How to build Mathematics

by

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Mathematics is often taught from the perspective of a pedagogy that treats mathematics as pattern matching. Here is a problem of type X, here is a formula of type Y, for which we have given you several examples; plug and play. Or as some engineers like to say; plug and chug.

I contend this pedagogy robs the reader of the true nature of mathematics. What math truly is; is a language. And thus it should be taught as such. And the best way to learn a language is simple - to speak it with other native speakers. Unfortunately, very few people truly 'speak' mathematics anymore.

So, this book is intended to be a 'how to' guide to speaking mathematics. As a result a long the way, hopefully, you will find a pedagogy for learning math that is obvious and all together new. That allows you to speak and converse. Instead of trivial memorization.

The alphabet of mathematics.

This is actually somewhat tricky since we don't actually know the full alphabet. Some contend the alphabet is limitless, infinite. Others contend that the alphabet is infinite, but with a finite basis, or set of sets that can be combined in an infinite number of ways. Still others claim the alphabet is indeed finite and we've simply 'made up' a bunch of useless letters with no real meaning or value.

Lets be specific now with some examples of 'letters' in math. Of course, we are talking about numbers. But more than that, we are talking about collections of numbers with very specific properties.

For instance, what is the meaning of the number 4? Well is it equal to 4.0? How about $4/1$? What about 4 mod 7?

The answer is sort of. Which may seem surprising! But becomes obvious when you think about it in context.

All sets, only make sense in some use case. And there are relationships between similar ideas across different sets. For instance, the natural numbers; $\mathbb{N}$ make sense for counting.

As a reminder:

$$\mathbb{N} := \{0, 1, 2, 3, \dots n, \dots, \infty\}$$

Some people don't include 0 or ∞ explicitly by the way. So make sure that you always define what you mean.

Now, if we are trying to 'count' things, why would we ever need negative numbers? As far as I'm aware there is no such thing as a negative amount of sheep or people. In absolute terms the least physical stuff you can have on the macro-physical scale is zero.

Sure, you can lose stuff.

So say I have ten sheep and five get eaten by wolves. Then the sentence:

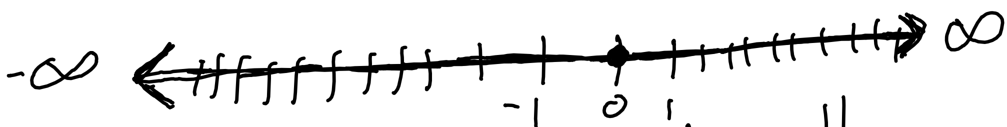
$$10 - 5 = 5$$

Makes sense. But is that negative five sheep? Of course not. It's 10 sheep, minus 5 sheep, which equals 5 sheep. Thus, "minus" here means "take away". This may seem like semantics or worse yet really pedantic semantics. But this sort of thing is at the heart of almost all mathematical <u>mis</u>understanding. And its also the great danger of plug and chug math. If all of the formula and patterns look familiar all of the time, then we can mechanically <u>do</u> the math. But we may not understand what or <u>why</u> we are doing it.

Continuing on to our next example we consider the integers, $\mathbb{Z}$.
As a reminder,

$$\mathbb{Z} := \{-\infty, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \infty\}$$

In this example everyone includes zero, no debate is even <u>possible</u> here. The inclusion of the tuple $(-\infty, \infty)$ is highly debatable however.
Many would contend that the integers are defined as such graphically:



And infinity sits <u>outside</u> the integers. At this point we need to introduce a new concept in order to effectively talk about this notion fully; intervals on a set.

Our fundamental assumption for all the sets that we have looked at thus far is that they are <u>ordered</u>. That is there is a largest and smallest number in each set. This notion of ordering is really easy for the sets we've already talked about, $\mathbb{N}$, $\mathbb{Z}$, and becomes hard to define if not impossible to define in other cases. It can even lead to some paradoxes which we will get to.

As an aside, if there are paradoxes in our mathematical objects? Is our logic flawed in some fundamental way? I have a personal answer, but you should think about this for yourself before I give you mine.

