Curves of Minimal Length and Energy on Manifolds: Geodesics and Their Approximations

Eric Rice
Math 273C

October 1, 2018

1 Abstract

In this project we will consider the problem of minimizing length and energy along curves on a Riemannian manifold. Viewed as a variational optimization problem, we will derive the first variation for a notion of an energy function and include examples of both exact and approximate solutions of the Euler-Lagrange (geodesic) equations.

2 Preliminaries

2.1 Notation

In the entirety of this project, M is an n-dimensional Riemannian manifold taken with its Riemannian connection ∇. Given a differentiable function $f : M \to N$ between two smooth manifolds, we denote the differential of f based at $p \in M$ as

$$df_p : T_pM \to T_{f(p)}N$$

Suppose that we have a chart $\mathbf{x} : U \subset \mathbb{R}^n \to M$ about a point $p \in M$, and we consider the coordinates on U denoted $(x_1, \ldots, x_n)$. Then, there is a natural choice of basis for T_pM

$$\frac{\partial}{\partial x_i} := d\mathbf{x}_p(e_i), \quad i = 1, \ldots, n,$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$.

If $p \in M$, then $\langle \cdot, \cdot \rangle_p$ is the Riemannian metric tensor on the tangent space T_pM, and we denote by $|v| := \sqrt{\langle v, v \rangle_p}$ the norm of a vector $v \in T_pM$ induced by the inner product $\langle \cdot, \cdot \rangle_p$. The subscript p denoting the base point of the metric may be omitted, if it is obvious by context.

Consider a curve $c : [0, 1] \to M$. If X is a smooth vector field along c, the covariant derivative of X along c is denoted $\frac{D}{dt}X$.

2.2 Definitions

Take two points $p, q \in M$. Define $\mathcal{C}_p^q$ as the class of all piecewise smooth curves $c : [0,1] \to M$ such that $c(0) = p$ and $c(1) = q$. The **length** and **energy**, respectively, of $c \in \mathcal{C}_p^q$ are defined to be

$$L[c] := \int_0^1 \left|c'(t)\right| dt$$
$$E[c] := \int_0^1 \left|c'(t)\right|^2 dt$$

In pursuit of variational methods, we make the following definition.

Definition 1. [1, 9.2.1] *Let $\varepsilon > 0$. A **variation** on a curve $c \in \mathcal{C}_p^q$ is a continuous map $f : (-\varepsilon, \varepsilon) \times [0,1] \to M$, $(s,t) \mapsto f(s,t)$, such that:*

1. *$f(0,t) = c(t)$ for all $t \in [0,1]$*
2. *For some $k \in \mathbb{N}$, there exists a partition $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1$ such that $f\big|_{(-\varepsilon,\varepsilon)\times[t_{i-1},t_i]}$ is differentiable for every $i = 1, \ldots, k$.*

*In addition, if $f(s,0) = c(0)$ and $f(s,1) = c(1)$ for every $s \in (-\varepsilon, \varepsilon)$, then we call f a **proper variation**.*

Now, given some curve $c \in \mathcal{C}_p^q$, we define the length and energy functions along a variation f of c, respectively, as

$$L[c,f](s) := \int_0^1 |f_t(s,t)|\, dt$$
$$E[c,f](s) := \int_0^1 |f_t(s,t)|^2\, dt$$

Here, f_t is the partial derivative of f with respect to t. We will see later that the minimizers of L and E are related, so for the time being we concern ourselves only with E. Suppose that $\gamma \in \mathcal{C}_p^q$ is an extremal curve of E, and let $f(s,t)$ be any proper variation of $\gamma(t)$. The curves $t \mapsto f(s,t)$ also lie within $\mathcal{C}_p^q$ for every $s \in (-\varepsilon, \varepsilon)$. So, it is necessary that

$$\frac{d}{ds} E[\gamma, f](s)\bigg|_{s=0} = 0$$

This motivates the following definition.

Definition 2. *Let $c \in \mathcal{C}_p^q$ and suppose that f is a variation of c. The **first variation** of the energy $E[c,f](s)$ is*

$$\delta E[c,f] := \frac{d}{ds} E[c,f](s)\bigg|_{s=0}$$

In the next section we compute $\delta E[c,f]$ with the objective of classifying the critical points of the energy of a variation of a curve.

