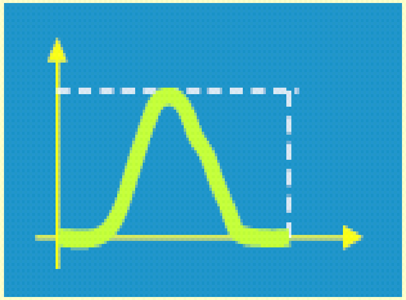
(b) **Trapezoidal MF** is specified by 4 parameters {**a,b,c,d**}:

$$
trp(x : \mathbf{a,b,c,d}) = \begin{cases} \mathbf{0,} & \mathbf{if\ x < a} \\ \mathbf{(x\text{-}a)/(b\text{-}a),} & \mathbf{if\ a \le x < b} \\ \mathbf{1,} & \mathbf{if\ b \le x < c} \\ \mathbf{(d\text{-}x)/(d\text{-}c),} & \mathbf{if\ c \le x < d} \\ \mathbf{0,} & \mathbf{if\ x \ge d} \end{cases}
$$



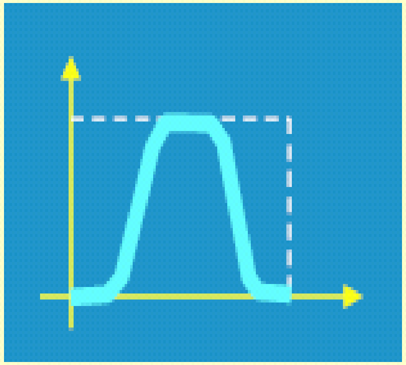
(c) **Gaussian MF** is specified by 2 parameters {**a**,δ}:

$$gsn(x : a, \delta) = \exp\left(\frac{-(x-a)^2}{\delta^2}\right)$$



(d) **Bell-shaped MF** is specified by 3 parameters {**a**,**b**,δ}:

$$bll(x : a, b, \delta) = \frac{1}{1 + \left|\frac{x-\delta}{a}\right|^{2b}}$$



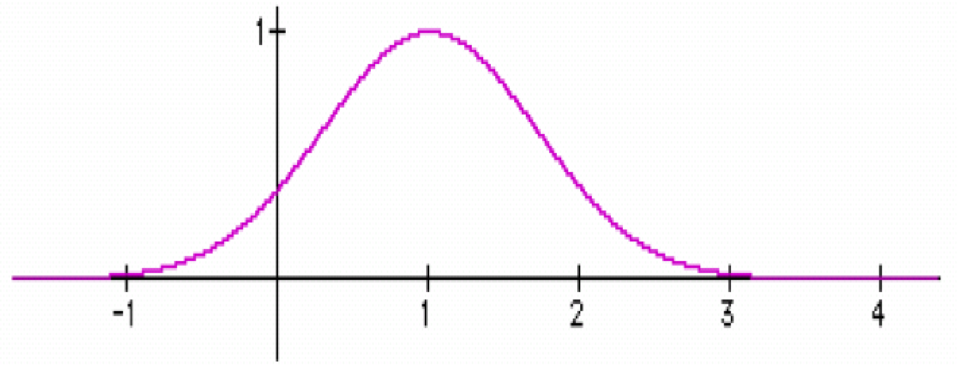
1.2.5 Some Examples of Fuzzy Sets:

The membership function of the fuzzy set of real *numbers "close to 1", can be defined as*

$$\mu_A(x) = e^{-\beta(x-1)^2}$$

where β is a positive real number.

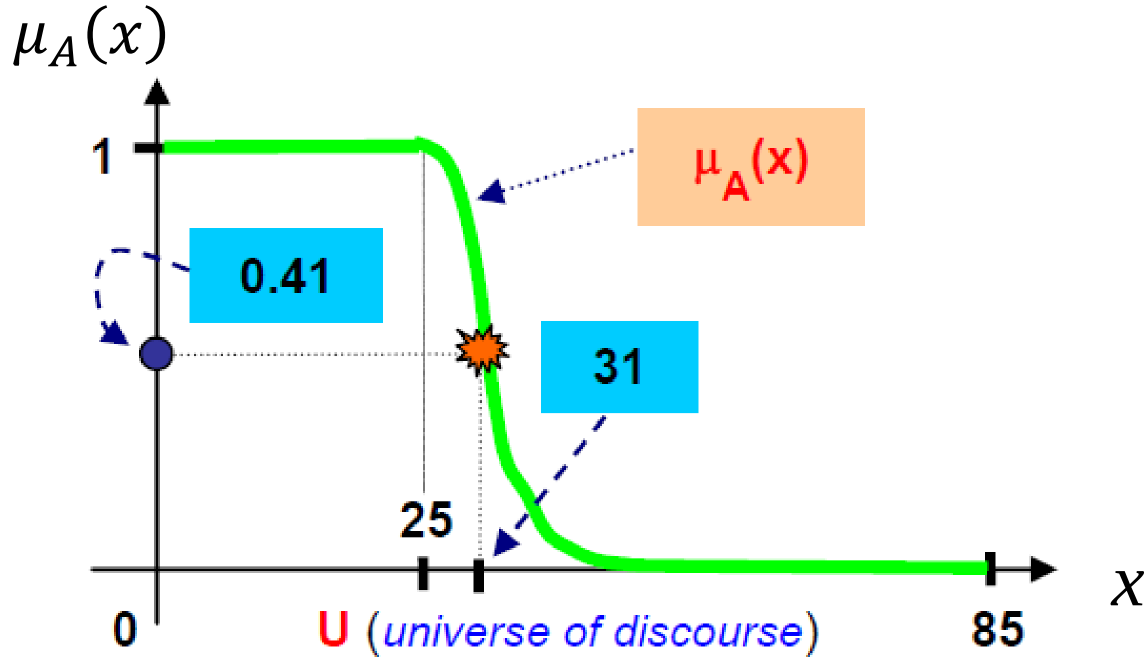
$\mu_A(x)$



x

*Suppose **X** is the interval [0, 85] representing the age of ordinary human beings, fuzzy set "young" can be defined as*

$$A = \text{"young"} = \int_0^{25} 1/x + \int_{25}^{85} \left(1 + \left(\frac{x-25}{5}\right)^2\right)^{-1} \Big/ x$$



1.2.6 Some Definitions and Properties of Fuzzy Sets

Definition (Height):

The height of a fuzzy set A is the highest (maximum) value of its membership function, i.e.

$$\textbf{height(A)} = \max_{x_i} \mu_{\mathbf{A}}(x_i)$$

Definition (Normal Fuzzy Set):

If a fuzzy set has a **height 1**, then it is called a **normal** fuzzy set; in contrast, if height(A) < 1, the fuzzy set is said to be **subnormal**.

In most applications fuzzy sets are normal, and during the reasoning process usually subnormal fuzzy sets are generated.

The operator norm(A) denotes normalization of a fuzzy set, i.e.,

$$A' = \mathrm{norm}(A) \Leftrightarrow \mu_{A'}(x) = \mu_A(x) / \mathrm{hgt}(A), \ \forall x$$

Definition (Support):

A set of all elements of the universal set X whose degree of membership in a fuzzy set A is nonzero is called the support of a fuzzy set A, i.e.

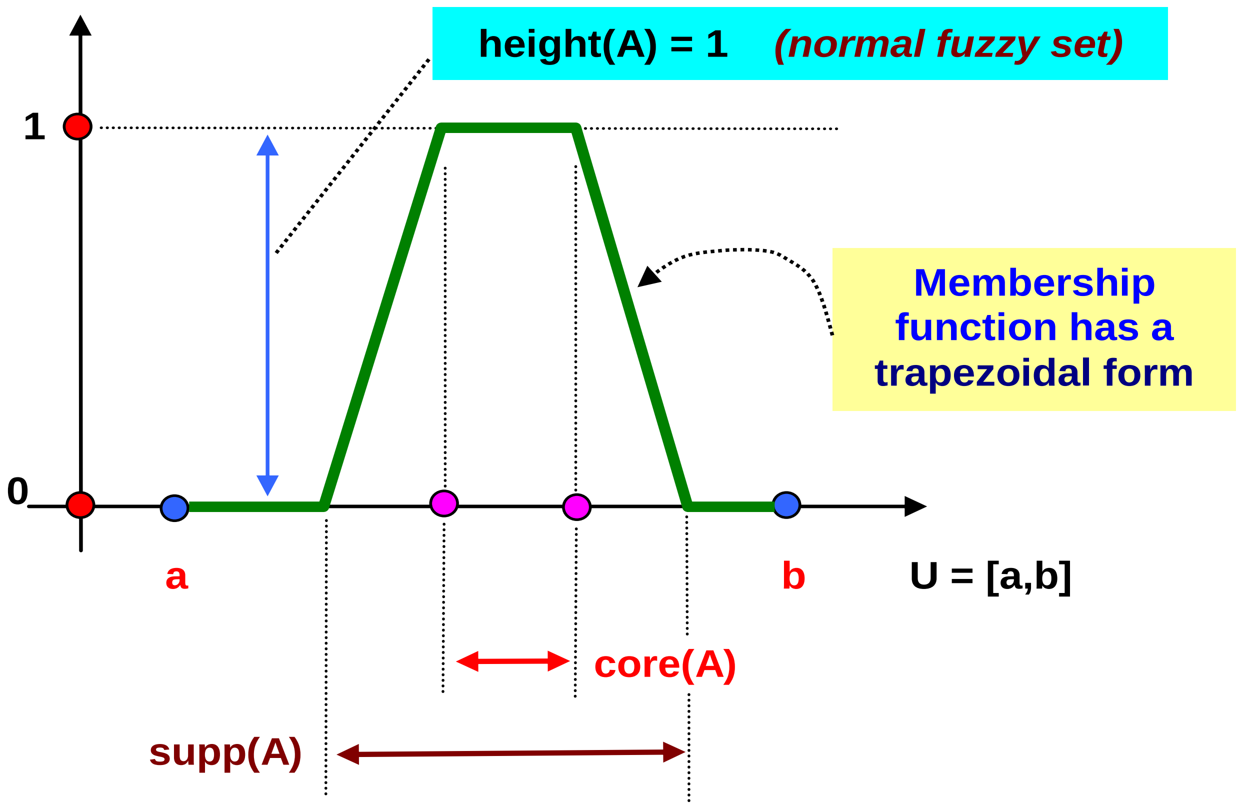
$$\textbf{supp(A)} = \{x \in X \ \mid \ \mu_{\mathbf{A}}(x) > 0\}$$

Definition (Core):

A set of all elements **x** of the universal set **X** with a property μ_A **(x) = 1** is called the core of a fuzzy set A (**core(A)**)

$$\mathrm{core}(A) = \{x \mid \mu_A(x) = 1\}$$

In the literature, the core is sometimes also denoted as the kernel, ker(A). The core of a subnormal fuzzy set is empty.

