# NFW Phase Space Distribution

Eric Putney[1]

[1]*Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08554, USA*

June 9, 2021

## 1 Introduction

We want to find the density function $f(\mathcal{E})$ for the Navarro-Frenk-White (NFW) profile, a spatial mass distribution commonly used to fit dark matter halos, using Eddington's formula.

### 1.1 Basic Statements

For any spherical stellar system confined by a known spherical potential, it is possible to derive a unique *ergodic* density function that depends on the phase-space coordinates only through the Hamiltonian $\mathcal{H}(\mathbf{x}, \mathbf{v})$. (Binney and Tremaine, Galactic Dynamics, page 288).

The probability density describing the spatial distribution of dark matter is called $\nu(r)$. It is defined in two ways, one of which involves the mass density $\rho(r)$:

$$\nu(r) = \frac{\rho(r)}{M} = \frac{\rho_0}{\frac{r}{a}\left(1 + \frac{r}{a}\right)^2} \frac{1}{4\pi\rho_0 a^3(\ln(1+c) - \frac{c}{1+c})} \tag{1}$$

where M is the halo mass enclosed at the cutoff radius $r_{200}$, the radius at which the mass density has fallen to 200 times the critical mass density of the universe. For many physical cases, this radius $r_{200}$ is between 6 to 16 times the NFW scale radius $a$ (Binney and Tremaine, Galactic Dynamics, page 70).

The other way to define $\nu$ is formally from the full density function $f(\mathcal{E})$:

$$\nu(r) = 4\pi \int dv\, v^2 f(\Psi - \frac{1}{2}v^2). \tag{2}$$

$\Psi$ is the relative potential, defined as $\Psi(r) = -\Phi(r)$ where $\Phi$ is the potential of the NFW profile. The relative potential $\Psi$ is always positive.

$\mathcal{E}$ is the relative energy, defined as $\mathcal{E} = \Psi - \frac{1}{2}v^2$. Since the system must be in a steady state to be ergodic, we only consider objects in these isolated systems that have a positive relative energy and are bound.

In principle, we should be able to invert the relative potential as a function of radius. This inversion lets us use some tricks (the Abel integral equation) to manipulate Equation 2 into an exact form for $f(\mathcal{E})$ in terms of $\nu$.

$$\frac{1}{\sqrt{8}\pi}\frac{d\nu}{d\Psi} = \int_0^{\Psi} d\mathcal{E} \frac{f(\mathcal{E})}{\sqrt{\Psi - \mathcal{E}}} \tag{3}$$

becomes

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}} \frac{d\Psi}{\sqrt{\Psi-\mathcal{E}}}\frac{d^2\nu}{d\Psi^2} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{d\nu}{d\Psi}\right)_{\Psi=0}\right]. \tag{4}$$

This is Eddington's formula, and we are going to use it to explicitly derive $f(\mathcal{E})$ for the NFW profile. As a note, the sign of $\Psi - \mathcal{E}$ in the square root of Equation 4 is different in both Binney & Tremaine and Bovy. I am convinced that this is a typo, since $\mathcal{E} < \Psi$ by definition of $\mathcal{E}$.

### 1.2 Dimensionless Quantities

This is all a lot easier to work with if we dump as many constants as possible in favor of dimensionless quantities. We can't eliminate them entirely, but we should be able to express radii and energies in terms of ratios. A good choice for a dimensionless radius is:

$$\tilde{r} = \frac{r}{r_{200}} \tag{5}$$

since it can only vary between 0 and 1. Note that the "concentration" $c$ of a dark matter halo is defined as $c = r_{200}/a$, meaning that the often-appearing ratio $r/a$ becomes:

$$\frac{r}{a} = c\tilde{r}. \tag{6}$$

The NFW relative potential is given by:

$$\Psi = 4\pi G\rho_0 a^2\left(\frac{\ln(1+c\tilde{r})}{c\tilde{r}} - \frac{\ln(1+c)}{c}\right) \tag{7}$$

We can group those prefactors into a constant called $\psi = 4\pi G\rho_0 a^2$ and define the dimensionless relative potential $\tilde{\Psi}$ as:

$$\tilde{\Psi} = \frac{\Psi}{\psi} = \frac{\ln(1+c\tilde{r})}{c\tilde{r}} - \frac{\ln(1+c)}{c} = \frac{\ln(1+c\tilde{r})}{c\tilde{r}} + \tilde{\Phi}_0. \tag{8}$$