3 Minimization of E and L over $\mathcal{C}_p^q$

3.1 Derivation of the First Variation of E

A similar derivation is given in [1, 9.2.4]. Let $c \in \mathcal{C}_p^q$ and let f be any variation of c. We begin by differentiating $E[c,f](s)$ with respect to s.

$$\begin{aligned}\frac{d}{ds}E[c,f](s) &= \frac{d}{ds}\int_0^1 |f_t(s,t)|^2\,dt \\ &= \int_0^1 \frac{d}{ds}\langle f_t(s,t), f_t(s,t)\rangle dt\end{aligned}$$

Applying compatibility of the Riemannian connection with the Riemannian metric gives

$$\frac{d}{ds}E[c,f](s) = 2\int_0^1 \left\langle \frac{D}{ds}f_t(s,t), f_t(s,t)\right\rangle dt$$

From here, we use the symmetry $\frac{D}{ds}f_t = \frac{D}{dt}f_s$ [1, 3.3.4] along with compatibility again to see that

$$\begin{aligned}\frac{d}{ds}E[c,f](s) &= 2\int_0^1 \left\langle \frac{D}{dt}f_s(s,t), f_t(s,t)\right\rangle dt \\ &= 2\int_0^1 \left[\frac{d}{dt}\langle f_s(s,t), f_t(s,t)\rangle - \left\langle f_s(s,t), \frac{D}{dt}f_t(s,t)\right\rangle\right] dt\end{aligned}$$

If $t_0 = 0$, $t_k = 1$, and $t_1 < \ldots < t_{k-1}$ are the points that c and f fail to be differentiable in t, we have

$$\frac{d}{ds}E[c,f](s) = \sum_{i=1}^{k}\langle f_s(s,t), f_t(s,t)\rangle\Big|_{t_{i-1}}^{t_i} - \int_0^1 \left\langle f_s(s,t), \frac{D}{dt}f_t(s,t)\right\rangle dt$$

Recall that $f_t(0,t) = c'(t)$ by the definition of a variation. Setting $c'(t_i^+) = \lim_{t\to t_i^+} c'(t)$ and $c'(t_i^-) = \lim_{t\to t_i^-} c'(t)$, we evaluate the above at $s = 0$ to complete the derivation.

$$\delta E[c,f] = \sum_{i=1}^{k}\langle f_s(0,t), c'(t)\rangle\Big|_{t_{i-1}}^{t_i} - \int_0^1 \left\langle f_s(0,t), \frac{D}{dt}c'(t)\right\rangle dt$$

$$\begin{aligned}\delta E[c,f] = {} & \langle f_s(0,1), c'(1)\rangle - \langle f_s(0,0), c'(0)\rangle + \sum_{i=1}^{k-1}\langle f_s(0,t_i), c'(t_i^-) - c'(t_i^+)\rangle \\ & - \int_0^1 \left\langle f_s(0,t), \frac{D}{dt}c'(t)\right\rangle dt\end{aligned} \tag{1}$$

The equation $\delta E[c,f] = 0$ is known as the **Euler-Lagrange** equation for E, where f ranges over all proper variations of $c \in \mathcal{C}_p^q$. The solutions to such an equation are the extremal curves of E.

3.2 Consequences of the Formula for δE

We begin with a definition.

Definition 3. *Let $I \subset \mathbb{R}$ be an interval. A curve $\gamma : I \to M$ is a* ***geodesic*** *on I if γ is smooth on all of I and $\frac{D}{dt}\gamma'(t) \equiv 0$.*

We remark here that any geodesic γ is parametrized proportional to its length. That is, using compatibility of ∇ with the Riemannian metric

$$\frac{d}{dt}|\gamma'(t)| = 2\left\langle \frac{D}{dt}\gamma'(t), \gamma'(t)\right\rangle = 0$$

So, $|\gamma'(t)| = \text{const}$. If $I = [a, b]$, the length of γ is then

$$L[\gamma] = \int_a^b |\gamma'(t)|dt = |\gamma'(t)|(b-a)$$

The goal of this subsection is to use (1) to prove the following characterization:

Proposition 4. [1, 9.2.5] *Let $c \in \mathcal{C}_p^q$. Then, $\delta E[c, f] = 0$ for every proper variation f of c if and only if c is a geodesic on $[0, 1]$.*

However, we do not yet have the tools needed to solve such a problem. Let $\varepsilon > 0$. Take any geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$. Suppose that $\mathbf{x} : U \subset \mathbb{R}^n \to M$ is a chart which locally trivializes γ, and write γ in local coordinates as

$$(\mathbf{x}^{-1} \circ \gamma)(t) = (x_1(t), \dots, x_n(t)).$$

The equation $\frac{D}{dt}\gamma'(t) = 0$ may be written under these local coordinates as a system of second-order ordinary differential equations using the Christoffel symbols $\Gamma_{ij} := (\Gamma_{ij}^1, \dots, \Gamma_{ij}^n) = \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$ [1, Ch. 2].

$$x_k'' + \sum_{i,j=1}^n \Gamma_{ij}^k x_i' x_j' = 0 \qquad (k = 1, \dots, n)$$

We comment here that the Christoffel symbols corresponding to the Levi-Civita connection on M are completely determined by the Riemannian metric using the equation below.

