Machine Vision

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More about 3D Rotation

$$\boldsymbol{R} = \begin{pmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{pmatrix}$$

The matrix $\boldsymbol{R_x}$ describes a rotation along the **X** axis by a certain angle α ***(pan angle)***. It holds that;

$$\boldsymbol{R_x}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

The matrix $\boldsymbol{R_y}$ describes a rotation along the **Y** axis by a certain angle β ***(tilt angle)***. It holds that;

$$\boldsymbol{R_y}(\beta) = \begin{pmatrix} \cos(\beta) & 0 & -sin(\beta) \\ 0 & 1 & 0 \\ sin(\beta) & 0 & \cos(\beta) \end{pmatrix}$$

The matrix $\boldsymbol{R_z}$ describes a rotation along the **Z** axis by a certain angle γ ***(roll angle)***. It holds that;

$$\boldsymbol{R_z}(\gamma) = \begin{pmatrix} \cos(\gamma) & sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Computation of 3D Rotation

- Rotation matrix is orthogonal

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$$

- The magnitude of each row or column is 1

$$r_1^2 + r_2^2 + r_3^2 = 1$$

Also

$\det(\boldsymbol{R}) = 1$	determinant
$\boldsymbol{R}^{-1} = \boldsymbol{R}^T$	the inverse rotation is the transpose
$\|\|\boldsymbol{Ra}\|\| = \|\|\boldsymbol{a}\|\|$	rotational transformation does not change the magnitude of a vector
$(\boldsymbol{Ra}, \boldsymbol{Rb}) = (\boldsymbol{a}, \boldsymbol{b})$	for arbitrary vectors **a** and **b** the length and angle are preserved by a rotation

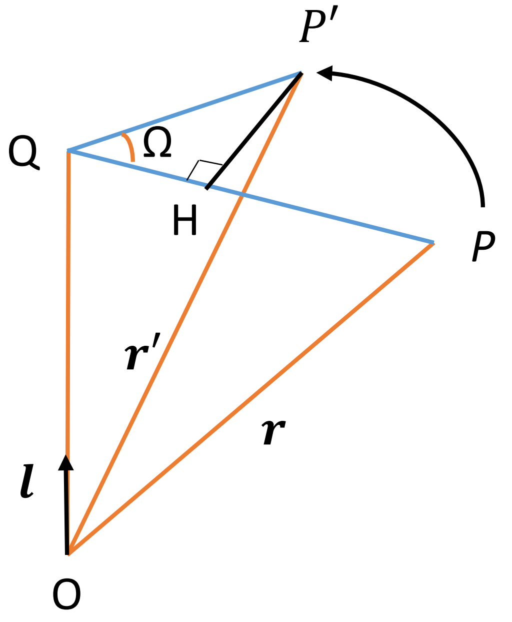
Computation of 3D Rotation – Euler's Theorem

1) Euler's theorem - Every rotation matrix represents a rotation around an axis by some angle

2) Let $\boldsymbol{l} = (l_1, l_2, l_3)^T$ denote the unit vector of the rotation axis, Ω denote the angle, the rotation matrix $\boldsymbol{R}$ can be expressed as;

$$\boldsymbol{R} = \begin{pmatrix} C + l_1^2 V & l_1 l_2 V - l_3 S & l_1 l_3 V + l_2 S \\ l_2 l_1 V + l_3 S & C + l_2^2 V & l_2 l_3 V - l_1 S \\ l_3 l_1 V - l_2 S & l_3 l_2 V + l_1 S & C + l_3^2 V \end{pmatrix}$$

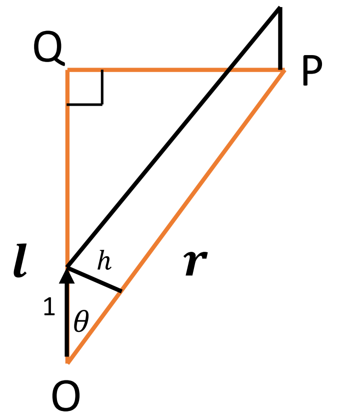
- Where $C = \cos(\Omega)$ and $V = 1 - \cos(\Omega)$



Computation of 3D Rotation

- *Proof.* Point $\boldsymbol{P}$ rotates by angle $\boldsymbol{\Omega}$ around $\boldsymbol{l}$, Let $\boldsymbol{P}'$ be the resulting new point. Let $\boldsymbol{Q}$ be the orthogonal projection of point $\boldsymbol{P}$ onto the axis. Then, $|\mathbf{OP}| = |\mathbf{OP}'|$ and $|\mathbf{QP}| = |\mathbf{QP}'|$. Let $\boldsymbol{H}$ be the orthogonal projection of point $\boldsymbol{P}'$ onto $\boldsymbol{QP}$. If we put $\mathbf{r} = \overrightarrow{\mathbf{OP}}$ and $\mathbf{r}' = \overrightarrow{\mathbf{OP}'}$ we see that

$$\boldsymbol{r}' = \overrightarrow{OQ} + \overrightarrow{QH} + \overrightarrow{HP'}$$



- Since $\overrightarrow{\mathbf{OQ}}$ is the orthogonal projection of vector $\boldsymbol{r}$ onto the axis by $\boldsymbol{l}$, we have

$$\overrightarrow{OQ} = (\boldsymbol{r}, \boldsymbol{l})\boldsymbol{l}$$

- Similarly, $\overrightarrow{\mathbf{QH}}$ is the orthogonal projection of vector $\overrightarrow{\mathbf{QP}'}$ onto $\overrightarrow{\mathbf{QP}}$. Noting that $|\mathbf{QP}| = |\mathbf{QP}'|$, we have

$$\overrightarrow{QH} = \frac{\overrightarrow{QP}}{|QP|}|QP'|cos\Omega = \overrightarrow{QP}cos\Omega = (\overrightarrow{OP} - \overrightarrow{OQ})cos\Omega = (\boldsymbol{r} - (\boldsymbol{r}, \boldsymbol{l})\boldsymbol{l})\,cos\Omega$$

Computation of 3D Rotation

- Vector $\overrightarrow{\mathbf{HP'}}$ is orthogonal to both $\boldsymbol{l}$ and $\boldsymbol{r}$, and has length of $|\mathbf{QP'}|\mathbf{sin\Omega}$. Noting that

$|QP'| = |QP| = ||\boldsymbol{l} \times \boldsymbol{r}||$, we have

$$\overrightarrow{HP'} = \frac{\boldsymbol{l} \times \boldsymbol{r}}{||\boldsymbol{l} \times \boldsymbol{r}||}|QP'|\text{sin}\Omega = \frac{|QP|}{||\boldsymbol{l} \times \boldsymbol{r}||}(\boldsymbol{l} \times \boldsymbol{r})sin\Omega$$

$$= \boldsymbol{l} \times \boldsymbol{r}\ sin\Omega$$

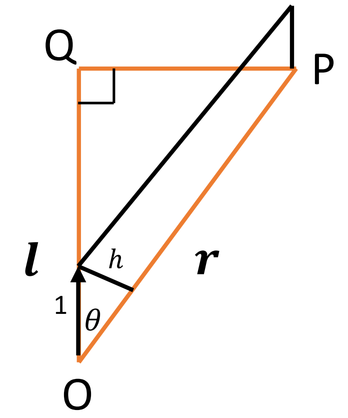
- Substituting $\overrightarrow{\boldsymbol{OQ}}$, $\overrightarrow{\boldsymbol{QH}}$ and $\overrightarrow{\mathbf{HP'}}$ into $\boldsymbol{r}' = \overrightarrow{OQ} + \overrightarrow{QH} + \overrightarrow{HP'}$, we have;

$$\boldsymbol{r}' = \boldsymbol{r}cos\Omega + \boldsymbol{l} \times \mathbf{r}\ \text{sin}\Omega + (1 - \cos\Omega)\ (\boldsymbol{l}, \boldsymbol{r})\boldsymbol{l}$$

- Rewriting this equation in matrix form as

$$\boldsymbol{r}' = \boldsymbol{R}\boldsymbol{r}$$

we obtain equation $\boldsymbol{R}$.



$$||\boldsymbol{l} \times \boldsymbol{r}|| = ||\boldsymbol{r}||h$$

$$\frac{h}{1} = sin\theta$$

$$||\boldsymbol{l} \times \boldsymbol{r}|| = ||\boldsymbol{r}||sin\theta = |QP|$$

Computation of 3D Rotation

Conversely, giving the rotation matrix, we can compute its axis $\boldsymbol{l}$ and the rotation angle.

3) The axis l and angle $\Omega(0\leq \Omega \leq \pi)$of rotation $R = (r_{ij})$ are given by

$$\Omega = \arccos \frac{trace\boldsymbol{R} \ - 1}{2}$$

$$l = N[\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}]$$

- *Proof*

$$\boldsymbol{R} = \begin{pmatrix} C + l_1^2 V & l_1 l_2 V - l_3 S & l_1 l_3 V + l_2 S \\ l_2 l_1 V + l_3 S & C + l_2^2 V & l_2 l_3 V - l_1 S \\ l_3 l_1 V - l_2 S & l_3 l_2 V + l_1 S & C + l_3^2 V \end{pmatrix} \longrightarrow trac\boldsymbol{R} = r_{11} + r_{22} + r_{33} = 1 + 2cos\Omega$$

$$\frac{1}{2}(\boldsymbol{R} - \boldsymbol{R}^T) = \begin{pmatrix} 0 & -l_3 \sin\Omega & l_2 \sin\Omega \\ -l_3 \sin\Omega & 0 & l_1 \sin\Omega \\ -l_2 \sin\Omega & l_1 \sin\Omega & 0 \end{pmatrix}$$

Computation of 3D Rotation

- Consider a rotational motion ***R(t)*** around a fixed axis l with a constant angular velocity ω. This rotational motion is specified by vector

$$\boldsymbol{w} = \omega \boldsymbol{l}$$

which is called the rotation velocity, the angular velocity is $\omega = ||\boldsymbol{w}||$, and the axis is $l = N[\boldsymbol{w}]$

QUATERNION REPRESENTATION

- We define a 4×1 vector $\mathbf{q} = (q_0, q_1, q_2, q_3)^T$ such that

$$||\mathbf{q}||^2 = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2 = 1$$

Rewrite (re-define) it as a pair

$$\mathbf{q} = (s, \mathbf{v})$$

and

$$s = q_0, \qquad \mathbf{v} = (q_1, q_2, q_3)^T$$

- A 3-D rotation can be represented as

$$s = \cos(\theta/2)$$
$$\mathbf{v} = \sin(\theta/2)\mathbf{u}$$

θ – angle rotated

$\mathbf{u}$ – rotation axis. (unit vector)

Note: $-\mathbf{q} = (-s, -\mathbf{v})$ represent the same rotation.

QUATERNION REPRESENTATION

- The conjugate of a quaternion is

$$\bar{\mathbf{q}} = (s, -\mathbf{v})$$

(similar to a complex number)

- The product of two quaternions $\mathbf{q}$, $\mathbf{q}'$ is defined as

$$\mathbf{q}\mathbf{q}' = (ss' - (\mathbf{v}, \mathbf{v}'),\ s\mathbf{v}' + s'\mathbf{v} + \mathbf{v} \times \mathbf{v}')$$

Given two rotations represented by $\mathbf{q}_1$ and $\mathbf{q}_2$, the product of the two rotations (apply rotation 2 first then rotation 1) corresponds to the products $\mathbf{q}_1\mathbf{q}_2$ and $-\mathbf{q}_1\mathbf{q}_2$.

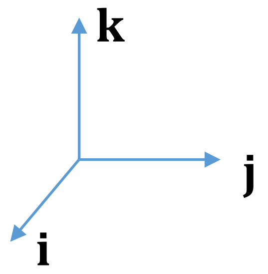
- Quaternion is not commutative:

$$\mathbf{q}_1\mathbf{q}_2 \neq \mathbf{q}_2\mathbf{q}_1$$

QUATERNION REPRESENTATION

- An easy way to perform the quaternion product is by the algebraic expression (Hamilton)

$$\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$



where

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

- Quaternion is associative:

$$(\mathbf{qq}')\,\mathbf{q}'' = \mathbf{q}(\mathbf{q}'\mathbf{q}'')$$

QUATERNION REPRESENTATION

- The inverse of quaternion is

$$\mathbf{q}^{-1} = \bar{\mathbf{q}} \,/\, ||\mathbf{q}||^2$$

- A rotation given by the orthogonal matrix $\boldsymbol{R}$ can be expressed using the quaternion notation

$$(0, \boldsymbol{R}\mathbf{m}) = \mathbf{q}(0, \mathbf{m})\bar{\mathbf{q}}$$

or in short

$$\boldsymbol{R}\mathbf{m} = \mathbf{q}\mathbf{m}\bar{\mathbf{q}}$$

The rotation matrix $\boldsymbol{R}$ is given as

$$\begin{pmatrix} {q_0}^2 + {q_1}^2 - {q_2}^2 - {q_3}^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_2q_1 + q_0q_3) & {q_0}^2 - {q_1}^2 + {q_2}^2 - {q_3}^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_3q_1 - q_0q_2) & 2(q_3q_2 + q_0q_1) & {q_0}^2 - {q_1}^2 - {q_2}^2 + {q_3}^2 \end{pmatrix}$$

QUATERNION REPRESENTATION

- **Problem**— Given two sets of unit vectors $\mathbf{m}_i$ and $\mathbf{m}'_i$, $i = 1, 2, \dots, N$, compute a rotation $\boldsymbol{R}$ such that

$$\mathbf{m}_i = \boldsymbol{R}\mathbf{m}'_i \, , i = 1, 2, \dots, N$$

This can be solved using least squares:

$$\sum_i^N W_i ||\mathbf{m}_i - \boldsymbol{R}\mathbf{m}'_i||^2 \rightarrow min$$

for positive weights W_i.

A close form solution can be found in terms of maximizing

$$trace(\boldsymbol{R}^T K) \rightarrow max$$

where K is the correlation matrix

$$K = \sum_i^N W_i \, \mathbf{m}_i \mathbf{m}'^T_i \qquad\qquad (3 \times 3)$$

QUATERNION REPRESENTATION

- Given correlation matrix K, define a four-dimensional symmetric matrix

$\widehat{K} =$

$$\begin{pmatrix} K11 + K22 + K33 & K32 - K23 & K13 - K31 & K21 - K12 \\ K32 - K23 & K11 - K22 - K33 & K12 + K21 & K31 + K13 \\ K13 - K31 & K12 + K21 & -K11 + K22 - K33 & K23 + K32 \\ K21 - K12 & K31 + K13 & K23 + K32 & -K11 - K22 + K33 \end{pmatrix}$$

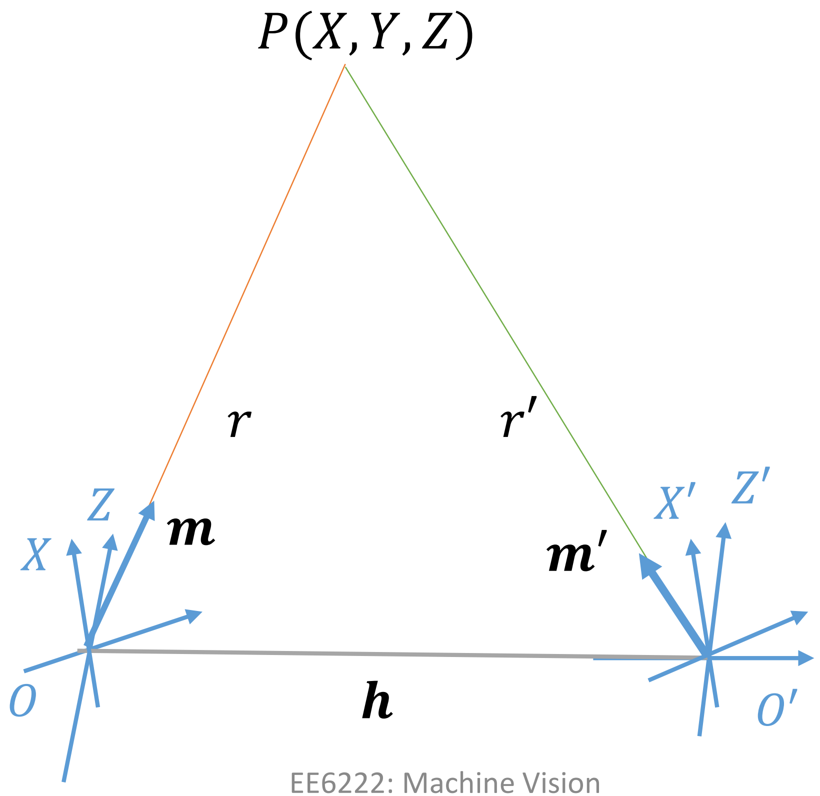
Let $\hat{\mathbf{q}}$ be the four-dimensional unit eigenvector of $\widehat{K}$ for the largest eigenvalue. Then,

$trace(\boldsymbol{R}^T K)$ is maximized by the rotation $\boldsymbol{R}$ represented by $\hat{\mathbf{q}}$.