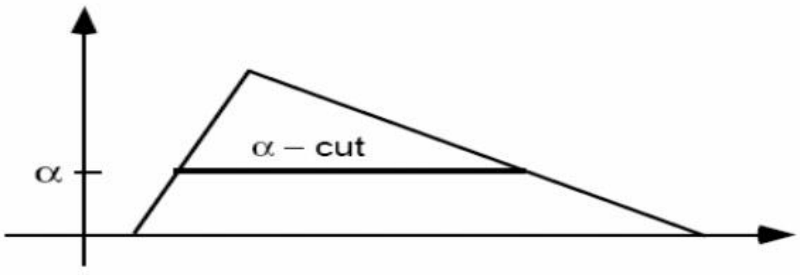

height(A) = 1 (normal fuzzy set)
1
0
Membership function has a trapezoidal form
a
b
U = [a,b]
core(A)
supp(A)

Definition (α-Cut):

For a given $\alpha \in (0,1]$ and fuzzy set A, the crisp set A^{α} contains those $\mathbf{x} \in \mathbf{U}$ for which

$$\mu_A(x) \ \geq \ \alpha$$

is called an α-cut of the fuzzy set A.



An α-cut of a triangular fuzzy number.

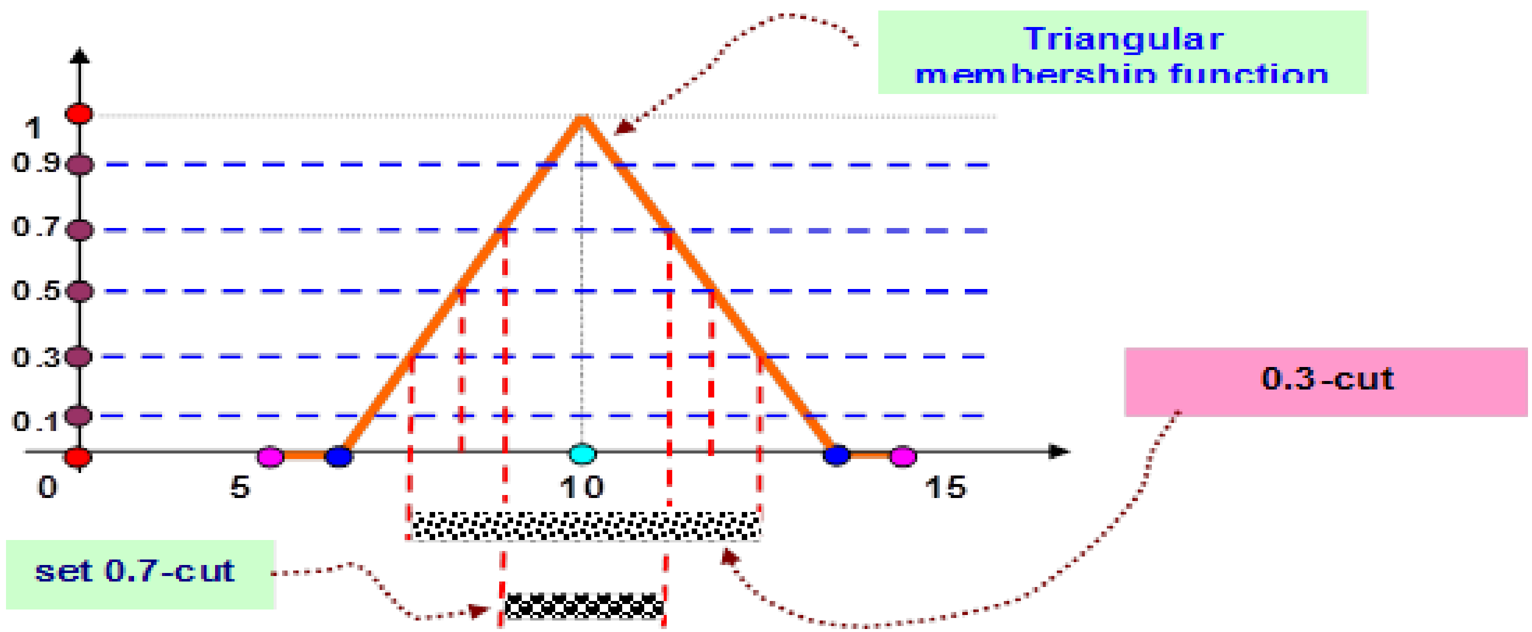
An α-cut A^{α} is strict if $\mu_A(x) \ > \ \alpha$ for each $x \in A^{\alpha}$

The value α is called the α-level.

Example:

Consider a fuzzy set A which is represented analytically in the universe of discourse U = [5,15] as follows:

$$\mu_A(x) = \begin{cases} 1 - (|x-10|)/4, & \text{if } 6 \le x \le 14 \\ 0, & \text{otherwise} \end{cases}$$



Example:

Consider the fuzzy set `High Score'

$$A=0.1/50+0.3/60+0.5/70+0.8/80+1/90+1/100$$

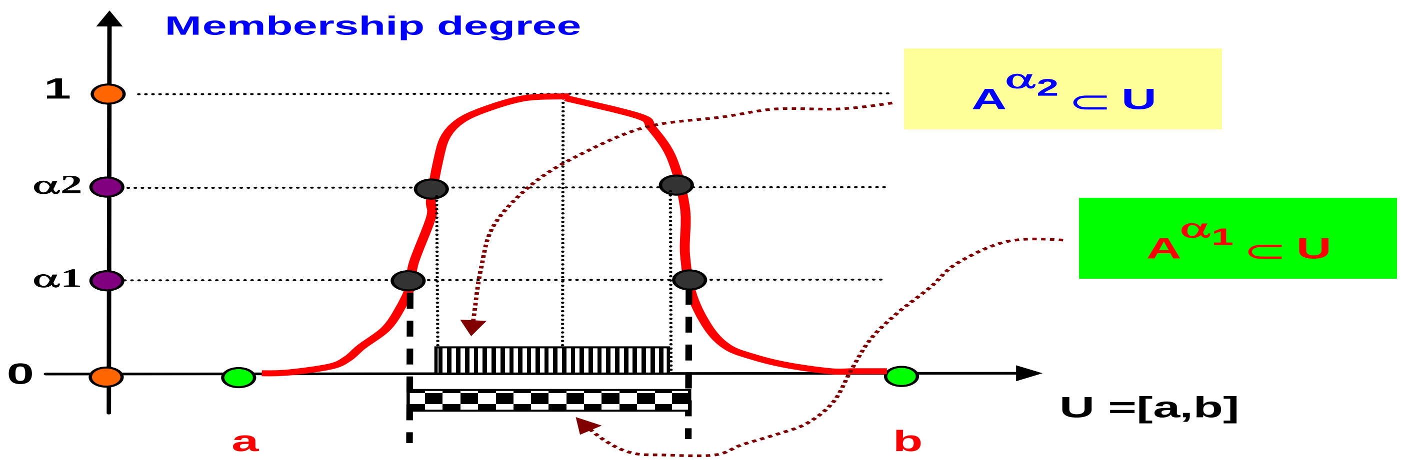
Then $A^{0.5}=\{70, 80, 90, 100\}$

A general property of α-cuts:

For any fuzzy set ***A*** and two values $\alpha_1, \alpha_2 \in [0,1]$ that satisfy the condition $\alpha_1 < \alpha_2$, the following is true:

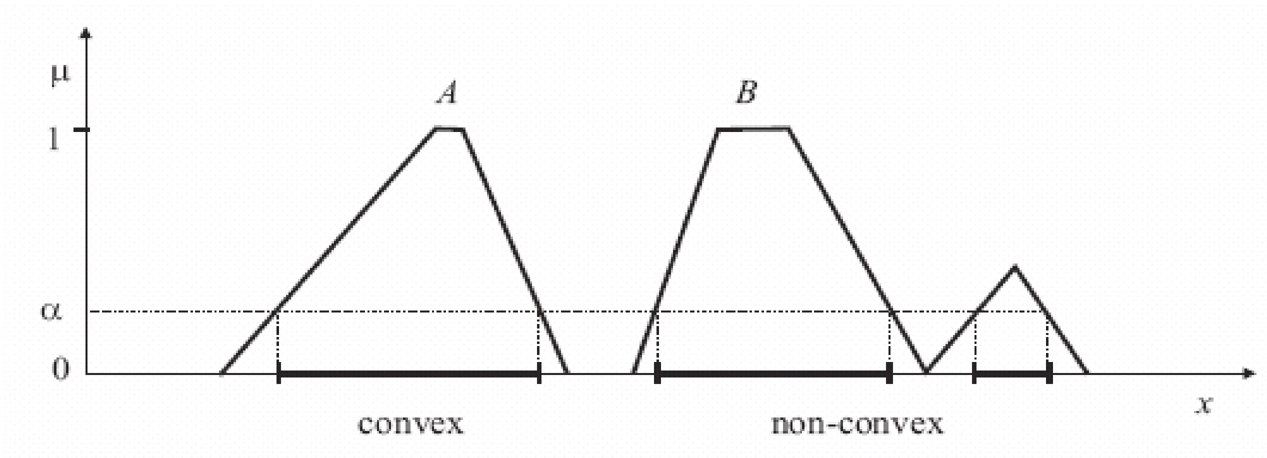
$A^{\alpha_1} \supseteq A^{\alpha_2}$, i.e. A^{α_2} is a subset of A^{α_1} and as a result

$$A^{\alpha_1} \cap A^{\alpha_2} = A^{\alpha_2} \qquad A^{\alpha_1} \cup A^{\alpha_2} = A^{\alpha_1}$$



Definition (Convex Fuzzy Set):