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{m=1}^n (\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij}) g^{mk} \tag{2}$$

Now, we rewrite the system of n second-order equations as a system of $2n$ first-order equations:

$$\begin{cases} x_k' = y_k \\ y_k' = -\sum_{i,j=1}^n \Gamma_{ij}^k y_i y_j \end{cases} \qquad (k = 1, \dots, n) \tag{3}$$

Such an equation admits a unique solution trajectory given initial conditions $x_i(0) = p_i$ and $x_i'(0) = y_i(0) = v_i$ for every i. Returning to the manifold, we write $\mathbf{x}(p_1, \dots, p_n) = p$ and $d\mathbf{x}_p(v_1, \dots, v_n) = v$. Thus, for any $p \in M$ and $v \in T_pM$, there is a unique geodesic γ which satisfies $\gamma(0) = p$ and $\gamma'(0) = v$.

The interval of existence of such a geodesic is a question that must be considered. However, there are properties outlined in [1, Ch. 3, sec. 2] which allow for the well-definition of the following map.

Definition 5. *There exists an open set $\Omega \subset TM$ where the* ***exponential map*** *on M is defined to be*

$$\exp : \Omega \to M, \quad \exp(p, v) = \gamma(1),$$

where p and v are the initial conditions corresponding to the unique geodesic γ whose local expression solves (3)*. We usually wish to fix p and consider* exp *as a map*

$$\exp_p : \mathcal{U} \subset T_pM \to M, \quad \exp_p(v) = \exp(p, v)$$

where $\mathcal{U}$ is an open neighborhood of the origin in T_pM.

A few basic properties of $\exp_p$ include:

- $\exp_p(0) = p$
- $d(\exp_p)_0 = id_{T_pM}$, where for any $v \in T_pM$ we identify $T_v(T_pM)$ with the original tangent space T_pM
- The radial line $tv \in T_pM$, where $t \geq 0$, corresponds to a geodesic $\gamma(t) = \exp_p(tv)$ satisfying $\gamma(0) = p$, $\gamma'(0) = v$, for all t such that $\exp_p(tv)$ is defined. In particular, $\gamma(1) = \exp_p(v)$.

We may now prove the proposition that we had stated earlier.

Proof. (of Proposition 4*)*
The "if" statement is obvious. Suppose that $\delta E[c, f] = 0$ for every proper variation f of c and define

$$V(t) = g(t)\frac{D}{dt}c'(t),$$

where $g(t)$ is a smooth function satisfying $g(t_i) = 0$ and $g(t) > 0$ if $t \neq t_i$, for $i = 0, 1, \ldots, k$. We will construct a proper variation f such that $f_s(0, t) = V(t)$. Since $c([0, 1])$ is compact, there exists $\delta > 0$ such that $\exp_{c(t)}$ is defined for every $v \in T_{c(t)}M$ with $|v| < \delta$. Set $N = \max_{t\in[0,1]} |V(t)|$ and take any $\varepsilon \in \left(0, \frac{\delta}{N}\right)$. Then, the following is well-defined:

$$f : (-\varepsilon, \varepsilon) \times [0, 1] \to M, \quad f(s, t) := \exp_{c(t)}(sV(t))$$

Notice that $f(s, 0) = c(0)$ and $f(s, 1) = c(1)$ since $V(0) = V(1) = 0$. Differentiating, we verify that

$$\frac{\partial}{\partial s} f(s, t)\Big|_{s=0} = d(\exp_{c(t)})_{sV(t)}(V(t))\Big|_{s=0} = V(t)$$

as desired. Using this proper variation, we have by assumption and our formula (1):

$$\begin{aligned}
0 &= \delta E[c, f] \\
&= \langle V(1), c'(1)\rangle - \langle V(0), c'(0)\rangle + \sum_{i=1}^{k-1} \langle V(t_i), c'(t_i^-) - c'(t_i^+)\rangle \\
&\quad - \int_0^1 \left\langle V(t), \frac{D}{dt}c'(t) \right\rangle dt \\
&= -\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} g(t) \left|\frac{D}{dt}c'(t)\right|^2 dt
\end{aligned}$$