The solution is unique if the largest eigen value of $\widehat{K}$ is a simple root.

Motion Parallax

- A Translation of the camera causes an effective translation of the object relative to the camera, and the resulting image motion of points and lines relative their 3D geometries. This fact is known as *motion parallax*.



Motion Parallax

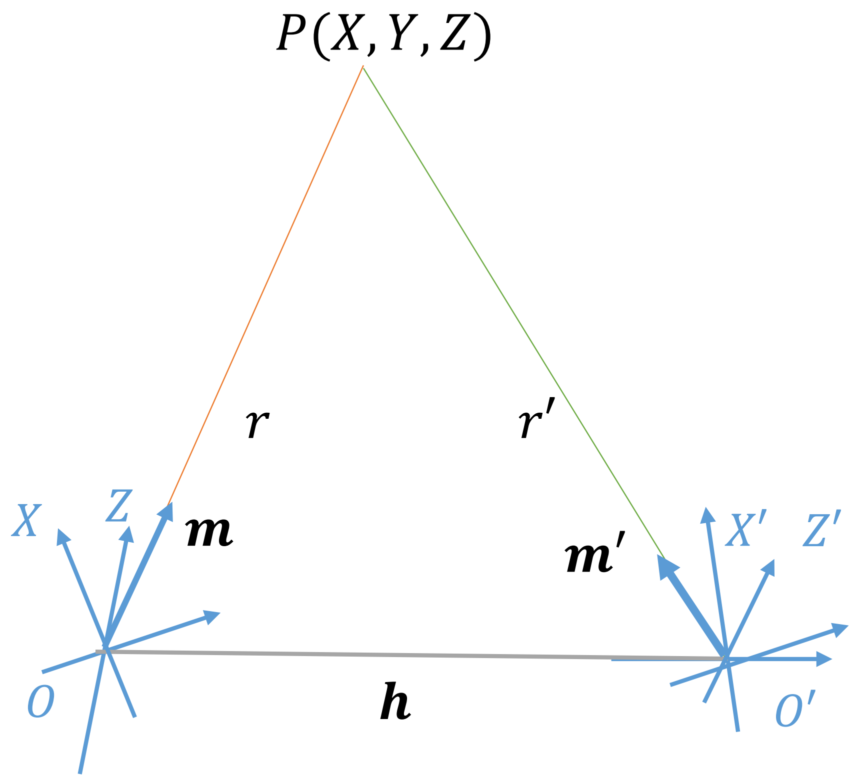
- The Camera Translates from ***O*** to ***O'*** with no rotation:

$$\overrightarrow{OO'} = \boldsymbol{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

- A world point ***P*** has observed twice as *(x,y)* and *(x',y')*
  For convenience, we convert the image points into unit vectors:

$$\boldsymbol{m} = \frac{1}{\sqrt{x^2 + y^2 + f^2}} \begin{pmatrix} x \\ y \\ f \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

$$\boldsymbol{m}' = \frac{1}{\sqrt{x'^2 + y'^2 + f'^2}} \begin{pmatrix} x' \\ y' \\ f' \end{pmatrix} = \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix}$$



Motion Parallax

- Now, we can recover the distance r or r' (scalars) from the following vector relation:

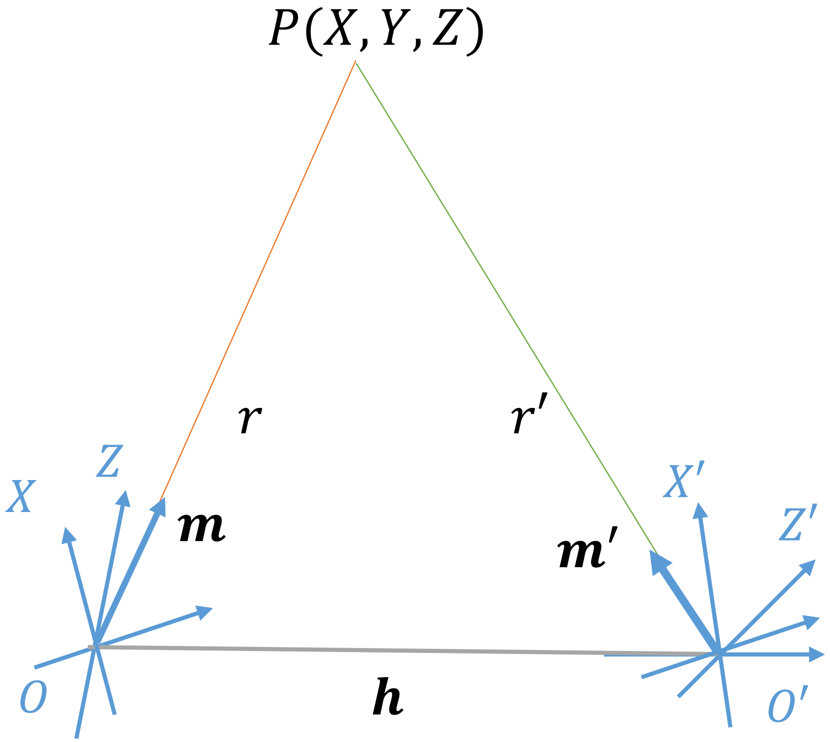
$$\overrightarrow{OO'} + \overrightarrow{O'P} = \overrightarrow{OP}$$

$$\boldsymbol{h} + r'\boldsymbol{m}' = r\boldsymbol{m}$$

- In fact there are three equations here, being:

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} + r' \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = r \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

- ***r*** (or ***r'***) can be solved by picking any 2 of the 3 equations



Motion Parallax - Parallel Stereo

- Parallel Stereo is a special case of motion parallax.
- The world point can be expressed as *(X,Y,Z)=(rm₁ ,rm₂ ,rm₃)*.

$$\overrightarrow{OO'} + \overrightarrow{O'P} = \overrightarrow{OP}$$

$$\boldsymbol{h} + r'\boldsymbol{m}' = r\boldsymbol{m}$$

- We are looking for the depth

$$Z = rm_3$$

- Let $\boldsymbol{h} = (h_1, 0{,}0)$, because the camera translated only along the X axis. Rewrite $\boldsymbol{h} + r'\boldsymbol{m}' = r\boldsymbol{m}$, we have,

$$\begin{cases} h_1 + r'm'_1 = rm_1 \\ r'm'_2 = rm_2 \\ r'm'_3 = rm_3 \end{cases} \qquad r' = r\frac{m_3}{m'_3}$$

$$h_1 = rm_1 - r'm'_1 = rm_1 - \left(r\frac{m_3}{m'_3}\right)m'_1 = rm_3\left(\frac{m_1}{m_3} - \frac{m'_1}{m'_3}\right) = Z\left(\frac{m_1}{m_3} - \frac{m'_1}{m'_3}\right) \Rightarrow Z = \frac{h_1}{\frac{m_1}{m_3} - \frac{m'_1}{m'_3}}$$

Parallel Stereo

- From the perspective projection equation we know,

$$\boldsymbol{x_{left}} = \frac{m_1}{m_3} f$$

$$\boldsymbol{x_{right}} = \frac{m'_1}{m'_3} f$$

$$Z = \frac{h_1 f}{x_{left} - x_{right}}$$

- With h_1 being the effective baseline.

Motion Parallax - using 3-equations

- We can compute motion parallax by using all 3 equations

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = r\boldsymbol{m} - r'\boldsymbol{m}' - \boldsymbol{h}$$

- In the presence of noise, ***a*** may not be a zero-vector (error). So, we proceed to look for the minimum value of $\|\boldsymbol{a}\|^{\mathbf{2}}$ in an attempt to find the optimal solution for *r* and *r'*. Define the residual E as,

$$E = \boldsymbol{a}^T\boldsymbol{a} = (a_1, a_2, a_3)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \ (= \|\boldsymbol{a}\|^{\mathbf{2}})$$

- Take the first derivative of E with respect to r :

$$\frac{\partial E}{\partial r} = \boldsymbol{a}^T \frac{\partial \boldsymbol{a}}{\partial r} + \frac{\partial(\boldsymbol{a}^T)}{\partial r}\boldsymbol{a} = \boldsymbol{a}^T\boldsymbol{m} + \boldsymbol{m}^T a = \boldsymbol{a}^T\boldsymbol{m} + \boldsymbol{a}^T\boldsymbol{m} = 2\ \boldsymbol{a}^T\boldsymbol{m}$$

Motion Parallax - using 3-equations

- Substitute $\boldsymbol{a} = r\boldsymbol{m} - r'\boldsymbol{m}' - \boldsymbol{h}$ into $\frac{\partial E}{\partial r} = 2\,\boldsymbol{a}^T\boldsymbol{m}$ and set it to zero, we get;

$$r - r'(\boldsymbol{m}, \boldsymbol{m}') - (\boldsymbol{h}, \boldsymbol{m}) = 0$$

- Repeat the above steps for r', we get;

$$r(\boldsymbol{m}, \boldsymbol{m}') - r' - (\boldsymbol{h}, \boldsymbol{m}') = 0$$

- r and r' can be solved from the two equations as:

$$r = \frac{(\boldsymbol{h}, \boldsymbol{m}) - (\boldsymbol{m}, \boldsymbol{m}')(\boldsymbol{h}, \boldsymbol{m}')}{1 - (\boldsymbol{m}, \boldsymbol{m}')^2} \qquad r' = \frac{(\boldsymbol{m}, \boldsymbol{m}')(\boldsymbol{h}, \boldsymbol{m}) - (\boldsymbol{h}, \boldsymbol{m}')}{1 - (\boldsymbol{m}, \boldsymbol{m}')^2}$$

Motion Parallax - with Rotation

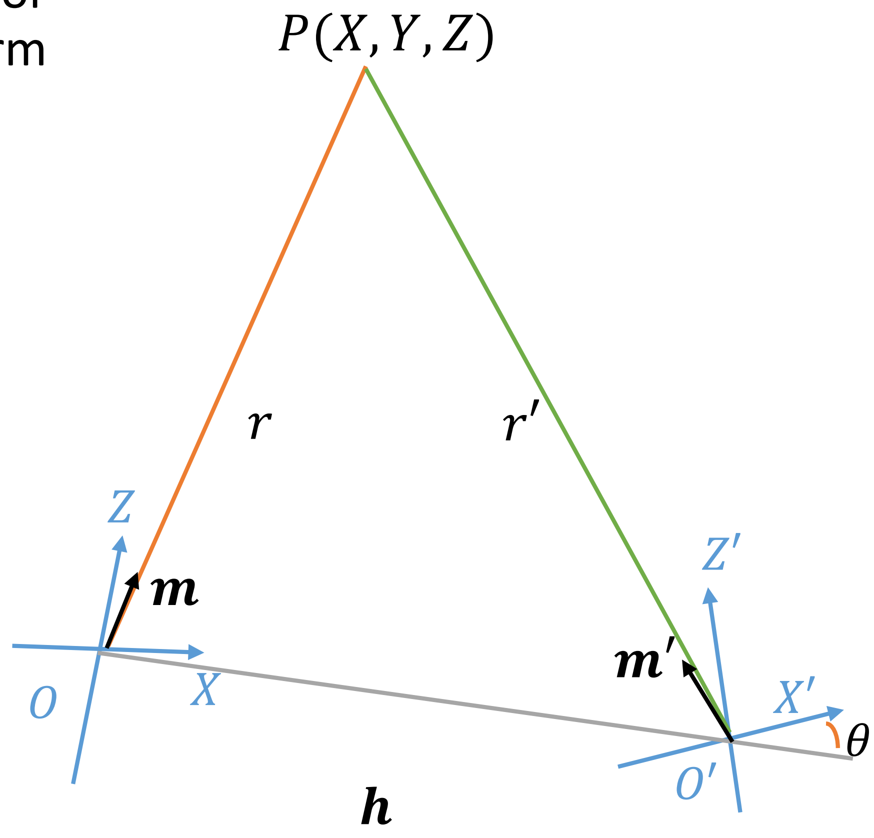
- Full 3D motion can be described by Translation and Rotation. For example, when the camera is mounted in a vehicle or on a robot arm (manipulator), the motion is no longer a pure translation.
- Suppose the camera undergoes translation ***h*** and a rotation ***R*** (3x3 matrix) , the following is true:

$$r'\boldsymbol{m}' = \boldsymbol{R}^{-1}(r\boldsymbol{m} - \boldsymbol{h})$$
$$\boldsymbol{R}(r'\boldsymbol{m}') = r\boldsymbol{m} - \boldsymbol{h}$$
$$r'(\boldsymbol{R}\boldsymbol{m}') = r\boldsymbol{m} - \boldsymbol{h}$$

- This is equivalent to substitute ***m'*** with ***Rm'*** in the motion parallax equation, hence

$$\mathrm{r} = \frac{(\boldsymbol{h},\boldsymbol{m}) - (\boldsymbol{m},\boldsymbol{R}\boldsymbol{m}')(\boldsymbol{h},\boldsymbol{R}\boldsymbol{m}')}{1-(\boldsymbol{m},\boldsymbol{R}\boldsymbol{m}')^2}$$

$$\mathrm{r}' = \frac{(\boldsymbol{m},\boldsymbol{R}\boldsymbol{m}')(\boldsymbol{h},\boldsymbol{m}) - (\boldsymbol{h},\boldsymbol{R}\boldsymbol{m}')}{1-(\boldsymbol{m},\boldsymbol{R}\boldsymbol{m}')^2}$$



*The camera undergoes translation **h** (with reference to the first frame) and rotation R. The simplified case in this figure shows that the rotation is against the X axis with angle θ. The vector $\overrightarrow{O'P} = r'm'$ is equally rotated R^{-1} with reference to the new frame.*

Vanishing points & vanishing lines

1. Projections of parallel space lines meet at a common "vanishing point" on the image plane; or the *vanishing point* of a space line is the limit of the projection of a point that moves along the space line indefinitely in one direction.
2. A space line extending along unit vector $\mathbf{m}$ has, when projected a *vanishing point* of N -vector $\pm\mathbf{m}$.
3. A planar surface of unit surface normal $\mathbf{n}$ has, when projected, a vanishing line of N -vector $\pm\mathbf{n}$.
4. Projections of planar surfaces that are parallel in the scene define a common vanishing line.

- In summary

1) if a vanishing point is detected on the image plane, its N -vector indicates the 3-D orientation of the corresponding space line;
2) if a vanishing line is detected on the image plane, its N -vector indicates the surface normal to the corresponding planar surface. This 3-D interpretation of vanishing points and vanishing lines plays an essential role in 3-D scene analysis for machine vision.