Going back to ordering; lets talk about what it means to be ordered in a general sense:

Suppose I have 2 elements of some set X such that $a \in X$, $b \in X$. Then I can always determine $a<b$, $b<a$, $a \leq b$, $b \leq a$ or $a=b$. And there is no other possability.

This definition is <u>very</u> strict. And it only applies to sets of certain kinds. We can look at thet behavior in a bit. First, lees consider the implication of this ordering and how it relates to intervals and infinity.

For intervals:

Suppose we are in $\mathbb{N}$, then we can define an interval as open:

(a, b)

Closed:

$[a, b]$

Clopen (half open):

$[a, b)$ or $(a, b]$.

An example of an open interval is:

$(4, 7) = \{5, 6\}$

That's because in a open interval we don't include the end points.

An example of a closed interval is:

$[4, 7] = \{4, 5, 6, 7\}$

An example of a clopen interval is:

$[4, 7) = \{4, 5, 6\}$

So most people define the set $\mathbb{Z}$ as

$\mathbb{Z} ::= (-\infty, \infty)$

That is going from $-\infty, \infty$ without the end points. The reason this is important is because of what it signifies about infinity, at least for the integers. Infinity is a book end. It's a terminal value. It bounds the set of integers. It's not really a part of the set. And it's something you can or really would ever work with. Other than as an absolute maximum or absolute minimum. This may appear limiting. Especially if you are used to working with the reals. But I find the definition quiet useful, powerful and intuitive. Also, the inflexability gives us a nice guarantee about what we mean when we say the "biggest" or "smallest" number.

This always us to use precise language about ordering. And it allows us the ability to make absolute statements about direction and magnitude without having to be specific about what we mean by infinity.

Lets take two examples:

Suppose we had two processes:

#1 An object at the center of the universe is slowly gathering all the mass of the universe towards it. Suppose that the more mass gathered the stronger the gravational pull. Further suppose the mass has already exceeded the point such that any mass within the gravational pull can ever escape.

thus we can completely describe the rate of increase of the mass by the successor function:

$$\text{successor}(x) = x+1$$

So at time t if mass $= x$.
At time time $t+1$ $x = x+1$.
where mass is on some normalized cosmic scale.

We can say for $t = \infty$, $x = \infty$.
Assuming the universe is infinite and there is infinite time, our mass still approaches a fixed point!

2

Suppose now instead we accumulate mass faster as x increases such that:

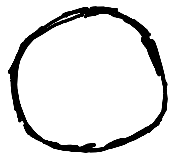
$$\text{accumulation}(x) = (x * x) + 1$$

obviously at $t = \infty$, $x = \infty$.
But the rates at which x grows will clearly be different!

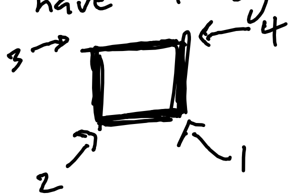
The important question then becomes, do we care? And for the purposes of this example, no we do not.

And that's the power of constructs like infinity. They allow us to talk intelligently and specifically without being overly detailed. This ability is not something we've always had in mathematics. And there was a time when if one wanted to talk about such things they would have to go into a great deal of detail and try very hard to specify something that was not easy to get your hands around.

In truth infinity is all around us. There are even 'finite' infinities. For instance, consider the circle of constant radius:



You can trace it again and again with pen or paper and never stop for an "infinite" amount of time. And you can look for an 'edge' for an 'infinite' amount of time. Here, informally, squares have 4 edges:



You can also move an infinitely small distance on a circle, if you know how. But we'll get to that last one.

We've talked about the ordering and size of $\mathbb{N}$ and $\mathbb{Z}$. Now let's talk about navigation.

Let's say we are at some number x and we want a number four "bigger" than x. For $\mathbb{N}, \mathbb{Z}$ this always well defined as:

$$x+4$$

Specifically, $x+4$ means start at x and count 4 to the right on the number line. We'll see later how this "breaks down" for the reals.

Additionally, we have,

$$x-4$$

means four to the left.

We also have "repected" addition
as multiplication;

$X * 4$

which means count "by x 4 times"
starting from zero.
And division,

$X/4$

which answers the following question:
Given a number x and a counting
scheme 4 (in this case), how many
times can we subtract 4 before get
to a value that is less than 4?
Note if we can get to zero, then
we say 4 divides x "evenly".