Since $g(t) > 0$ on each interval $[t_{i-1}, t_i]$, it follows that $\frac{D}{dt}c'(t) \equiv 0$. A similar construction to f using V allows us to construct a proper variation $\overline{f}(s,t)$ of c from $\overline{V}(t) = \overline{f}_s(0,t)$, where we take $\overline{V}(t)$ to be any smooth function satisfying $\overline{V}(0) = \overline{V}(1) = 0$ and $\overline{V}(t_i) = c'(t_i^-) - c'(t_i^+)$ for $i = 1, \dots, k-1$. Applying the first result $\frac{D}{dt}c'(t) \equiv 0$ in the formula (1) gives

$$\begin{aligned}
0 &= \delta E[c, \overline{f}] \\
&= \langle \overline{V}(1), c'(1)\rangle - \langle \overline{V}(0), c'(0)\rangle + \sum_{i=1}^{k-1} \langle \overline{V}(t_i), c'(t_i^-) - c'(t_i^+)\rangle \\
&= \sum_{i=1}^{k-1} \left| c'(t_i^-) - c'(t_i^+) \right|^2
\end{aligned}$$

It follows that $c'(t_i^-) = c'(t_i^+)$ for every i. That is, c is smooth. $\square$

3.3 Conditions for Minimization

We will see that, given some local assumptions on p and q, there is always a unique geodesic in $\mathcal{C}_p^q$ which is the minimizer of both L and E.

Definition 6. *Let $p \in M$ and suppose that there is a neighborhood of the origin $\mathcal{U} \subset T_pM$ such that $\exp_p|_{\mathcal{U}}$ is a diffeomorphism onto its image. Then, we call $U = \exp_p(\mathcal{U})$ a* ***normal neighborhood*** *of p. For any $\varepsilon > 0$ such that the closure of the open ball of radius ε centered at the origin, $\overline{B_\varepsilon(0)}$, is a subset of $\mathcal{U}$, we call $B_\varepsilon(p) := \exp_p(B_\varepsilon(0)) \subset U$ the* ***normal ball*** *of radius ε centered at p.*

We make a note here that a normal ball $B_\varepsilon(p)$ can be covered by a single coordinate chart using the composition of $\exp_p$ with the isomorphism which identifies the bases $\{e_i\}_{i=1}^n$ and $\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^n$ of $\mathbb{R}^n$ and T_pM. This system is known as the ***normal coordinates*** *about p.*

The existence of a normal ball about each $p \in M$ is established in [1, Prop. 2.9]. The conditions for minimization of L come from the results [1, Prop. 3.6, Cor. 3.9]. We state a modified version below.

Proposition 7. *Let $p \in M$ and let $B_\varepsilon(p)$ be a normal ball about p. Take any $q \in B_\varepsilon(p)$, which corresponds to a unique $v = \exp_p^{-1}(q) \in B_\varepsilon(0) \subset T_pM$. Suppose that $\gamma : [0,1] \to B_\varepsilon(p)$ is the unique geodesic segment such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then, $\gamma \in \mathcal{C}_p^q$, and it is true that*

$$L[\gamma] \leq L[c], \quad \forall c \in \mathcal{C}_p^q$$

with equality if and only if $\gamma([0,1]) = c([0,1])$. Conversely, if $\gamma \in \mathcal{C}_p^q$ satisfies $|\gamma'(t)| = \text{const.}$ and

$$L[\gamma] = \min_{c \in \mathcal{C}_p^q} L[c],$$

then γ is the unique geodesic such that $\gamma'(0) = v$.

So, as long as we take q inside of a normal ball at p, the critical points of the first variation δE will minimize length from p to q. We still wish to show that these critical points of E are in fact the minimizers of E. Suppose that q is in a normal ball at p and $\gamma \in \mathcal{C}_p^q$ is a geodesic. Then, we notice that

$$\begin{aligned} E[\gamma] &= \int_0^1 |\gamma'(t)|^2 dt \\ &= |\gamma'(t)|^2 \int_0^1 dt \\ &= \left(|\gamma'(t)| \int_0^1 dt \right)^2 \\ &= L[\gamma]^2 \\ &= \min_{c \in \mathcal{C}_p^q} L[c]^2 \end{aligned} \tag{4}$$

Furthermore, through an application of the Cauchy-Schwarz inequality using the $L^2([0,1])$ inner product, we have for an arbitrary $c \in \mathcal{C}_p^q$:

$$L[c]^2 = \left(\int_0^1 |c'(t)| dt \right)^2 \leq \left(\int_0^1 |c'(t)|^2 dt \right) \left(\int_0^1 dt \right) = E[c] \tag{5}$$

With this, we state the proposition that completes our goal for this section.

Proposition 8. [1, 9.2.3] *Let $p \in M$ and take any point q within a normal ball at p. Let $\gamma \in \mathcal{C}_p^q$. Then, the following are equivalent:*

(i) γ is the unique geodesic such that $\gamma'(0) = \exp_p^{-1}(q)$.