Vanishing points & vanishing lines

Example: Show that if a planar surface in the scene with unit normal $\mathbf{n} = (n_1, n_2, n_3\ )^T$ is not parallel to the image plane, its vanishing line is

$$n_1 x + n_2 y + n_3\ f\ = 0$$

The equation of the planar surface passing through (X_0, Y_0, Z_0) and having unit surface normal $\mathbf{n}$ is

$$n_1(X - X_0) + n_2(Y - Y_0) + n_3\ (Z - Z_0) = 0$$

The scene coordinates (X, Y, Z) and image coordinates (x, y) are related by the projection equations, that is

$$X = xZ/f, \qquad\qquad Y = yZ/f.$$

Substituting these into the surface equation, we obtain

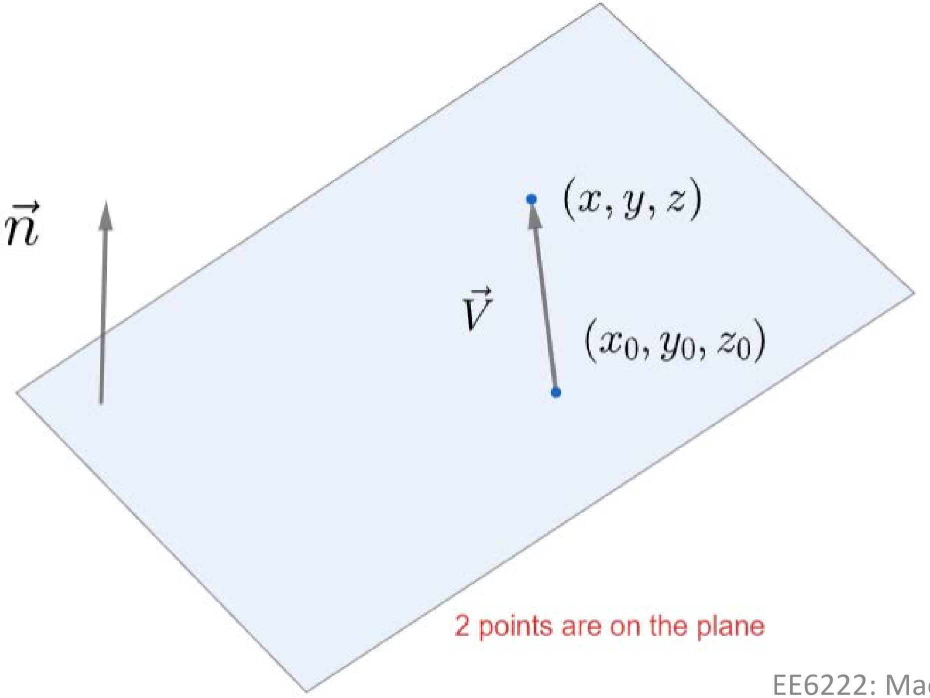
$$n_1 x + n_2 y + n_3\ f\ = f\,\frac{n_1 X_0 + n_2 Y_0 + n_3 Z_0}{Z}$$

Vanishing points & vanishing lines

The image coordinates (x, y) of all the image points for which the Z-coordinate of the corresponding space points is infinity satisfy $(Z \ \rightarrow \pm\infty)$ satisfy, irrespective of (X_0, Y_0, Z_0),

$$n_1 x + n_2 y + n_3\, f = 0$$

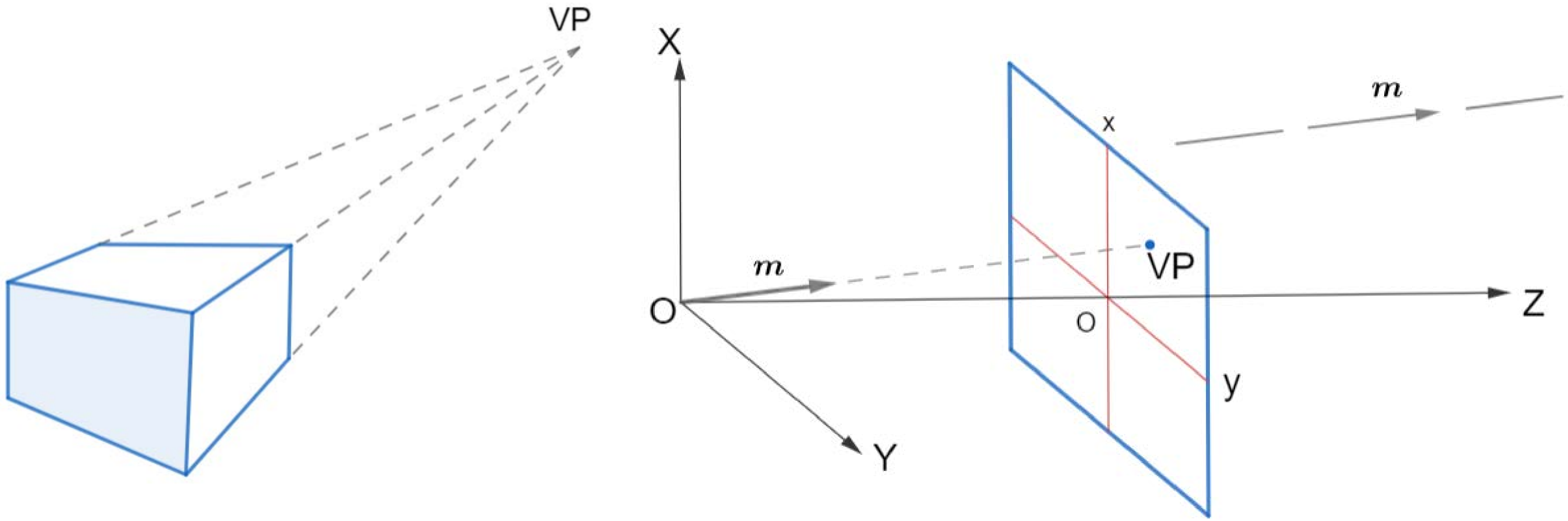
which defines the vanishing line on the image plane. Since the plane is not parallel to the image plane, n_1 and n_2 are not both zero.



$$\vec{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\vec{V} \perp \vec{n}$$

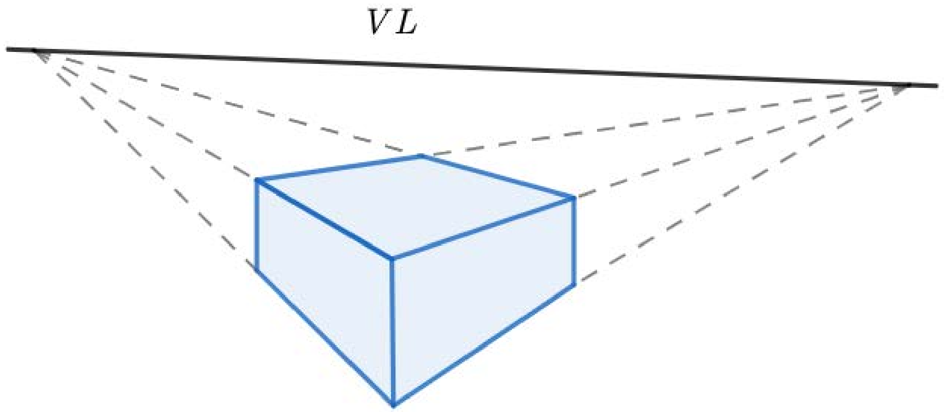
Vanishing points & vanishing lines



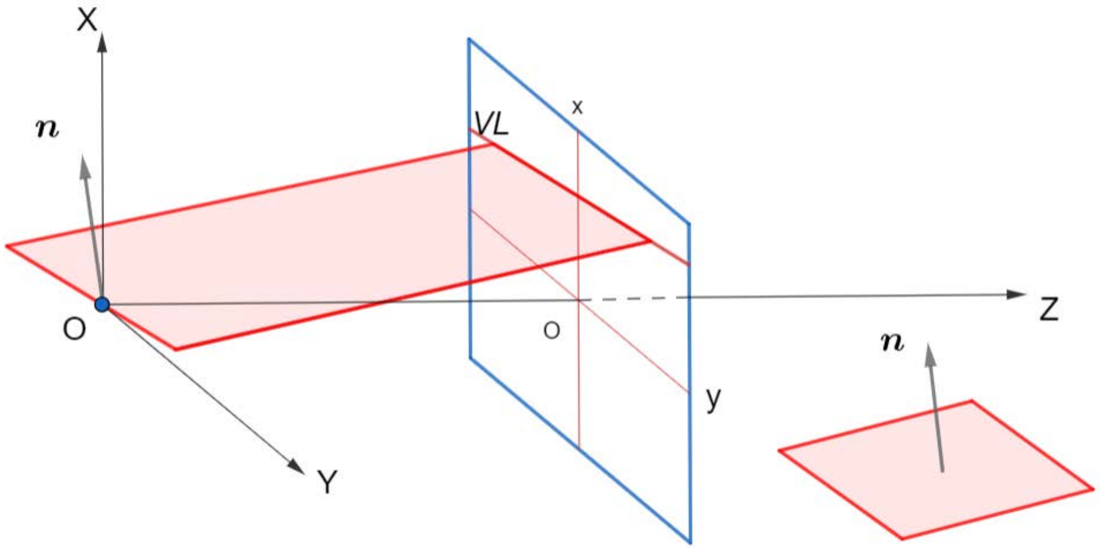
(a) (b)

Fig. 2.3 (a) Vanishing point. (b) The N-vector m of the vanishing point indicates the 3-D orientation of the line.

Vanishing points & vanishing lines



(a)



(b)

Fig. 2.4 (a) Vanishing line. (b) The N-vector n of the vanishing line indicates the unit normal to the surface.

Cross Ratio

- Let A , B , C and D be distinct points on line l. Their *cross ratio* $[ABCD]$ is defined by

$$[ABCD] = \frac{AC}{BC} / \frac{AD}{BD}$$

where AC, BC... are signed distances with respect to an arbitrary fixed orientation of the line l, hence

$$AC = -CA, \ldots$$

- The following relations are obvious

$$[ABCD] = [BADC] = [CDAB] = [DCBA]$$

$$[ABDC] = 1/[ABCD]$$

$$[ACBD] = 1 - [ABCD]$$

$$[ACDB] = \frac{1}{1 - [ABCD]}$$

$$[ADBC] = \frac{[ABCD] - 1}{[ABCD]}$$

- The cross ratio of four collinear space points is equal to the cross ratio of their projections on the image plane. (perspective invariance of cross ratio)

Cross Ratio

Show: $[ACBD] = 1 - [ABCD]$

1) $[ACBD] = \frac{AB}{CB} / \frac{AD}{CD} = \frac{AB \cdot CD}{CB \cdot AD}$

2) 1- $[ABCD] = 1 - \frac{AC}{BC} / \frac{AD}{BD}$

$$= 1 - \frac{AC \cdot BD}{BC \cdot AD} = 1 + \frac{AC \cdot BD}{CB \cdot AD} = \frac{CB \cdot AD + AC \cdot BD}{CB \cdot AD}$$

$$= [\mathrm{CB} \cdot (\mathrm{AB} + \mathrm{BD}) + (\mathrm{AB} + \mathrm{BC}) \cdot \mathrm{BD}] / \mathrm{CB} \cdot \mathrm{AD}$$

$$= [\mathrm{AB} \cdot (\mathrm{CB} + \mathrm{BD}) + \mathrm{BD}(\mathrm{CB} + \mathrm{BC})] / \mathrm{CB} \cdot \mathrm{AD}$$

$$= [\mathrm{AB} \cdot (\mathrm{BD} - \mathrm{BC})] / \mathrm{CB} \cdot \mathrm{AD}$$

$$= [\mathrm{AB} \cdot \mathrm{CD}] / \mathrm{CB} \cdot \mathrm{AD}$$

Cross Ratio

$$1 - \frac{AC \cdot BD}{BC \cdot AD}$$

$$= \frac{BC \cdot AD - AC \cdot BD}{BC \cdot AD}$$

$$= \frac{CB \cdot AD + AC \cdot BD}{CB \cdot AD}$$

$$= \frac{CB \cdot (AB + BD) + (AB + BC) \cdot BD}{CB \cdot AD}$$

$$= \frac{CB \cdot AB + CB \cdot BD + AB \cdot BD + BC \cdot BD}{CB \cdot AD}$$

$$= \frac{AB \cdot (CB + BD) + (CB + BC) \cdot BD}{CB \cdot AD}$$

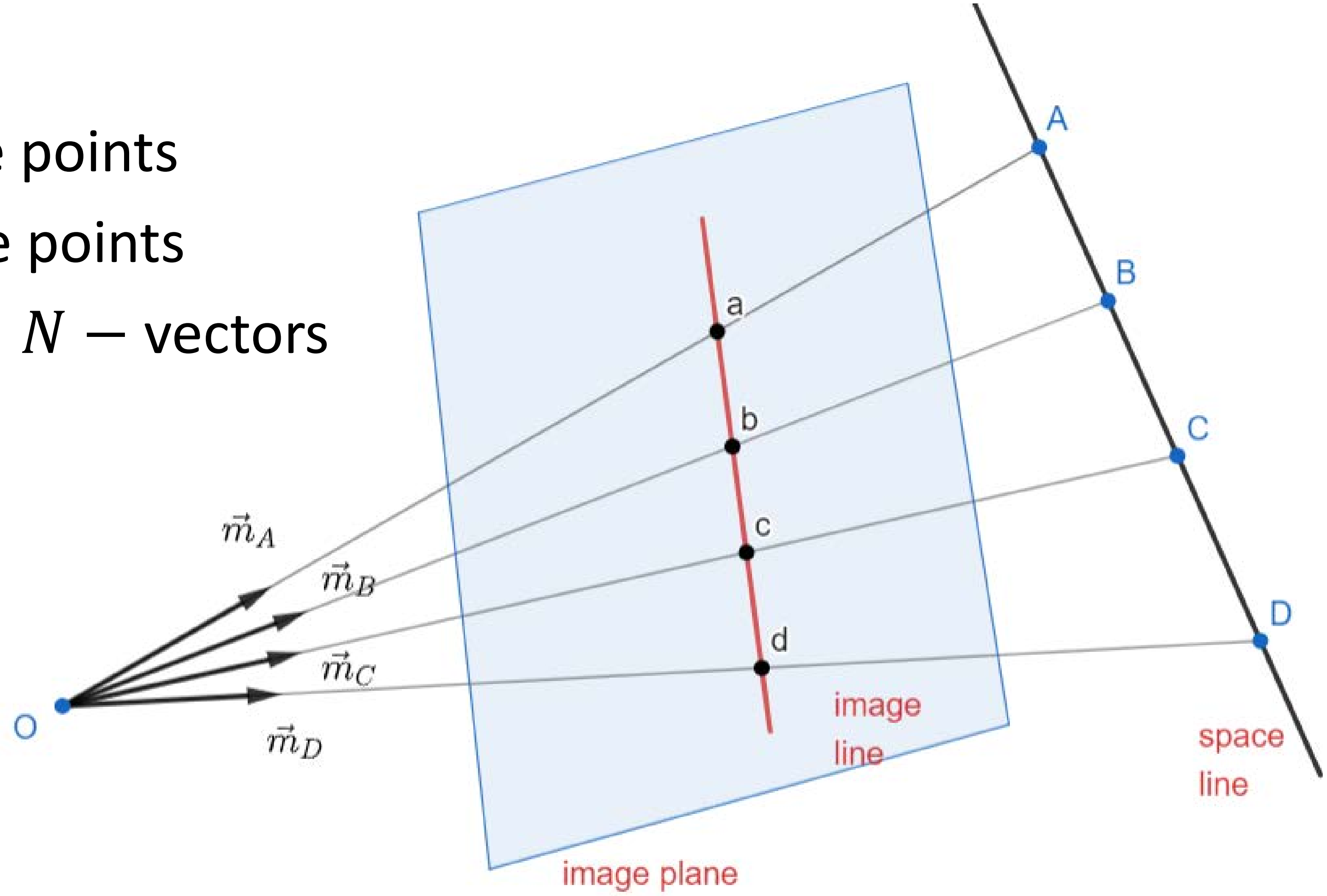
$$= \frac{AB \cdot CD}{CB \cdot AD}$$

Computation of Cross Ration

* A, B, C & D space points

  a, b, c & d image points

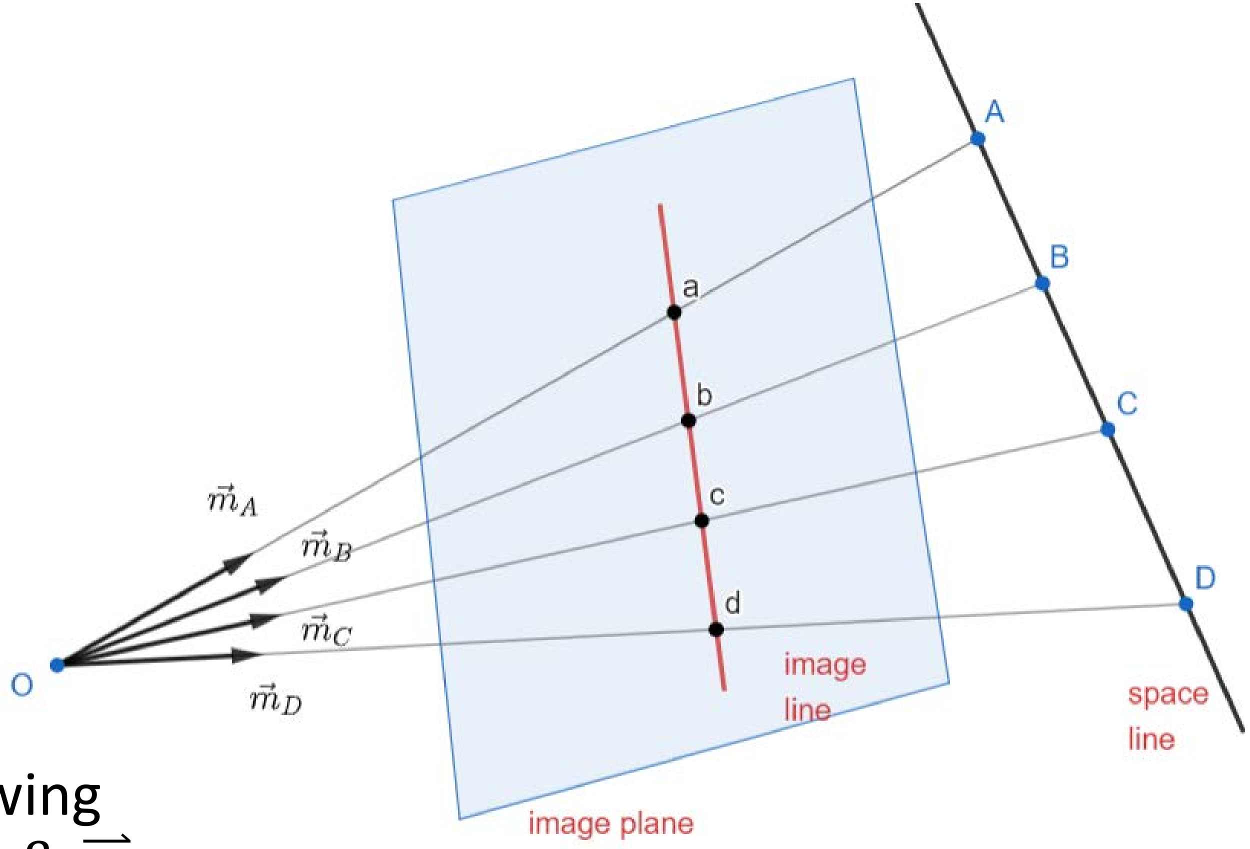
  $\vec{m}_A, \vec{m}_B, \vec{m}_C$ & $\vec{m}_D$ $N-$vectors



Computation of Cross Ration

- Given 4 collinear points on the image
- The cross ratio can be found as

$$[ABCD] = \frac{||\vec{m}_A \times \vec{m}_C||}{||\vec{m}_B \times \vec{m}_C||} / \frac{||\vec{m}_A \times \vec{m}_D||}{||\vec{m}_B \times \vec{m}_D||}$$



1) Knowing a, b, c & d is equivalent to knowing $\vec{m}_A, \vec{m}_B, \vec{m}_C$ & $\vec{m}_D$

using the normalization operator $N[\cdot]$

Computation of Cross Ration

2) Draw a line $\mathrm{O}p$; make sure p is on the image line and $\mathrm{O}p \perp ad$

Let $\mathrm{O}p = h$

3) Consider the area of the triangle $\mathrm{O}ac$

$$\frac{1}{2} h \cdot |ac| = \frac{1}{2}\ ||\overrightarrow{\mathrm{O}a} \times \overrightarrow{\mathrm{O}c}||$$

where $||\cdot||$ represents the magnitude of the vector.