Note that all derivatives $\frac{d}{d\Psi}$ can be turned into derivatives of $\tilde{\Psi}$ via the chain rule:

$$\frac{d}{d\Psi} = \frac{d\tilde{\Psi}}{d\Psi}\frac{d}{d\tilde{\Psi}} = \frac{1}{\psi}\frac{d}{d\tilde{\Psi}} \tag{9}$$

## 2 Inverting the Relative Potential

Much of the algebra involved in inverting $\Psi(r)$ can be found **here** (click-able) in Appendix H. $c\tilde{r}(\Psi)$ is found to be

$$c\tilde{r}(\Psi) = -1 - \frac{W_{-1}(-(\frac{\Psi}{\psi})e^{-(\frac{\Psi}{\psi})})}{\frac{\Psi}{\psi}} \tag{10}$$

where $W_{-1}$ is the first negative branch of the Lambert W function (also known as the Product Log function). We can define a convenience factor $A(\Psi)$ (it is dimensionless) which simplifies Equation 10 greatly

$$A(\Psi) = W_{-1}(-(\frac{\Psi}{\psi})e^{-(\frac{\Psi}{\psi})}). \tag{11}$$

Derivatives in terms of $\tilde{\Psi}$ of A will be needed, here is the first derivative:

$$\frac{dA}{d\tilde{\Psi}} = \left(\frac{\psi}{\Psi} - 1\right)\frac{A}{1+A}. \tag{12}$$

Since we were able to write the first derivative of A purely in terms of A, we can express any higher derivatives of A in terms of A using this relation recursively.

# 3 Eddington's formula for the NFW profile

## 3.1 The Probability Density

The spatial probability density $\nu(\tilde{r}$ for an NFW profile is given by:

$$\nu(\tilde{r}) = \rho(\tilde{r})\frac{1}{M} = \frac{\rho_0}{c\tilde{r}(1+c\tilde{r})^2}\frac{1}{4\pi\rho_0 a^3(\ln(1+c) - \frac{c}{1+c})} = \frac{\rho_0}{4\pi\rho_0 a^3 c\tilde{r}(1+c\tilde{r})^2 g(c)} \tag{13}$$

where $g(c) = \ln(1+c) - \frac{c}{1+c}$. Substituting Eq. 10 into this, we obtain $\nu(\Psi)$. When simplified, $\nu(\Psi)$ becomes:

$$\nu(\Psi) = \frac{-1}{4\pi a^3 g(c)}\frac{\left(\frac{\Psi}{\psi}\right)^3}{A^2\left(\left(\frac{\Psi}{\psi}\right) + A\right)}. \tag{14}$$

## 3.2 The Second Term

The second term of $f(\tilde{\mathcal{E}})$ is

$$f_2(\tilde{\mathcal{E}}) = \frac{1}{\pi^2\psi^{3/2}\sqrt{8\tilde{\mathcal{E}}}}\left(\frac{d\nu}{d\tilde{\Psi}}\right)_{\tilde{\Psi}=0}. \tag{15}$$

We need to find the first derivative of $\nu$ and then evaluate it at $\tilde{\Psi} = 0$. The algebra will be included in an appendix later (and I'll clean up my accompanying Mathematica notebook), but for now we have found that $\frac{d\nu}{d\tilde{\Psi}}$ is:

$$\frac{d\nu}{d\tilde{\Psi}} = \frac{\nu}{1+A}\left(\frac{3A\psi}{\Psi} + 2\right). \tag{16}$$

Mathematica plots of $\frac{d\nu}{d\tilde{\Psi}}$ with some reasonable test values of $\psi$ and $c$ suggest that it vanishes at $\tilde{\Psi} = 0$. Therefore $f_2(\tilde{\mathcal{E}}) = 0$ for non-zero values of $\tilde{\mathcal{E}}$.

## 3.3 The First Term

The first term of $f(\tilde{\mathcal{E}})$ is

$$f_1(\tilde{\mathcal{E}}) = \frac{1}{\pi^2\psi^{3/2}\sqrt{8}}\int_0^{\tilde{\mathcal{E}}}\frac{d\tilde{\Psi}}{\sqrt{\tilde{\Psi} - \tilde{\mathcal{E}}}}\frac{d^2\nu}{d\tilde{\Psi}^2} \tag{17}$$

which involves the second derivative of $\nu$. Luckily, we were able to express the first derivative of $\nu$ in terms of $\nu$ itself, so we may easily obtain the second derivative of $\nu$ in terms of $\nu$. We have found that $\frac{d^2\nu}{d\tilde{\Psi}^2}$ is:

$$\frac{d^2\nu}{d\tilde{\Psi}^2} = \frac{\nu}{(1+A)^2}\left(1 + \frac{A\psi}{\Psi}\right)\left(2 + \left(1 + \frac{A}{1+A}\right)\left(\frac{3A\psi}{\Psi} + 2\right)\right). \tag{18}$$

This term is much more complex, especially since it sits inside the integral over $\Psi$. We can evaluate this integral numerically for any choice of $\mathcal{E}(\mathbf{r},\mathbf{v})$ and pray that no divergences show up.

