Quantitative Finance, 2019
https://doi.org/10.1080/14697688.2019.1662079

Personalized goal-based investing via multi-stage stochastic goal programming

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(*Received 29 May 2018; accepted 16 August 2019; published online 31 October 2019*)

In this paper, we propose a goal-based investment model that is suitable for personalized wealth management. The model only requires a few intuitive inputs such as size of wealth, investment amount, and consumption goals from individual investors. In particular, a priority level can be assigned to each consumption goal and the model provides a holistic solution based on a sequential approach starting with the highest priority. This allows strict prioritization by maximizing the probability of achieving higher priority goals that are not affected by goals with lower priorities. Furthermore, the proposed model is formulated as a linear program that efficiently finds the optimal financial plan. With its simplicity, flexibility, and computational efficiency, the proposed goal-based investment model provides a new framework for automated investment management services.

Keywords: Goal-based Investing; Goal Programming; Multi-stage Stochastic Programming; Automated Investment Management; Robo-advisor; Financial Technology (FinTech); Retirement planning; Defined contribution pension plan

1. Introduction

For institutional investors, modern portfolio theory (MPT) propounded by Markowitz (1952) is considered the fundamental approach for quantitative investment management (Kolm *et al.* 2014). Since the mean-variance framework is an asset-only approach (Fabozzi *et al.* 2002), asset-liability management (ALM) models have been developed to address investments that involve future liabilities (Ziemba and Mulvey 1998). ALM models have been heavily applied to pension management because a key requirement for pension funds is meeting future pension payments (Bauer *et al.* 2006).

Among individual investors, high-net-worth individuals can be distinguished from others based on their investment objectives. Wealth management for these high-net-worth individuals is focused on increasing their wealth while maintaining a reasonable exposure to risk. Therefore, wealth management for high-net-worth individuals share many similarities with asset-only management for institutions. However, the asset-only approach of MPT becomes a crucial shortcoming for managing wealth of average investors because average individuals are tightly constrained to their financial requirements (Brunel 2003, Nevins 2004, Chhabra 2005, Das *et al.* 2010). For example, their investment objectives may focus on securing the basic cost of living after retirement or paying for their children's education.

Future consumption requirements can be seen as liabilities that individuals hope to fulfill throughout their lifetime. Therefore, individual wealth management should be addressed within an ALM framework instead of concentrating only on assets. There are several approaches for including financial goals to individual ALM: personalized models for a single goal at the final stage of investment, financial planning models for multiple goals throughout several life stages, and models that even include priority levels of goals. Consiglio

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et al. (2007) introduce a scenario-based ALM optimization for a target goal at the final stage, where a target rate of return is computed based on the target goal and the investment amount of an investor. Market uncertainty is accounted for by adjusting the target return with various scenarios for inflation rate, and the model attempts to match the portfolio value to the projected target amount in each period. The formulation finds the financial plan that maximizes the expected final surplus while controlling shortfall risk. Höcht *et al.* (2008) further focus on individual investment management under uncertainty in both assets and liabilities and propose a formulation that minimizes weighted sum of downside risk with stochastic liabilities.

One of the first studies that present ALM for individuals is introduced by Berger and Mulvey (1998). They consider a financial plan with multiple financial goals and the model allows decisions to postpone goals. They address the trade-off among savings, consumption, and wealth accumulation with a multi-stage formulation that maximizes the weighted sum of four utilities, which includes goal achievement, bonus consumption, and surplus of wealth over goals. Consiglio *et al.* (2004, 2007) discuss a personalized ALM model that separately manages segmented goals. They also present an approach that accounts for an individual's future income for optimizing a plan that maximizes consumption while minimizing required savings. Moreover, Consigli (2007) proposes a model that maximizes the utility of terminal wealth while penalizing deviation from target wealth.

Some individual ALM models also incorporate priorities of financial goals in order to put more focus on higher priority goals. Specifically, Medova *et al.* (2008), Dempster and Medova (2011), and Dempster *et al.* (2016) formulate optimization models that maximize expected utility of lifetime consumption where the utility function is piece-wise linear with its slopes representing goal priorities. Taking a different approach, Fowler and de Vassal (2006) suggest sequential optimization to first solve for the first priority goal, then solve for the second priority goal, and so on. The lower priority goals are conditioned on meeting the higher priority goals with certain probability, which provides a holistic approach.

The importance of collecting multiple financial goals with varying priorities for personalized financial planning is the main idea of goal-based investing (GBI), with its basis in behavioral portfolio theory. GBI states that individuals often have multiple spending goals and investors typically view these goals with different levels of aspiration or risk-aversion (Shefrin and Statman 2000). It provides an intuitive individual wealth management framework that can successfully reflect personalized goals.†

Most individual ALM models introduced above are formulated as multi-stage stochastic programs (MSP) in order to incorporate some degree of uncertainty using scenarios.‡ However, there has not been an MSP model for individual financial planning that comprehensively optimizes for prioritized goals. While Fowler and de Vassal (2006) present a holistic method as mentioned above, they discuss a conceptual framework without considering changes in wealth due to income and savings, and demonstrate goal achievements assuming normally distributed asset returns. Therefore, in this paper, we propose a multi-stage stochastic goal programming (MSGP) model for holistically optimizing personalized financial planning, which is a GBI model based on MSP for inclusively considering an individual's current and future cash flows along with prioritized financial goals. We note that an individual's initial wealth, income, savings, and consumptions are included in previous models (Berger and Mulvey 1998, Consiglio *et al.* 2007, Medova *et al.* 2008), but the proposed model in this paper fully addresses strict priority of goals.

The proposed MSGP model has several advantages that make it effective for individual wealth management. First, the model only requires three types of intuitive inputs: current wealth, investment amount, and consumption goals along with their priorities. Note that all the inputs are values that individuals can prepare without financial expertise.§ Second, while the model incrementally optimizes for a single priority level at each iteration, it finds a holistic optimal financial plan for all consumption goals that strictly prioritizes higher priority goals. Third, the model combines multi-stage stochastic programming and goal programming approaches while maintaining a linear program structure. In other words, the problem can be solved very efficiently, and linear constraints on allocation and downside risk can be easily incorporated into the model. As we further explain in the following sections, the model is applicable for financial planning in a trial-and-error fashion with instant feedback due to its optimality and efficiency, which allows investors to search for an ideal financial plan by adjusting their future savings and goals.