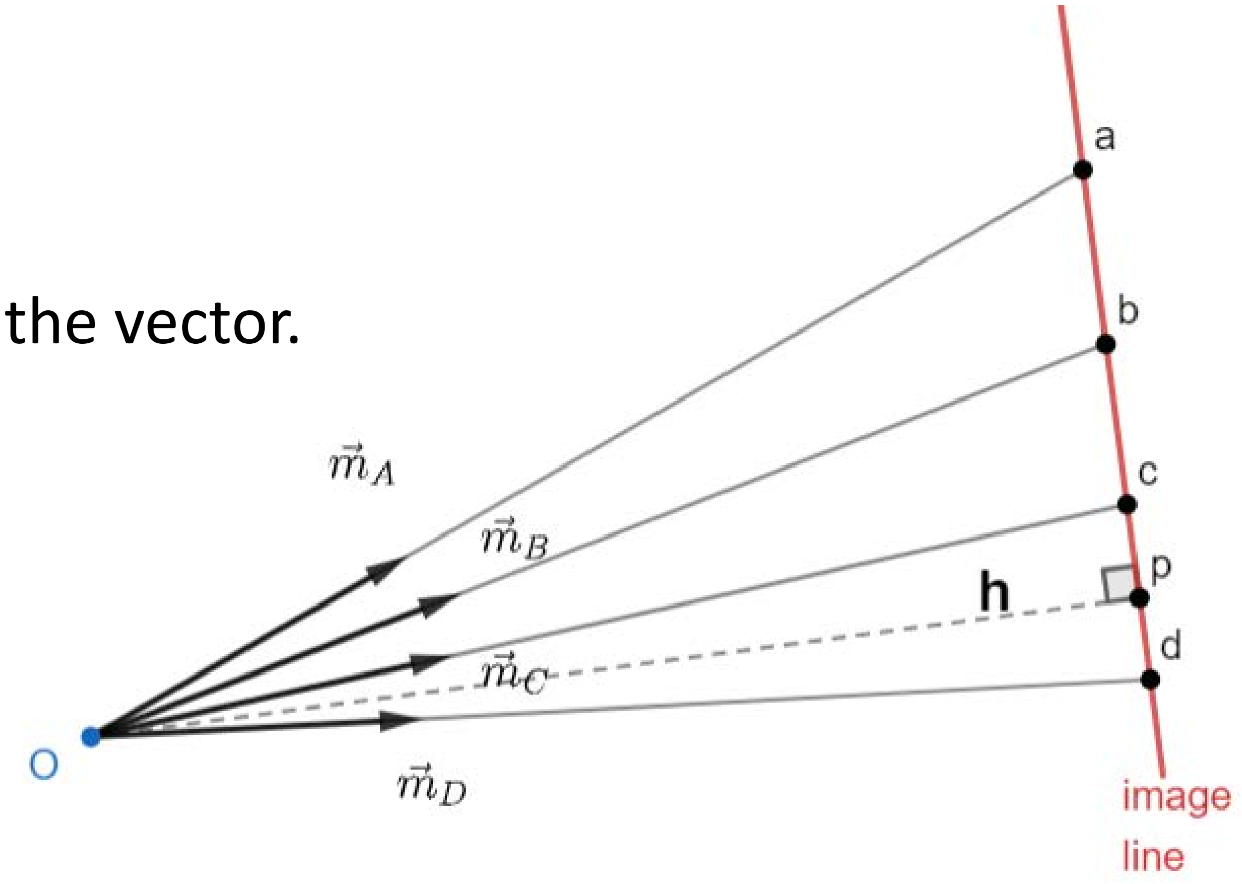
4) $|ac| = \frac{1}{h}\ ||\overrightarrow{\mathrm{O}a} \times \overrightarrow{\mathrm{O}c}||$

similarly

$$|bc| = \frac{1}{h}\ ||\overrightarrow{\mathrm{O}b} \times \overrightarrow{\mathrm{O}c}||$$

$$|ad| = \ \ldots$$

$$|bd| = \ \ldots$$



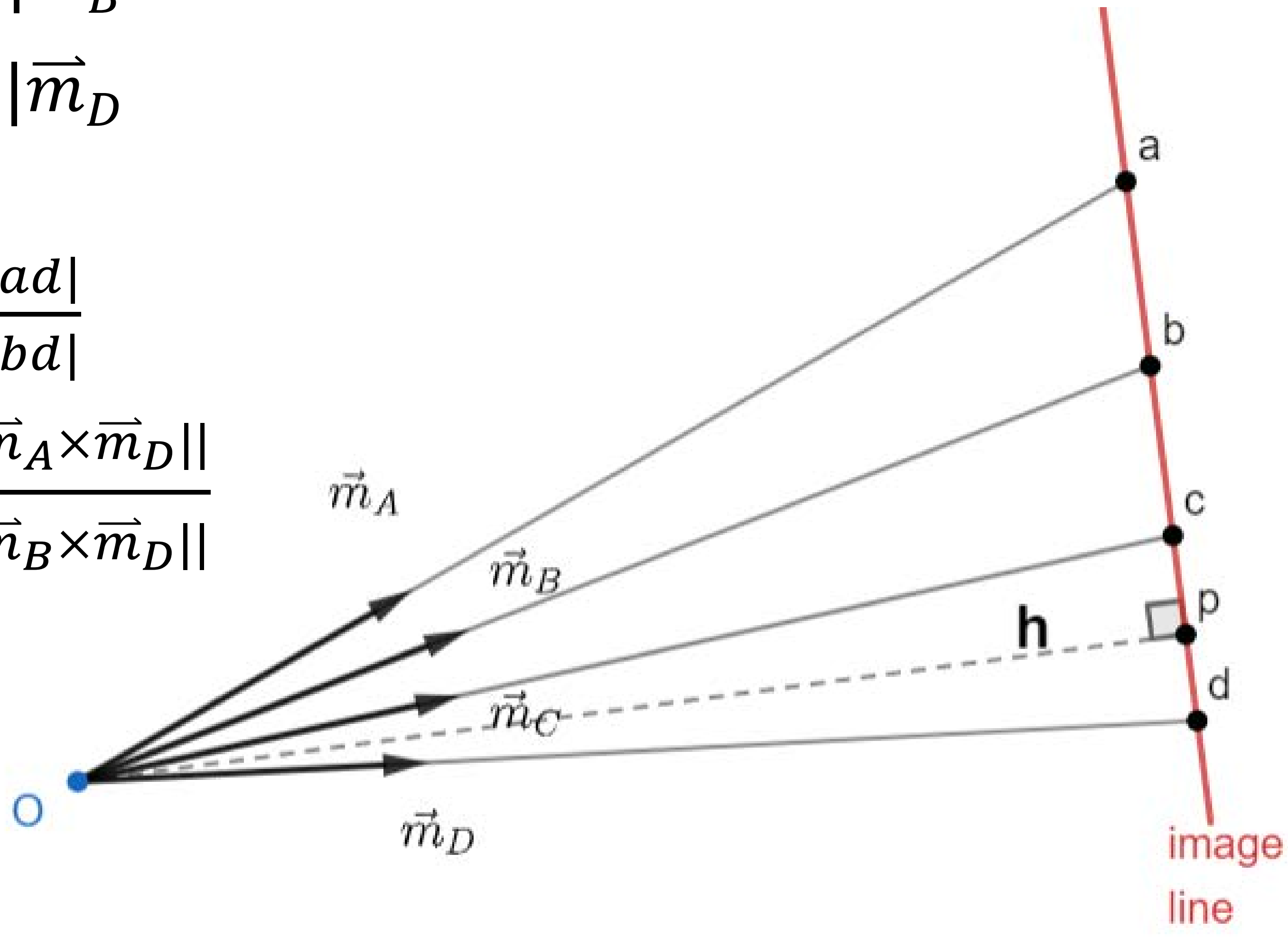
Computation of Cross Ration

5) $\overrightarrow{Oa} = |Oa|\vec{m}_A \quad , \quad \overrightarrow{Ob} = |Ob|\vec{m}_B$

$\quad \overrightarrow{Oc} = |Oc|\vec{m}_c \quad , \quad \overrightarrow{Od} = |Od|\vec{m}_D$

6) $[ABCD] = [abcd] = \frac{|ac|}{|bc|} / \frac{|ad|}{|bd|}$

$$= \frac{\frac{|Oa|\cdot|Oc|}{h}\,||\vec{m}_A \times \vec{m}_C||}{\frac{|Ob|\cdot|Oc|}{h}\,||\vec{m}_B \times \vec{m}_C||} \Big/ \frac{\frac{|Oa|\cdot|Od|}{h}\,||\vec{m}_A \times \vec{m}_D||}{\frac{|Ob|\cdot|Od|}{h}\,||\vec{m}_B \times \vec{m}_D||}$$



FOCUS OF EXPANSION

- Projections of translating space points seem to be moving on the image plane away from (or toward) a fixed point, this is known as the *focus of expansion (FOE)*.

- Since the focus of expansion is simply the "vanishing point" of the trajectories in the scene. Thus,

1) A space point translating in the direction of unit vector $\boldsymbol{u}$ has, when projected onto the image plane, a focus of expansion whose N-vector is $\pm\boldsymbol{u}$.

2) Projections of rigidly translating space points have a common focus of expansion.

FOCUS OF EXPANSION

3) If two image points of N-vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ in the first frame move to image points of N-vectors $\boldsymbol{m}_1'$ and $\boldsymbol{m}_2'$ in the second frame, respectively, the N-vector of the focus of expansion is given by:

$$\boldsymbol{u} = \pm N\left[N[\boldsymbol{m}_1 \times \boldsymbol{m}_1'] \times N[\boldsymbol{m}_2 \times \boldsymbol{m}_2']\right]$$

provided that the four image points are all distinct.

Proof. Let P_1, P_2, P_1', P_2' be the image points of N-vectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_1', \boldsymbol{m}_2'$, respectively.

The N-vector of the trajectory defined by P_1 and P_1' is $\pm N[\boldsymbol{m}_1 \times \boldsymbol{m}_1']$ and the N-vector of the trajectory defined by P_2 and P_2' is $\pm N[\boldsymbol{m}_2 \times \boldsymbol{m}_2']$. The N-vector of their intersection is given by the cross product of these two lines.

FOCUS OF EXPANSION

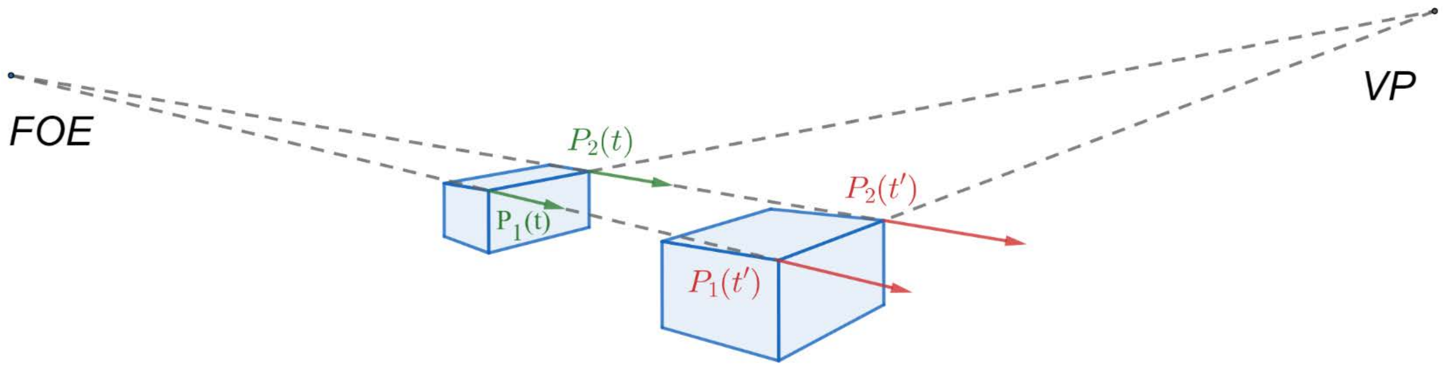
4) If two image points of N-vectors $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ are moving on the image plane with N-velocities $\dot{\boldsymbol{m}}_1$ and $\dot{\boldsymbol{m}}_2$, respectively, the N-vector of the focus of expansion is given by:

$$\boldsymbol{u} = \pm N\big[N[\boldsymbol{m}_1 \times \dot{\boldsymbol{m}}_1] \times N[\boldsymbol{m}_2 \times \dot{\boldsymbol{m}}_2]\big] \quad (8.2)$$

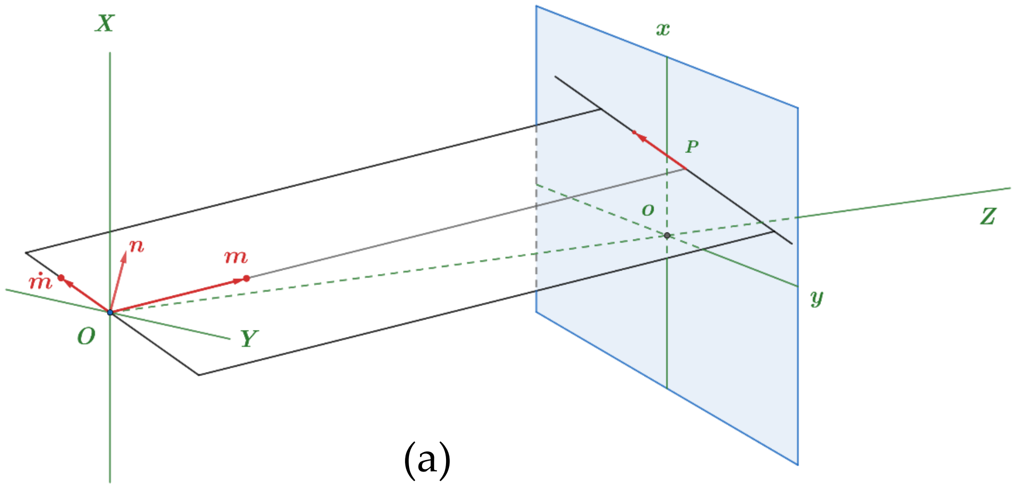
provided that the trajectories of the two image points are distinct.

Proof. The focus of expansion is the intersection of the trajectories on the image plane. The N-vectors of the trajectories of the two image points are $\pm N[\boldsymbol{m}_1 \times \dot{\boldsymbol{m}}_1]$ and $\pm N[\boldsymbol{m}_2$

FOCUS OF EXPANSION



(b)



(a)

Fig. 8.1. (a) The N-velocity $\dot{\boldsymbol{m}}$ of a moving image point and the N-vector $\boldsymbol{n}$ of its trajectory. b) The focus of expansion (*FOE*) of a translational motion and the vanishing point (*VP*) of a translating line segment.

