Discrete II

Principles of counting

Theorem 1: If an operation contains two steps, of which the first can be done in n_1 ways and for each of these the second can be done in n_2 ways, then the entire operation can be done in $n_1 n_2$ ways

Example 1: given two fair dice, one die is red and the other is blue, what are the total number of outcomes/ways the pair of dice can be tossed?

$n_1 = 6,\ n_2 = 6 \}\ 6(6) = 36$ ways

Theorem 2: If an operation consists of k steps, one of which can be done in n_1 ways, for each of these the 2nd step can be done in n_2 ways for each of these two steps the 3rd step can be done in n_3 ways, and so on, then the entire operation can be done in $n_1 n_2 \ldots n_k$ ways

Example 2: at a certain restraunt, there are 4 soups, 6 entrees, 5 desserts, and 8 different beverages. What are the total number of ways to order a meal?

$n_1 = 4,\ n_2 = 6,\ n_3 = 5,\ n_4 = 8 \}\ 4(6)(5)(8) = 960$

Example 3: given a 15 question T/F exam, what are the total # of possibilities? $2^{15} = 32,768$

Theorem 3: The number of permutations of n distinct objects is $n!$

Example 4: What are total number of ways to introduce the nine batters in a baseball lineup?

$9! = 362,880$

Theorem 4: The number of permutations of n distinct objects taken r at a time is given by $\rightarrow$ ${}_nP_r = \frac{n!}{(n-r)!}$ $r = 0, 1, 2 \ldots n$

Proof: For $r = 0 \rightarrow {}_nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

For $1 \leq r \leq n \rightarrow {}_nP_r = n(n-1)(n-2)\ldots(n-r+1)$

$${}_nP_r = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{(n-r)!}$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

Example 5: From a group of 24 individuals, what are the total number of ways 4 people can be selected to rep the group?

$$\frac{24!}{(24-4)!} = 255,024$$

Theorem 5: The number of permutations of n distinct objects is arranged in a circle is $(n-1)!$

Theorem 6: The number of permutations of n objects of which n_1 are of the first kind, n_2 are of the second kind..., and n_k are of the k^{th} kind, where $n_1 + n_2 + \ldots n_k = n$, is given by:

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}$$

Example 6: What are the total number of ways to arrange the letters in the word "APOPLECTIC"?

$n = 10,\ n_1 = 2,\ n_2 = 2$

$$\frac{10!}{2!\,2!} = 907,200$$

Definition: A permutation of n distinct objects is taken r at a time with disregard for order is a combination

Theorem 7: The number of combos of n distinct objects taken r at a time is given by; ${}_nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$

Example 7: If a fair coin is flipped eight times, what are the total number of ways you can have a result of 3 heads and 5 tails?

same thing

$${}_8C_3 = \frac{8!}{3!(8-3)!} = \frac{8!}{3!\,5!} = 56$$

$${}_8C_5 = \frac{8!}{5!(8-5)!} = \frac{8!}{5!\,3!} = 56$$

2/4

Theorem 8: The number of ways in which a set of n distinct objects can be partioned into k subsets with n_1 objects in the 1st subset, n_2 objects in the 2nd subset, and ... n_k objects in the k^{th} subset is given by:

$$\frac{n!}{n_1!\,n_2!\ldots n_k!}$$

example 8: At a certain college, one particular residence hall is composed of suites containing 3 bedrooms, with each suite housing 8 students. What are the total number of ways the 8 students may be assigned to 2 triple bedrooms and one double bedroom?

$$\frac{8!}{3!\,3!\,2!} = \frac{40,320}{72} = 560 \text{ ways}$$

Theorem 9:

$$(x+y)^n = \sum_{r=0}^{n} \binom{c}{n\,r} x^{n-r} y^r$$ for any $n > 0$

(generates binomial coeff)

Elementary probability

Definition 2: given an event ~~which~~ having prob. measure with N total outcomes and n favorable outcomes, the prob. of the event is given by the ratio

$\frac{n}{N}$, $n \leq N$, $N > 0$

Basics: ① A fair coin is a coin that when tossed and 2 and only 2 possible outcomes equally likely to occur, heads, H, tails, T

② A fair die...

③ A standard deck of cards...