Finally, we note that average individuals without large savings are facing growing responsibilities for managing their wealth due to changes in pension plan structure where many pensions are defined-contribution plans instead of defined-benefit plans.¶ Consequently, the importance of investment management is escalating because average individuals need to take more control of their savings. With recent advances in financial technology, more affordable wealth management services have emerged amid surging demand. Web-based investment management or advisory services, also known as robo-advisors, aim to serve all individuals by significantly lowering the entry barrier to professional wealth management. The intuitive inputs, holistic optimal solution, efficient computation, and flexibility of the proposed model allow it to be well-suited for automated investment management.

† Early GBI models by Brunel (2003), Nevins (2004), and Chhabra (2005) create a sub-portfolio for each consumption goal and the sub-portfolios are ultimately combined with appropriate proportions depending on an investor's risk appetite. Das *et al.* (2010), Wang *et al.* (2011), and Deguest *et al.* (2015) have also proposed GBI models based on similar methods.

‡ Other studies include Geyer *et al.* (2009), Amenc *et al.* (2009), and Pedersen *et al.* (2013). Furthermore, individual pension problems within an ALM framework are addressed by Consigli *et al.* (2011, 2012, 2018) and Kopa *et al.* (2016).

§ Among the five major features of personal asset allocation models pointed out by Consiglio *et al.* (2007), the first one to be emphasized is simple language. In other words, typical individuals should easily understand the inputs required by the model, and the use of simple language is particularly important when there is only limited interaction with investors.

¶ See Chhabra (2005) for a more detailed discussion on changes in investment circumstances for individuals.

The remainder of the paper is organized as follows. Section 2 introduces our model including detailed explanation on notation and formulation. Incorporating additional constraints on risk and transaction costs are also discussed in Section 2. Implementation of the model as a personalized wealth management service is illustrated in Section 3 where practical concerns are also addressed. Section 3 also demonstrates a few example investor cases. Finally, Section 4 concludes.

2. Multi-stage stochastic goal programming

In this section, we formulate our GBI model by combining multi-stage stochastic programming and goal programming. Multi-stage stochastic programming solves multi-period optimization problems that include uncertain random variables in its constraints, and possible outcomes are represented as a scenario tree.† Goal programming is a sequential optimization procedure that solves multi-objective problems. In particular, it uses priority level of goals rather than importance weights of goals for solving a problem with multiple objectives. Goal programming sequentially optimizes the objectives starting from the highest priority goal and the goals with lower priorities utilize the remaining resources after optimizing the higher goals. We take a goal programming approach because it uses input parameters that are simple and more intuitive. The multi-stage stochastic programming and goal programming are adopted to address the randomness in the return of assets and to sequentially satisfy an individual's prioritized goals, respectively. We refer to the combined model as multi-stage stochastic goal programming (MSGP).

2.1. Parameters, decision variables, and notations

The parameters of the MSGP model can be categorized into two groups: the parameters for individual investors and scenario trees. The first set of parameters can be provided by the investors, which includes their initial savings, future investments (contributions), and consumption goals along with their priorities. The second set represents a scenario tree that defines multiple market scenarios at each stage in order to incorporate market uncertainty.

We introduce the notation used in the model before presenting its detailed formulation. We denote the set of scenarios by S, the set of assets by A, the number of goal priorities by P, and the number of stages (time periods) by T. The set of parameters and decision variables used in the problem are defined below for each $s \in S, i \in A, p \in \{1, \ldots, P\}$, and $t \in \{0, \ldots, T\}$. The first asset ($i = 1$) is assumed to represent cash (or cash equivalent). An investor's initial wealth is fully allocated in cash (but can be easily generalized to cases when the investor is holding a portfolio), and the investment starts in stage 0. The MSGP model solves P number of sub-problems, one for each priority level.

† See Bertocchi *et al.* (2011) for more details of multi-stage stochastic programming and its applications.

Parameters for scenario tree:

$r_{i,t,s}$	Return of asset i from stage $t-1$ to stage t under scenario s
$f_{t,s}$	Inflation rate from stage $t-1$ to stage t under scenario s
$\pi_{t,s}$	Probability of scenario s in stage t
$d_{t,s}$	Discount factor from stage 0 to stage t under scenario s

Parameters set by investor (all in monetary values):

$x^{\mapsto}_{1,0}$	Cash (or cash equivalent) savings at time 0
G^p_t	Consumption goal in stage t with priority p expressed in monetary value
I_t	Additional investment in stage t

Decision variables at step p:

$c^p_{t,s}$	Cumulative consumption for the goals with priorities 1 to p in stage t under scenario s
$x^{p,+}_{i,0}$	Purchase amount of asset i in stage 0 at step p
$x^{p,-}_{i,0}$	Sell amount of asset i in stage 0 at step p
$x^{p}_{i,0}$	Final amount of asset i in stage 0 at step p
$x^{p,\mapsto}_{i,t,s}$	Amount of asset i at the beginning of stage t under scenario s at step p
$x^{p,+}_{i,t,s}$	Purchase amount of asset i in stage t under scenario s at step p
$x^{p,-}_{i,t,s}$	Sell amount of asset i in stage t under scenario s at step p
$x^{p}_{i,t,s}$	Final amount of asset i in stage t under scenario s at step p

Decision from step $p-1$:

$c^{p-1}_{t,s}$	Cumulative consumption for the goals with priorities 1 to $p-1$ in stage t under scenario s

2.2. Formulation

As illustrated in Figure 1, MSGP is sequentially performed for P steps. In step 1, MSGP only considers the goal with the highest priority and maximizes the probability of achieving the highest goal. The model keeps track of the maximum achievement level for the highest goal (the consumption amount $c^1_{t,s}$ for each stage t and scenario s). In step 2, MSGP considers the first and second priority goals, but for each stage t and scenario s, the consumption $c^2_{t,s}$ in the second step cannot be lower than $c^1_{t,s}$. So in the pth step, the model maximizes the probability of achieving the pth goal, while maintaining the achievement of goals with priorities 1 to $p-1$ attained in steps 1 to $p-1$. This is repeated until the Pth step. While the MSGP formulation for a single step maximizes linear utility on consumption, the overall utility for P steps becomes a concave piece-wise linear function. We note that Figure 1 illustrates the case when all goals are placed in the same stage, but MSGP allows investors to place their goals in different stages.