FOCUS OF EXPANSION

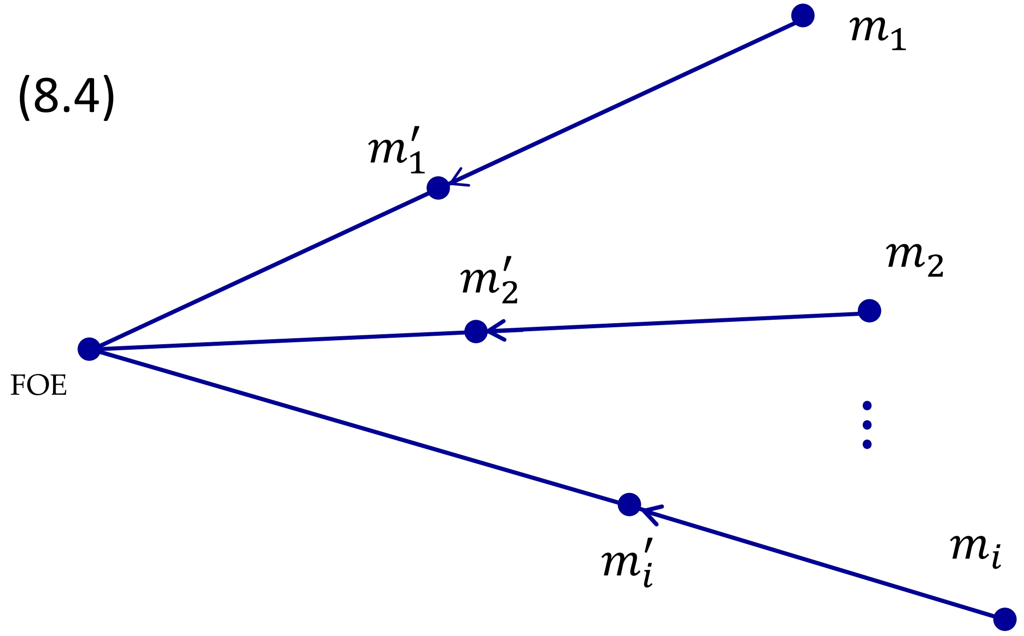
5) If image points of N-vectors $\boldsymbol{m}_i$ move to image points of N-vectors $\boldsymbol{m}_i'$, $i = 1, \ldots, N$, and if $\boldsymbol{u}$ is the N-vector of the focus of expansion, then

$$|\boldsymbol{u}, \boldsymbol{m}_i, \boldsymbol{m}_i'| = 0, \;\; i = 1, \ldots, N. \qquad (8.3)$$

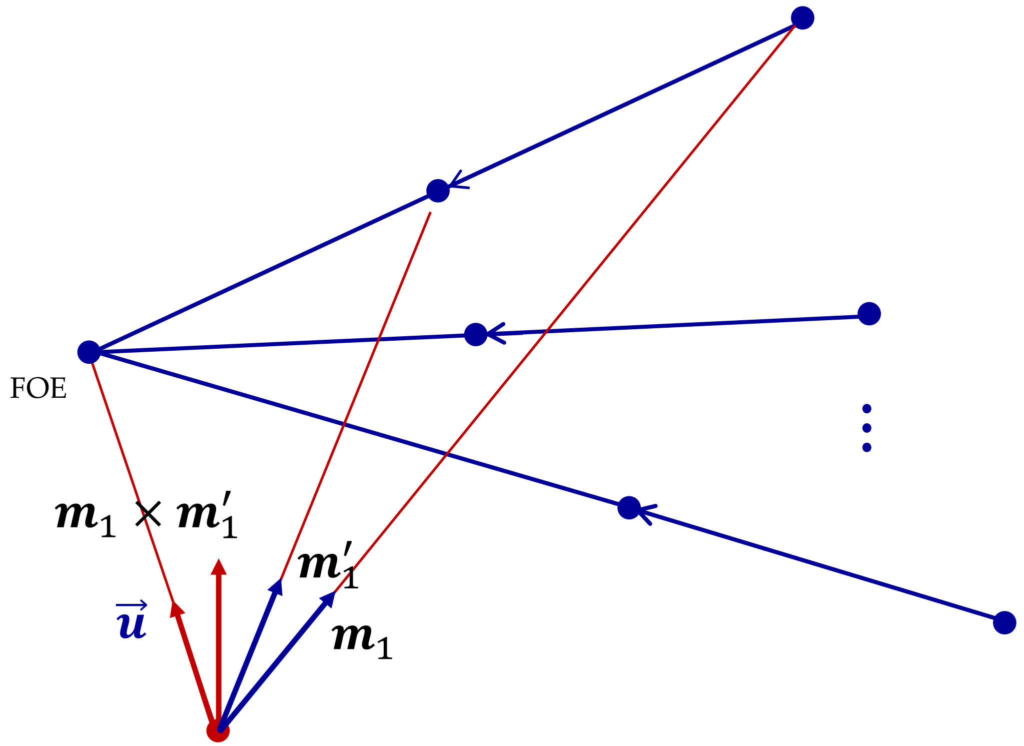
This is called the *epipolar equation*. From this we can robustly compute the N-vector $\boldsymbol{u}$ of the focus of expansion by

$$\sum_{i=1}^{N} \boldsymbol{W}_i |\boldsymbol{u}, \boldsymbol{m}_i, \boldsymbol{m}_i'|^2 \;\longrightarrow\; min, \qquad (8.4)$$

where $\boldsymbol{W}_i$ are positive weights.



FOCUS OF EXPANSION



FOCUS OF EXPANSION

We can re-arrange the above equation (8.4)

$$\sum_{i=1}^{N} \boldsymbol{W}_i |\boldsymbol{u}, \boldsymbol{m}_i, \boldsymbol{m}'_i|^2 = \sum_{i=1}^{N} \boldsymbol{W}_i (\boldsymbol{u}, \boldsymbol{m}_i \times \boldsymbol{m}'_i)^2$$

$$= \sum_{i=1}^{N} \boldsymbol{W}_i u^{\mathrm{T}} (\boldsymbol{m}_i \times \boldsymbol{m}'_i)(\boldsymbol{m}_i \times \boldsymbol{m}'_i)^{\mathrm{T}} \boldsymbol{u}$$

$$= \left(\boldsymbol{u}, \left(\underbrace{\sum_{i=1}^{N} \boldsymbol{W}_i (\boldsymbol{m}_i \times \boldsymbol{m}'_i)(\boldsymbol{m}_i \times \boldsymbol{m}'_i)^{\mathrm{T}}}_{\mathbf{A}}\right) \boldsymbol{u}\right)$$

$$= (\boldsymbol{u}, A\boldsymbol{u}). \qquad (8.5)$$

where $\boldsymbol{A}$ is a 3×3 matrix. The problem is now reduced to :

$$(\boldsymbol{u}, A\boldsymbol{u}) \longrightarrow \ min. \qquad (8.6)$$

The solution of $\boldsymbol{u}$ is given by the unit eigenvector of A for the smallest eigenvalue.

FOCUS OF EXPANSION

$$\mathbf{B} = \boldsymbol{m}_i \times \boldsymbol{m}_i' = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$(\boldsymbol{u}, \mathbf{B}) = \boldsymbol{u}^{\mathrm{T}}\mathbf{B} = \mathbf{B}^{\mathrm{T}}\boldsymbol{u}$$

$$\begin{aligned}(\boldsymbol{u}, \mathbf{B})^2 &= (\boldsymbol{u}^{\mathrm{T}}\mathbf{B})(\mathbf{B}^{\mathrm{T}}\boldsymbol{u}) \\ &= \boldsymbol{u}^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\boldsymbol{u} \\ &= \boldsymbol{u}^{\mathrm{T}}(\mathbf{B}\mathbf{B}^{\mathrm{T}})\boldsymbol{u}\end{aligned}$$

$$\boxed{\mathbf{A}\,\boldsymbol{u} = \lambda\,\boldsymbol{u}}$$

Eigen vector

Eigen value

FOCUS OF EXPANSION

$$\mathbf{A}^{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\mathbf{A}\boldsymbol{u} = \lambda\boldsymbol{u} = \lambda \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{\mathbf{I}} \boldsymbol{u}$$

$$= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \boldsymbol{u}$$

$$\mathbf{A}\boldsymbol{u} = \lambda\boldsymbol{u}$$

$$\Rightarrow \left[\mathbf{A} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right] \boldsymbol{u} = 0$$

$$\Rightarrow \begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

Solve $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$

$\Rightarrow \lambda_1, \lambda_2, \lambda_3$

According to eigen values we can obtain eigen vectors $\boldsymbol{u}$

Representation of Space Line

- Define H the centre of a space line l as the point closest to the viewpoint O on l, $\boldsymbol{u}$ (unit vector) as its orientation, and $\boldsymbol{n}$ as the N-vector of l.
- The sign of $\boldsymbol{u}$ is chosen so that the three vectors $\{\boldsymbol{n},\ \overrightarrow{OH},\ \ \boldsymbol{u}\}$ form a right-handed system; or a space line is oriented so that it "positively circulates" around its N-vector.
- *Note:* Lines passing through the viewpoint O is invisible and therefore are not considered.

$$\boldsymbol{p} = \frac{\boldsymbol{u}}{|OH|} \tag{9.1}$$

is called the *P-vector.* The space line l is completely defined by {**n, p**}:

$$\boldsymbol{u} = N[\boldsymbol{p}],\ \|\boldsymbol{p}\| = \frac{1}{|OH|} \tag{9.2}$$

- and H is found in the direction of $\boldsymbol{p} \times \boldsymbol{n}$. The 3-D position of the centre ***H*** is given by

$$\overrightarrow{OH} = \frac{\mathrm{N}[\boldsymbol{p}\times\boldsymbol{n}]}{\|\boldsymbol{p}\|} = \frac{\boldsymbol{p}\times\boldsymbol{n}}{\|\boldsymbol{p}\|^2} \tag{9.3}$$

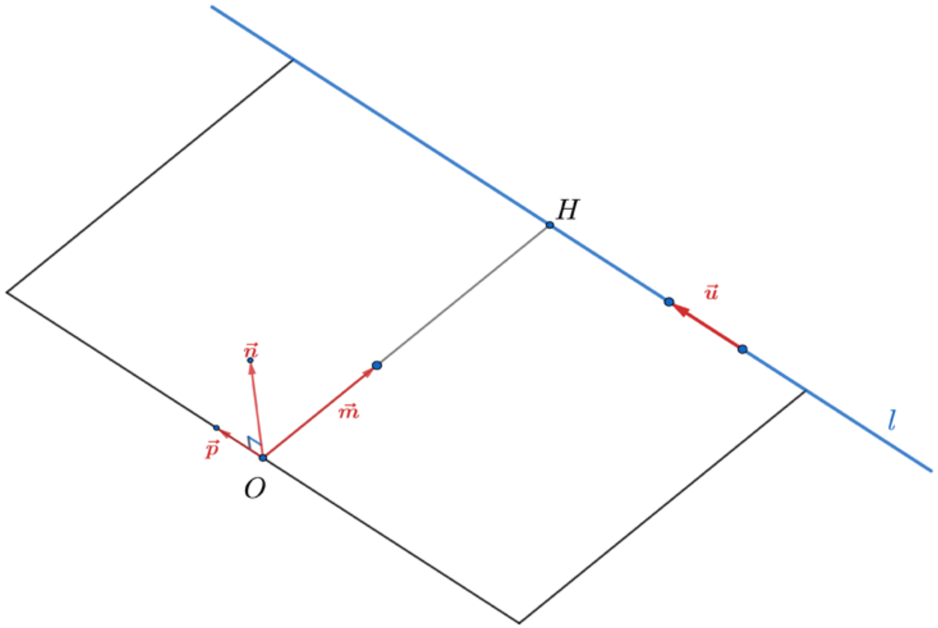


Fig. 9.1 Definition of space line and P-vector

Representation of Space Line

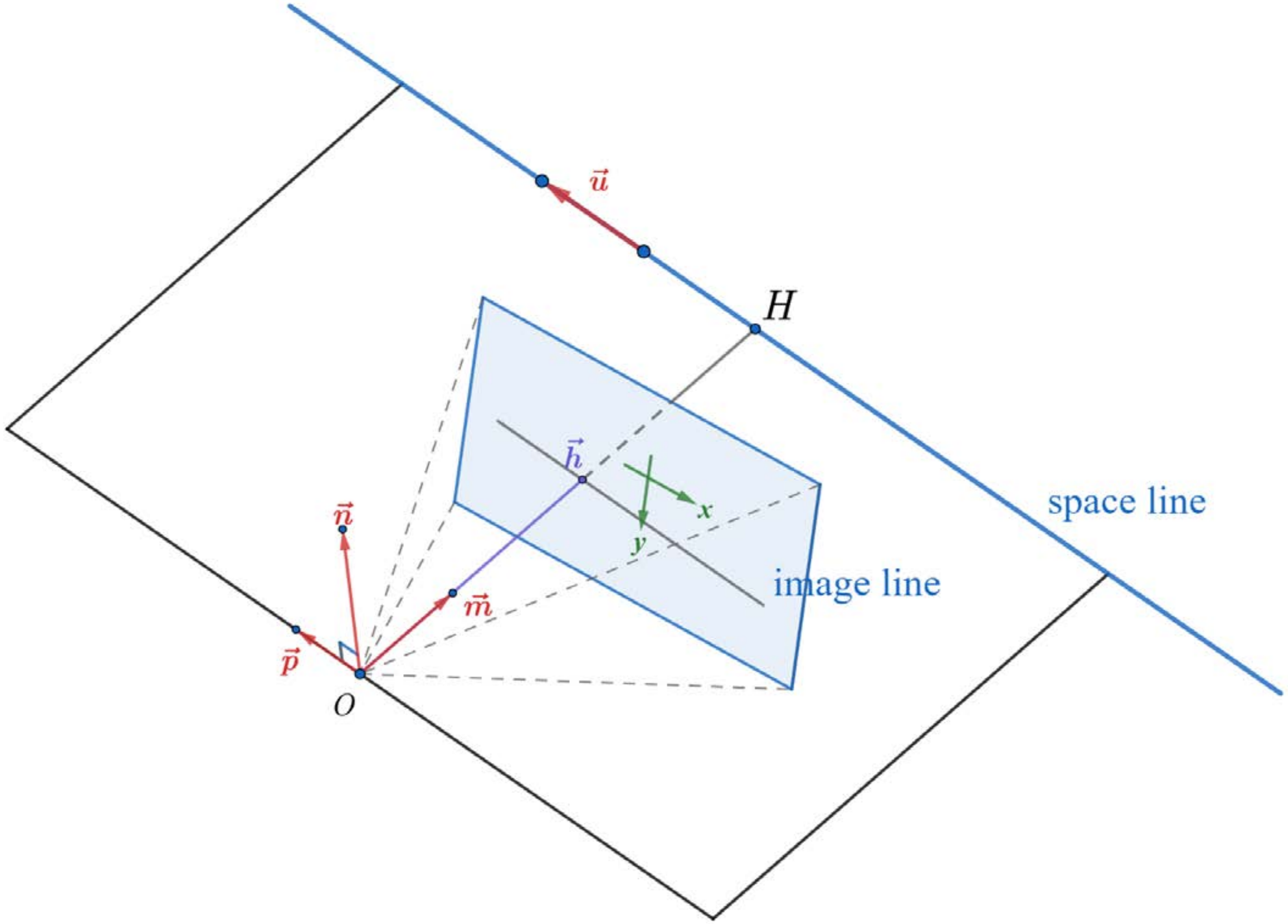


Fig. 9.1 Definition of space line and P-vector

Representation of Space Line

Proof:

1) By definition: $\mathrm{N}[\boldsymbol{p} \times \boldsymbol{n}] = \frac{\boldsymbol{p}\times\boldsymbol{n}}{\|\boldsymbol{p}\times\boldsymbol{n}\|}$

2) Let $\boldsymbol{m} = N[\boldsymbol{p} \times \boldsymbol{n}]$, then:

$$\overrightarrow{OH} = \frac{1}{\|\boldsymbol{p}\|}\boldsymbol{m} = \frac{\mathrm{N}[\boldsymbol{p}\times\boldsymbol{n}]}{\|\boldsymbol{p}\|}$$
$$= \frac{\boldsymbol{p} \times \boldsymbol{n}}{\|\boldsymbol{p}\| \cdot \|\boldsymbol{p} \times \boldsymbol{n}\|}$$

3) $\|\boldsymbol{p} \times \boldsymbol{n}\| = \|\boldsymbol{p}\|$, because $\|\boldsymbol{n}\|$=1 and $\boldsymbol{p} \perp \boldsymbol{n}$ (fundamental identity), as shown in Figure 9.2. Therefore Equation 9.3 is proved.