④ Roulette wheel → 38 numbers,

00, 0, 1, 2, 3 ... 35, 36

00, 0 → green, 1-36 red/black, odd/even

⑤ Mega Millions ⇒ select any 5 distinct ways

$70 \cdot 69 \cdot 68 \cdot 67 \cdot 66 = 1,452,361,680$

"Mega ball" → 1-25

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to select the winning 5 #s

$$P(\text{win jackpot}) = \frac{120}{1,452,361,680} \cdot \frac{1}{25} = 1/302,575,350$$

2/6 Notes:

Definition 3 - Any process of observations or measurements is called an experiment. The obtained results of an experiment are called outcomes of the experiment.

Definition 4 - The set of all possible outcomes of an experiment is called the sample space. If a sample space contains a finite number (countable) or infinitely countable events, then the sample space is considered discrete.

Example 9: You roll a pair of fair dice, then add together the number appearing on each face. What is the prob. that the sum is divisible by 3?

$n = 36$

Sum	Freq
3	2
6	5
9	4
12	1

$P(3 \mid \text{sum}) = 12/36$

Definition 5: The prob. of an event is a non-negative real number; that is for any subset A of S, $P(A) \geq 0$

Definition 6: $P(S) = 1$

Definition 7: If $A_1, A_2 \ldots$ is a finite or infinite sequence of mutually exclusive events of S, then $P(A_1 \text{ or } A_2 \text{ or} \ldots) = P(A_1) + P(A_2) + \ldots)$

$P(A_1 \cup A_2 \cup \ldots)$

Theorem 10: If A is an event in a discrete sample space S, then the prob. of A = sum of the prob. of the individual outcomes comprising A.

Example 10: A die is loaded such that an odd number is three times as likely to appear as an even number. What is the prob. that the roll of the die results in a # greater than 3?

Let X = the probability the die lands on an even number

$3X$ = the probability the die lands on an odd number

$3X + X + 3X + X + 3X - X = 1$; $12X = 1$ $X = 1/12$

P(even number) = 1/12

P(odd number) = 3/12

P(die is > 3) = 5/12

2/11 Notes

Theorem 11 → If an experiment can result in any one of N equally likely, different outcomes and if n of these outcomes together constitute event A, then the probability of event A is given by:

$P(A) = \frac{n}{N}$, $n \leq N$

Proof: Let $O_1 O_2 \ldots O_n$ represent the individual outcomes in S, each with prob. $\frac{1}{N}$. If A is the union of n of these equally exclusive outcomes, and it doesn't matter which ones, then $P(A) = P(O_1 O_2 \ldots O_n)$ or $P(A) = (O_1 \cup O_2 \cup \ldots \cup O_n)$, $P(O_1) + P(O_2) + \ldots + P(O_n) = \frac{1}{N} + \frac{1}{N} + \ldots + \frac{1}{N} = \frac{n}{N}$

Example 11: A 5 card poker hand is delt from a standard deck of cards, what is the probability of being delt three of a kind and a pair?

$N = {}_{52}C_5 = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 (\cancel{52} - 5)}$ = Total # of ways to deal cards

$n = \binom{4}{3}(13)\binom{4}{2}(12) = 3744$

P(full house) = $\frac{3744}{2{,}598{,}960}$

Theorem 12: If A and A' are complementary events in a sample space S, the $P(A') = 1 - P(A)$

Proof: $A \cup A' = S$ $\{P(S) = 1\}$ $1 = P(A \cup A') = P(A \text{ or } A') = P(A) + P(A')\}$

$P(A') = 1 - P(A)$

Theorem 13: $P(\{\}) = 0$ for any sample spaces S

Proof: $S \cup (\{\}) = S$, $\therefore P(S \text{ or } \{\}) = P(S)$ or $P(\{\}) = P(S)$

Theorem 14: If A and B are events in a sample space S and $A \subset B$, then $P(A) \leq P(B)$

Proof: since $A \subset B$, $B = A \cup (A' \cap B)$

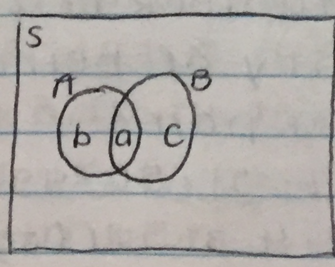
$P(B) = P(A) + P(A' \text{ and } B) \geq P(A)$

Theorem 15: For any event A, $0 \leq P(A) \leq 1$

Proof: $\{\} \subset A \subset S$ $\{P(\{\}) \leq P(A) \leq P(S)\}$ $0 \leq P(A) \leq 1$