I need to be absolutely sure that Eqs. 16 and 18 are correct. I've checked my result for these derivatives against what Mathematica claims they should be, and they match when I plot them. However, Mathematica was unable to simplify the second derivative into what I had. This is probably because it is heavily and non-trivially factored.

### 3.4 Eddington's Formula for the NFW Profile

The full form of Eddington's formula for the NFW profile may now be written as:

$$f(\tilde{\mathcal{E}}) = \frac{1}{\sqrt{128}\pi^3 a^3 \psi^{3/2} g(c)} \int_0^{\tilde{\mathcal{E}}} \left[ \frac{d\tilde{\Psi}}{\sqrt{\tilde{\Psi}-\tilde{\mathcal{E}}}} \frac{-(\tilde{\Psi}-\tilde{\Phi}_0)^3}{A^2(1+A)^2(\tilde{\Psi}-\tilde{\Phi}_0+A)} \right.$$
$$\left. \left(1+\frac{A}{\tilde{\Psi}-\tilde{\Phi}_0}\right)\left(2+\left(1+\frac{A}{1+A}\right)\left(\frac{3A}{\tilde{\Psi}-\tilde{\Phi}_0}+2\right)\right)\right].$$

As a sanity check, we can verify that the dimensions of this expression are correct. $f(\tilde{\mathcal{E}})$ should have inverse dimensions of 3D phase space volume. The $\frac{1}{a^3}$ gives us 1/length$^3$, and the $\frac{1}{\psi^{3/2}}$ gives us 1/velocity$^3$.

## 4 Maxwell-Boltzmann Approximation

### 4.1 Obtaining the velocity distribution, following the halotools documentation

This work expands on what's given in the halotools documentation on their implementation of the NFW profile. *Click here to get to the docs.*

We can suppose that the radial velocities of the tracers are given by a one-dimensional MB-distribution such as:

$$f(v_r) = \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_r^2}{2kT}} \tag{19}$$

The first moment of this distribution is clearly zero, but its second moment is quite useful. This is given by:

$$\sigma_{v_r}^2 = \frac{kT}{m}. \tag{20}$$

For an NFW profile, the halotools documentation claims that $\sigma_{v_r}^2$ is also given by:

$$\sigma_{v_r}^2 = V_{virial}^2 \frac{c^2\tilde{r}(1+c\tilde{r})^2}{g(c)} \int_{c\tilde{r}}^{\infty} \frac{g(y)}{y^3(1+y)^2} dy. \tag{21}$$

Note that this integral is the worst, here it is in analytic form:

$$\frac{1}{2(c\tilde{r})^2((c\tilde{r})+1)^2}\Big[6(c\tilde{r}+1)^2(c\tilde{r})^2\mathrm{Li}_2(-(c\tilde{r}))+\pi^2(c\tilde{r})^4+3(c\tilde{r})^4\log^2((c\tilde{r})+1)+$$
$$(c\tilde{r})^4\log((c\tilde{r})+1)+2\pi^2(c\tilde{r})^3-7(c\tilde{r})^3+6(c\tilde{r})^3\log^2((c\tilde{r})+1)-$$
$$4(c\tilde{r})^3\log((c\tilde{r})+1)+\pi^2(c\tilde{r})^2-9(c\tilde{r})^2+3(c\tilde{r})^2\log^2((c\tilde{r})+1)-$$
$$((c\tilde{r})+1)^2(c\tilde{r})^2\log((c\tilde{r}))-8(c\tilde{r})^2\log((c\tilde{r})+1)-(c\tilde{r})-2(c\tilde{r})\log((c\tilde{r})+1)+\log((c\tilde{r})+1)\Big]$$

I've compared evaluating this directly vs. numerical integration, and numerical integration is much faster while retaining excellent accuracy. So we'll use numerical integration instead!

Although we ultimately want the speed distribution, we looked at the distribution of radial velocities since we know how to calculate $\sigma_{v_r}^2$. Now we can relate $\sigma_{v_r}^2$ to temperature T and plug this into the speed-version of the Maxwell-Boltzmann distribution to get:

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-\frac{mv^2}{2kT}} = \left(\frac{1}{2\pi\sigma_{v_r}^2}\right)^{3/2} 4\pi v^2 e^{-\frac{v^2}{2\sigma_{v_r}^2}}. \tag{22}$$

$\sigma_{v_r}^2$ must be re-calculated at each radius, which is the biggest bottleneck in generating tracers that sample from this distribution.