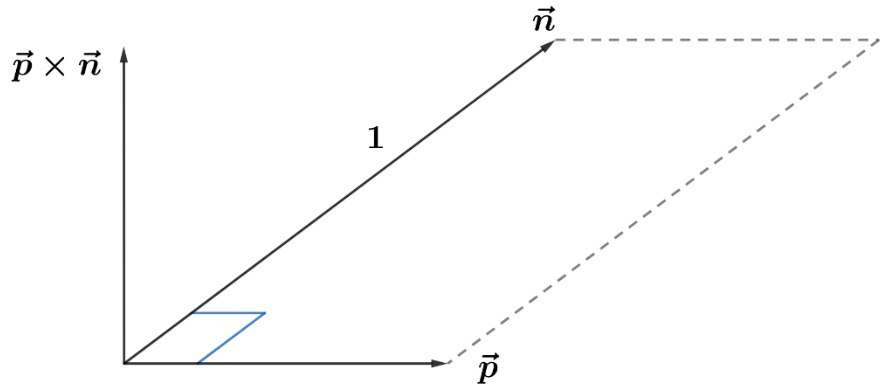


Fig. 9.2 Illustration of fundamental identity

Representation of Space Line

Another way of proving:

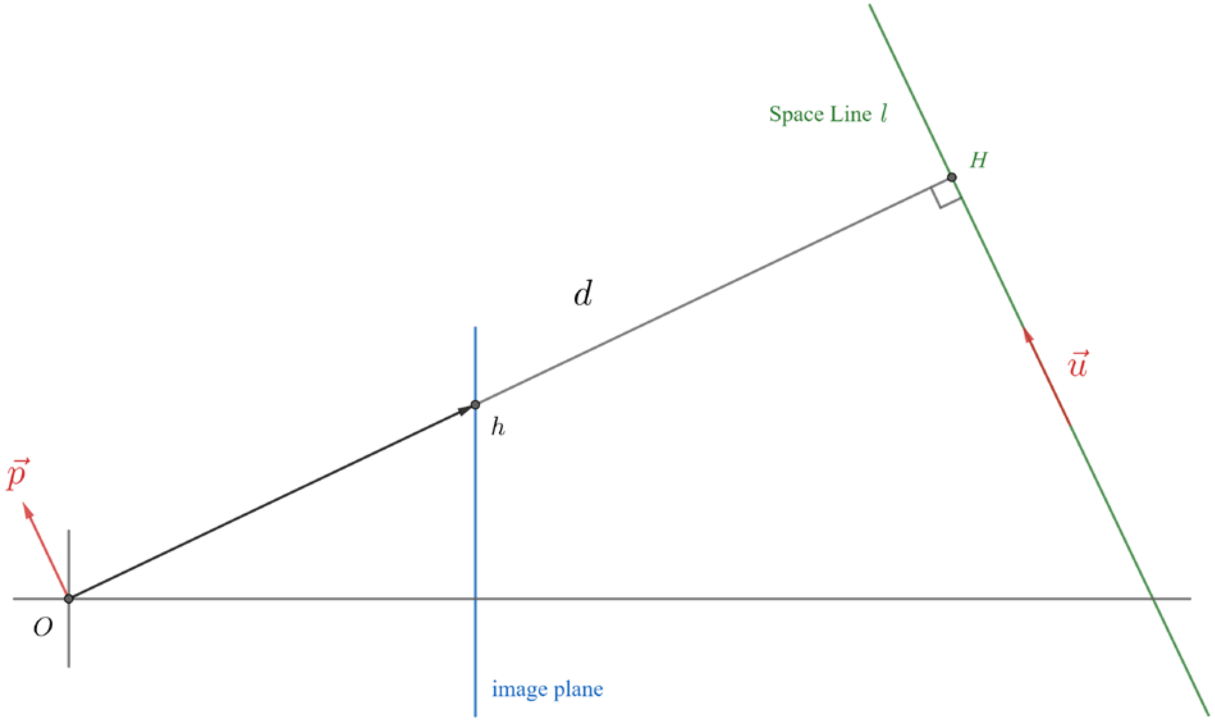
$$\overrightarrow{OH} = |OH| \cdot \mathrm{N}[\boldsymbol{p} \times \boldsymbol{n}]$$

$$= \frac{\mathrm{N}[\boldsymbol{p} \times \boldsymbol{n}]}{||\boldsymbol{p}||}$$

$$= \frac{1}{||\boldsymbol{p}||}\left[\frac{\boldsymbol{p}}{||\boldsymbol{p}||} \times \boldsymbol{n}\right]$$

$$= \left[\frac{1}{||\boldsymbol{p}||^2}\right] \cdot \boldsymbol{p} \times \boldsymbol{n}$$

We set the length $|OH|$ as d where $||\boldsymbol{p}|| = {}^{1}/_{d}$



Representation of Space Line

Property (1):

For any space line $\{\boldsymbol{n},\ \boldsymbol{p}\}$, its N-vector $\boldsymbol{n}$ and P-vector $\boldsymbol{p}$ are mutually orthogonal.

$$(\boldsymbol{n}, \boldsymbol{p}) = 0 \qquad (9.4)$$

This is known as the *fundamental identity* of a space line in algebraic geometry.

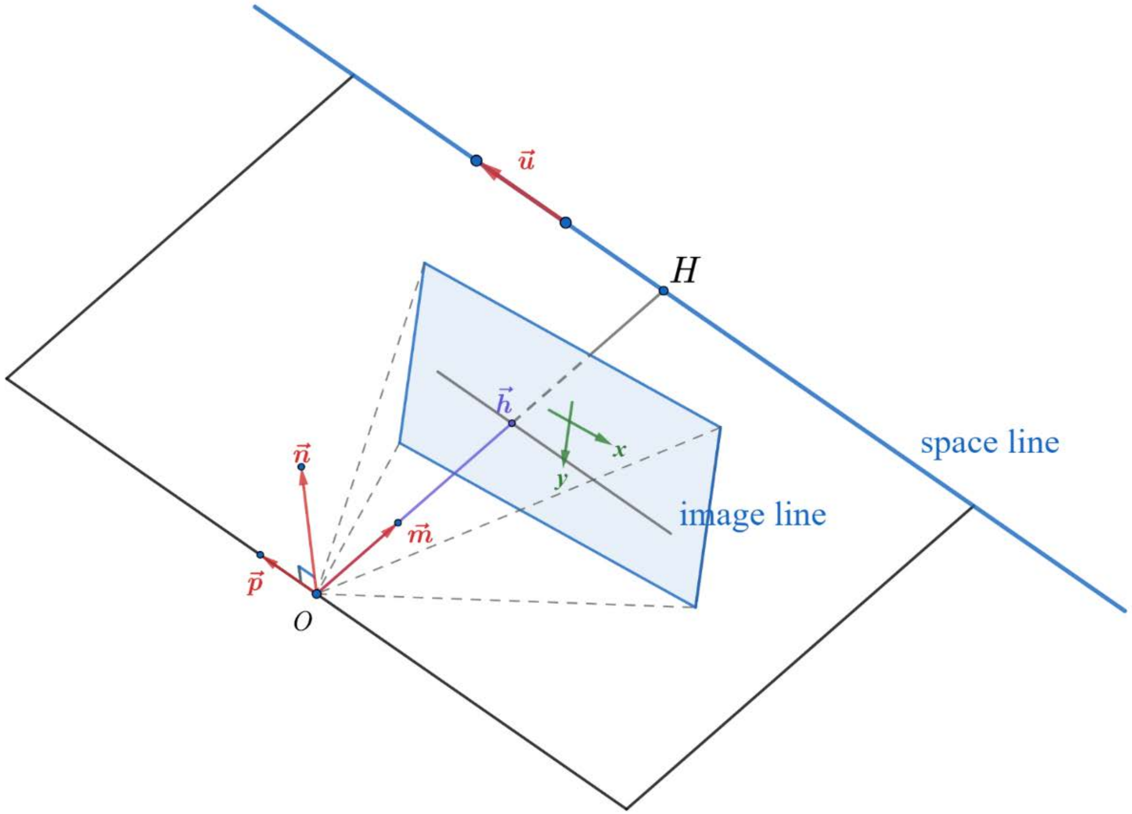


Fig. 9.1 Definition of space line and P-vector

Representation of Space Line

<u>Property (2)</u>:

The equation of space line $\{\boldsymbol{n},\ \ \boldsymbol{p}\}$ is

$$\boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{n} \qquad (9.10)$$

Proof. According to Figure 9.3, let H be the centre of space line $\{\boldsymbol{n}, \boldsymbol{p}\}$. Let $d = |OH|, \boldsymbol{m} = \overrightarrow{OH}/d$, and $\boldsymbol{u} = d\boldsymbol{p}$.

a. If P is a space point on the space line l, we set

$$\overrightarrow{OP} = \boldsymbol{r} \qquad (9.5)$$

b. $$\overrightarrow{OP} = \overrightarrow{OH} + \overrightarrow{HP} \qquad (9.6)$$

c. $$\overrightarrow{OH} = d\boldsymbol{m} \qquad (9.7)$$

d. $$\overrightarrow{HP} = t\boldsymbol{u} \qquad (9.8)$$

Note that $\boldsymbol{r} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

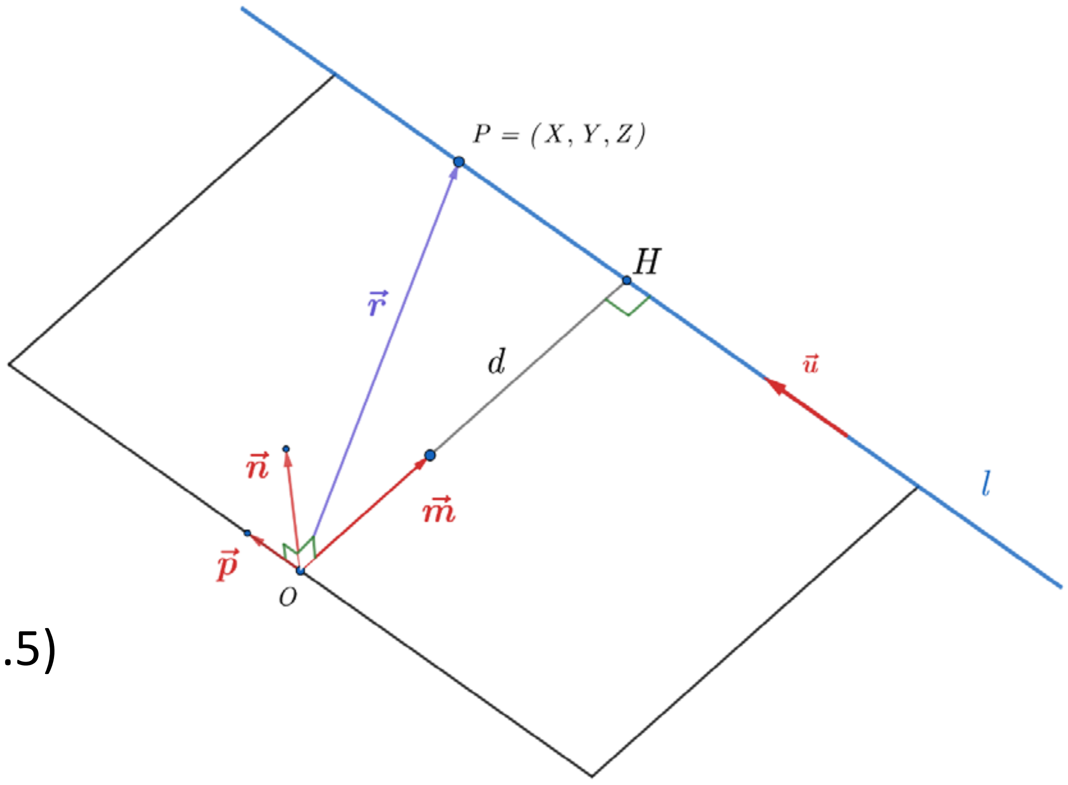


Fig. 9.3 Definition of $\boldsymbol{r}$ vector

Representation of Space Line

So a point P is on this line iff vector $\boldsymbol{r} = \overrightarrow{OP}$ satisfies

$$\boldsymbol{r} = d\boldsymbol{m} + t\boldsymbol{u} \qquad (9.9)$$

for some number t, or

$$\boldsymbol{r} \times \boldsymbol{p} = (d\boldsymbol{m} + t\boldsymbol{u}) \times \boldsymbol{p}$$

$$\boldsymbol{r} \times \boldsymbol{p} = d\boldsymbol{m} \times \boldsymbol{p} + \boldsymbol{0}$$

$$\boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{m} \times d\boldsymbol{p}$$

$$\boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{m} \times \boldsymbol{u}$$

Since $\boldsymbol{n}$ is orthogonal to $\boldsymbol{u}$ and $\boldsymbol{m}$, $\boldsymbol{u}$ are all unit vectors,

$$\boldsymbol{m} \times \boldsymbol{u} = \boldsymbol{n}$$

Hence

$$\boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{n}$$

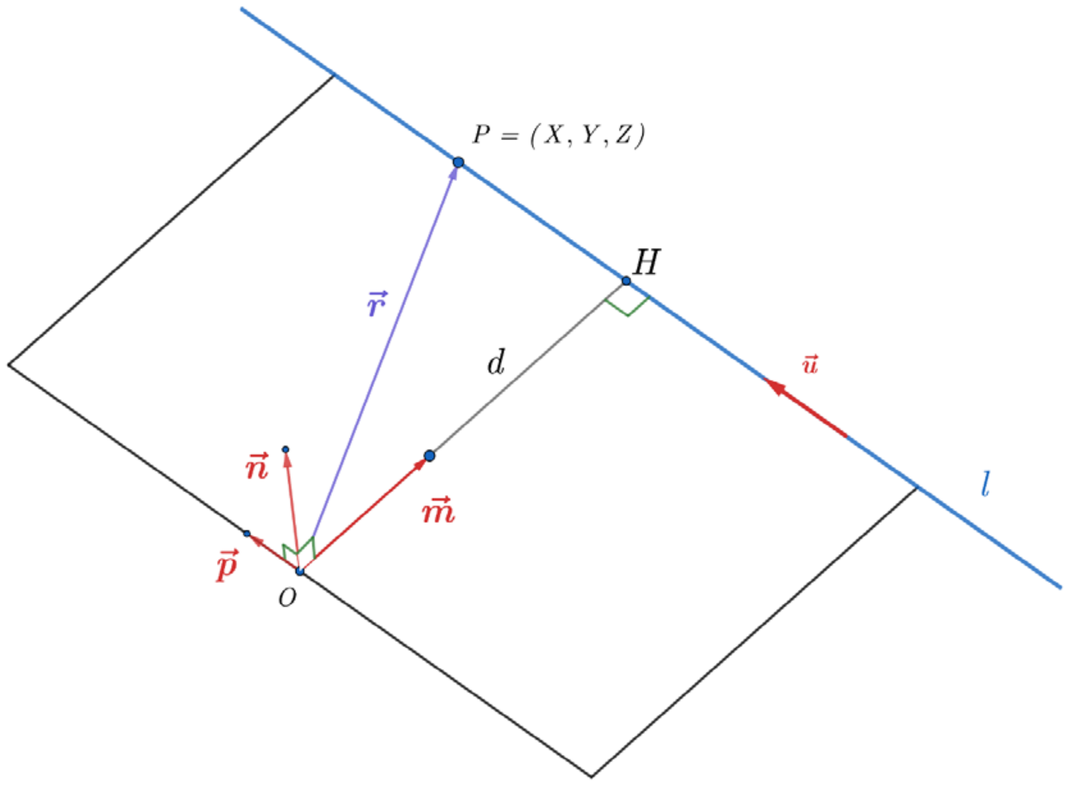


Fig. 9.3 Definition of $\boldsymbol{r}$ vector

$$d = |OH|, \boldsymbol{m} = \overrightarrow{OH}/d, \text{ and } \boldsymbol{u} = d\boldsymbol{p}.$$

Representation of Space Line

Property (3):

Space lines $l_1: \{\boldsymbol{n}_1, \boldsymbol{p}_1\}$ and $l_2: \{\boldsymbol{n}_2, \boldsymbol{p}_2\}$ intersect iff

$$(\boldsymbol{n}_1, \boldsymbol{p}_2) + (\boldsymbol{n}_2, \boldsymbol{p}_1) = 0 \tag{9.11}$$

Proof. The projections of these space lines intersect at an image point of N-vector $\boldsymbol{m} = \pm N[\boldsymbol{n}_1 \times \boldsymbol{n}_2]$. The two space lines intersect iff there exist a real number s such that the end point of vector $\mathbf{r} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 / s$ is on both lines; if $s = 0$, the intersection is interpreted to be at infinity. From property (2), this condition is written as:

$$\frac{\boldsymbol{n}_1 \times \boldsymbol{n}_2}{s} \times \boldsymbol{p}_1 = \boldsymbol{n}_1 \tag{9.12}$$

$$\frac{\boldsymbol{n}_1 \times \boldsymbol{n}_2}{s} \times \boldsymbol{p}_2 = \boldsymbol{n}_2 \tag{9.13}$$

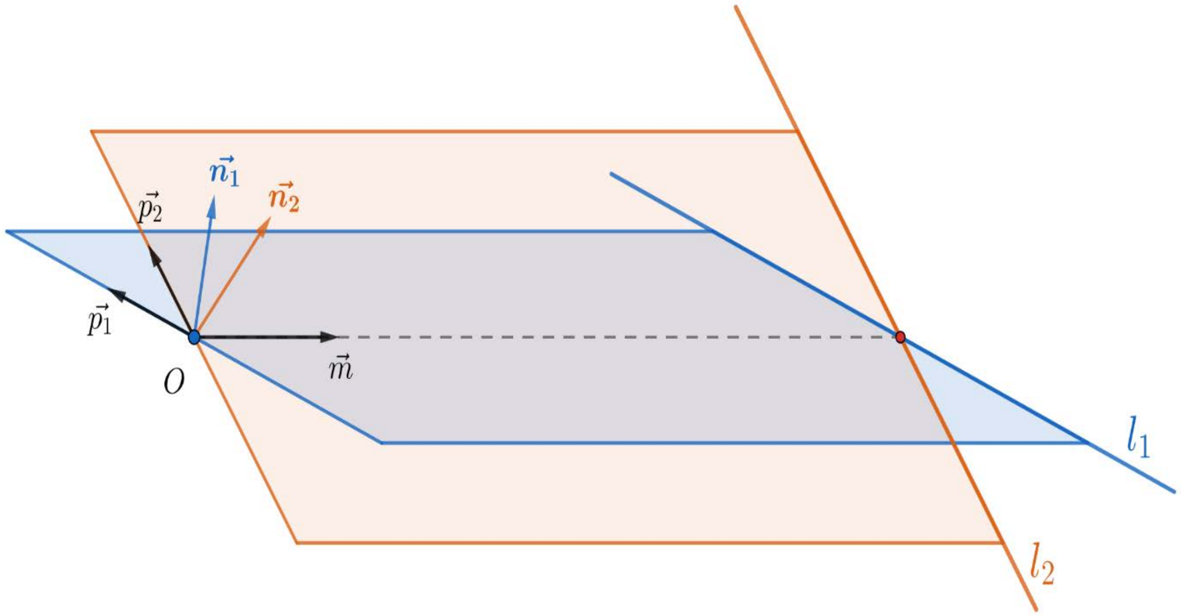


Fig. 9.4 Intersection of space lines.