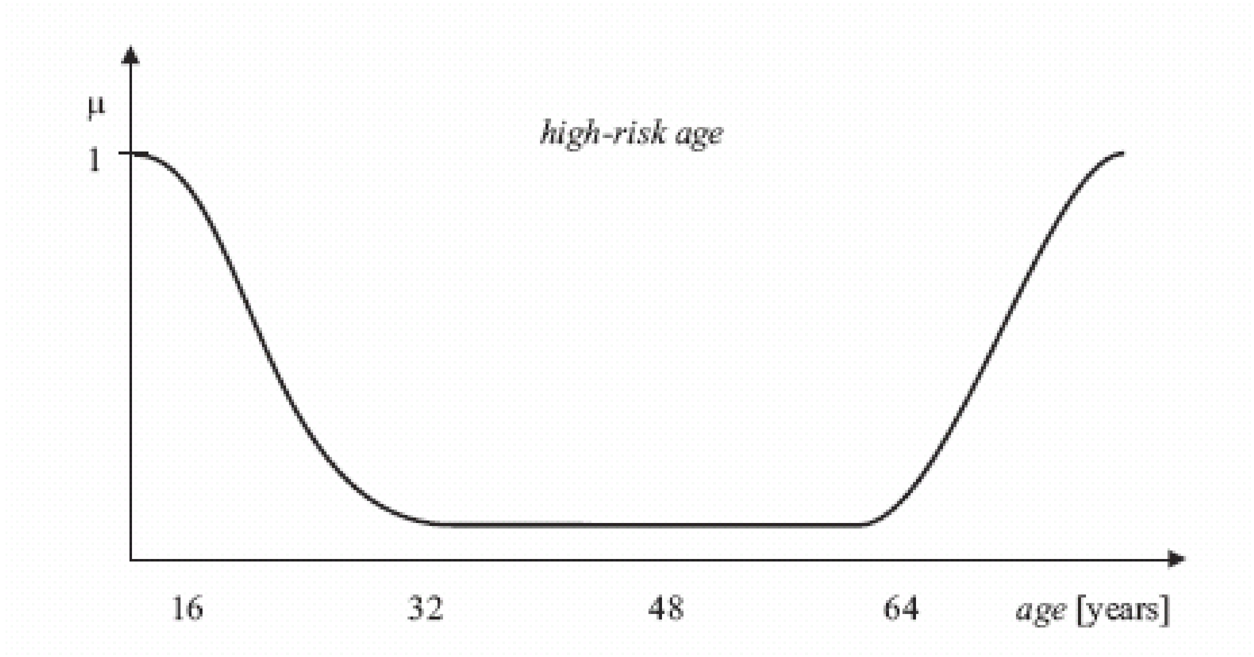
A fuzzy set defined in R^{n} *is convex if each of its α-cuts is a convex set.*



Example (Non-convex Fuzzy Set)

The figure below gives an example of a non-convex fuzzy set representing "high-risk age" for a car insurance policy.

Drivers who are too young or too old present higher risk than middle-aged drivers.



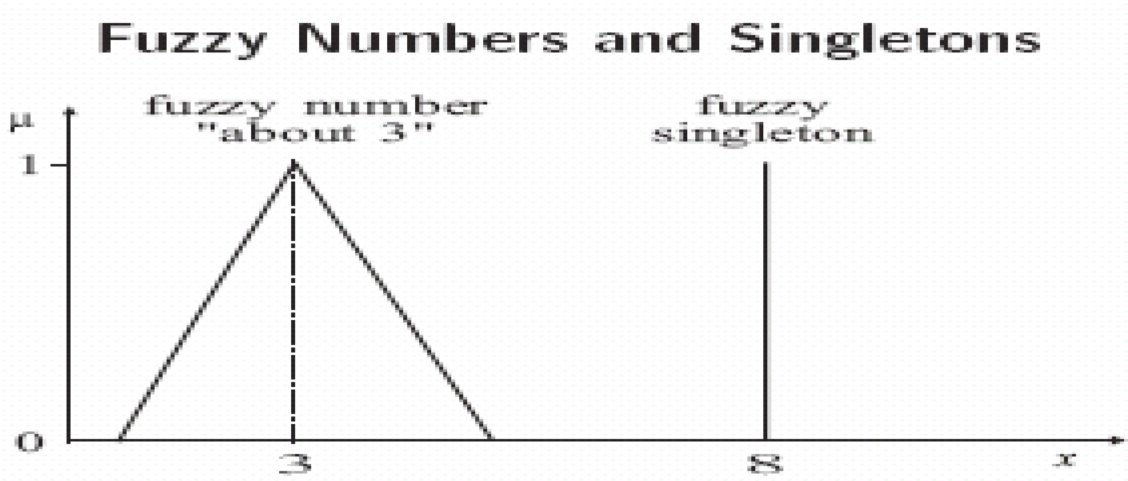
Definition (fuzzy singleton):

A fuzzy set whose support is a single point in the universe of discourse **U** is called a fuzzy singleton.

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = x_0, \\ 0, & \text{otherwise}. \end{cases}$$

Definition: *a fuzzy number* is a normal, convex fuzzy set which is defined on the real line R.

Example:



1.2.7 Operations of Fuzzy Sets

Assume two *fuzzy sets* **A** and **B** are defined on the universe of discourse **U.** Several basic operations and definitions can be represented as follows.

(1) **Standard Fuzzy Union**

$$\mathbf{C} = \mathbf{A} \vee \mathbf{B} = \mu_{\mathbf{A} \cup \mathbf{B}}(x) = \mathbf{max}\,[\, \mu_{\mathbf{A}}(x),\ \mu_{\mathbf{B}}(x)\,]$$

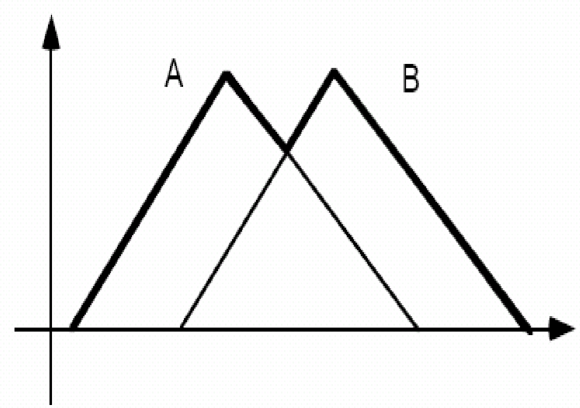
The maximum operator is also denoted by \`$\vee$'.

For example

$$\mu_C(x) = \max[\mu_A(x), \mu_B(x)]$$

can be represented as

$$\mu_C(x) = \mu_A(x) \vee \mu_B(x).$$



Union of two triangular fuzzy numbers.

(2) Standard Fuzzy Intersection

$$\mathbf{A} \wedge \mathbf{B} = \mu_{A\cap B}(x) = \textbf{min}\,[\,\mu_{\mathbf{A}}(x),\ \mu_{\mathbf{B}}(x)\,]$$

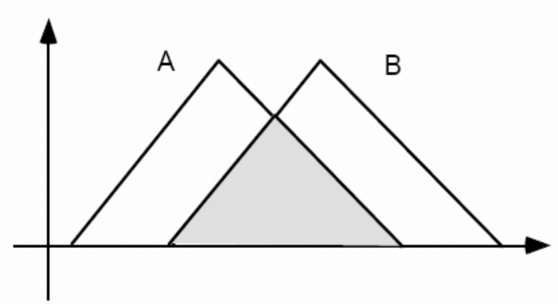
The minimum operator is also denoted by '$\wedge$'

For example,

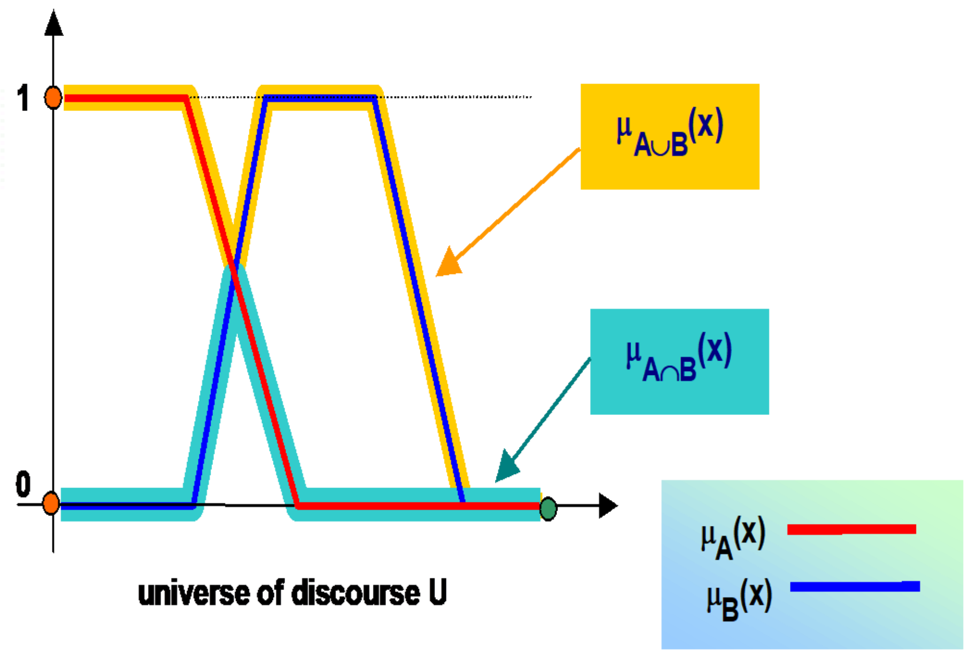
$$\mu_C(x) = \min[\mu_A(x), \mu_B(x)]$$

can be represented as

$$\mu_C(x) = \mu_A(x) \wedge \mu_B(x)$$



Intersection of two triangular fuzzy numbers



Example

Let A and B be fuzzy sets of $X = \{-2, -1, 0, 1, 2, 3, 4\}$.

$A = 0.6/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.4/4$

$B = 0.1/-2 + 0.3/-1 + 0.9/0 + 1.0/1 + 1.0/2 + 0.3/3 + 0.2/4$

Then $A \cap B = 0.1/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.2/4$.

$A \cup B = 0.6/-2 + 0.3/-1 + 0.9/0 + 1.0/1 + 1.0/2 + 0.3/3 + 0.4/4$.

T-norms and T-conorms

Fuzzy intersection of two fuzzy sets can be specified in a more general way by a binary operation on the unit interval, i.e., a function of the form:

$$T: [0,1] \times [0,1] \to [0,1]$$

Definition (t**-Norm/Fuzzy Intersection):**

A t-*norm* (triangular norms) T *is a binary operation on the unit interval that satisfies at least the following axioms for all* $a, b, c \in [0,1]$

$$\begin{array}{ll} T(a,1) = a & \textit{(boundary condition)}, \\ b \le c \ \textit{ implies } \ T(a,b) \le T(a,c) & \textit{(monotonicity)}, \\ T(a,b) = T(b,a) & \textit{(commutativity)}, \\ T(a,T(b,c)) = T(T(a,b),c) & \textit{(associativity)}. \end{array}$$

The standard (Zadeh) intersection T(a, b) = min(a, b) is a t-norm.