(ii) $|\gamma'(t)| = \text{const.}$ and γ is the minimizer for L over $\mathcal{C}_p^q$.

(iii) γ is the minimizer for E over $\mathcal{C}_p^q$.

Proof. The equivalence of *(i)* and *(ii)* has already been established Proposition 7. Suppose *(i)* holds. Then, by (4), (5), and *(ii)*

$$E[\gamma] = L[\gamma]^2 = \min_{c \in \mathcal{C}_p^q} L[c]^2 \leq \min_{c \in \mathcal{C}_p^q} E[c]$$

Thus, *(iii)* follows. Conversely, suppose that *(iii)* is true. By Proposition 4, γ is a geodesic connecting p and q. Assume by way of contradiction that γ is not length minimizing. Then, set $\tilde{\gamma} \in \mathcal{C}_p^q$ as the unique geodesic such that $\gamma'(0) = \exp_p^{-1}(q)$. Then,

$$L[\tilde{\gamma}]^2 < L[\gamma]^2$$

However, applying (4) and noting that both γ and $\tilde{\gamma}$ are parametrized proportional to length gives the contradiction

$$E[\tilde{\gamma}] = L[\tilde{\gamma}]^2 < L[\gamma]^2 = E[\gamma]$$

$\square$

4 Strategy for Approximation

Our goal is to use a numerical integrator to produce a coordinate approximation of the minimizing geodesic between two points $p, q \in M$ within a normal ball about p. In this scenario, the initial value problem (3) becomes a boundary value problem. It is a standard approach, then, to apply a shooting method on the initial velocities in conjunction with a root-finding method to approximate the solution to the boundary value problem. We describe this precisely in the following paragraphs.

Let $p \in M$ and take any q within a normal ball $B_\varepsilon(p)$. We denote by $\gamma : [0, 1] \to M$ the unique minimizing geodesic such that $\gamma(0) = p$, $\gamma(1) = q$. We will use the $\hat{}$ accent to refer to the (normal) coordinate expression of any of these objects.

At the core of our strategy is our ability to numerically integrate the (first order) geodesic equation (3) in a coordinate representation given an initial position $\hat{p}$ and an initial velocity $v = \hat{v}$. To accomplish this, we apply the common fourth-order Runge-Kutta method (RK4) [4, 2.4] whose flow map with time step h is denoted Ψ_h. Given an initial iterate z_0 and vector field f, we define this recursively:

$$\begin{cases} Z_1 = z_k \\ Z_2 = z_k + \frac{h}{2} f(Z_1) \\ Z_3 = z_k + \frac{h}{2} f(Z_2) \\ Z_4 = z_k + h f(Z_3) \\ \Psi_h(z_k) = z_{k+1} = z_k + \frac{h}{6}\left[f(Z_1) + 2f(Z_2) + 2f(Z_3) + f(Z_4)\right] \end{cases} \qquad (k \in \mathbb{N})$$

We remark that the flow of the geodesic equation (3) is actually Hamiltonian, and a symplectic integrator may be more appropriate in many cases. However, the Hamiltonian function of such a system is non-separable, in general, which complicates the application of such an integrator [3]. We do not have our sights set on the long-time approximation of geodesics, so RK4 is accurate enough for our purposes.

If $h > 0$ is a time step and $N \in \mathbb{N}$ such that $Nh = 1$, then the composition $\Psi_h^N((\hat{p}, v)) = (\Psi_h \circ \overset{N}{\cdots} \circ \Psi_h)((\hat{p}, v))$ will give an approximation of $\widehat{\exp_p}(v)$. For any $v \in T_pM$ such that $|v| < \varepsilon$, we define

$$F(v) := \widehat{\exp_p}(v) - \hat{q}$$

This reduces the boundary value problem $\frac{D}{dt}\gamma'(t) = 0$, $\gamma(0) = p$, $\gamma(1) = q$ to the problem of finding a root $\tilde{v}$ of F. That is, γ must be the geodesic obtained by integrating (3) with the initial velocity $\tilde{v}$.

Proceeding via Newton's method, we have the following Newton update, given some initial guess vector $v_0 \in T_pM$, $|v| < \varepsilon$:

$$v_{k+1} = v_k - dF_{v_k}^{-1}(F(v_k))$$

Calculating this update requires that we approximate the matrix dF_{v_k} via a finite difference method and then invert the approximation. Repeating the Newton update until our current iterate v_k satisfies $|F(v_k)| < \delta$, for some error bound $\delta > 0$, we integrate one last time via $\Psi_h^N((\hat{p}, v_k))$ to achieve the approximation of γ.