Representation of Space Line

Proof. (Continued from last page)

Note: $(a \times b) \times c = (a, c)b - (b, c)a \qquad (9.14)$

From equation (9.12) and (9.13) we have:

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 \times \boldsymbol{p}_1 = s\boldsymbol{n}_1$$

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 \times \boldsymbol{p}_2 = s\boldsymbol{n}_2$$

Using equation (9.14)

$$(\boldsymbol{n}_1, \boldsymbol{p}_1)\boldsymbol{n}_2 - (\boldsymbol{n}_2, \boldsymbol{p}_1)\boldsymbol{n}_1 = s\boldsymbol{n}_1$$

$$(\boldsymbol{n}_1, \boldsymbol{p}_2)\boldsymbol{n}_2 - (\boldsymbol{n}_2, \boldsymbol{p}_2)\boldsymbol{n}_1 = s\boldsymbol{n}_2$$

Since $(\boldsymbol{n}_1, \boldsymbol{p}_1) = 0$, $(\boldsymbol{n}_2, \boldsymbol{p}_2) = 0$, (the fundamental identities), these are equivalently written as

$$\cancel{(\boldsymbol{n}_1, \boldsymbol{p}_1)\boldsymbol{n}_2} - (\boldsymbol{n}_2, \boldsymbol{p}_1)\boldsymbol{n}_1 = s\boldsymbol{n}_1$$

$$(\boldsymbol{n}_1, \boldsymbol{p}_2)\boldsymbol{n}_2 - \cancel{(\boldsymbol{n}_2, \boldsymbol{p}_2)}\boldsymbol{n}_1 = s\boldsymbol{n}_2$$

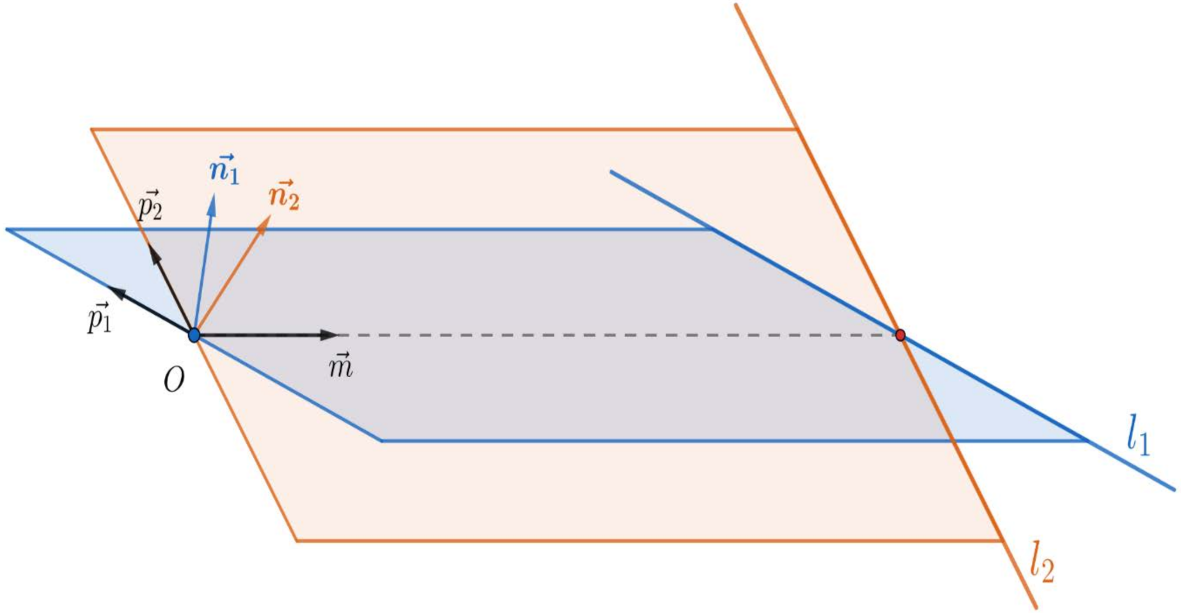


Fig. 9.4 Intersection of space lines.

Representation of Space Line

Proof. (Continued from last page)

Equations (9.12) and (9.13) are equivalently written as:

$$-(\boldsymbol{n}_2, \boldsymbol{p}_1)\, \boldsymbol{n}_1 = s\, \boldsymbol{n}_1 \qquad (9.14)$$

$$(\boldsymbol{n}_1, \boldsymbol{p}_2)\, \boldsymbol{n}_2 = s\, \boldsymbol{n}_2 \qquad (9.15)$$

Or

$$s = -(\boldsymbol{n}_2, \boldsymbol{p}_1) \qquad (9.16)$$

$$s = (\boldsymbol{n}_1, \boldsymbol{p}_2) \qquad (9.17)$$

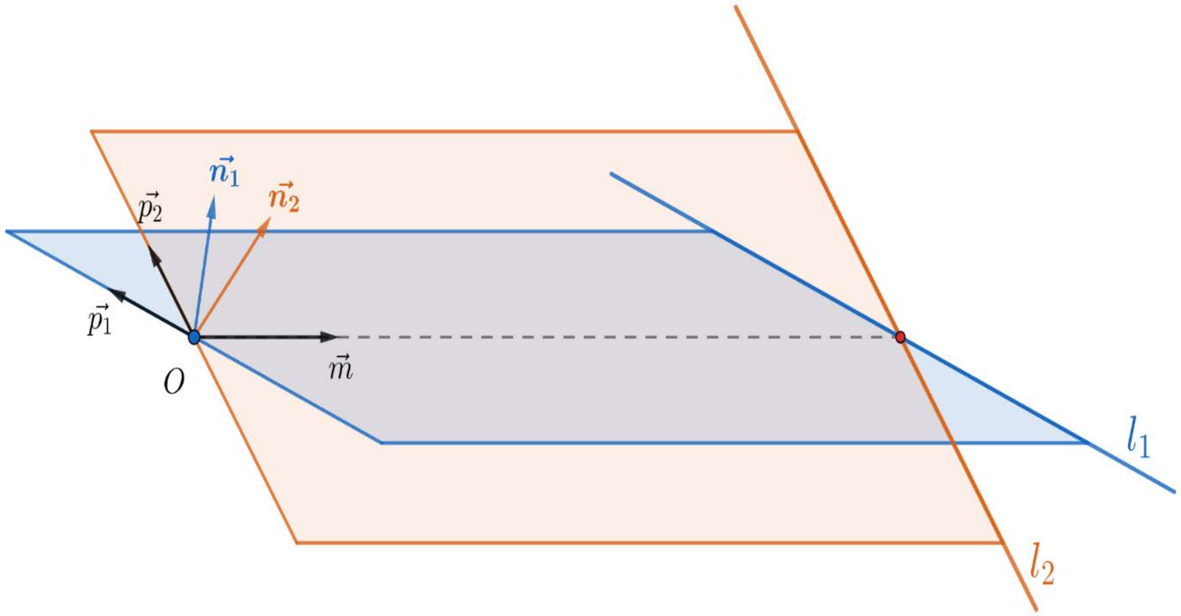


Fig. 9.4 Intersection of space lines.

Such an s exists iff $(\boldsymbol{n}_1, \boldsymbol{p}_2) + (\boldsymbol{n}_2, \boldsymbol{p}_1) = 0$.

Representation of Space Line

Property (4):

The space lines $\{\boldsymbol{n}, \boldsymbol{p}\}$ that passes through two space points at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ $(\boldsymbol{r}_1 \times \boldsymbol{r}_2 \neq 0)$ is given by

$$\boldsymbol{n} = \pm N[\boldsymbol{r}_1 \times \boldsymbol{r}_2] \qquad (9.18)$$

$$\boldsymbol{p} = \mp \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\|\boldsymbol{r}_1 \times \boldsymbol{r}_2\|} \qquad (9.19)$$

Proof. Since the N-vector should be orthogonal to both $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, we obtain the first equation

$$\boldsymbol{n} = \pm N[\boldsymbol{r}_1 \times \boldsymbol{r}_2]$$

See Figure 9.5, if d is the distance of the space line from the viewpoint O, as $\boldsymbol{u} = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|}$,we have P-vector

$$\boldsymbol{p} = \mp \frac{N[\boldsymbol{r}_1 - \boldsymbol{r}_2]}{d} = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\| d} \qquad (9.20)$$

according to our sign convention.

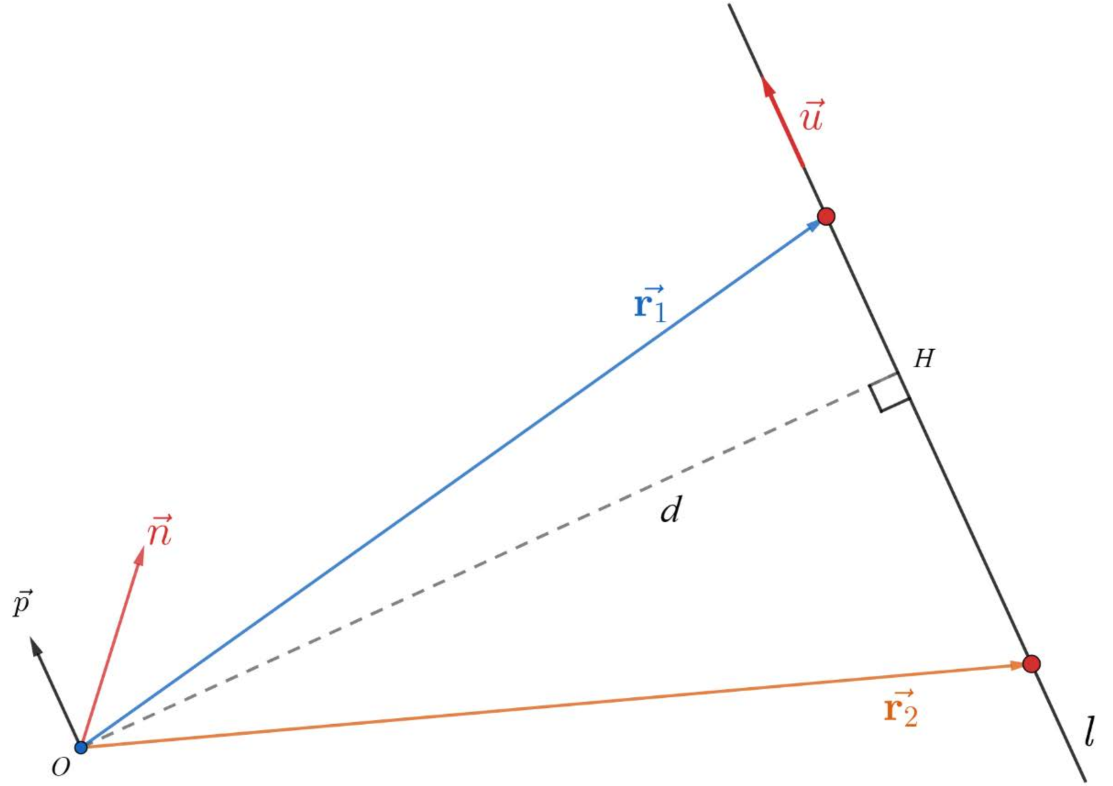


Fig. 9.5 Intersection of space lines.

Representation of Space Line

Proof. (Continued from last page)

Consider the triangle defined by the two points at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and the view-point O. The triangle area is $\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|d/2$.

As $\|\boldsymbol{r}_1 \times \boldsymbol{r}_2\|$ is defined as the area of parallelogram that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ span, the area of the triangle is also equal to $\|\boldsymbol{r}_1 \times \boldsymbol{r}_2\|/2$. Hence,

$$d = \frac{\|\boldsymbol{r}_1 \times \boldsymbol{r}_2\|}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|} \qquad (9.21)$$

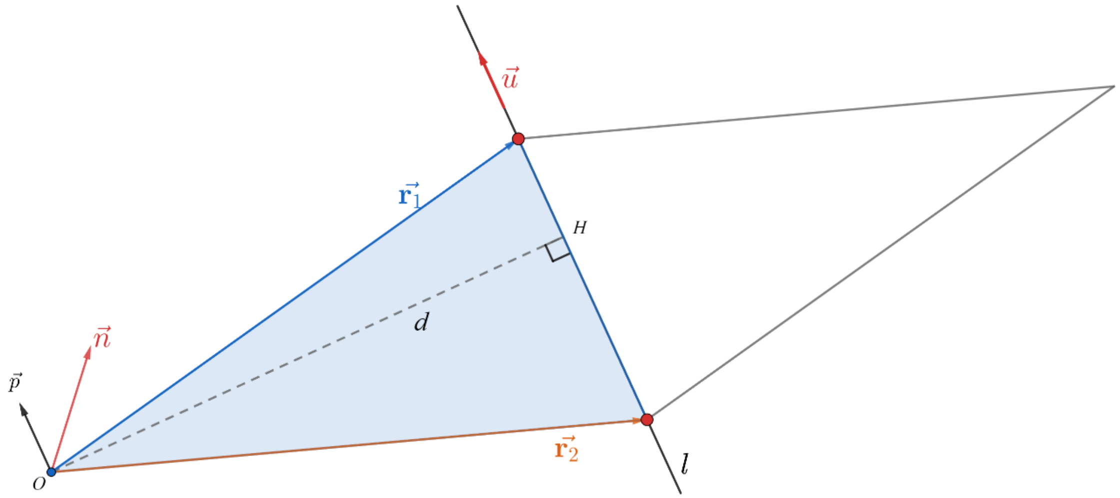


Fig. 9.6 Intersection of space lines. The area of the triangle is half of the parallelogram spanned by $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

Motion Parallax of A Line

Property (1):

If the camera is translated by $\boldsymbol{h}$, the representation $\{\boldsymbol{n}, \boldsymbol{p}\}$ of a space line changes into $\{\boldsymbol{n}', \boldsymbol{p}'\}$ in the form

$$\boldsymbol{n}' = N[\boldsymbol{n} - \boldsymbol{h} \times \boldsymbol{p}], \qquad (10.1)$$

$$\boldsymbol{p}' = \frac{\boldsymbol{p}}{\|\boldsymbol{n} - \boldsymbol{h} \times \boldsymbol{p}\|} \qquad (10.2)$$

Proof. Let l be the space line in question. Its orientation $\boldsymbol{u} = \boldsymbol{p}/\|\boldsymbol{p}\|$ is the same for both frames. Let O and O' be the origins of the first and second frames, respectively. Let H be the centre of l. The N-vector $\boldsymbol{n}'$ of l for the second frame is orthogonal to both $\boldsymbol{u}$ and $\overrightarrow{O'H}$. Noting that $\{\boldsymbol{n}', \overrightarrow{O'H}, \boldsymbol{u}\}$ is a right-handed system, we obtain

$$\boldsymbol{n}' = N\left[\overrightarrow{O'H} \times \boldsymbol{u}\right]. \qquad (10.3)$$