Other Intersection Operators (T-norms)

Algebraic product (probabilistic intersection) : T(a, b) = ab

$$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Łukasiewicz (bold) intersection: T(a, b) = max(0, a + b − 1)

$$\mu_{A\cap B}(x) = \max(0, \mu_A(x) + \mu_B(x) - 1)$$

Definition (t-Conorm/Fuzzy Union):

A t-conorm S is a binary operation on the unit interval that satisfies at least the following axioms for all $a, b, c \in [0, 1]$

$$
\begin{array}{ll}
S(a,0) = a & \textit{(boundary condition)}, \\
b \le c \;\; \textit{implies} \;\; S(a,b) \le S(a,c) & \textit{(monotonicity)}, \\
S(a,b) = S(b,a) & \textit{(commutativity)}, \\
S(a,S(b,c)) = S(S(a,b),c) & \textit{(associativity)}.
\end{array}
$$

The standard (Zadeh) union: S(a, b) = max(a, b) is a t-conorm

Other Union Operators

algebraic sum S(a, b) = a + b – ab:

$$\mu_{A \cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Łukasiewicz (bold) union

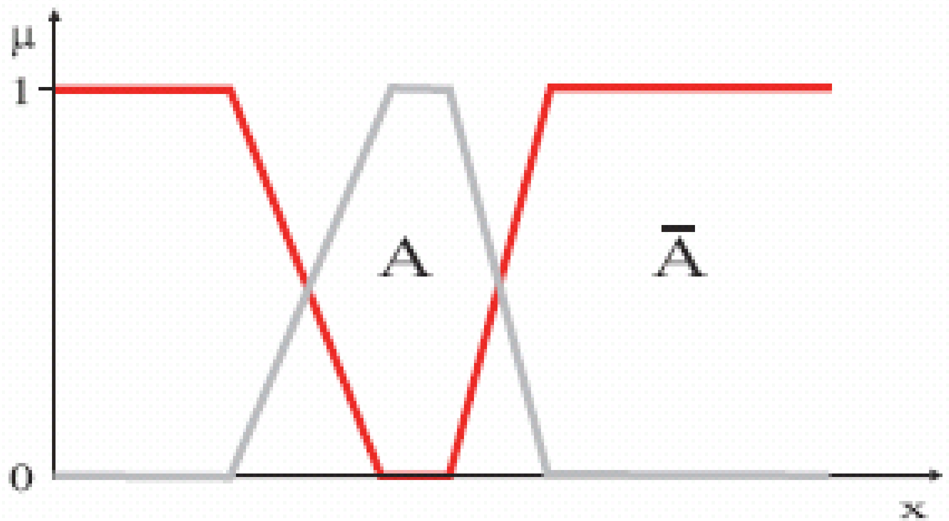
S(a, b) = min(1, a + b):

$$\mu_{A \cup B}(x) = \min(1, \mu_A(x) + \mu_B(x))$$

(3) Complement (Negation) of a Fuzzy Set

Complement (Negation) of a fuzzy set A, denoted as $\overline{A}$ (or $\neg A$), is defined by its membership function as

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$



Fuzzy sets overlap with their complements (an element may *partially belong* to both fuzzy set and set's complement). In contrast, **classical (crisp) sets never overlap** with their complements.

Thus the law of **Contradiction** $A \cap \overline{A} = \varnothing$ is *violated* in **Fuzzy Set Theory**

Example

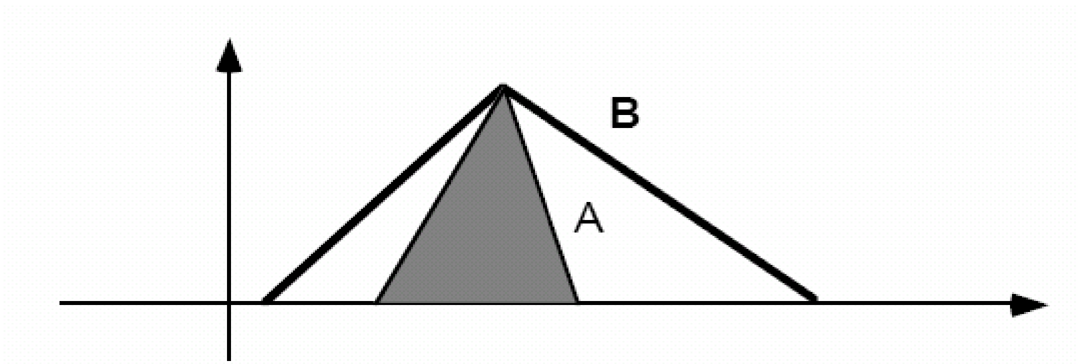
Let $\mu_A(x) = 1/2,\ \forall x \in R$. Then it is easy to see that

$$\overline{A} \wedge A = \min\{\overline{A}, A\} = \min\{1 - 1/2, 1/2\}$$

$$= 1/2 \neq 0$$

(4) Subset

Let A and B be fuzzy sets in the universe of discourse X.
Then, if μ_A ***(x)*** $\leq$ μ_B ***(x)*** for all x $\in$X,
A is contained in set B (or a subset of B) denoted as $A \subseteq B$

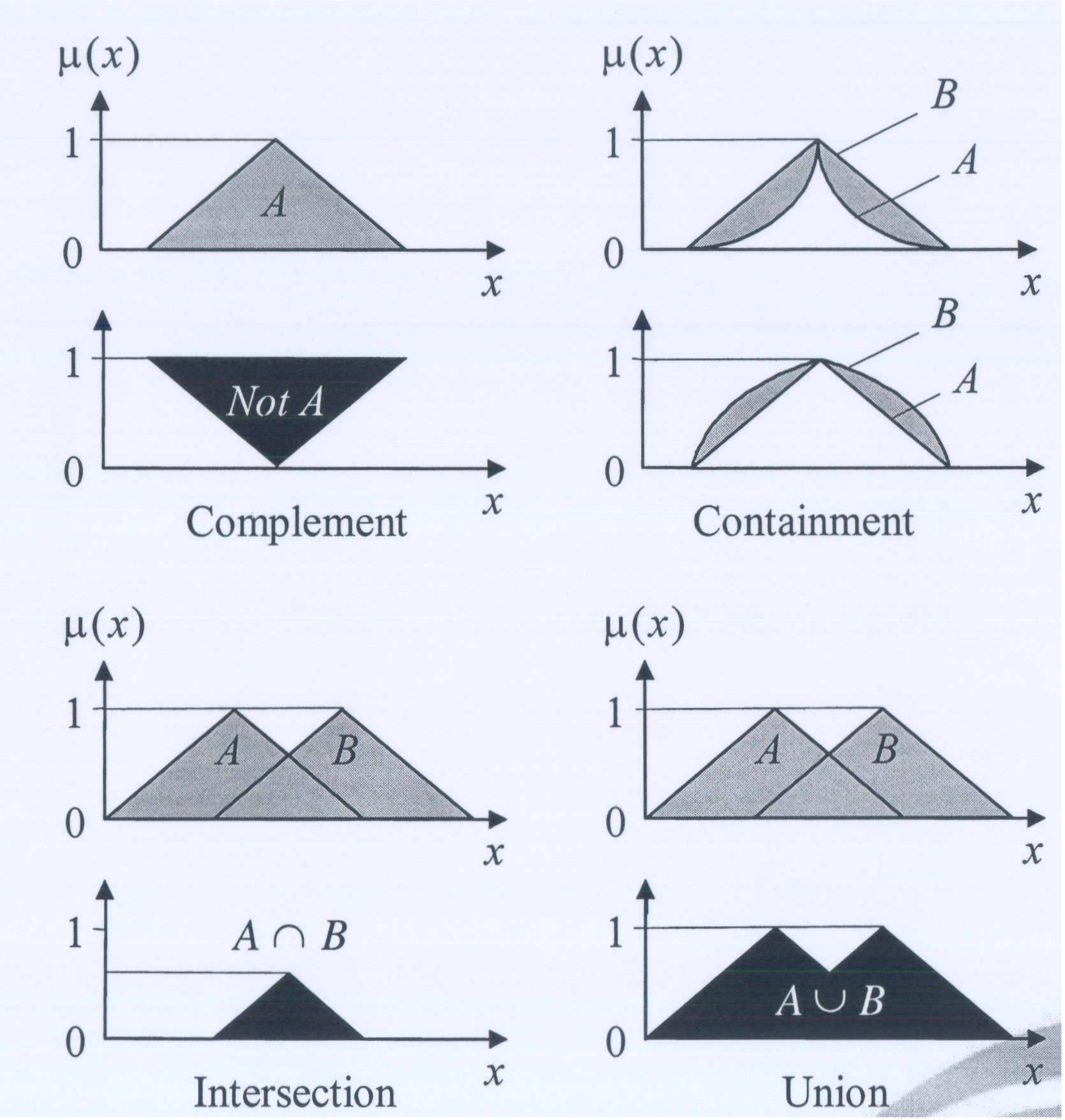


Example:
Let A and B be fuzzy sets of X = {−2,−1, 0, 1, 2, 3, 4}.

A = 0.0/ −2 + 0.3/ −1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4

B = 0.1/ −2 + 0.3/ −1 + 0.9/0 + 1.0/1 + 1.0/2 + 0.3/3 + 0.2/4

It is easy to check that $A \subseteq B$ holds.


μ(x)
1
0
A
x
1
0
Not A
x
Complement
μ(x)
B
1
A
0
x
B
1
A
0
x
Containment
μ(x)
1
A
B
0
x
1
A ∩ B
0
x
Intersection
μ(x)
1
A
B
0
x
1
A ∪ B
0
x
Union

(5) DeMorgan's Law

$$\overline{A \vee B} = \bar{A} \wedge \bar{B}$$

$$\overline{A \wedge B} = \bar{A} \vee \bar{B}$$

(6) Equality

Fuzzy set **A** is equal to fuzzy set **B** if and only if

$\mu_A(x) = \mu_B(x), \forall x \in \mathbf{U}$

(7) Empty Fuzzy Set

A fuzzy set is *empty* if and only if its membership function is identically zero on *U*.

(8) Cartesian Product

Let $A_1, A_2, \ldots, A_n$ be fuzzy sets in $\mathbf{U_1}, \mathbf{U_2}, \ldots, \mathbf{U_n}$, respectively. The Cartesian product of $A_1, A_2, \ldots, A_n$ is a fuzzy set in the product space $\mathbf{U_1} \times \mathbf{U_2} \times \ldots \times \mathbf{U_n}$ with membership function as

$$\mu_{A_1 \times A_2 \times \ldots \times A_n}(x_1, x_2, \ldots x_n) = min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots \mu_{A_n}(x_n)\}$$

$$x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$$

Example

Let A={0.5/3,1/5,0.6/7}, B={1/3,0.6/5}.

Then A×B={0.5/(3,3),1/(5,3),0.6/(7,3),0.5/(3,5), 0.6/(5,5),0.6/(7,5)}

1.2.8 Fuzzy Relations

Fuzzy relations are very important because they can describe interactions between variables.

