Portfolio Selection in Goals-Based Wealth Management

Hungjen Wang, Anil Suri, David Laster, and Himanshu Almadi

Hungjen Wang
is a vice president at Merrill Lynch Global Wealth Management, a division of Bank of America in New York, NY.
hungjen.wang@bankofamerica.com

Anil Suri
is a managing director at Merrill Lynch Global Wealth Management, a division of Bank of America in New York, NY.
anil_suri@ml.com

David Laster
is a director at Merrill Lynch Global Wealth Management, a division of Bank of America in New York, NY.
david.laster@ml.com

Himanshu Almadi
is a vice president at Merrill Lynch Global Wealth Management, a division of Bank of America in New York, NY.
himanshu_almadi@ml.com

The mean–variance framework developed in Markowitz's [1952] groundbreaking paper, "Portfolio Selection," has in recent decades become the workhorse model for wealth and investment managers. In that framework, each investor weighs the expected return on her overall portfolio against its variance (or standard deviation) to identify the efficient portfolio that delivers the highest expected return for the level of risk the investor is willing to bear.

While logical and normative, this approach takes no explicit account of whether the portfolio helps the investor achieve her goals. It also fails to account for the array of well-documented behavioral propensities that people exhibit. This approach is appropriate for investors who seek to achieve all their goals by investing in a single mean–variance efficient portfolio. However, as Thaler [1985] points out, investors typically do not focus on overall portfolio performance. Rather, they are prone to mental accounting and to making investment decisions based on the specific goal to be met. Shefrin and Statman [2000] propose "behavioral portfolio theory," which posits that investors do not view their portfolios holistically but, rather, associate each goal with a subportfolio. From the investor's perspective, each goal has its own aspiration level, or acceptable level of risk with respect to meeting the goal. Thus, someone nearing retirement might be risk averse with the portion of her wealth associated with retirement goals, much less risk averse regarding the portion associated with home renovations, and even risk seeking with respect to the portion devoted to the goal of amassing substantial wealth.

An emerging consensus in the wealth management industry favors a goals-based approach to advising clients on asset allocation and wealth management; see Brunel [2003, 2006], Chhabra [2005], Chhabra, et al., [2008] and Nevin [2004], among others. More fundamentally, best practices for financial planning, as advocated by the Certified Financial Planner (CFP) Board and encapsulated in international financial planning standards, identify eliciting clients' financial goals as the foundation of sound wealth management.[1] A goals-based wealth management approach reflects investor concerns in a practical manner and, to paraphrase Brunel [2010], helps the investor "bond" with her portfolio. This improved understanding could help the investor stick with her overall financial plan in the inevitable periods of market stress.

Under the goals-based wealth management framework, investors first specify their goals and priorities. Each investment goal, with its associated "subportfolio problem,"[2] is treated separately and solved independently (Barberis and Huang [2001]). Because each goal is likely to be met with some acceptable degree of uncertainty, investors are less prone to overreact to extreme market conditions.

The mean–variance approach, by contrast, guides investors to portfolios that often fail to reflect their ambitions and fears. If the expectations and risk tolerances for each goal are not taken into account, many investors will not commit to their original asset allocation decisions. Even those who do will become especially reluctant to adhere to them during times of market stress. This may happen partly because, as Kahneman and Tversky [1979] note, investors experience losses far more intensely than gains. Additionally, this "loss aversion" is usually simultaneously operative with other behavioral propensities, such as mental accounting (Thaler [1980]) and recency (Ebbinghaus [1885]; Kahneman and Tversky [1973]).

Das, Markowitz, Sheid, and Statman [2009] study goals-based investment in a mean–variance framework. They argue that their goals-based wealth management methodology does not introduce material inefficiency and show that the optimal subportfolios it leads to are mean–variance efficient in an unconstrained setting and nearly efficient otherwise.

This article's contribution is to construct a portfolio that minimizes the wealth required to achieve an investor's goals with specified probabilities of success. We begin by considering a one-period, single-goal case in a stylized two-asset market. We derive a closed-form solution to the problem and show how it relates to the mean–variance efficient frontier. Then we consider a single, multi-period goal, such as a retirement plan, and demonstrate that it can be viewed as a collection of single-period problems. This is similar to the "lockbox separation" concept proposed by Sharpe [2007]. We then extend our result to a market with multiple assets, where portfolios are given exogenously, as in Das, Markowitz, Sheid, and Statman [2009]. Finally, we discuss the general case of multiple goals, multi-period cash flows, and many assets. The next section characterizes the goals-based models mentioned above. Then we present an illustrative case study. The last section concludes.