We now detail the formulation of our model. The model for a single priority level p is described and the same problem is solved repeatedly for each priority level. The objective function of the optimization problem is to maximize the expected

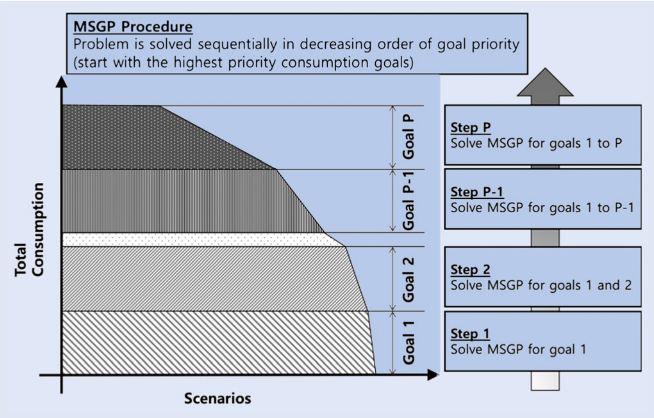

Figure 1. Goal programming approach of MSGP to prioritize goals.

total consumption (in present value) of all scenarios as given by equation (1).

$$\max \sum_t \mathbb{E}\left[d_{t,s} c_t^p\right] = \max \sum_t \sum_s d_{t,s} c_{t,s}^p \pi_{t,s} \tag{1}$$

equations (2) and (3) enforce strict priority of goals. Prior to the pth step, MSGP finds the optimal consumption for the goals with priorities 1 to $p-1$. Therefore, in the pth step, the model decides the consumption for the pth priority goal, while guaranteeing the optimal consumption levels for goals with higher priorities that are computed in the previous steps. As shown in equations (2) and (3), the value of $c_{t,s}^p$ is restricted to be at least equal to $c_{t,s}^{p-1}$ and at most equal to the sum of $c_{t,s}^{p-1}$ and the pth priority goal G_t^p, where $c_{t,s}^{p-1}$ denotes the solution attained by solving the $p-1$th step with $c_{t,s}^0 = 0$ for all t and s. Note that we represent the consumption goal G_t^p in inflation-adjusted terms using the compounded inflation rate $\prod_{\tau=1}^t (1+f_{\tau,s})$. Due to the upper limit on $c_{t,s}^p$, further consumption does not increase the utility once the pth goal is achieved. This allows extra wealth to be consumed for the remaining goals in the following steps.

$$c_{t,s}^p \le c_{t,s}^{p-1} + \prod_{\tau=1}^t (1+f_{\tau,s}) G_t^p,\ \forall s \in S,\ t = 1, \dots, T \tag{2}$$

$$c_{t,s}^{p-1} \le c_{t,s}^p,\ \forall s \in S,\ t = 1, \dots, T \tag{3}$$

In stage 0, we assume that the investor begins with full investment in cash ($i = 1$). In equation (4), $x_{1,0}^{\mapsto}$ represents the total capital in stage 0, and equations (4–6) express allocation of the initial capital in financial assets ($x_{i,0}^{p,+}$ for $i \neq 1$). The assumption of initial cash holdings can be generalized to any investment portfolios by modifying (4–6).

$$x_{1,0}^p = x_{1,0}^{\mapsto} - x_{1,0}^{p,-} \tag{4}$$

$$x_{i,0}^p = x_{i,0}^{p,+},\ i \in A \setminus \{1\} \tag{5}$$

$$x_{1,0}^{p,-} = \sum_{i \neq 1} x_{i,0}^{p,+} \tag{6}$$

Equations (7) and (8) reflect purchasing and selling assets in stage $t > 0$. In equation (8), the net purchase amount of all assets should be equal to the net income of investments, where net income is computed by subtracting the consumption for the pth goal from the additional investment amount.

$$x_{i,t,s}^p = x_{i,t,s}^{p,\mapsto} + x_{i,t,s}^{p,+} - x_{i,t,s}^{p,-},\quad \forall s \in S,\quad \forall i \in A,\quad t = 1, \dots, T \tag{7}$$

$$I_t - c_{t,s}^p = \sum_i x_{i,t,s}^{p,+} - \sum_i x_{i,t,s}^{p,-},\quad \forall s \in S,\quad t = 1, \dots, T \tag{8}$$

Next, equations (9) and (10) model the evolvement of asset prices according to the scenarios of asset returns in stages 0–1 and stages t to $t+1$ with $t > 0$, respectively.

$$x_{i,1,s}^{p,\mapsto} = (1 + r_{i,1,s}) x_{i,0}^p,\quad \forall s \in S,\quad \forall i \in A \tag{9}$$

$$x_{i,t,s}^{p,\mapsto} = (1 + r_{i,t,s}) x_{i,t-1,s}^p,\quad \forall s \in S,\quad \forall i \in A,\quad t = 2, \dots, T \tag{10}$$

Furthermore, equation (11) incorporates the non-anticipative policy, making sure that scenarios with the same ancestor node share the same decision in the ancestor node.

$$x_{i,t-1,s_1}^{p,+} = x_{i,t-1,s_2}^{p,+},\ x_{i,t-1,s_1}^{p,-} = x_{i,t-1,s_2}^{p,-},\ c_{t-1,s_1}^p = c_{t-1,s_2}^p,$$
$$\forall s_1, s_2 \in S \text{ sharing same parent node},\ \forall i \in A,\ t = 2, \dots, T \tag{11}$$

Finally, in equation (12), the net purchase amounts $x_{i,0}^p$ (in stage 0) and $x_{i,t,s}^p$ (in stage t) are set to be greater than or equal to 0 to prevent short selling. In addition, all purchase and sale amounts are restricted to take non-negative values.

$$x_{i,0}^p, x_{i,0}^{p,+}, x_{1,0}^{p,-}, x_{i,t,s}^p, x_{i,t,s}^{p,+}, x_{i,t,s}^{p,-} \ge 0,$$
$$\forall s \in S,\ \forall i \in A,\ t = 1, \dots, T \tag{12}$$

Combining all equations give the following formulation for the pth step of the MSGP model. Note that the objective function and all constraints are linear, so the problem becomes a linear program. Hence, the global optimal solution of the problem can be found in polynomial time with respect to the number of optimization variables.

For step $p \in \{1, \dots, P\}$

$$\underset{x,c}{\text{maximize}} \sum_t \mathbb{E}[d_{t,s} c_t^p] = \sum_t \sum_s d_{t,s} c_{t,s}^p \pi_{t,s} \tag{13}$$

$$\text{subject to } c_{t,s}^p \le c_{t,s}^{p-1} + \prod_{\tau=1}^t (1+f_{\tau,s}) G_t^p,$$
$$\forall s \in S,\quad t = 1, \dots, T \tag{14}$$

$$c_{t,s}^{p-1} \le c_{t,s}^p,\quad \forall s \in S,\quad t = 1, \dots, T \tag{15}$$

$$x_{1,0}^p = x_{1,0}^{\mapsto} - x_{1,0}^{p,-} \tag{16}$$

$$x_{i,0}^p = x_{i,0}^{p,+},\quad i \in A \setminus \{1\} \tag{17}$$

$$x_{1,0}^{p,-} = \sum_{i \neq 1} x_{i,0}^{p,+} \tag{18}$$

$$x^p_{i,t,s} = x^{p,\mapsto}_{i,t,s} + x^{p,+}_{i,t,s} - x^{p,-}_{i,t,s}, \quad \forall s \in S, \ \forall i \in A, \ t = 1, \dots, T \tag{19}$$