Theorem 16: If A and B are any two events in a sample space S, then the $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Proof:



Let A and B = a

A and ¬B = b

¬A and B = c

$P(A \text{ or } B) = a + b + c$

$P(A \text{ or } B) = (a+b) + (c+a) - a$

$= P(A) + P(B) - P(A \text{ and } B)$

Theorem 17: $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \text{ and } B) - P(A \text{ and } C) - P(B \text{ and } C) + P(A \text{ and } B \text{ and } C)$

2/18 Notes

Definition 10: If A and B are two events in a sample space S and A and B are not independent events, then A and B are considered to be dependent events.

Theorem 21: If A and B are independent events, then A and B' are also ind. events

Proof: Let A = (A and B) or (A and B'), where (A and B) and (A and B') are mutually exclusive. Since A and B are ind. events,

$P(A) = P[(A \text{ and } B) \text{ or } (A \text{ and } B')]$

$P(A) = P(A)P(B) + P(A \text{ and } B')\Big\} P(A \text{ and } B') = P(A) - P(A)P(B)$

$P(A \text{ and } B') = P(A)[1 - P(B)]\Big\} P(A \text{ and } B') = \underbrace{P(A)P(B')}_{\text{independent}}$

Definition 11: Events $A_1 A_2 \ldots A_k$ are ind. if and only if the prob. of the intersection of any 2, 3 ..., k of these events equals the product of their respective prob.

Example 15: A fair coin is tossed three times.

Let A = a head occurs on the first 2 tosses

B = a tail occurs on the third toss

C = exactly two tails occur in 3 tosses

What events are independent or dependent?

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

$P(A) = 2/8$

$P(B) = 4/8$

$P(C) = 3/8$

$P(A \text{ and } B) = \frac{1}{8}$

$P(A)P(B) = \frac{2}{8} \cdot \frac{4}{8} = \frac{1}{8}$

A and B are ind. events

$P(A \text{ and } C) = 0$

$P(A)P(C) \neq 0$

A and C are not ind. events

$P(B \text{ and } C) = 2/8$

$P(B)P(C) = \frac{4}{8} \cdot \frac{3}{8} = \frac{12}{64} \neq \frac{2}{8}$ } B and C are dependent *good test?*

Example 16: Due to a worker strike, a construction may now not be completed by its deadline. The respective probabilities are .6 ⇒ there will be a strike, .35 → the job is completed on time if there is a strike and .85 → the job is completed on time if there is no strike. What is the probability the job is completed on time?

Let B = there is a strike, A = job is completed on time

$P(A) = P(A \text{ and } B) \text{ or } P(A \text{ and } B')$

$P(B) = .6,\ P(A|B) = .35,\ P(A|B') = .85$

$P(A) = (.6)(.35) + (.4)(.85) = .55$

Definition 12: If S is a sample space with probability measure and X is a real-valued function defined over the elements of S, then X is called a random variable

2/13 Notes

At a certain restaurant, a particular meal is offered costing \$20. After the meal you have the opportunity to roll a pair of dice. If you roll a

1) "2" the meal is 1/2 price
2) "7" it's free
3) "11" it's \$12

What can you expect to pay for your meal?

$P(2) = 1/36$, $P(7) = 6/36$, $P(11) = 2/36$

$E(x) = (\$2)(1/36) + 0(6/36) + (10)(2/36) + 20(27/36) =$ \$15.61

Definition 8: If A and B are two events in a sample space S and $P(A) = 0$, then the conditional prob. of B given A is given by:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Theorem 18: If A and B are any two events in a sample space S, and $P(A) \neq 0$, then

$$P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$$

Example 13:

In Business \ service	good under warranty	poor under warranty
10 or more years	16	4
Less than 10 years	10	20

Let G = good service

Let LT = less than 10 yrs

$P(LT \text{ and } G) = \frac{10}{50}$, $P(LT) = \frac{30}{50}$

$P(G|LT) = \frac{10/50}{30/50} = 10/30$

Example 14: With regard to ex. 10, what is the pro

a) the face of the die is perfect square?

b) the face of the die is a perfect square provided the die is > than 3?