GOALS-BASED INVESTMENT ENVIRONMENT AND MODELS

The Market

This section presents a standard continuous-time model of asset prices. Consider an economy with $K+1$ assets traded without transaction costs. One asset is riskless, with returns given by

$$dS_0(t) = r(t)S_0(t)\,dt \tag{1}$$

where $r(t)$ is the risk-free rate at time t. Prices of remaining assets 1 through K follow the Ito process

$$dS_k(t) = \mu_k(t)S_k(t)dt + \sum_{j=1}^{K}\sigma_{kj}(t)S_k(t)dB_j(t), \quad k = 1,\ldots,K \tag{2}$$

where the drift $\mu_k(t)$ and volatility $\sigma_{kj}(t)$ are known, and $B_j(t)$ denotes standard Brownian motion (see Shreve [2004] for details). A standard result is that the price of each asset, S_k, at time $t > 0$ is

$$S_k(t) = S_k(0)\exp\left\{\int_0^t\left(\mu_k(s) - \frac{1}{2}\sum_{j=1}^{K}\sigma_{kj}^2(s)\right)ds + \int_0^t\sum_{j=1}^{K}\sigma_{ki}(s)dB_j(s)\right\}$$

Since the rate $r(t)$, drift $\mu_k(t)$ and volatility $\sigma_{kj}(t)$ are known, we can, over any given time period, replace them with constants. For example, over the period $[0, T]$, one can find a constant $\bar{\sigma}_k$ that satisfies

$$\bar{\sigma}_k^2 T = \sum_{j=1}^{K}\int_0^T\sigma_{kj}^2(s)ds$$

Joshi [2008] has more details. Thus, for simplicity and clarity of presentation, the rest of this analysis will use a constant risk-free rate r, drift μ, and volatility σ.

Characterizing a Goal

To make our discussion concrete, we formally define goals. In a goals-based investment framework, each goal has three elements: 1) when it is to be fulfilled, 2) its associated target, or "aspiration" level, and 3) the acceptable probability of success with which it is to be achieved (see Shefrin and Statman [2000] for details).

In a single-period, single-goal model, let T denote the time at which the goal is to be fulfilled, $W(T)$ wealth at time T, $\bar{W}$ the aspiration level and $(1-\alpha)$ the desired probability of success (in other words, α is the probability of failure). The goal can be characterized as[3]

$$P(W(T) \le \bar{W}) \le \alpha$$

Single-Period, Two-Asset Case

First, assume there are only two assets, a risk-free asset S_0 and a risky asset S_1. The investor must determine how much wealth is required and how to allocate that wealth to achieve her goals with an acceptable level of uncertainty. If she allocates her entire portfolio to the risk-free asset, returns will be modest, necessitating a high level of wealth to fund the goal. If, on the other hand, she invests in only the risky asset, although the cost of achieving her goal might be significantly less, she would be uncertain of achieving the goal.

Let θ_0 and θ_1 denote the number of units of the risk-free and risky asset, respectively, that an investor purchases to build a subportfolio associated with her goal. The corresponding stochastic process describing the subportfolio's value is

$$dW(t) = \theta_0 dS_0(t) + \theta_1 dS_1(t)$$

so that at time T it is worth

$$W(T) = \theta_0 S_0(T) + \theta_1 S_1(T)$$

This description of the wealth process assumes that once an investor has made her initial investment in units of the risk-free and risky assets at time 0, she will hold those units until time T. Note, however, that if the risky asset S_1 is a fund that is periodically rebalanced, then the subportfolio is implicitly rebalanced as well.

Applying the martingale methodology of Cox and Huang [1989], we formulate the investor's problem (P1) as

$$\begin{aligned} & \underset{\theta_0, \theta_1, W}{\text{minimize}} \; W \\ \text{subject to} \quad & P(W(T) \le \bar{W}) \le \alpha \\ & \theta_0 S_0(0) + \theta_1 S_1(0) \le W \\ & \theta_0, \theta_1, W(t) \ge 0, \; t \in [0, T] \end{aligned} \tag{P1}$$

The problem is to find the minimum level of initial wealth W needed to fulfill the investor's goal at time T. The first inequality in problem (P1) states the requirement that, at time T, the probability of the subportfolio's value $W(T)$ being less than the aspiration level $\bar{W}$ is α. The second inequality is a budget constraint stating that the initial value of all the subportfolios cannot exceed initial wealth W. The last inequality is a no short-sale constraint.

Proposition 1: *The optimal solution to the investment decision problem (P1) is*

$$\theta_0^* = \begin{cases} \dfrac{\bar{W}}{S_0(0)} \exp(-rT), & \text{if } \left(r - \mu + \dfrac{1}{2}\sigma^2\right)T - \sigma\sqrt{T}\Phi^{-1}(\alpha) \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\theta_1^* = \begin{cases} \dfrac{\bar{W}}{S_1(0)} \exp\left(-\left(\mu - \dfrac{1}{2}\sigma^2\right)T - \Phi^{-1}(\alpha)\sigma\sqrt{T}\right), & \text{if } \left(r - \mu + \dfrac{1}{2}\sigma^2\right)T - \sigma\sqrt{T}\Phi^{-1}(\alpha) \le 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\Phi^{-1}(\cdot)$ denotes the inverse cumulative distribution function of the standard normal distribution, and the minimum required initial wealth is

$$W^* = \theta_0^* S_0(0) + \theta_1^* S_1(0)$$

The proof, provided in the Appendix, involves transforming the investment problem (P1) into a linear programming problem.