$$I_t - c^p_{t,s} = \sum_i x^{p,+}_{i,t,s} - \sum_i x^{p,-}_{i,t,s}, \quad \forall s \in S, \quad t = 1, \dots, T \tag{20}$$

$$x^{p,\mapsto}_{i,1,s} = (1 + r_{i,1,s}) x^p_{i,0}, \quad \forall s \in S, \quad \forall i \in A \tag{21}$$

$$x^{p,\mapsto}_{i,t,s} = (1 + r_{i,t,s}) x^p_{i,t-1,s}, \quad \forall s \in S, \ \forall i \in A, \quad t = 2, \dots, T \tag{22}$$

$$x^{p,+}_{i,t-1,s_1} = x^{p,+}_{i,t-1,s_2},\ x^{p,-}_{i,t-1,s_1} = x^{p,-}_{i,t-1,s_2},\ c^p_{t-1,s_1} = c^p_{t-1,s_2}, \\ \forall s_1, s_2 \in S \text{ sharing same parent node}, \ \forall i \in A, \ t = 2, \dots, T \tag{23}$$

$$x^p_{i,0}, x^{p,+}_{i,0}, x^{p,-}_{1,0}, x^p_{i,t,s}, x^{p,+}_{i,t,s}, x^{p,-}_{i,t,s} \geq 0, \\ \forall s \in S, \ \forall i \in A, \ t = 1, \dots, T \tag{24}$$

2.3. Incorporating risk constraints

GBI models need to address two kinds of risk: the risk of not achieving goals and the risk in investment returns. The first type of risk is addressed by the model, as the purpose of GBI is to minimize the probability of underachieving goals. Nonetheless, additional downside protection of goal achievements can be incorporated into the MSGP model. For example, constraints on conditional value-at-risk (CVaR) of the unachieved consumptions in each stage can be included as linear constraints based on portfolio selection models with CVaR as a risk measure developed by Rockafellar and Uryasev (2000).

Suppose we want to constrain the average underachievement of the pth (inflation-adjusted) goal in stage t, $\tilde{G}^p_t = \prod_{\tau=1}^t (1 + f_{\tau,s}) G^p_t$, in the worst $(1-\alpha) \times 100\%$ of all scenarios to be less than or equal to $\rho \tilde{G}^p_t$, where $0 \leq \rho \leq 1$. In other words, we define the loss L_t as the underachievement of the pth goal $\tilde{G}^p_t$, $L_t = \tilde{G}^p_t - (c^p_t - c^{p-1}_t)$, and want to add the following CVaR constraint.

$$\text{CVaR}_\alpha(\tilde{G}^p_t - (c^p_t - c^{p-1}_t)) \leq \rho \tilde{G}^p_t \tag{25}$$

Krokhmal *et al.* (2001) show that equation (25) can be addressed by adding the following equivalent linear constraints

$$\begin{aligned} &\zeta_t + \frac{1}{1-\alpha} \sum_s L_{t,s} \pi_{t,s} \leq \rho \tilde{G}^p_t \\ &L_{t,s} \geq (\tilde{G}^p_t - (c^p_{t,s} - c^{p-1}_{t,s})) - \zeta_t, \ \forall s \in S \\ &L_{t,s} \geq (\tilde{G}^p_t - (c^p_{t,s} - c^{p-1}_{t,s})) - \zeta_t, \ \forall s \in S \end{aligned} \tag{26}$$

where α represents the quantile level, which is usually 0.9, 0.95, or 0.99, ζ_t is the value-at-risk (VaR) of the unachieved goal in stage t with the quantile level α, and $L_{t,s}$ is the excessive unachieved consumption over the VaR ζ_t in stage t under scenario s. In addition, ζ_t and $L_{t,s}$ for each scenario s are included as optimization variables. By the constraints expressed as equation (26), the worst $1-\alpha$ portion of scenarios at time t should have an expected margin of underachievement lower than some ratio ρ of goal $\tilde{G}^p_t$.

As for the second type of risk, financial risk of investments, MSGP implicitly incorporates an investor's risk appetite through spending goals specified by the investor. The MSGP formulation recommends risky portfolios if the goals are ambitious and safe portfolios if the goals are easily achieved. However, investors might wish to reduce portfolio risk while slightly sacrificing the possibility of achieving goals. Although various portfolio risk constraints can be incorporated into the model, we show how to constrain the CVaR of portfolio returns by modifying equation (26). It is possible to also apply similar techniques to incorporate other risk measures such as mean absolute deviation or lower semi-absolute deviation discussed in Konno (1990) and Konno and Yamazaki (1991).

First, define the measure of loss in stage t to be $L_t = -x_t^T r_{t+1}$, where x_t is a vector of portfolio allocation on n assets in stage t, and r_t is a vector of n asset returns from stages t to $t+1$. Consider the following CVaR constraint

$$\text{CVaR}_\alpha(-x_t^T r_{t+1}) \leq \rho \tag{27}$$

where $-1 \leq \rho \leq 1$. The average loss of the worst $(1-\alpha) \times 100\%$ of all scenarios should be less than $\rho \times 100\%$. If ρ is set as a negative value, the average of worst-case returns should be positive. The transformation of CVaR suggested by Krokhmal *et al.* (2001) cannot be applied directly to the CVaR constraint on portfolio returns at each stage, but the following steps enable conversion to a set of linear constraints.

In stage 0, portfolio returns are computed based on the initial allocation $x_{i,0}$ on asset $i = 1, \dots, n$. Therefore, (27) can be converted into the following set of constraints.

Stage 0

$$\begin{aligned} &\zeta_0 - \frac{1}{1-\alpha} \sum_s L_{0,s} \pi_{0,s} \leq \rho \\ &L_{0,s} \geq -\frac{\sum_i x^p_{i,0} r_{i,1,s}}{\sum_i x^p_{i,0}} - \zeta_0, \ \forall s \in S \\ &L_{0,s} \geq 0, \ \forall s \in S \end{aligned} \tag{28}$$

Note that the second constraint of (28) is non-linear. However, multiplying $\sum_i x^p_{i,0}$ on both sides of all inequalities leads to linear constraints given by (29), where loss $\tilde{L}_{0,s}$ and VaR $\tilde{\zeta}_0$ are represented in monetary values, not in rates of return.

