Solving $Ax = b$ Using Iterative Methods

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Source code for all computations is given at the end of this submission.

 # 1. *Consider the matrix $A \in \mathbb{R}^{4\times 4}$ and the vector $b \in \mathbb{R}^4$ given by*

$$A = \begin{bmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ 1 \\ -5 \\ -8 \end{bmatrix}.$$

Use the starting vector $x^{(0)} = 0$ and conduct five steps each of the Jacobi iteration, the forward Gauss-Seidel iteration, and compute the Euclidean norms of the errors.

Let $x^* = (-2, 1, 3, -1)$. It is easily verified that x^* is an exact solution to $Ax = b$. We use this x^* in our calculation of the errors at each step of the Jacobi and Gauss-Seidel iterations.

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Jacobi Calculation for (1):

x^(1) = { -1, 0.5, 2.5, -2 }
    with error ||x* - x^(1)|| = 1.58114

x^(2) = { -1.75, 1.25, 2.75, -1.5 }
    with error ||x* - x^(2)|| = 0.661438

x^(3) = { -2, 1, 3.125, -1.125 }
    with error ||x* - x^(3)|| = 0.176777

x^(4) = { -2.03125, 1.0625, 3, -1 }
    with error ||x* - x^(4)|| = 0.0698771

x^(5) = { -2.01563, 0.984375, 3.03125, -0.984375 }
    with error ||x* - x^(5)|| = 0.0413399


Gauss-Seidel Calculation for (1):

x^(1) = { -1, 0, 2.5, -1.5 }
    with error ||x* - x^(1)|| = 1.58114
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x^(2) = { −1.625, 0.9375, 2.96875, −1.1875 }
    with error ||x* − x^(2)|| = 0.425046

x^(3) = { −1.97656, 0.996094, 2.99805, −1.01172 }
    with error ||x* − x^(3)|| = 0.0265654

x^(4) = { −1.99854, 0.999756, 2.99988, −1.00073 }
    with error ||x* − x^(4)|| = 0.00166034

x^(5) = { −1.99991, 0.999985, 2.99999, −1.00005 }
    with error ||x* − x^(5)|| = 0.000103771
```

\# **2.** *Consider the symmetric positive definite matrix* $A \in \mathbb{R}^{4\times 4}$ *and the vector* $b \in \mathbb{R}^4$ *given by*

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & 2 & 2 \\ -1 & 2 & 3 & 1 \\ -1 & 2 & 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 1 \\ 6 \\ -7 \end{bmatrix}.$$

Use the starting vector $x^{(0)} = 0$ *and conduct two steps of the steepest descent algorithm. Compute the errors for each approximate solution.*

Let $x^* = (-1, 1, 2, -3)$. It is easily verified that x^* is the exact solution to $Ax = b$. We use this x^* in our calculation of the errors at each step of the steepest descent iteration.

```
Steepest Descent Calculation for (2):

x^(1) = { −0.39726, 0.39726, 2.38356, −2.78082 }
    with error ||x* − x^(1)|| = 0.960078

x^(2) = { −0.493414, 0.493414, 2.45349, −2.69341 }
    with error ||x* − x^(2)|| = 0.901616
```

 # 3. *Consider the symmetric positive definite matrix $A \in \mathbb{R}^{3\times3}$ and the vector $b \in \mathbb{R}^3$ given by*

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 9 & 1 \\ 2 & 1 & 16 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 18 \\ 16 \end{bmatrix}.$$

Use the starting vector $x^{(0)} = 0$ and conduct three steps of the steepest descent algorithm with and without Jacobi preconditioning. Compute the errors for each approximate solution.

Let $x^* = (-1, 2, 1)$. It is easily verified that x^* is the exact solution to $Ax = b$. We use this x^* in our calculation of the errors at each step of the steepest descent iteration (preconditioned and otherwise).

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Steepest Descent Calculation for (3):

x^(1) = { 0, 1.37586, 1.22298 }
    with error ||x* − x^(1)|| = 1.1997

x^(2) = { −0.358687, 1.78827, 0.759016 }
    with error ||x* − x^(2)|| = 0.717066

x^(3) = { −0.538975, 1.93327, 1.02728 }
    with error ||x* − x^(3)|| = 0.466627


(Jacobi) Preconditioned Steepest Descent Calculation for (3):

x^(1) = { 0, 1.85714, 0.928571 }
    with error ||x* − x^(1)|| = 1.01267

x^(2) = { −0.90562, 1.89584, 0.885032 }
    with error ||x* − x^(2)|| = 0.181586

x^(3) = { −0.915909, 1.99672, 0.988976 }
    with error ||x* − x^(3)|| = 0.0848735
```