### 4.2 The (Approximate) Phase Space Distribution Function

Binney and Tremaine describe a function $f(\mathbf{x}, \mathbf{v}, t)$ that satisfies:

$$\int d^3\mathbf{x} d^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v}, t) = 1. \tag{23}$$

We can factorize $f$ into a PDF generated by the halo mass distribution $\nu(\mathbf{x})$ and a conditional velocity distribution $h(v, r)$. The PDF from the mass distribution is defined (in terms of f) as the integral over all velocities at a fixed spatial position:

$$\nu(\mathbf{x}) = \int d^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v}). \tag{24}$$

Writing $\nu(\mathbf{x})$ in terms of the mass distribution, we have:

$$\nu(\mathbf{x}) = \nu(r) = \frac{1}{4\pi a^2 g(c) r (1 + \frac{r}{a})^2}. \tag{25}$$

Since our velocity distribution is being approximated as a Maxwell-Boltzmann distribution, the conditional velocity distribution $h(v, r)$ must be:

$$h(v, r) = \left(\frac{1}{2\pi\sigma^2(r)}\right)^{\frac{3}{2}} e^{-\frac{v^2}{2\sigma^2(r)}}. \tag{26}$$

We can combine these to get f, our phase space density function:

$$f(r, v) = \frac{\rho(r) e^{\frac{-v^2}{2\sigma(r)^2}}}{\sqrt{128\pi^5}\rho_0 a^3 \sigma(r)^3 g(c)}. \tag{27}$$

## 5 Comparing Maxwell-Boltzmann and Eddington

After enforcing a cutoff at the kinematic limit (so that there aren't any stars moving fast enough to escape the potential and fly off), we can compare these two phase space distributions. This is for the following choice of paramters: the scale radius a is 15 kpc, the scale NFW halo density is $3.8 \cdot 10^{-2}$ solar masses per cubic parsec. The cutoff scale is set to $\Delta = 200$. Figure 1 shows both phase space distributions (multiplied by the 6D volume element $(4\pi rs)^2$ where $s$ is speed.

We can also take slices of these distributions at particular radii to compare their velocity distributions. This is shown in Figure 2.

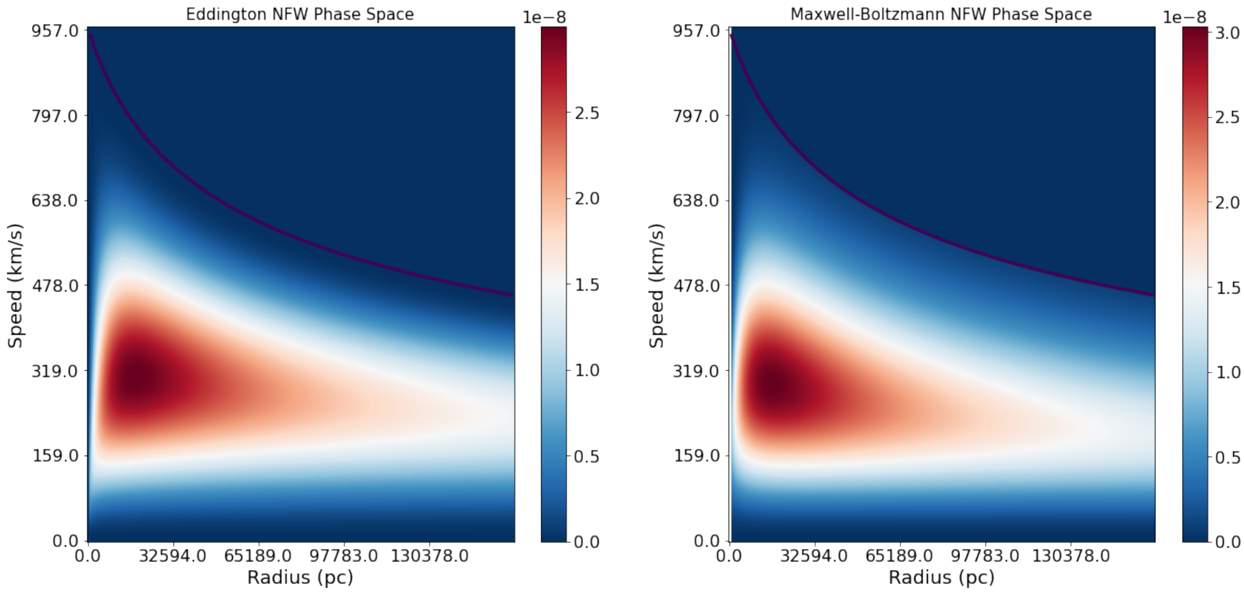


Figure 1: Comparison of Eddington and Maxwell-Boltzmann phase space distribution functions, multiplied by the 6D volume element. The dark line across the center of each figure marks the kinematic cutoff. The units here are inverse parsecs cubed times inverse kilometers per second.