Stage 0

$$\begin{aligned} &\tilde{\zeta}_0 - \frac{1}{1-\alpha} \sum_s \tilde{L}_{0,s} \pi_{0,s} \leq \rho \sum_i x^p_{i,0} \\ &\tilde{L}_{0,s} \geq -\sum_i x^p_{i,0} r_{i,1,s} - \tilde{\zeta}_0, \ \forall s \in S \\ &\tilde{L}_{0,s} \geq 0, \ \forall s \in S \end{aligned} \tag{29}$$

In stage $t > 0$, there will be more than one node. For N_t number of nodes in stage t, the return from stages t to $t+1$ should be calculated based on each node in stage t. The expected return from stages t to $t+1$ can be written as

$$\mathbb{E}[x_t^T r_{t+1}] = \frac{1}{N_t} \sum_{\eta=1}^{N_t} \sum_{s\in\eta} \frac{\sum_i x_{i,t,s}^p r_{i,t+1,s}}{\sum_i x_{i,t,s}^p}$$

where η denotes a node in stage t, and $s \in \eta$ means that scenario s belongs to node η. Here, linearity cannot be achieved with respect to x_t by simply multiplying a common factor. Therefore, in stage $t > 0$, a CVaR constraint should be incorporated for each node. For scenarios that come out from node η, it is possible to add a CVaR constraint with linear equations given in (30).

Stage t, for node η,

$$\begin{aligned}
&\tilde{\zeta}_t - \frac{1}{1-\alpha} \sum_{s\in\eta} \tilde{L}_{t,s}\pi_{t,s} \le \rho \sum_i x_{i,t,s}^p \\
&\tilde{L}_{t,s} \ge -\sum_i x_{i,t}^p r_{i,t+1,s} - \tilde{\zeta}_t,\ \forall s \in \eta \\
&\tilde{L}_{t,s} \ge 0,\ \forall s \in \eta
\end{aligned} \tag{30}$$

2.4. *Incorporating additional constraints*

One advantage of the proposed model is its flexibility in incorporating various constraints without sacrificing the linear structure of the formulation. For example, transaction costs can be added to the problem by modifying the constraints given by (6) and (8) to (31) and (32),

$$(1-\delta^-)x_{1,0}^{P,-} - (1+\delta^+) \sum_{i\neq 1} x_{i,0}^{P,+} = 0 \tag{31}$$

$$\begin{aligned}
&(I_t - c_{t,s}^P) + (1-\delta^-) \sum_i x_{i,t,s}^{P,-} - (1+\delta^+) \sum_i x_{i,t,s}^{P,+} = 0, \\
&\forall s \in S,\ t = 1,\dots,T
\end{aligned} \tag{32}$$

where δ^- and δ^+ are transaction costs incurred by sales and purchases, respectively. These adjustments also require strictly increasing utility functions, and incorporating transaction costs are further discussed by Zenios (2002). In addition, turnover constraints can be easily added by setting upper bounds for the decision variables $x_{i,0}^{P,+}$, $x_{1,0}^{P,-}$, $x_{i,t,s}^{P,+}$, and $x_{i,t,s}^{P,-}$.

We note that the main objective of the paper is to introduce the MSGP model, which is a unique approach for personalized financial planning. Therefore, we have focused on presenting the basic formulation along with key extensions on managing risk. There are some limitations of the current paper since the aim is not to present a complex formulation with an exhaustive list of notations and equations. For example, the formulation can be extended to incorporate borrowing (e.g. mortgage), investment goals (non-spending), or initial wealth invested in financial assets. The framework we suggest allows additional components to be added without affecting the key properties and advantages of the model.

Table 1. Preliminary inputs from investor.

Panel A. Time stages

Current age	30	Number of stages	4	

	Stage 0	Stage 1	Stage 2	Stage 3	Stage 4
Decisions	Asset allocation	Asset allocation, Consumption	Asset allocation, Consumption	Asset allocation, Consumption	Consumption
Number of years (Age)	0 (30)	10 (40)	10 (50)	10 (60)	20 (80)

Panel B. Asset universe

Asset class	Data	Expected return* (%)	Standard deviation* (%)
3-month bond	3-month Treasury Bill	3.13	2.46
10-year bond	10-year Treasury Bond	7.13	9.28
Commodity	S&*P* GSCI Commodity	7.45	26.37
Real estate	FTSE NAREIT All REITs	11.62	19.43
Developed market	MSCI EAFE	7.12	17.77
U.S. market	S&*P* 500	11.52	17.92
Emerging market	MSCI EM	13.70	24.66

*Expected returns and standard deviations are estimated from 1989.01 to 2015.12 and annualized

Panel C. Initial wealth and additional investment (savings)

	Stage 0 (30)	Stage 1 (40)	Stage 2 (50)	Stage 3 (60)	Stage 4 (80)
Initial wealth	\$30,000	–	–	–	–
Additional investment (savings)	–	\$40,000	\$50,000	–	–

Panel D. Additional constraints

Maximum allocation	45% for each asset
Minimum allocation	0% for each asset

3. Illustration of personalized financial planning

The proposed model uses scenario trees and this has several practical benefits for personalized financial planning, especially for automated investment management services. Most importantly, various return distributions of asset returns can be represented with scenario trees, and the problem can be solved with prebuilt scenario trees for reducing computation time. These advantages allow investment optimization and market uncertainty modeling to be performed separately and also allow investors to interact iteratively with the model. In this section, we illustrate how MSGP can be utilized in practice with several numerical examples. In the appendix, we discuss some computational issues on implementing the MSGP model for large scenario trees.

3.1. Case study

We illustrate how an average investor can use the MSGP framework for finding an optimal lifetime investment plan and consumption decisions that are fully customized to the investor's financial condition. We introduce several cases with varying inputs and financial goals in order to demonstrate a trial-and-error approach for optimizing a financial plan, which is especially important when there is minimum assistance from human advisors in an automated setting.

In this case study, we assume investment in seven asset classes that cover bonds, stocks, and alternative investments: 3-month Treasury Bill, 10-year Treasury Bond, S&P GSCI Commodity, FTSE NAREIT All REITs, MSCI EAFE, S&P 500, and MSCI EM. The asset list is summarized in Panel B of Table 1, but the asset universe is not restricted to this list in practice.