A ***fuzzy relation*** is ***a fuzzy set*** in the Cartesian product $\mathbf{X_1} \times \mathbf{X_2} \times \mathbf{.\ .\ .} \times \mathbf{X_n}$. The membership grades represent the ***degree of association***.

Definition (Fuzzy Relation)

An n-*ary fuzzy relation R is a mapping*

R: $\mathbf{X_1} \times \mathbf{X_2} \times \mathbf{.\ .\ .} \times \mathbf{X_n} \rightarrow [0, 1]$,

which assigns membership grades to all n-tuples

$(x_1, x_2, \ldots, x_n)$ from the Cartesian product $\mathbf{X_1} \times \mathbf{X_2} \times \mathbf{.\ .\ .} \times \mathbf{X_n}$.

A ***Binary Fuzzy Relation*** is a fuzzy relation between two sets *X* and *Y. It is* a fuzzy set in X ×Y, which map every element in X ×Y into a membership grade between 0 and 1. Hence, ***Binary Fuzzy Relations*** are 2-dimensional MFs

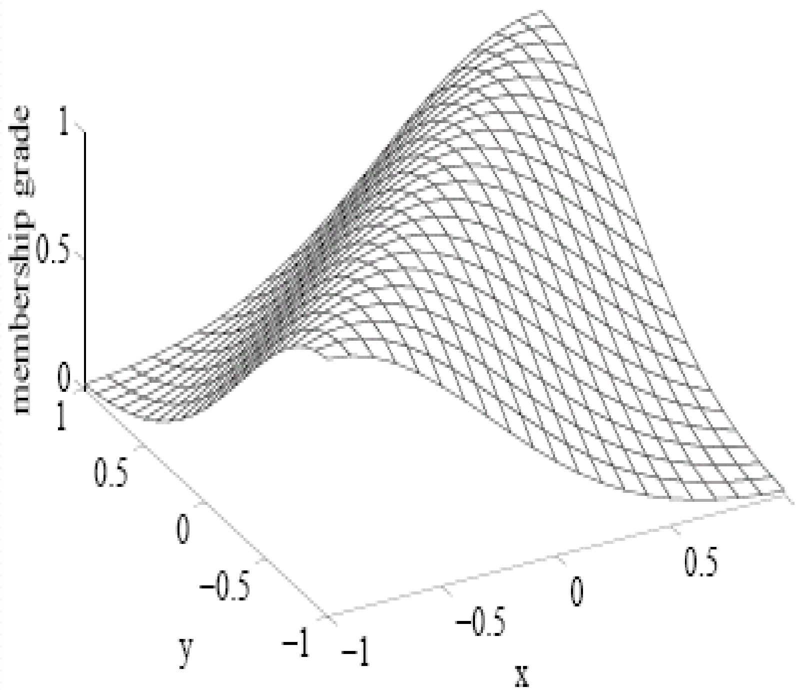
$$\boxed{R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}}$$

- Examples:
  - x is close to y (x and y are numbers)
  - x depends on y (x and y are events)
  - x and y look alike (x and y are persons or objects)
  - If x is large, then y is small (x is an observed reading and y is a corresponding action)

Example:

Consider a fuzzy relation *R* describing the relationship x ≈ y (“x is approximately equal to y”) by means of the following membership function

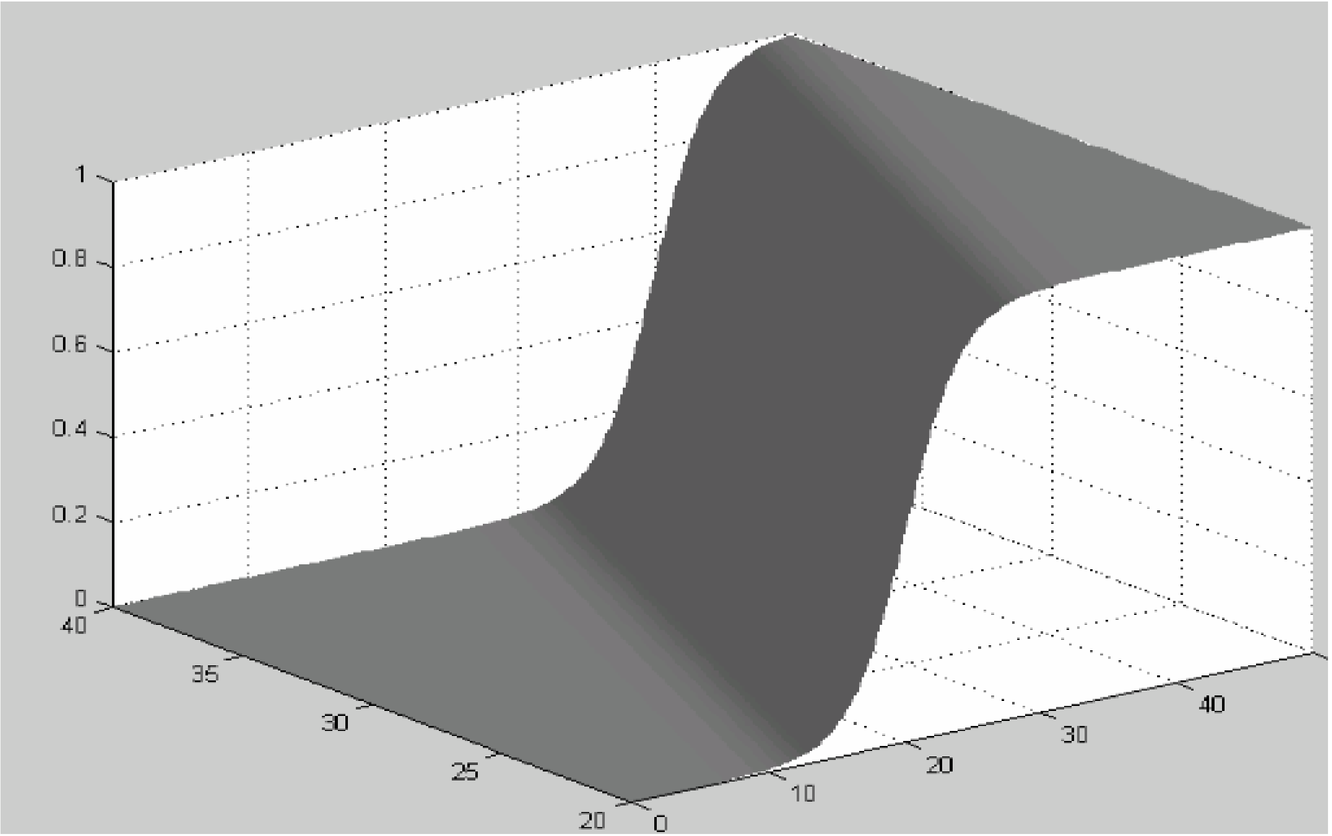
$$\mu_R(x, y) = e^{-(x-y)^2}$$



Example:

A fuzzy relation «x_1 ***is much larger than*** x_2» may be defined by the membership function

$$\mu_R = \frac{1}{1+e^{-0.5\cdot(x_1-x_2)}}$$



A binary fuzzy relation *R(X,* Y) can also be represented as a fuzzy matrix.

Let X={$x_1, x_2, \dots x_n$}and Y={$y_1, y_2, \dots y_m$}.
The fuzzy relation *R (X,* Y) can be expressed by an *n* ×*m* matrix as

$$R(X,Y) = \begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) & \cdots & \mu_R(x_1, y_m) \\ \mu_R(x_2, y_1) & \mu_R(x_2, y_2) & \cdots & \mu_R(x_2, y_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_R(x_n, y_1) & \mu_R(x_n, y_2) & \cdots & \mu_R(x_n, y_m) \end{bmatrix}.$$

Example:

Suppose X = {1, 2, 3} and Y = {1, 2, 3}. The binary fuzzy relation ``approximately equal" has a membership function

$$R(u,v) = \begin{cases} 1 & \mathit{if}\ u = v \\ 0.8 & \mathit{if}\ |u-v| = 1 \\ 0.3 & \mathit{if}\ |u-v| = 2 \end{cases}$$

Then

$$R(1,1) = R(2,2) = R(3,3) = 1$$

$$R(1,2) = R(2,1) = R(2,3) = R(3,2) = 0.8$$

$$R(1,3) = R(3,1) = 0.3$$

In matrix notation it can be represented as

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{pmatrix} \end{array}$$

Example:

Let $X=Y=R^+$ (+ve real line).
R = "y is much larger than x". The MF of the fuzzy relation R can be subjectively defined as:

$$\mu_R(x,y)=\begin{cases}\dfrac{y-x}{x+y+2}, & if\ y>x.\\ 0, & if\ y\le x.\end{cases}$$

If X = {3,4,5}, and Y = {3,4,5,6,7}, then *R* may be expressed as a relation matrix:

$$R=\begin{bmatrix}0 & 0.111 & 0.200 & 0.273 & 0.333\\ 0 & 0 & 0.091 & 0.162 & 0.231\\ 0 & 0 & 0 & 0.077 & 0.143\end{bmatrix}$$

Definition:

The domain of a binary fuzzy relation *R(X,Y)* is the fuzzy set *dom R(X,Y)* with the membership

$$\mu_{\mathrm{domR}}(x) = \max_{y \in Y} \mu_R(x, y) \qquad \text{for each } x \in X.$$

Definition:

The *range* of a binary fuzzy relation *R (X, Y)* is the fuzzy set "ran *R (X, Y)"* with the membership function

$$\mu_{\mathrm{ranR}}(y) = \max_{x \in X} \mu_R(x, y) \qquad \text{for each } y \in Y.$$

Definition:

The *height* of a fuzzy relation *R* is a number *H (R)* defined by

$$H(R) = \sup_{y \in Y} \sup_{x \in X} \mu_R(x, y).$$

1.2.9 Projection and Cylindrical Extension

Projection reduces a fuzzy set defined in a multi-dimensional domain (such as R^2) to a fuzzy set defined in a lower-dimensional domain (such as *R*).