Recall: $P(\text{odd}) = \frac{3}{12}$, $P(\text{even}) = \frac{1}{12}$ $\Big\}$ $A = \{1, 4\}$, $B = \{4, 5, 6\}$

a) $P(\text{perfect square}) = P(A) = 4/12$, $P(B) = 5/12$,
$P(A \text{ and } B) = 1/12$

b) $1/12 = 4/12 \, P(A \mid B) \Big\}$ $P(A \mid B) = \frac{1}{4}$

Theorem 20: If A, B, and C are three events in a sample space S such that $P(A \text{ and } B) \neq 0$, then:

$P(A \text{ and } B \text{ and } C) = P(A)\,P(B \mid A)\,P(C \mid A \text{ and } B)$

Proof: $P(A \text{ and } B \text{ and } C) = P[(A \text{ and } B) \text{ and } C]$

$P(A \text{ and } B \text{ and } C) = P(A \text{ and } B)\,P(C \mid A \text{ and } B)$

$= P(A)\,P(B \mid A)\,P(C \mid A \text{ and } B)$

Example 15: A box contains $n = 20$ fuses, of which 5 are defective. 3 of the fuses are randomly chosen. What is the probability that, if the fuses are selected w/o replacement are all defective?

Let A, B and C correspond to fuse 1, fuse 2, and fuse 3

$P(A) = \frac{5}{20}$ $\quad P(B \mid A) = \frac{4}{19}$ $\quad P(C \mid A \text{ and } B) = \frac{3}{18}$

$P(A \text{ and } B \text{ and } C) = \left(\frac{5}{20}\right)\left(\frac{4}{19}\right)\left(\frac{3}{18}\right) = \frac{60}{6840}$

Definition 9: Two events A and B are independent events if and only if, $P(A \text{ and } B) = P(A)\,P(B)$

called a random variable

2/20 Notes:

Example 17: A dresser drawer contains five brown socks and three green socks. Two socks are randomly selected from the drawer. Let X be the discrete random variable corresponding to the number of brown socks selected. What is the probability distribution, $f(X)$, for the random variable X?

★this type of problem will be on the exam★

solution: possible values for $x = 0, x = 1, x = 2$

$$\left.\begin{aligned} P(X=2) &= \left(\tfrac{5}{8}\right)\left(\tfrac{4}{7}\right) = \tfrac{20}{56} \\ P(X=0) &= \left(\tfrac{3}{8}\right)\left(\tfrac{2}{7}\right) = \tfrac{6}{56} \\ P(X=1) &= \left(\tfrac{5}{8}\right)\left(\tfrac{3}{7}\right)(2) = \tfrac{30}{56} \end{aligned}\right\} \text{must total } 1$$

Two ways to pull out a brown sock
green + brown or brown + green

$$f(x) = \begin{cases} 0 & x > 2 \\ \frac{20}{56} & x = 2 \\ \frac{30}{56} & 1 \le x < 2 \\ \frac{6}{56} & 0 \le x < 1 \\ 0 & x < 0 \end{cases}$$

cumulative Distribution

$$F(x) = \begin{cases} 1 & x \ge 2 \\ \frac{36}{56} & 1 \le x < 2 \\ \frac{6}{56} & 0 \le x < 1 \\ 0 & x < 0 \end{cases}$$

If X is a discrete random variable:

a) $f(x)$ is a probability distribution

b) $F(x)$ is a cumulative distribution

If X is a continuous random variable:

a) $f(x)$ is a probability density function

b) $F(x)$ is a cumulative density function (requires use of integral calculus)

Example 18: You flip a biased coin four times. Heads are three times as likely to appear than tails. Let X be the discrete random variable denoting the number of heads that appear. What is the probability distribution, F(X) for X?

Solution: $X = 0, 1, 2, 3, 4 \} \; P(H) = \frac{3}{4}, \; P(T) = \frac{1}{4}$

$P(X=4) = \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \frac{81}{256}$

$P(X=0) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{256}$

$P(X=3) = \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)(4) = \frac{108}{256}$

$P(X=1) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)(4) = \frac{12}{256}$

$P(X=2) = \qquad = \frac{54}{256}$

$$F(X) = \begin{cases} 0 & X > 4 \\ \frac{81}{256} & X = 4 \\ \frac{108}{256} & 3 \le X < 4 \\ \frac{54}{256} & 2 \le X < 3 \\ \frac{12}{256} & 1 \le X < 2 \\ \frac{1}{256} & 0 \le X < 1 \\ 0 & X < 0 \end{cases}$$