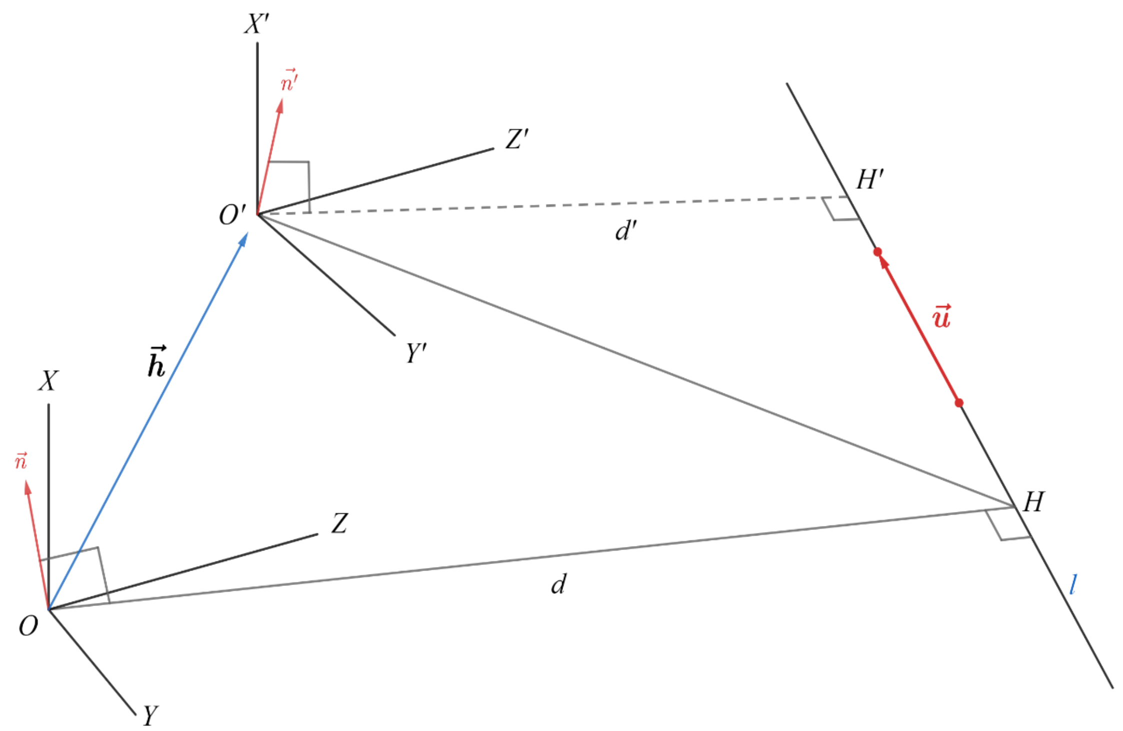


Fig. 9.7 Definition of motion parallax of a line l. The translation of two origins O and O' is $\boldsymbol{h} \in \mathbb{R}^3$.

Motion Parallax of A Line

From Figure 9.8, the area of the blue triangle is $\frac{d'}{2} \cdot \|\boldsymbol{u}\| = d'/2$. It is also equivalent to $\left\|\overrightarrow{O'H} \times \boldsymbol{u}\right\|/2$, so it is easy to see that the distance d' of l from O' is given by $\left\|\overrightarrow{O'H} \times \boldsymbol{u}\right\|$.

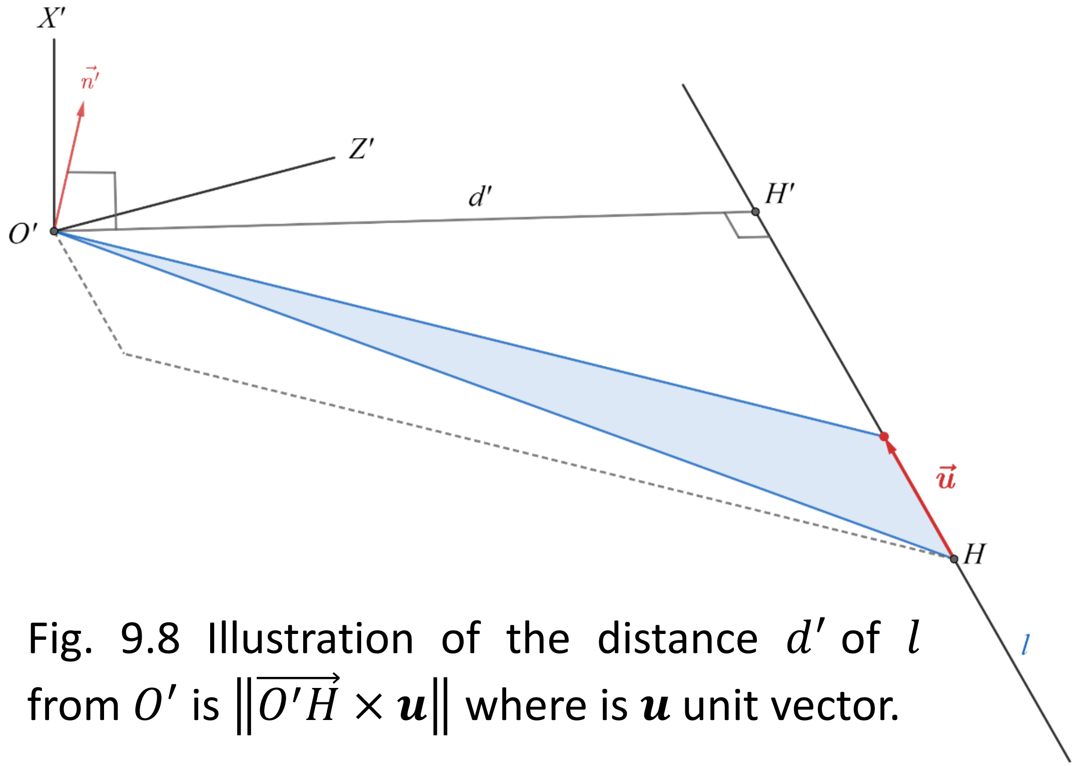


Fig. 9.8 Illustration of the distance d' of l from O' is $\left\|\overrightarrow{O'H} \times \boldsymbol{u}\right\|$ where is $\boldsymbol{u}$ unit vector.

As $\boldsymbol{u} = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}$, we have

$$\boldsymbol{p}' = \frac{\boldsymbol{u}}{\left\|\overrightarrow{O'H} \times \boldsymbol{u}\right\|} = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\| \cdot \left\|\overrightarrow{O'H} \times \boldsymbol{u}\right\|}. \tag{10.4}$$

Further,

$$\overrightarrow{O'H} \times \boldsymbol{u} = \left(\overrightarrow{OH} - \boldsymbol{h}\right) \times \boldsymbol{u}$$
$$= \left(\overrightarrow{OH} - \boldsymbol{h}\right) \times \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}$$

The distributive property of cross product

According to equation (9.10) $\boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{n}$, it is easy to obtain $\overrightarrow{OH} \times \boldsymbol{p} = \boldsymbol{n}$

Replace it into equation (10.4), the result is obtained.

Motion Parallax of A Line

Property (2):

If a space line of N-vector $\boldsymbol{n}$ moves to a space line of N-vector $\boldsymbol{n}'$ by a camera translation $\boldsymbol{h}$ and if $(\boldsymbol{h},\ \boldsymbol{n}) \neq 0$, and $(\boldsymbol{h},\ \boldsymbol{n}') \neq 0$, the P-vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ for the first and second are respectively given by

$$\boldsymbol{p} = \frac{\boldsymbol{n}\times\boldsymbol{n}'}{(\boldsymbol{h},\ \boldsymbol{n}')},\quad \boldsymbol{p}' = \frac{\boldsymbol{n}\times\boldsymbol{n}'}{(\boldsymbol{h},\boldsymbol{n})}. \qquad (10.5)$$

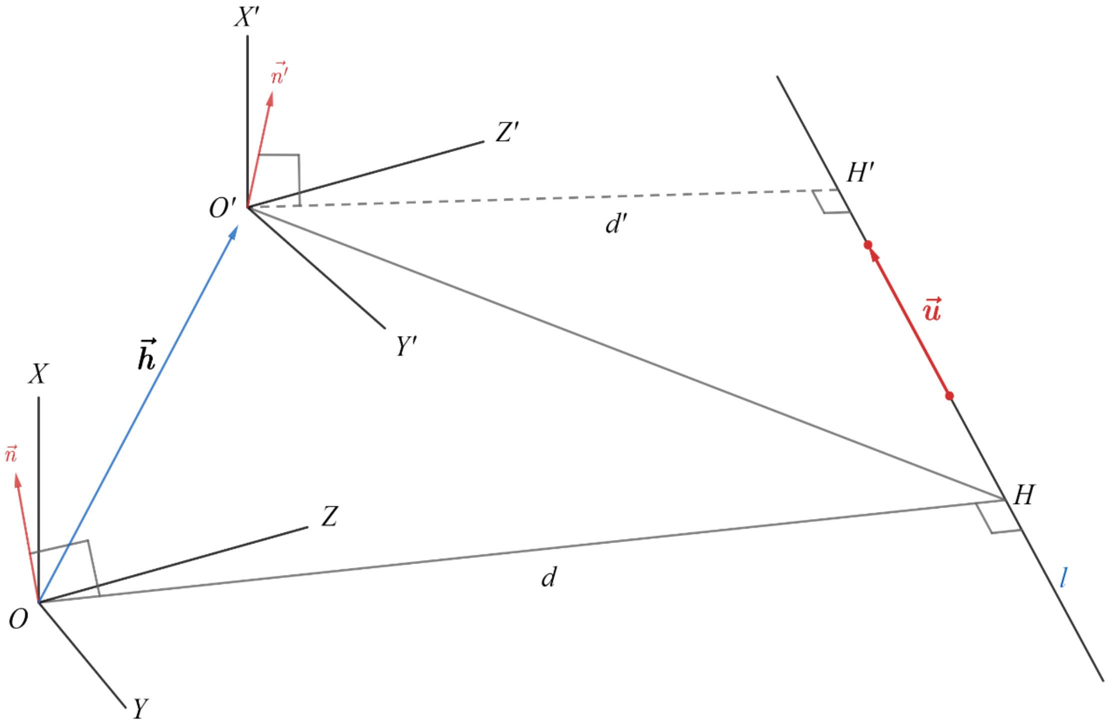


Fig. 9.7 Definition of motion parallax of a line l.

Motion parallax equations of a space line.

Proof.

(1) As unit vector $\boldsymbol{u}$ is mutually orthogonal to $\boldsymbol{n}$ and $\boldsymbol{n}'$, respectively

$$\boldsymbol{u} = \frac{\boldsymbol{n}\times\boldsymbol{n}'}{\|\boldsymbol{n}\times\boldsymbol{n}'\|} \qquad (10.6)$$

(2) According to equation (9.3) $\overrightarrow{OH} = \frac{\boldsymbol{p}\times\boldsymbol{n}}{\|\boldsymbol{p}\|^2}$, rewrite:

$$\overrightarrow{OH} = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}\times\frac{\boldsymbol{n}}{\|\boldsymbol{p}\|} = \boldsymbol{u}\times\frac{\boldsymbol{n}}{\|\boldsymbol{p}\|} = \frac{1}{\|\boldsymbol{p}\|}(\boldsymbol{u}\times\boldsymbol{n})$$

$$= d\boldsymbol{u}\times\boldsymbol{n} \qquad (10.7)$$

Motion Parallax of A Line

(3) Subject to equation (10.7), we obtain

$$\overrightarrow{O'H} = \overrightarrow{OH} - \boldsymbol{h} = d\boldsymbol{u} \times \boldsymbol{n} - \boldsymbol{h} \qquad (10.8)$$

(4) As $\overrightarrow{O'H} \perp \boldsymbol{n}'$, $\left(\overrightarrow{O'H},\ \boldsymbol{n}'\right) = 0$

So from equation (10.8), we obtain

$$(d\boldsymbol{u} \times \boldsymbol{n} - \boldsymbol{h},\ \boldsymbol{n}') = 0$$

$$(d\boldsymbol{u} \times \boldsymbol{n},\ \boldsymbol{n}') - (\boldsymbol{h},\ \boldsymbol{n}') = 0$$

$$d(\boldsymbol{u} \times \boldsymbol{n},\ \boldsymbol{n}') = (\boldsymbol{h},\ \boldsymbol{n}')$$

$$d = \frac{(\boldsymbol{h},\ \boldsymbol{n}')}{(\boldsymbol{u} \times \boldsymbol{n},\ \boldsymbol{n}')} = \frac{(\boldsymbol{h},\ \boldsymbol{n}')}{(\boldsymbol{u},\ \boldsymbol{n} \times \boldsymbol{n}')} = \frac{(\boldsymbol{h},\ \boldsymbol{n}')}{\|\ \boldsymbol{n} \times \boldsymbol{n}'\|} \qquad (10.9)$$

The distributive property of dot product

The scalar multiplication property of dot product

Identity

Subject to eq. (10.6), $(\boldsymbol{u},\ \boldsymbol{n} \times \boldsymbol{n}') = \left(\frac{\boldsymbol{n} \times \boldsymbol{n}'}{\|\boldsymbol{n} \times \boldsymbol{n}'\|}, \boldsymbol{n}\ \times\ \boldsymbol{n}'\right) = \|\ \boldsymbol{n} \times \boldsymbol{n}'\|$

(5) Subject to eq. (10.6) and (10.9)

$$\boldsymbol{p} = \frac{\boldsymbol{u}}{\|\overrightarrow{OH}\|} = \frac{\boldsymbol{u}}{d} = \frac{\boldsymbol{n} \times \boldsymbol{n}'}{(\boldsymbol{h},\ \boldsymbol{n}')} \qquad (10.10)$$

Motion Parallax of A Line

Property (3): Extension of property (2) with rotation

If a space line of N-vector $\boldsymbol{n}$ moves to a space line of N-vector $\boldsymbol{n}'$ by a camera translation $\boldsymbol{h}$ and Rotation R, and if $(\boldsymbol{h},\ \boldsymbol{n}) \neq 0$, and $(\boldsymbol{h},\ \boldsymbol{n}') \neq 0$, the P-vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$ for the first and second are respectively given by

$$\boldsymbol{p} = \frac{\boldsymbol{n} \times \mathbf{R}\,\boldsymbol{n}'}{(\boldsymbol{h},\ \mathbf{R}\,\boldsymbol{n}')}, \quad \boldsymbol{p}' = \frac{\mathbf{R}^{\mathrm{T}}(\boldsymbol{n} \times \mathbf{R}\,\boldsymbol{n}')}{(\boldsymbol{h}, \boldsymbol{n})}. \tag{10.11}$$

Motion parallax equations of a space line with $\mathbf{R} \in \mathbb{R}^{3\times3}$ and $\boldsymbol{h} \in \mathbb{R}^{3}$.

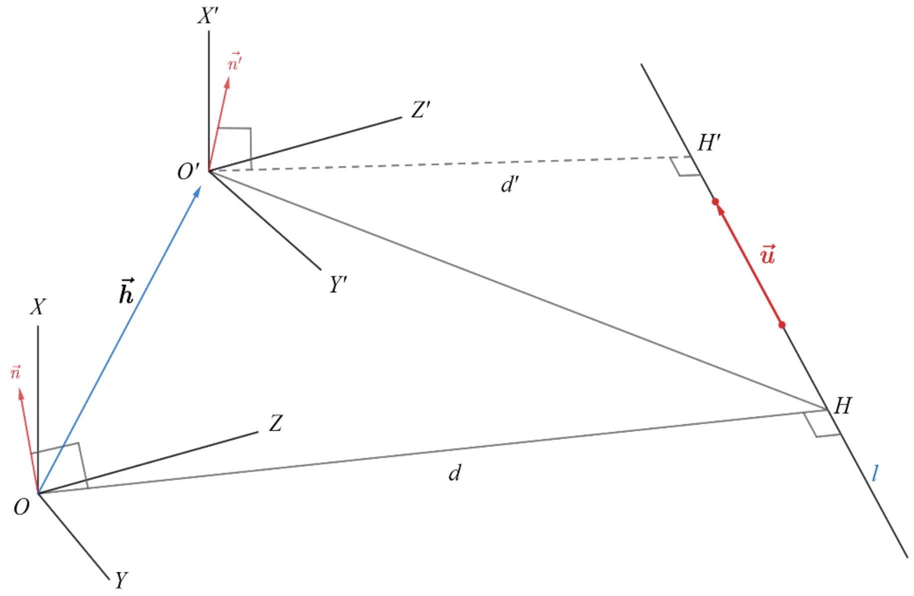


Fig. 9.7 Definition of motion parallax of a line l.

Research:

How to find the translation and rotation $\{\mathbf{R},\ \boldsymbol{h}\}$from the space line correspondences.