3.1.1. Investor. Suppose a 30-year old investor wants to optimize a financial plan for the next 50 years, where the 50 years is divided into four stages of 10-, 10-, 10-, and 20-year-long periods. Thus, the investment horizons ends at age 80 and investment decisions are made at ages 30, 40, 50, and 60. Panel A of Table 1 summarizes these inputs from the investor. Based on the candidate assets and investment periods, a scenario tree is generated to represent market

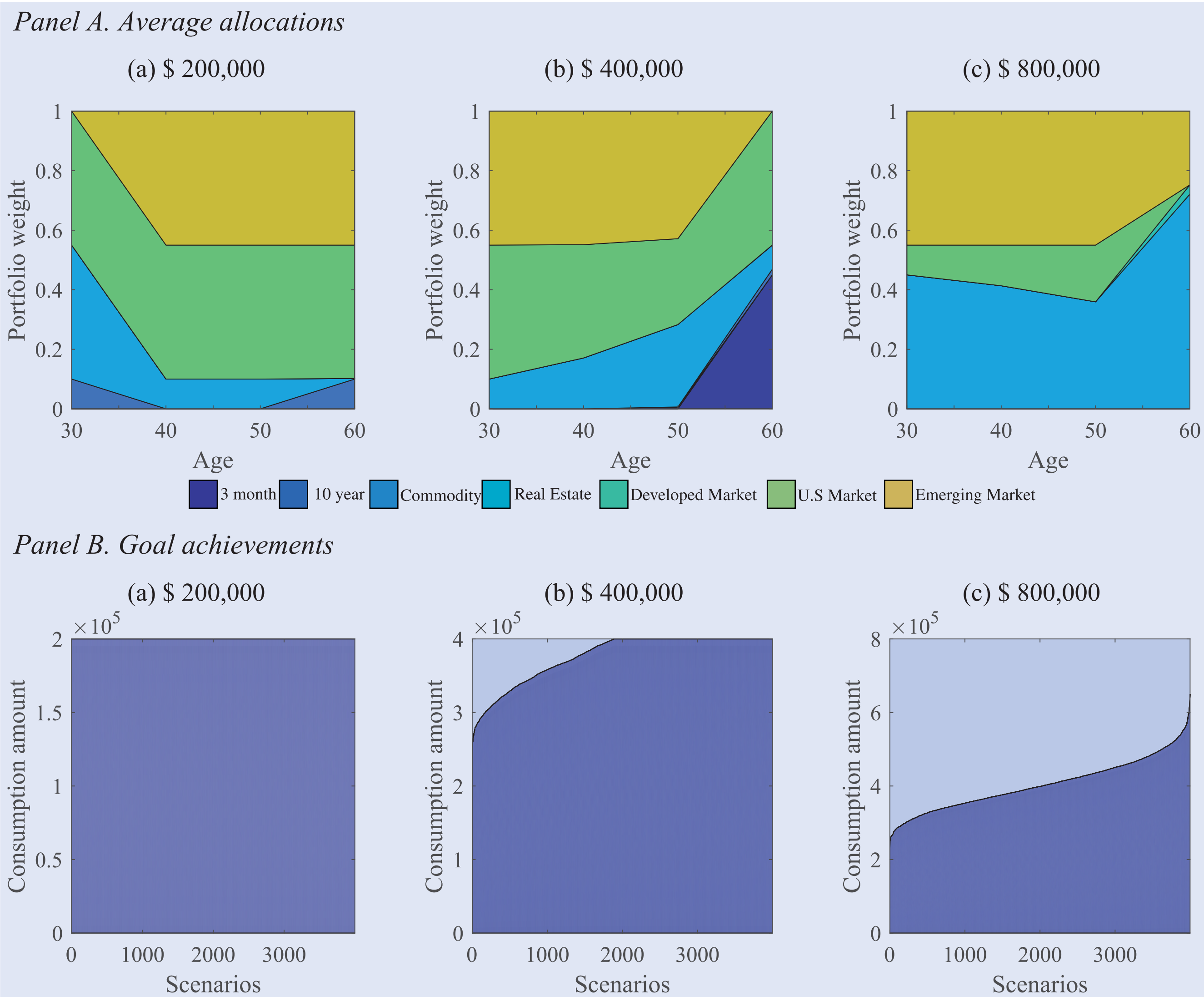

Figure 2. Average allocation and goal achievement results in case of a single consumption goal at age 60 (Case 1: (a) $ 200,000, (b) $ 400,000, and (c) $ 800,000).

dynamics. We generate scenario trees using k-means clustering in this example, but other scenario generation methods can be applied.†

The investor should also provide current wealth and planned future investment as shown in Panel C of Table 1, and the future investment may be specified for every stage. Finally, additional constraints can be placed by the investor as shown in Panel D of Table 1. We assume that the investor specifies maximum weight of 45% for each asset class in order to prevent allocation in only a few assets. Note that no short-selling constraint is already included in the model, and one may consider other constraints such as transaction cost, turnover, or portfolio risk constraints.

3.1.2. Case 1: single goal. We first consider an investor with a single goal: cost of living in retirement. This is similar to target-date funds that gradually reallocate wealth from risky assets to safe assets as the target date (retirement age) approaches. However, the MSGP model can be regarded as a customized version of target-date funds because MSGP incorporates personalized information such as future contributions and spending plans.

Figure 2 presents asset allocation and goal achievement outcomes of three alternative consumption goals at age 60 (stage 3): (a) \$ 200,000, (b) \$ 400,000, and (c) \$ 800,000. Panel A shows average asset allocations from stages 0–4, and Panel B plots goal achievements across all scenarios. Goal (a) is attainable in every scenario, and the model continues to invest in safe assets in all stages. On the other hand, Goal (c) is hardly achievable in most scenarios. Therefore, the model has no choice but to allocate to risky assets. Goal (b), which is in between Goals (a) and (c), is reasonably attainable and the average allocations resemble a conventional glidepath that invest aggressively in early stages and gradually increase allocation in safe assets. Therefore, investors should increase (reduce) their goals when they have highly conservative (aggressive) portfolios similar to the ones for Goal (a) (Goal (c)).

3.1.3. Case 2: multiple goals with three priority levels. Next, we assume that the same investor has multiple goals. The investor has multiple goals with three priority levels: the highest priority is retirement savings, the second priority is paying for children's education, and the third priority is extra savings for retirement. The specific consumption goal amounts are summarized in Table 2. Figure 3 shows average allocations, and Figure 4 plots the average accumulation of wealth along with consumption amounts.

As illustrated in Figure 1, MSGP first solves the optimization problem only for goals with priority level 1 (Step 1), then solves the problem for goals with priority levels 1 and 2 (Step 2), and so on until Step P. In this example, there are three goals with three different priorities in stage 3 (age 60), and we will denote them as Goals 1, 2, and 3. Figure 5 presents goal achievements in stage 3 when the problem is solved for (a) only Goal 1, (b) Goals 1 and 2, and (c) all three goals. It can be seen from the figure that additional goals have absolutely no influence on the achievements of higher priority goals.

Table 2. Consumption goals of the investor for Case 2.