Exmples:

$$\frac{12}{4} \Rightarrow 12 \xrightarrow{1} 8 \xrightarrow{2} 4 \xrightarrow{3} 0$$

$$\Rightarrow 12/4 = 3$$

$$\frac{16}{4} \Rightarrow 16 \xrightarrow{1} 12 \xrightarrow{2} 8 \xrightarrow{3} 4 \xrightarrow{4} 0$$

$$\Rightarrow 16/4 = 4$$

Example where $4 \nmid x$:

$$\frac{15}{4} \Rightarrow 15 \xrightarrow{1} 11 \xrightarrow{2} 7 \xrightarrow{3} 3$$

$$\Rightarrow \frac{15}{4} = 3 \text{ Remainder } 3.$$

Since 3 is "smaller" than 4, we cannot get back to zero.

It is now worth while to talk about set construction. Clearly, not all numbers will evenly divide one another. And yet division is a terribly useful operation. For instance, suppose we wished to know how much everyone owes on a bill if we all went out to dinner. Or perhaps we wanted to know how to many instances of a specific note we could play on the string of a standard guitar.

Of course the number of useful examples of division is probably only bounded by the imagination of the individual in question. Which in theory is "infinite" assuming eternal life is possible.

Construction of the rational numbers & a few other atypical examples

The rational numbers are exactly what you have probably guessed them to be - ratios of numbers from $\mathbb{Z}$ or $\mathbb{N}$.

The way we do this is by defining the rationals as the "quotient" of two numbers from $\mathbb{Z}$:

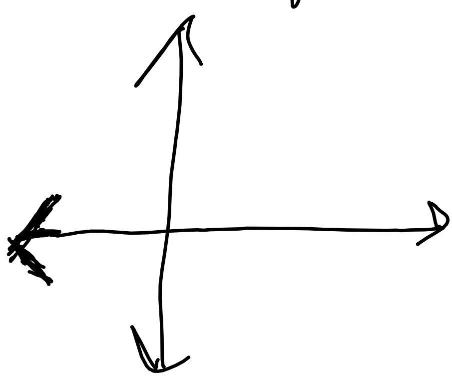
$$\mathbb{Q} := \left\{ \frac{a}{b} \text{ s.t. } a \in \mathbb{Z}, b \in \mathbb{Z} \text{ s.t. } b \neq 0 \right\}$$

Notice this is the first set that hard to write down explicitly

So now we can talk about $\frac{15}{4}$ explicitly as $3\frac{3}{4}$.

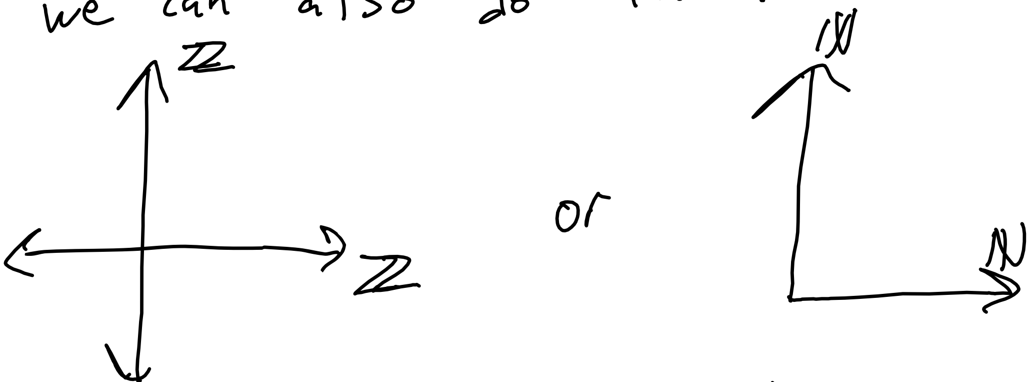
The "construction" we did was defining $\mathbb{Q}$ <u>in terms</u> of $\mathbb{Z}$. We'll have more to say on the rational numbers soon, but for now lets move onto making more new objects.

Next let's constuuct our first coordinate system. No doubt you've seen this picture before:



But in it, the number lines you we looking at were almost certainly copies of <u>real</u> numbers.