Definition (Projection of a Fuzzy Set)

Let $U \subseteq U_1 \times U_2$ be a subset of a Cartesian product space, where U_1 and U_2 can themselves be Cartesian products of lower dimensional domains. The projection of fuzzy set *A* defined in U onto U_1 is the mapping defined by

$$\mathrm{proj}_{U_1}(A) = \left\{ \sup_{U_2} \mu_A(u)/u_1 \middle| u_1 \in U_1 \right\}$$

Example: Assume a fuzzy set *A* defined in $U \subset X \times Y \times Z$ with

$$X = \{x_1, x_2\}, Y = \{y_1, y_2\} \quad Z = \{z_1, z_2\}$$

as follows

$$A = \{\mu_1/(x_1, y_1, z_1),\ \mu_2/(x_1, y_2, z_1),\ \mu_3/(x_2, y_1, z_1),$$
$$\mu_4/(x_2, y_2, z_1),\ \mu_5/(x_2, y_2, z_2)\}$$

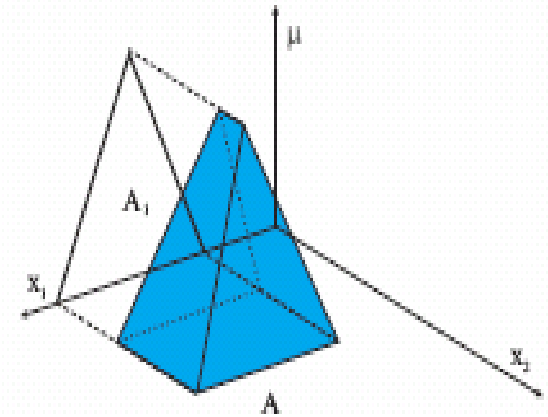
The projections of A onto X, Y and X × Y :

$$\mathrm{proj}_X(A) = \{\max(\mu_1, \mu_2)/x_1,\ \max(\mu_3, \mu_4, \mu_5)/x_2\},$$
$$\mathrm{proj}_Y(A) = \{\max(\mu_1, \mu_3)/y_1,\ \max(\mu_2, \mu_4, \mu_5)/y_2\},$$
$$\mathrm{proj}_{X \times Y}(A) = \{\mu_1/(x_1, y_1),\ \mu_2/(x_1, y_2),$$
$$\mu_3/(x_2, y_1),\ \max(\mu_4, \mu_5)/(x_2, y_2)\}.$$

Example:

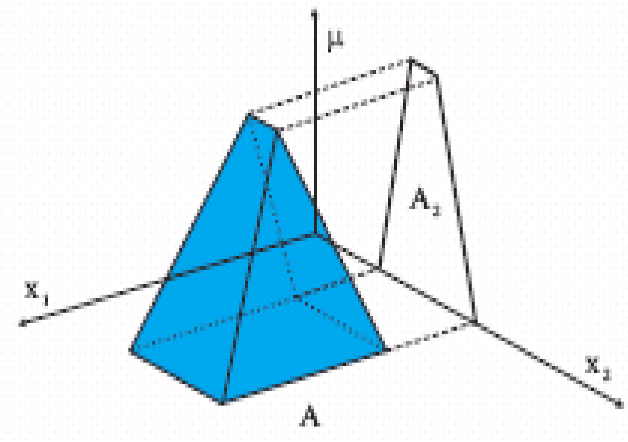
Projection onto X_1

$$\mathrm{proj}_{x_1}(A) = \{ \sup_{x_2 \in X_2} \mu_A(x_1, x_2))/x_1 \mid x_1 \in X_1 \}$$



Projection onto X_2

$$\mathrm{proj}_{x_2}(A) = \{ \sup_{x_1 \in X_1} \mu_A(x_1, x_2)/x_2) \mid x_2 \in X_2 \}$$



Since fuzzy relation is also a fuzzy set, so Projection also applies to fuzzy relation.

Definition

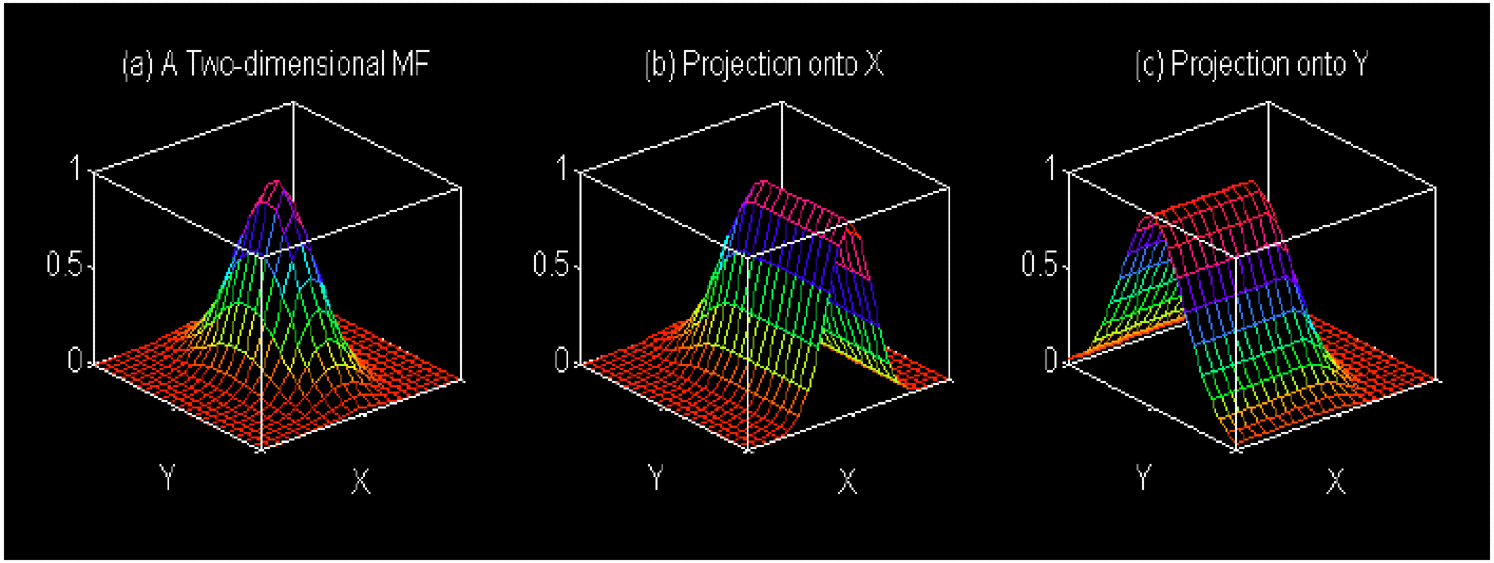
Let R be a binary fuzzy relation on X × Y . The projection of R with MF $\mu_R(x, y)$ onto X and Y are defined as

$$R_X = \int_X [\max_y \mu_R(x, y)] / x$$

$$R_Y = \int_Y [\max_x \mu_R(x, y)] / y$$

Example:

Two-dimensional MF | **Projection onto *X*** | **Projection onto *Y***

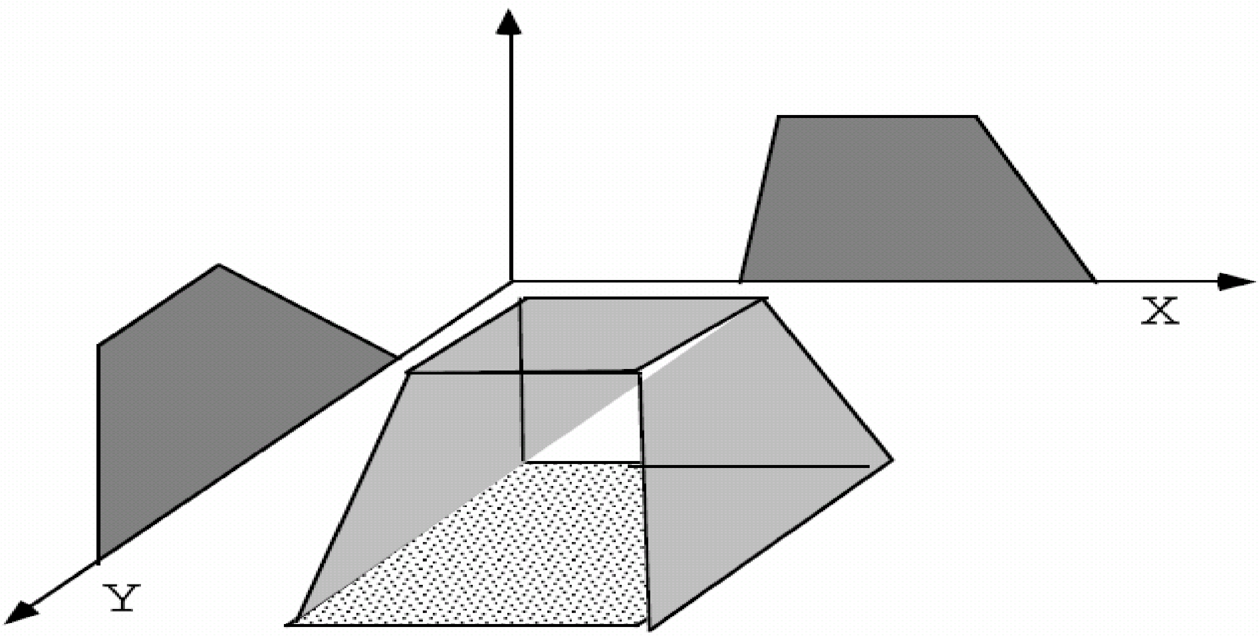


$\mu_R(x, y)$

$\mu_A(x) = \max_y \mu_R(x, y)$

$\mu_B(y) = \max_x \mu_R(x, y)$

Example:



Shadows of a fuzzy relation.

***Example**: Consider the relation*

$$R = \text{"}x \textit{ is considerable larger than } y\text{"}$$

$$= \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \left(\begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ 0.8 & 0.1 & 0.1 & 0.7 \\ 0 & 0.8 & 0 & 0 \\ 0.9 & 1 & 0.7 & 0.8 \end{array} \right)$$

Then the projection on X means that

- x_1 *is assigned the highest membership degree from the tuples* $(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4)$, *i.e.* $\Pi_X(x_1) = 0.8$, *which is the maximum of the first row.*
- x_2 *is assigned the highest membership degree from the tuples* $(x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4)$, *i.e.* $\Pi_X(x_2) = 0.8$, *which is the maximum of the second row.*
- x_3 *is assigned the highest membership degree from the tuples* $(x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4)$, *i.e.* $\Pi_X(x_3) = 1$, *which is the maximum of the third row.*

Cylindrical extension is the opposite operation, i.e., the extension of a fuzzy set defined in low-dimensional domain into a higher-dimensional domain.