	Stage 1 (40)	Stage 2 (50)	Stage 3 (60)	Stage 4 (80)
Priority level 1	0	0	\$ 200,000	\$ 10,000
Priority level 2	0	\$ 20,000	\$ 120,000	\$ 5,000
Priority level 3	0	0	\$ 40,000	0

3.1.4. Case 3: downside risk protection on goal achievement. We next present a case when the investor wants downside protection for a consumption goal. Let us assume that the investor wants downside protection for Goal 2 in stage 3. Then we may add the following CVaR constraint

$$\mathrm{CVaR}_{90\%}(\tilde{G}_3^2 - (c_3^2 - c_3^1)) \leq 80\% \times \tilde{G}_3^2$$

by reformulating it into linear constraints as discussed in Section 2.3.

Figure 6 compares goal achievements in stage 3 for Cases 2 and 3 (without and with downside protection). In both cases, the investor can fully achieve Goal 2 in stage 3 in more than 3000 out of a total of 4000 scenarios. Without the CVaR constraint, goal achievements gradually decrease in the remaining unachieved scenarios. However, addition of the CVaR constraint shows that Goal 2 is almost achieved in a lot more scenarios. Therefore, the results clearly show the effectiveness of controlling downside risk of achieving a consumption goal.

3.1.5. Case 4: investment downside risk protection. Investors may wish to control investment risk, often measured as portfolio return volatility. Figure 7(a) shows the allocation results of an investor who has a relatively ambitious goal at age 60. Note that the allocations are quite risky in the early stages (no bond investment until age 50) in order to increase the chance of achieving the goal. Suppose the investor is worried about the outlook on emerging markets and wants to diversify the allocation even if it slightly reduces the possibility of achieving the goal. Then, it is possible to add CVaR constraints on investment risk as given by formulations (29) and (30) in Section 2.3. Figures 7(b) and (c) illustrate when CVaR constraints are imposed at ages 30 and 40, which are stages with no allocation in bonds in the original case of Figure 7(a). The allocations after applying (relatively) weaker and stronger CVaR constraints support that the constraints can successfully guide the model to control investment risk by reducing allocation in risky assets.

3.2. *Discussion on implementation and results*

The four cases above clearly demonstrate the advantage of the MSGP model. The MSGP formulation for a single goal maximizes goal consumption for each scenario but the formulation does not consider over-consumption and only focuses

† See, for example, Høyland and Wallace (2001), Høyland *et al.* (2003), Ji *et al.* (2005), Xu *et al.* (2012), and Consiglio *et al.* (2016).

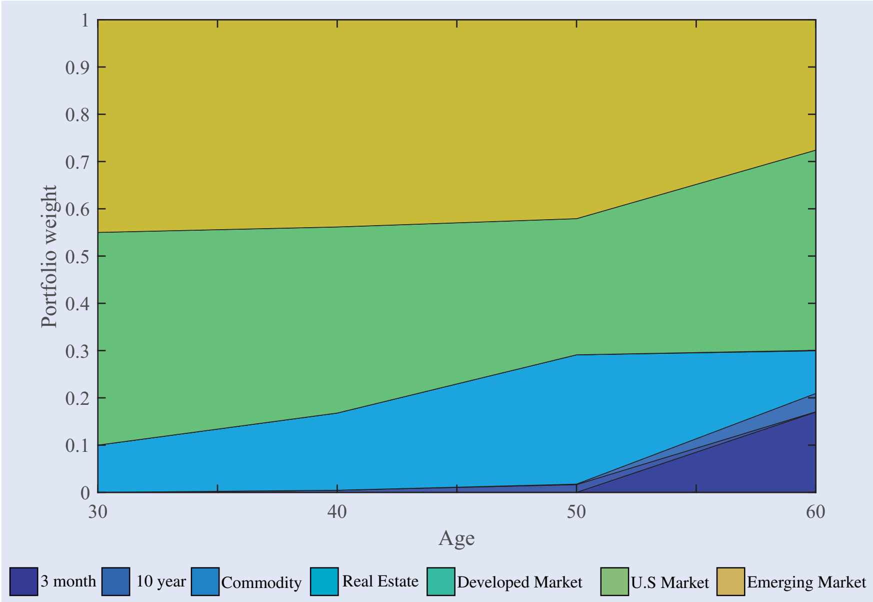

Figure 3. Average allocation results of Case 2.

Figure 4. Average wealth accumulation results of Case 2.

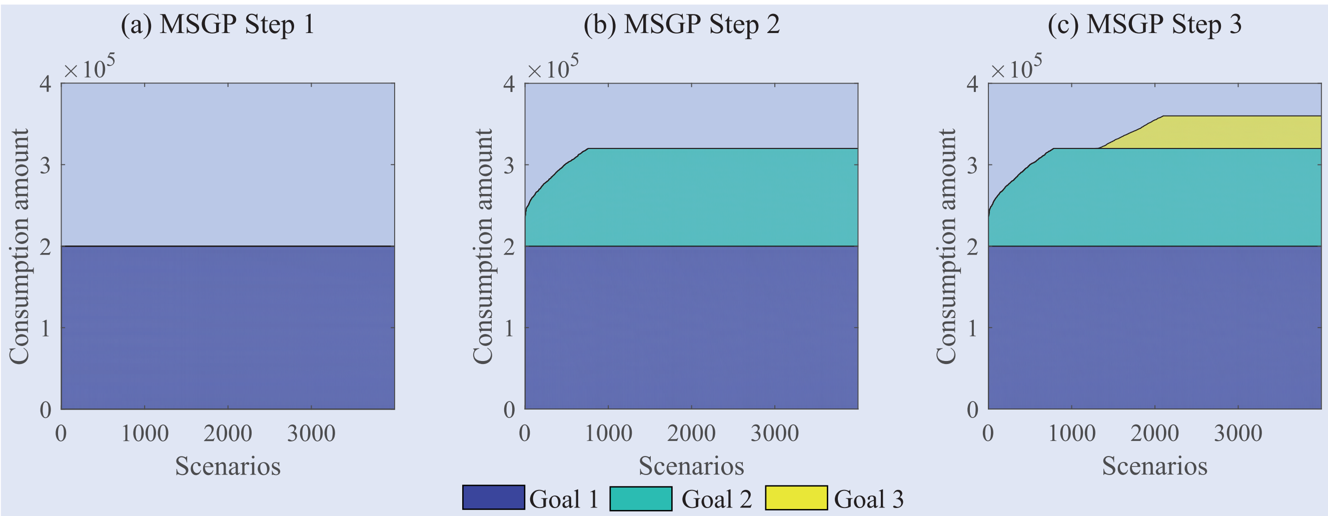

Figure 5. Incremental goal achievements in Stage 3 of Case 2.