5 The Sphere S^n

In the case of the sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, an explicit classification of all geodesics is not too difficult.

Take a point $p \in S^n \subset \mathbb{R}^{n+1}$ and a nonzero tangent vector $v \in T_pS^n$. There is a natural identification $T_pS^n \leftrightarrow (\text{span}\{p\})^\perp$, so we view v as a vector in $\mathbb{R}^{n+1}$ orthogonal to p. Then, $\text{span}\{p, v\}$ is a plane in $\mathbb{R}^{n+1}$ containing the origin, whose intersection with S^n is a great circle G. Parametrize a segment of G via

$$\gamma : [0,1] \to G, \quad \gamma(t) = \cos(|v|t)p + \sin(|v|t)\frac{v}{|v|} \tag{6}$$

We remark that $\gamma(t)$ is indeed a point of G for every t because $\gamma(t) \in \text{span}\{p, v\}$ and

$$\begin{aligned}
|\gamma(t)|^2 &= \left\langle \cos(|v|t)p + \sin(|v|t)\frac{v}{|v|}, \cos(|v|t)p + \sin(|v|t)\frac{v}{|v|} \right\rangle \\
&= \cos^2(|v|t)|p|^2 + \frac{2\cos(|v|t)\sin(|v|t)}{|v|}\langle p, v\rangle + \frac{\sin^2(|v|t)}{|v|^2}|v|^2 \\
&= \cos^2(|v|t) + \sin^2(|v|t) \\
&= 1
\end{aligned}$$

The first and second derivatives of γ are

$$\gamma'(t) = -|v|\sin(|v|t)p + \cos(|v|t)v, \quad \gamma''(t) = -|v|^2\cos(|v|t)p - |v|\sin(|v|t)v$$

Below, we show that γ' and γ'' are orthogonal for all t.

$$\begin{aligned}
\langle \gamma'(t), \gamma''(t)\rangle &= \langle -|v|\sin(|v|t)p + \cos(|v|t)v, -|v|^2\cos(|v|t)p - |v|\sin(|v|t)v\rangle \\
&= |v|^3\sin(|v|t)\cos(|v|t)|p|^2 + |v|^2\left[\sin^2(|v|t) - \cos^2(|v|t)\right]\langle p, v\rangle \\
&\qquad - |v|\sin(|v|t)\cos(|v|t)|v|^2 \\
&= |v|^3\sin(|v|t)\cos(|v|t) - |v|^3\sin(|v|t)\cos(|v|t) \\
&= 0
\end{aligned}$$

Since $\frac{D}{dt}\gamma'(t)$ is the projection of $\gamma''(t)$ onto $T_\gamma(t)S^n$, this shows that $\frac{D}{dt}\gamma'(t) = 0$. That is, γ is the unique geodesic satisfying the initial conditions $\gamma(0) = p$, $\gamma'(0) = v$.

Now, recall the definition $\exp_p(v) = \gamma(1)$, and suppose for the moment that $|v| \leq \pi$. Since $\gamma([0,1])$ is a segment of a half great circle, and any two distinct great circles intersect only at antipodal points of the sphere, it follows that $\exp_p$ may only fail to be injective at the value $-p$. That is, when $|v| = \pi$. So, for any p, the set $S^n \setminus \{-p\} = \bigcup_{\varepsilon>0} B_{(\pi-\varepsilon)}(p)$ is a normal ball about p.

We now wish to adapt this framework to give a parametrization of a geodesic γ between two points $p, q \in S^n$, where $q \neq -p$. Let $\alpha = \cos^{-1}(\langle p, q\rangle)$ be the angle between p and q, taken as a number between 0 and π. The projection of q onto the subspace T_pS^n is $q - \langle p, q\rangle p$. With this, we take v to be the vector of length α in the direction of the projection. That is,

$$v = \alpha\left(\frac{q - \langle p, q\rangle p}{|q - \langle p, q\rangle p|}\right) \tag{7}$$

Basic geometry gives us the equations $\sin(\alpha) = |q - \langle p, q\rangle p|$ and $\cos(\alpha) = \langle p, q\rangle$. If $\gamma : [0,1] \to S^n$ is the unique geodesic with initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$ as in (6), then

$$\begin{aligned}
\gamma(1) &= \cos(|v|)p + \sin(|v|)\frac{v}{|v|} \\
&= \cos(\alpha)p + \sin(\alpha)\left(\frac{q - \langle p, q\rangle p}{|q - \langle p, q\rangle p|}\right) \\
&= \langle p, q\rangle p + q - \langle p, q\rangle p \\
&= q
\end{aligned}$$

Thus, using (6) and (7), we can determine a unique (exact) length- and energy-minimizing geodesic for any two (non-antipodal) points $p, q \in S^n$. We will now use this exact formula to compare with our approximation results.