Definition (Cylindrical Extension)

Let $U \subseteq U_1 \times U_2$ be a subset of a Cartesian product space, where U_1 and U_2 can themselves be Cartesian products of lower dimensional domains. The cylindrical extension of fuzzy set A defined in U_1 onto U is the mapping

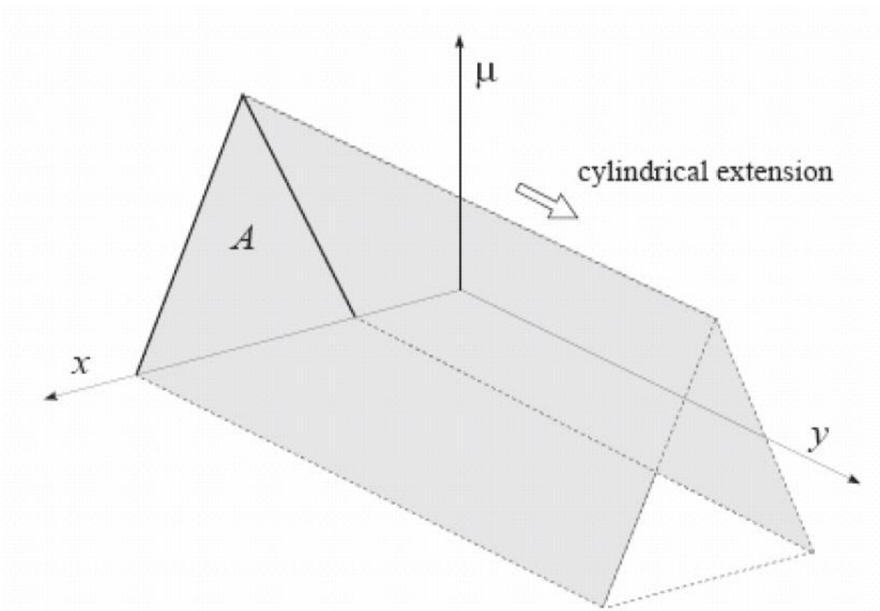
$$\text{ext}_U(A) = \left\{ \mu_A(u_1)/u \,\middle|\, u \in U \right\}$$

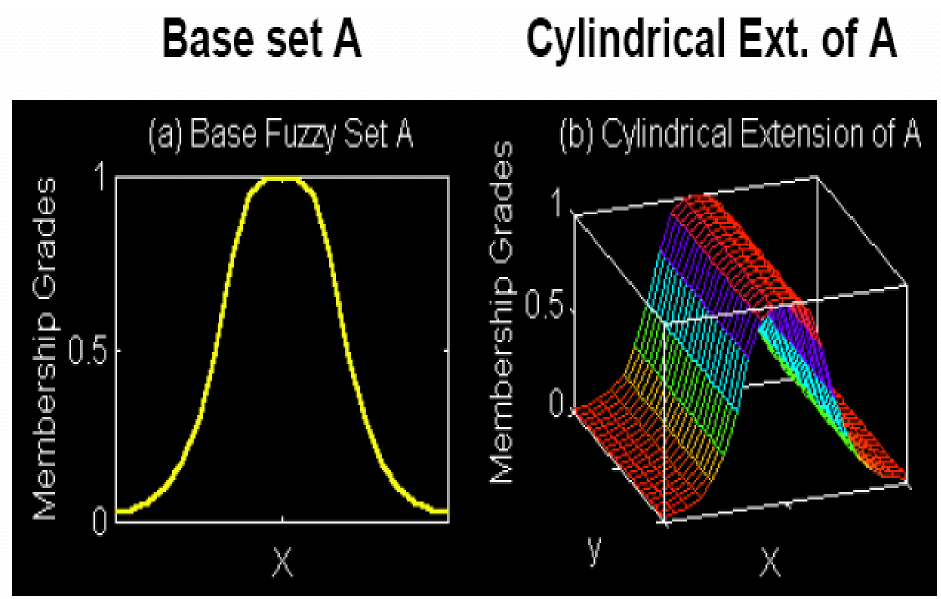
A is referred to as the base set.

Cylindrical extension thus simply replicates the membership degrees from the existing dimensions into the new dimensions from R to R^2.

Example:

The following figures show the cylindrical extension





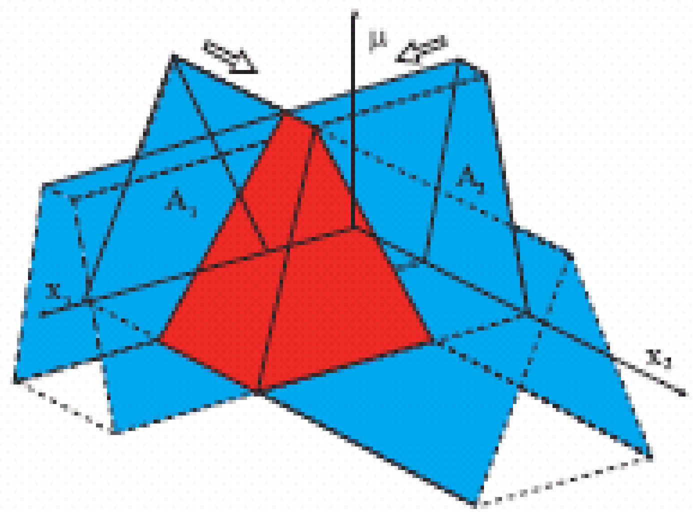
1.2.10 Intersection on Cartesian Product Space

- ***Set-theoretic operations*** such as the union or intersection applied to fuzzy sets defined in **different domains** result in a **multi-dimensional** fuzzy set in the **Cartesian product** of those domains.
- The operation is in fact performed by **first extending** the original fuzzy sets into the Cartesian product domain and **then computing the operation** on those multi-dimensional sets.

Example:

Consider two fuzzy sets A_1 and A_2 defined in domains X_1 and X_2, respectively. The intersection $A_1 \cap A_2$, also denoted by $A_1 \times A_2$ is given by:

$$A_1 \times A_2 = \text{ext}_{X_2}(A_1) \cap \text{ext}_{X_1}(A_2)$$



The cylindrical extension is usually considered *implicitly* and it is not stated in the notation:

$$\mu_{A_1 \times A_2}(x_1, x_2) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)$$

1.2.11 Operations on Fuzzy Relations

Definition *(Intersection of Relations)*

The intersection of *R* and *S* is defined by

$$(R \wedge S)(u, v) = \min\{R(u, v), S(u, v)\}$$

Note that *R*: X × Y → [0, 1], i.e. the domain of *R* is the whole Cartesian product X × Y .

Definition *(Union of Relations)*

The union of R and S is defined by

$$(R \vee S)(u,v) = \max\{R(u,v), S(u,v)\}$$

Example: Let us define two binary relations:

$R =$ *"x is considerable larger than y"*

$$= \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.8 & 0.1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{array}$$

$S =$ *"x is very close to y"*

$$= \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.4 & 0 & 0.9 & 0.6 \\ x_2 & 0.9 & 0.4 & 0.5 & 0.7 \\ x_3 & 0.3 & 0 & 0.8 & 0.5 \end{array}$$

The intersection of R and S means that
“x is considerable larger than y” **and** *“x is very close to y”.*

$$(R \wedge S)(x,y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ 0.4 & 0 & 0.1 & 0.6 \\ 0 & 0.4 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0.5 \end{array}$$

The union of R and S means that
"x is considerable larger than y" **or** *"x is very close to y".*

$$(R \vee S)(x,y) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.8 & 0.1 & 0.9 & 0.7 \\ x_2 & 0.9 & 0.8 & 0.5 & 0.7 \\ x_3 & 0.9 & 1 & 0.8 & 0.8 \end{array}$$

1.2.12 Composition of Relations

Let $R \in F(X \times Y)$ and $S \in F(Y \times Z)$ be two binary relations that share a common set Y. The ***composition*** of R and S is a relation in $X \times Z$. Using the membership functions of relations , we have an equivalent definition for composition that is given below.

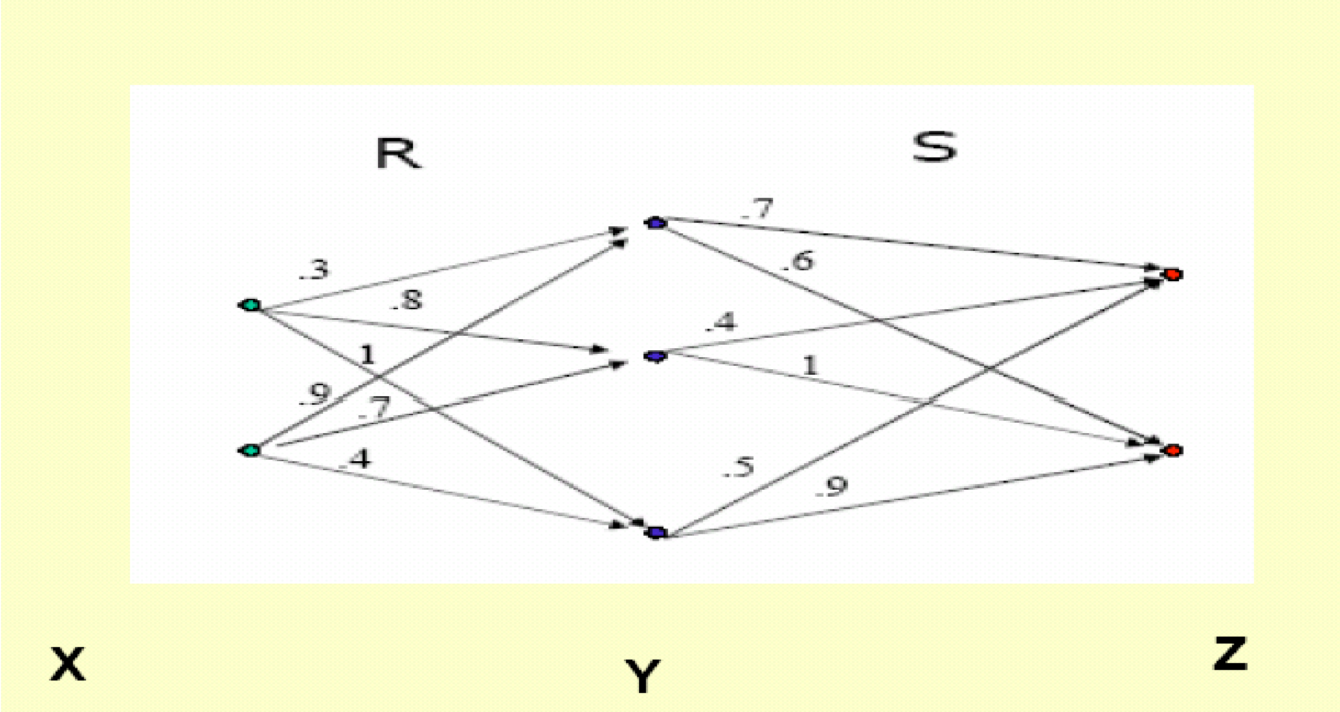
Definition *(sup-min composition of fuzzy relations)* Let $R \in F(X \times Y)$ and $S \in F(Y \times Z)$. The sup-min composition of R and S, denoted by $R \circ S$ is defined as