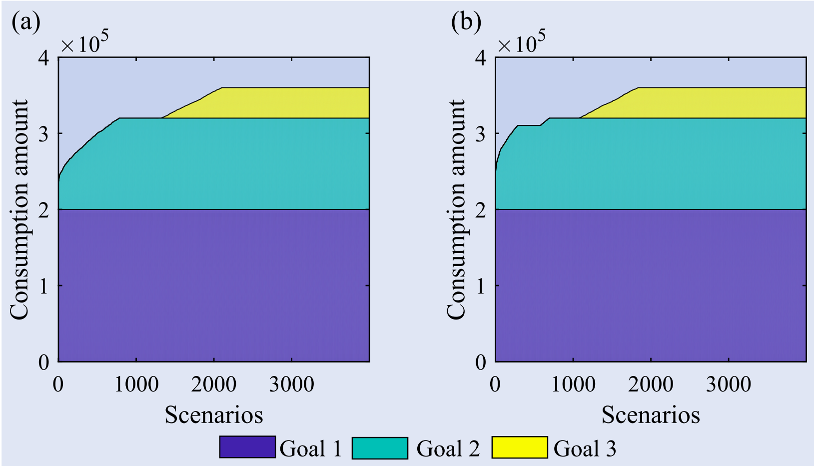

Figure 6. Goal achievements with and without goal achievement downside protection.

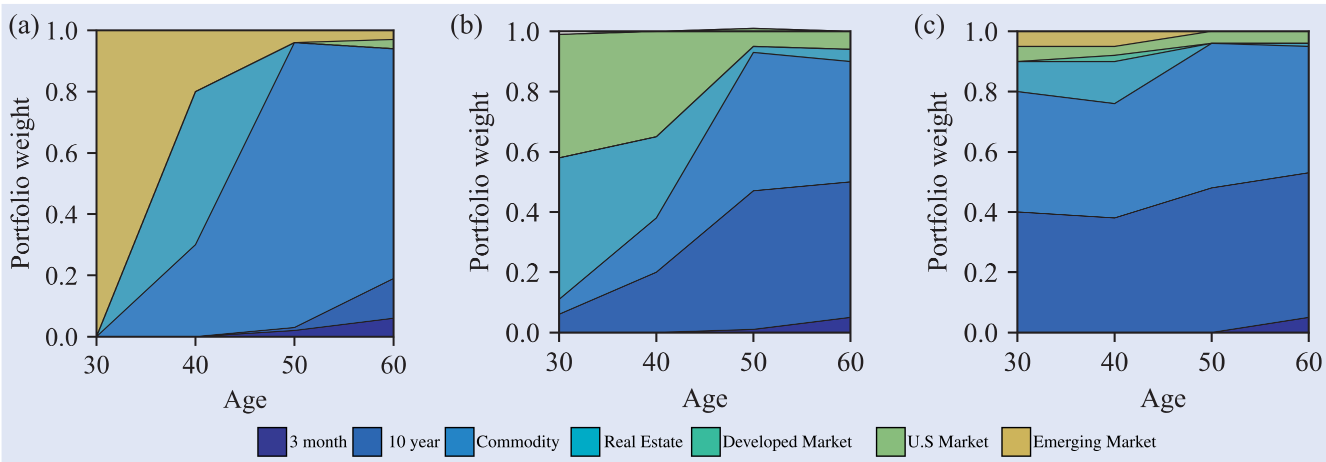

Figure 7. Average allocations with and without investment downside risk protection.

on reaching the target consumption level for each scenario. As shown in Case 2, investors can also specify multiple goals with various priority levels and higher priority goals are optimized with strict preference. By using consumption levels from higher priority level solutions, prioritization is satisfied regardless of the timing and amount of goals. Even though the objective of maximizing consumption does not directly penalize risk, Cases 3 and 4 illustrate how CVaR constraints on consumption and portfolio return can reduce the downside risk of goal achievement and investment return, respectively. Due to the use of scenario trees in MSGP, computing the outcome in each node allows CVaR calculations for controlling downside risk. The basic MSGP formulation may result in an aggressive portfolio when target goals are not easily achieved, but imposing risk constraints will maximize goal achievement while managing risk.

4. Conclusion

In this study, we propose a GBI framework that finds the optimal financial plan for an individual aiming to achieve multiple consumption goals with various priority levels, which is applicable for automated financial advising services also known as robo-advisors. In order for a GBI-based automated model to provide individual investors with proper and personalized financial advice, the model should exhibit two important features: simple language and quasi-real-time interaction. In this regard, we combine multi-stage stochastic programming and goal programming approaches, so that the proposed model only requires inputs of current wealth, future investment, and prioritized consumption goals. In addition, the proposed MSGP model has no potential sub-optimality issues since it is formulated as a linear program that finds a holistic plan for all goals.

Numerical analyses support the strength of MSGP that comes primarily from its flexibility. First, individual investors can easily provide their wealth profiles and quickly make adjustments by interacting with the model, and, thus, investors can receive well-customized financial advice. Second, the model allows much freedom for incorporating market uncertainty because scenario trees used in the model can be constructed to represent various market situations. Furthermore, our model is shown to be computationally efficient even when increasing the number of candidate assets or the total number of scenarios of asset returns, as further discussed in the appendix.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix A. Further Discussion on Implementation: Structuring the MSGP Problem

While the proposed portfolio problem given by equations (1–12) in Section 2.2 is a linear program, it is a large optimization problem. Thus, it is important to reduce computation time in order to provide investors with quasi-real-time interaction. Most optimization solvers require linear problems to be in the following structure,

$$\max Fx \text{ s.t. } Ax = b, lb \le x \le ub$$

where x is the decision vector that includes consumption decisions and asset allocation decisions, F corresponds to equation (1), A and b are formed to reflect equations (4–11), and vectors lb and ub represent inequalities (2), (3), and (12).

In the above case, the dimension of A becomes too large when considering large scenario trees and constructing matrix A would be time and memory consuming task. However, the large dimension of

A is mainly due to scenario trees and this part of the matrix is very sparse. Furthermore, the portion of matrix A that is independent of investor information, such as the part representing scenario trees, does not need to change for different investors unless the market representation needs an update. In fact, it is possible to separate matrix A between investor-independent details and investor-specific details. Therefore, for fast execution, we recommend to prebuild matrix A only with investor-independent information and load it each time the problem is solved. Saving a sparse matrix is memory efficient and spending most of the computation time on solving the linear programming problem will contribute much to the quasi-real-time interaction between the service and investors. We empirically find that computational time grows linearly with respect to the number of scenarios.