For $n = 2$, we represent $S^2 \subset \mathbb{R}^3$ (without the north and south poles) with the chart

$$\mathbf{x} : (0, \pi) \times [0, 2\pi) \to S^2, \quad \mathbf{x}(\theta, \varphi) = \begin{bmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{bmatrix}$$

Let e_1 and e_2 be the standard basis vectors of $\mathbb{R}^2$. Taking the Euclidean metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$, the metric g induced on S^2 is defined by

$$g_{ij} = \langle d\mathbf{x}(e_i), d\mathbf{x}(e_j)\rangle$$

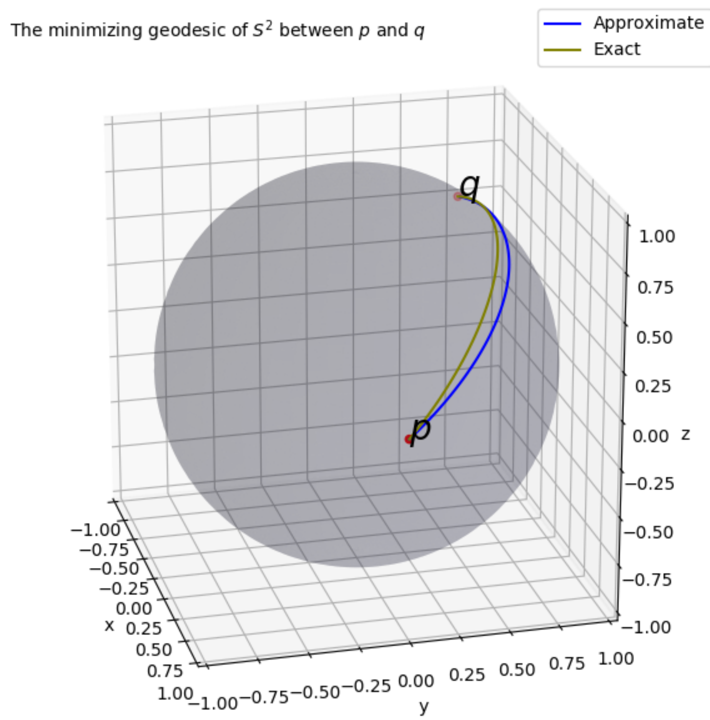
The differential of $\mathbf{x}$ is given below:

$$d\mathbf{x}_{(\theta,\varphi)} = \begin{bmatrix} \cos(\theta)\cos(\varphi) & -\sin(\theta)\sin(\varphi) \\ \cos(\theta)\sin(\varphi) & \sin(\theta)\cos(\varphi) \\ -\sin(\theta) & 0 \end{bmatrix}$$

With this, we derive the matrix $g = [g_{ij}]$ and its inverse $g^{-1} = [g^{ij}]$ as

$$g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2(\theta)} \end{bmatrix}$$

With this, we can compute the Christoffel symbols via (2) and integrate the geodesic equation. For our example, we choose $\hat{p} = \left(\frac{\pi}{2}, 0\right)$ and $\hat{q} = \left(\frac{\pi}{4}, \frac{2\pi}{3}\right)$. The comparison of the exact and approximate geodesics between $p = (1, 0, 0)$ and $q = \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{2}\right)$ is given below.



6 The Torus T^2

Consider the Torus T^2 as a submanifold of $\mathbb{R}^3$ with inner and outer radii r and R ($0 < r < R$), respectively, parametrized by

$$\mathbf{x} : [0, 2\pi) \times [0, 2\pi) \to T^2, \qquad \mathbf{x}(\theta, \varphi) = \begin{bmatrix} (R + r\cos(\theta))\cos(\varphi) \\ (R + r\cos(\theta))\sin(\varphi) \\ r\sin(\theta) \end{bmatrix}$$

Let e_1 and e_2 be the standard basis vectors of $\mathbb{R}^2$. Taking the Euclidean metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$, the metric g induced on T^2 is defined by

$$g_{ij} = \langle d\mathbf{x}(e_i), d\mathbf{x}(e_j) \rangle$$

The differential of $\mathbf{x}$ is given below:

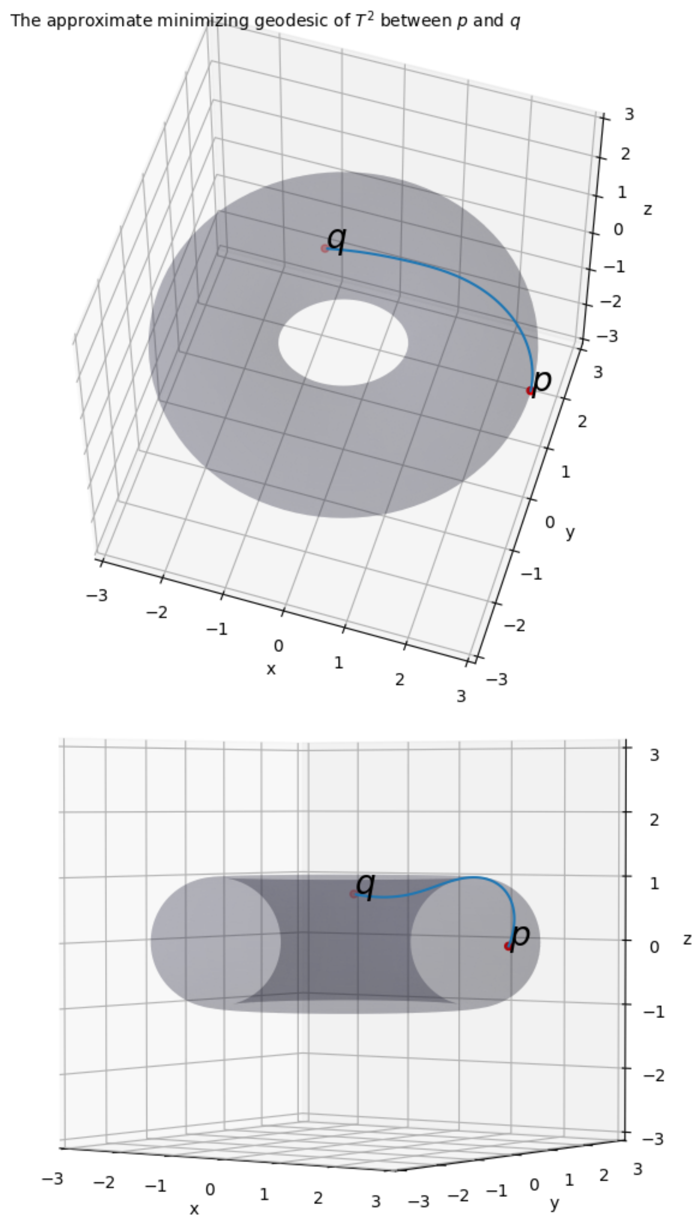
$$d\mathbf{x}_{(\theta,\varphi)} = \begin{bmatrix} -r\sin(\theta)\cos(\varphi) & -(R + r\cos(\theta))\sin(\varphi) \\ -r\sin(\theta)\sin(\varphi) & (R + r\cos(\theta))\cos(\varphi) \\ r\cos(\theta) & 0 \end{bmatrix}$$

With this, we derive the matrix $g = [g_{ij}]$ and its inverse $g^{-1} = [g^{ij}]$ as

$$g = \begin{bmatrix} r^2 & 0 \\ 0 & (R + r\cos(\theta))^2 \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{(R + r\cos(\theta))^2} \end{bmatrix}$$

Computing the Christoffel symbols with the formula (2), we can integrate the geodesic equation.

For our example, we choose $R = 2$, $r = 1$, $\hat{p} = (0, 0)$, and $\hat{q} = \left(\frac{3\pi}{4}, \frac{2\pi}{3}\right)$. A figure showing the approximate minimizing geodesic between $p = (3, 0, 0)$ and $q = \left(-1 + \frac{\sqrt{2}}{4}, \sqrt{3} - \frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{2}\right)$ is given below.



7 Further Topics

There are many more results that we did not have time to cover which provide additional information about minimizing geodesics. We list a few of them here:

- The Hopf-Rinow Theorem
- The Cut Locus and Injectivity Radius
- Jacobi Fields, Conjugate Points, and Rauch's Theorem

For the purpose of providing graphs as a visualization of our approximate geodesics, we have chosen examples that embed within $\mathbb{R}^3$. However, we remark that the approximation framework we have developed in Python will apply (though, possibly less efficiently) to higher dimensional manifolds.

8 Python Code

The complete source code may be downloaded at

https://github.com/EricRiceUCSD/MinimizingGeodesics

References

[1] do Carmo, Manfredo Perdigão; *Riemannian Geometry*, 1992.

[2] P.A. Absil, R. Mahony, and R. Sepulchre; *Optimization Algorithms on Matrix Manifolds*, 2008.

[3] J.P. Dedieu, D. Nowicki; *Symplectic methods for the approximation of the exponential map and the Newton iteration on Riemannian submanifolds*, via sciencedirect.com, 2004.

[4] B. Leimkuhler, S. Reich; *Simulating Hamiltonian Dynamics*, 2004.