Now we could do that but we can also do this:



If we restrict our selves to these "smaller" coordinate systems what's still true?

Well for starters, our elements will look like this:

$$(\mathbb{Z},\mathbb{Z}) := \{(a,b) \text{ s.t. } a\in\mathbb{Z}, b\in\mathbb{Z}\}$$

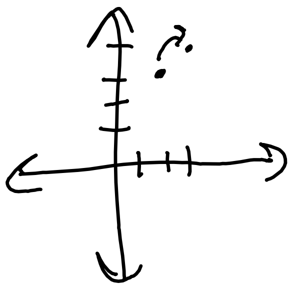
$$(\mathbb{N},\mathbb{N}) := \{(a,b) \text{ s.t. } a\in\mathbb{N}, b\in\mathbb{N}\}$$

This implies every action we take now has a geometric interpretation or in other words, we can visually draw how our operation will work.

Let's start with addition:

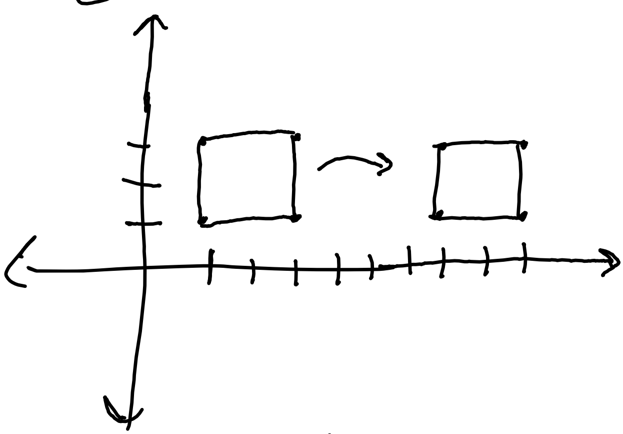
$(2,3) + (1,1) = (3,4)$

Visually:



Here we've added 1 to the 1st coordinate, and 1 to the 2nd coordinate.

We can also do this with shapes



Here we have a box with coordinates: $(1,1)$, $(1,3)$, $(3,1)$, $(3,3)$

If we "add 7" to the 1st coordinate we get a box with coordinates: $(8,1)$, $(8,3)$, $(11,1)$, $(11,3)$

This means we can define new operations which add to positional arguements like so

$$+_1 := \{(a+b, -) \text{ st } (a,c),(b,d) \in (\mathbb{Z},\mathbb{Z})\}$$

Here the _ in the second element means "leave" unchanged. This could also be signified by the identity function id,

$$id := x \to x \quad \text{or}$$

$$id(x) = x \quad \forall x, \, x \in \mathbb{Z}$$

$\forall$ means for all.

We can also define an addition that acts only on the second position:

$$+_2 := \{(-, c+d) \text{ st } (a,c),(b,d) \in (\mathbb{Z},\mathbb{Z})\}$$

finally we can define an addition which acts on all elements:

$$+_{all} := \{(a+b, c+d) \text{ st } (a,c),(b,d) \in (\mathbb{Z},\mathbb{Z})\}$$

As an aside, lets work out how many operators we can define for addition, given a certain number of dimensions:

2-D: there are 3 ways to define addition (as we saw)

For 3-D lets list them:

$(a+b,-,-)$, $(a+b,c+d,-)$,
$(-,c+d,-)$, $(-,c+d,e+f)$,
$(a+b,-,e+f)$, $(-,-,e+f)$
$(a+b,c+d,e+f)$

So there are 7 ways to define addition.

For N-D there are

$$2^N - 1$$ ways to define addition. This is because there are two possible "choices" for each element – added or not added. And there are 'N' possible choices overall. So, to get a given *addition space,' you need only multiply all the possible choices together. We subtract 1 because there must be

at least 1 addition defined.

Let's take a step back for a moment to think about why counting this way works at all.

Recall that addition is just repeated counting. And further recall that multiplication is just repeated addition. Thus, multiplication is simply counting, repeated "twice" in a sense. Therefore it is a type of counting. Specifically it counts "by" some number. In this case, we are counting by 2's.

At this point it's worth it to take a quick side track and define exponentiation and it's inverse, finding the exponential root.