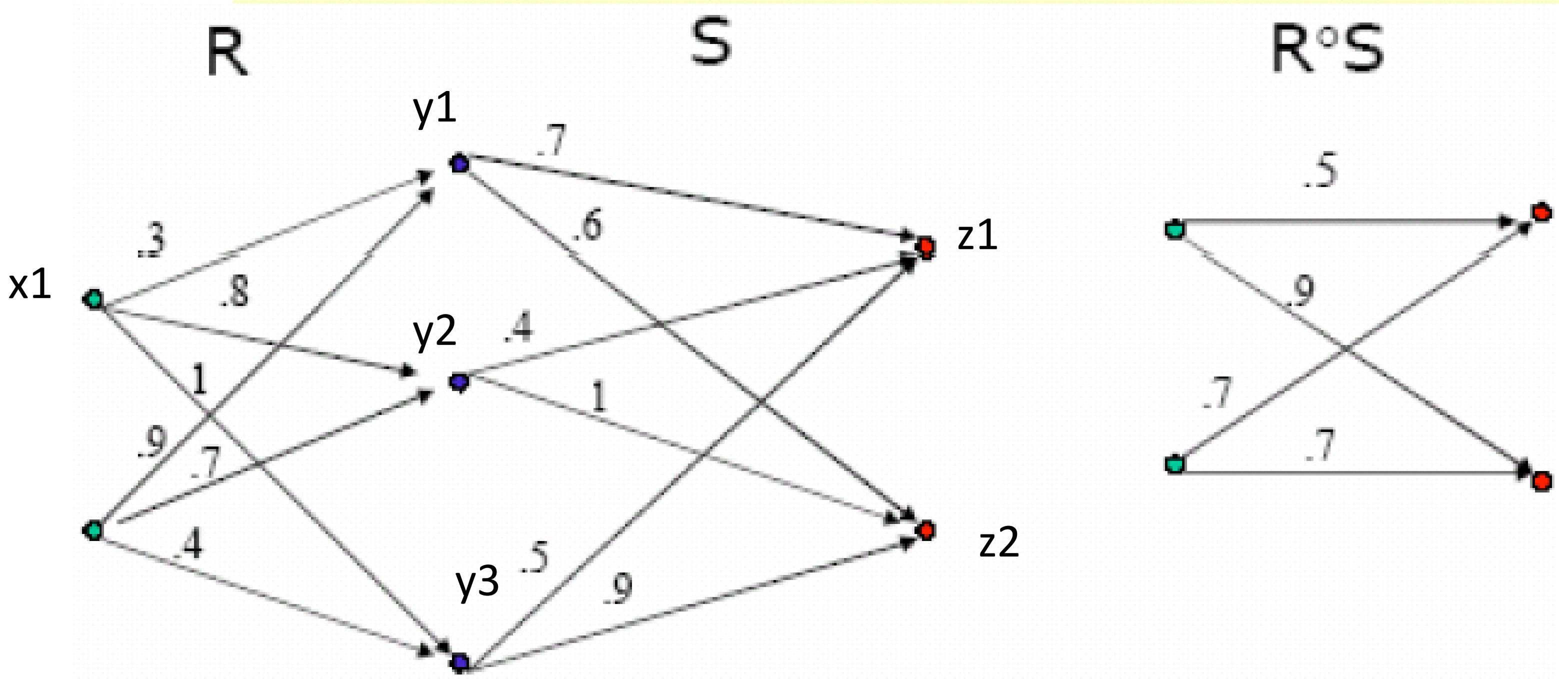
$$(R \circ S)(u, w) = \sup_{v \in Y} \min\{R(u, v), S(v, w)\}$$

The general rule when combining or ***composing*** fuzzy relations is:

- to pick the minimum fuzzy value in a `series connection’ ;
- and the maximum value in a `parallel connection.

Example





1.2.13 Composition of Fuzzy Relation with Fuzzy set

The *composition* is defined as follows:

Suppose there exists a fuzzy relation *R* in X × Y and *A* is a fuzzy set in X. Then, fuzzy subset *B* of Y can be induced by A through the composition of A and R:

$$B = A \circ R$$

i.e. the composition is defined by:

$$B = \mathrm{proj}_Y\left(R \cap \mathrm{ext}_{X\times Y}(A)\right).$$

The composition can be regarded in two phases: ***combination*** (intersection) and ***projection***.

This is actually the *sup-min* composition. Assume that *A* is a fuzzy set with membership function $\mu_A(x)$ and *R* is a fuzzy relation with membership function $\mu_R(x, y)$. Then

$$\mu_B(y) = \sup_x \min\left(\mu_A(x), \mu_R(x, y)\right),$$

***Example* :**

Consider a fuzzy relation *R* which represents the relationship “x is *approximately equal* to y”:

$$\mu_R(x, y) = \max(1 - 0.5 \cdot |x - y|, 0)$$

Further, consider a fuzzy set *A “approximately 5”*:

$$\mu_A(x) = \max(1 - 0.5 \cdot |x - 5|, 0)$$

$$\mu_B(y) = \overbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{\mu_A(x)} \circ \overbrace{\begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}}^{\mu_R(x,y)}$$

$$
= \max_x \overbrace{\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}}^{\min(\mu_A(x), \mu_R(x,y))}
$$

$$
= \overbrace{\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}}^{\max_x \min(\mu_A(x), \mu_R(x,y))}
$$

This results in a fuzzy set, defined in *Y* can be interpreted as *“approximately 5”*.

Note, however, that it is broader (more uncertain) than the set from which it was induced. This is because the uncertainty in the input fuzzy set was combined with the uncertainty in the relation.

12.14 Linguistic Variables and Hedges

- **Definition.** If a variable can take words in natural languages as its values, it is called a linguistic variable, where the words are characterized by fuzzy sets defined in the universe of discourse.

Example: The speed of a car is a variable x that takes values in the interval $[0, Vmax]$, where $Vmax$ is the maximum speed of the car. We can define three fuzzy sets "slow," "medium," and "fast" in $[0, Vmax]$. If we view x, $i.e$ speed, as a linguistic variable, then it can take "slow," "medium" and "fast" as its values called *linguistic values*.

Linguistic Hedges

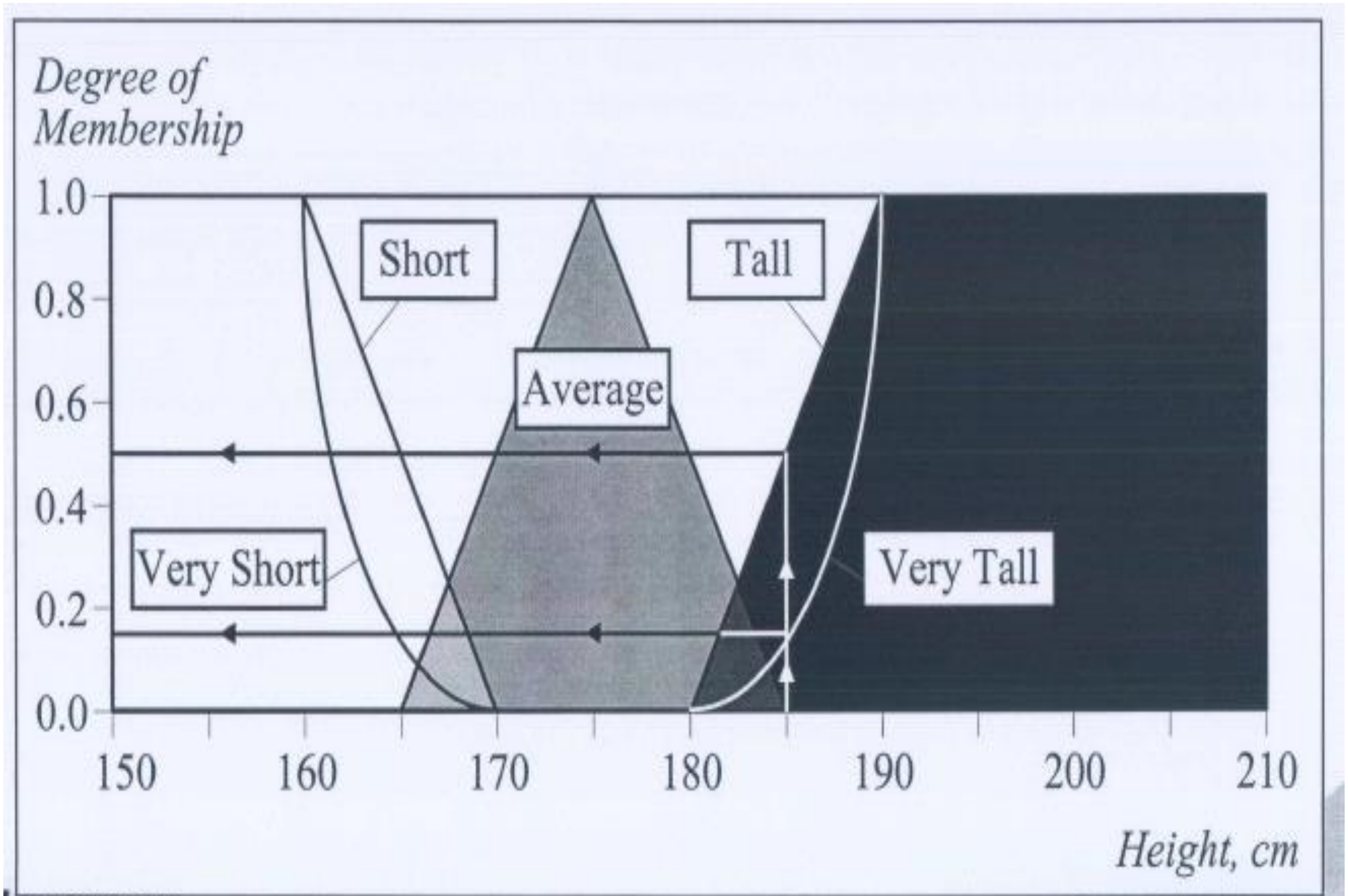
With the concept of linguistic variables, we are able to take words as values of (linguistic) variables. In our daily life, we often use more than one word to describe a variable. For example, if we view the speed of a car as a linguistic variable, then its values might be "not slow," "very slow," "slightly fast," "more or less medium," etc.

In general, these terms may be classified into three groups:

- Primary terms, which are labels of fuzzy sets, for examples `slow’, `medium’ and `fast’ ect.
- Complement "not" and connections "and" and "or.“
- Hedges, such as `very’, `slightly,’ `more or less,’ etc.

So Linguistic Hedges are terms that modify the shape of fuzzy sets. They include adverbs such as *very, somewhat, quite, more or less, slightly* .

Hedge	*Mathematical Expression*	*Graphical Representation*
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^{2}$	
Extremely	$[\mu_A(x)]^{3}$	


Degree of Membership
1.0
0.8
0.6
0.4
0.2
0.0
Short
Tall
Average
Very Short
Very Tall
150
160
170
180
190
200
210
Height, cm