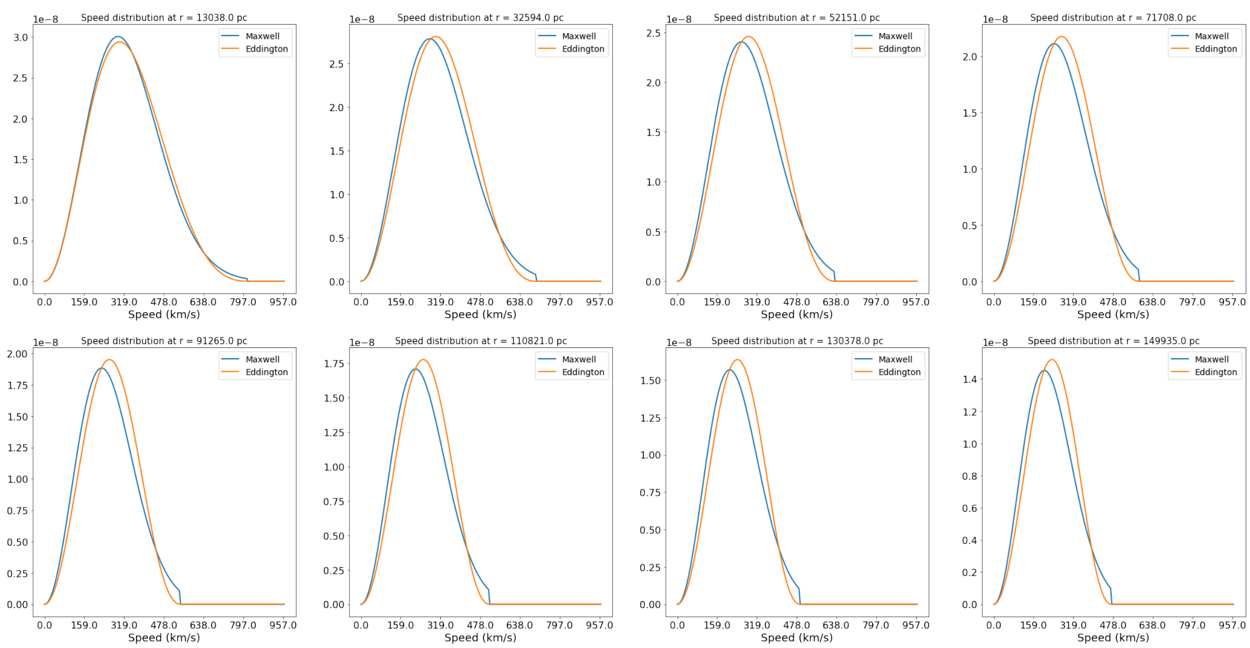


Figure 2: Velocity distributions from the Eddington and Maxwell-Boltzmann NFW phase space densities at different radii.

Maxwell-Boltzmann and Eddington seem similar enough, but there are some key differences. The transition to the kinematic cutoff is much smoother in the Eddington profile, whereas the Maxwell-Boltzmann distribution is cut off sharply along that line. The shape of the peak region (shown in red in Figure 1) is visibly different between the two as well.

There are also computational differences between the two. The Eddington phase space density is a complicated integral with no clean analytic form. It needs to be numerically integrated again and again every time it is evaluated at a different radius or speed. Meanwhile, the only "slow" step when evaluating the Maxwell-Boltzmann phase space density is obtaining the velocity dispersion. While this requires numerical integration, it is a much simpler function and only needs to be re-calculated at different radii. If subsequent calculations are required at different speeds but identical radii, the numerical integration does not need to be performed again.

To address this, my implementation of the Eddington phase space density involves a lookup table. Whenever a halo is generated I evaluate the Eddington phase space density over a 2D grid of points (radius vs. speed). I generate a cubic spline at each radius which interpolates along speed. This allows me to evaluate the phase space density at new speeds without having to numerically integrate again. When I need a radius in between radii that I generated a spline at, I linearly interpolate between the nearest two splines. This hundreds of times faster than calling the Eddington phase space distribution function every time I need it.