Exponentiation should hold no "real" Surprise, it's just "repeated" multiplication:

<u>Example:</u>

$$2^4 = 2\cdot 2\cdot 2\cdot 2 = 16$$

In general:

$$a^b = \underbrace{a\cdot a\cdot a\cdots a}_{b \text{ times}}$$

Interestingly there is a direct geometric interpretation for exponentiation. Suppose we represented the number 3 as dots:

$$\bullet \;\; \bullet \;\; \bullet$$

And now we find the "square" of 3:

$$3^2 = 9, \text{ thus } (\bullet\;\bullet\;\bullet)^2 = \begin{matrix}\bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet\end{matrix}$$

To "square" a number litterally means to write it as a set of dots and then stack that many copies on top of each other. So, if we want 4 squared, we do

• • • •

and then we create 3 more copies, and so there are 4 total copies and stack them together:

• • • •
• • • •
• • • •
• • • •

The same holds true for the square of <u>any</u> number. Furthermore, this procedure is easy to check for small numbers and doesn't require any explicit multiplication. Instead it leverages the fact that humans are good at seeing shapes, and the fact that multiplication is just "counting".

It turns out this geometric representation is also insanely useful for numbers that are perfect squares. Suppose you had the number 4:

And you wanted to find the square root. Well that's easy:

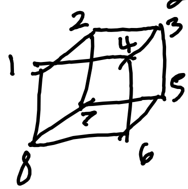
Square root

All you need to do is find a representation of the number such that it can be stacked as a square, and then you simply count 1 row. Granted this mechanism is impractical for large square roots. But for small ones it's incredibly useful. And it brings a strong geometric association to "counting".

You may have already guessed the next "trick" but we'll go through it anyway:

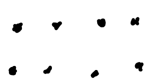
You can "cube" a number just as easily as you can "square" one:

$\cdot\ \cdot = 2$

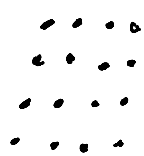


$= 2^3 = 8$

It's pretty hard to draw a "cube" with dots. But a proper cube is also just 2 copies of a square so the dot representation looks like:



The nice thing about this is it also creates a 2-D "representation" for 2^4:



$= 16$

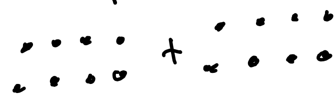
The drawing procedure even "feels" lik multiplication. Draw a second

$2^2 = 4 =$  then draw

copy to get $2^3 = 8 =$  then

Make a second copy of $2^3 =$ 

then combine the two copies

$$2^3 + 2^3 = 2^4 \Rightarrow$$ 

Algebriacally this checks because:

$$2^3 + 2^3 = 2^3(1+1) = 2^3(2) = 2^4$$

As long as we are in this digression lets see where it takes us!

For instance, with this representation we can look for a pattern or "rule" to build up "squares" from "simpler" ones:

If the general rule isn't obvious by now, lets write it down. We can "Generate" perfect squares by adding the "next" odd number. Thus if we know that some number is a perfect square and we know it's square rot we can simply count odd numbers "up to" it's square root and then add the odd number to the squre to get the next perfect square.

Lets write a little algorithm to make this explicit.

Suppose we know x is a perfect square and we wish to know the next largest square.

We can do:

$$\sqrt{x} = k$$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & k \\ 1 & 3 & 5 & 7 & \cdots & y \end{array}$$

$x + y$ = next largest square.

Notice we need to index the odd numbers by the natural numbers, starting at 1 in order to make this work.

Let's try an example:
What's the next largest square after 256?

$256 = 16^2 = 4^4 = 2^8$

aside: All powers of 2 are perfect squares of some number.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	5	7	9	11	13	15	17	21	23	25	27	29	31	33

So the next largest square is:

$$\begin{array}{r} 256 \\ +\ 33 \\ \hline 289 \end{array} = 17^2$$

This aside is only for programmers:
This implies you can easily build a generator function to check for perfect squares and square roots and then use a look up table for performance.

Now that we have a good sense of addition, subtraction, multiplication, division, exponentiation, and root finding we are ready to start building some "new" operators. Or at least ones you're less likely to have seen before in $\mathbb{Z}$ or $\mathbb{N}$.