One implication of this solution is that an extremely risk-averse investor seeking to attain her goal with near certainty should invest all her wealth in the risk-free asset. Her optimal solution is therefore $\theta_1^* = 0$ (invest nothing in the risky asset). This is because the inequality

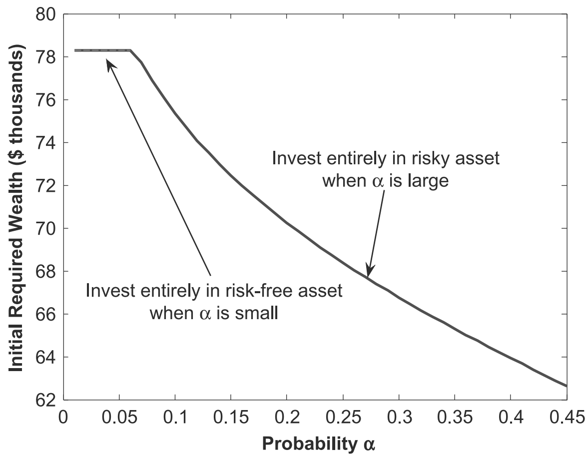
$$\left(r - \mu + \frac{1}{2}\sigma^2\right)T - \sigma\sqrt{T}\Phi^{-1}(\alpha) \ge 0$$

will always hold for a sufficiently small value of α.[4] As Exhibit 1 illustrates, when the probability α is small, the initial wealth is invested entirely in the risk-free asset.

As the acceptable probability of failure α increases, the necessary initial wealth decreases. This is because an investor who tolerates more risk with respect to a goal can allocate more to the risky asset, which earns a higher expected return.

Another implication of Proposition 1 is that, holding the risk-free rate r and volatility σ fixed, a higher value of drift μ (which implies higher expected returns to the risky asset) will mean a higher allocation to the risky asset and lower required wealth, W^*. If, on the other hand, rate r and drift μ are fixed, lower levels of volatility σ will

EXHIBIT 1
Initial Required Wealth versus Probability α



Note: The parameter values assumed in this example are $T = 10$, $\bar{W} = \$100{,}000$, $S_0 = S_1 = 1$, $r = 2.45\%$, $\mu = 5.27\%$ and $\sigma = 5.33\%$.

lead to a higher allocation to the risky asset and a lower required wealth, W^*.

This trade-off between risk and return is the fundamental theme of the mean–variance framework. Recall that a portfolio ω is called mean–variance efficient, if there exists no other portfolio υ, such that $\mu_\omega \le \mu_\nu$ and $\sigma_\omega \ge \sigma_\nu$, where at least one of the inequalities is strict. We define efficiency for goals-based investment in a similar manner.

Definition 1: A portfolio ω is called *goals-based efficient,* if there exists no other portfolio υ, such that $W_\omega^* \ge W_\nu^*$, $\mu_\omega \le \mu_\nu$, and $\sigma_\omega \ge \sigma_\nu$, where at least one of the three inequalities is strict.

Corollary 1: When $\alpha < 0.5$, (i.e., the probability of success is greater than 50%), a portfolio is goals-based efficient if—and only if—it is also mean–variance efficient. (For proof, see Appendix).

The corollary implies that a mean–variance efficient portfolio is goals-based efficient, and vice versa. This supports Das, Markowitz, Sheid, and Statman's [2009] use of exogenously given mean–variance efficient portfolios for goals-based investment analysis.

Multi-Period, Single-Goal Case

Now consider a goal requiring a stream of cash flows. For example, an investor might set a goal to receive a fixed amount of money annually, with a prespecified probability of success, for the 30 years after she retires. Sharpe [2007] develops the *lockbox separation* concept for a multi-period goal in a complete, discrete-time market. He argues that a multi-period goal is just a collection of single-period goals. More specifically, the cash flows required in each period can be treated independently, and the wealth needed for the goal is the sum of the minimum wealth requirements for each of its component single-period goals. We extend Sharpe's [2007] result to the case of a complete, continuous-time market.[5]

Characterizing a Multi-Period Goal

Consider a goal that for each period $i = 1, \ldots, N$, has an aspiration level $\bar{W}_i$ with success probability $1-\alpha_i$ at time T_i.

Applying the martingale methodology discussed in the previous section, an investor with this multi-period goal in mind formulates her investment decision problem (P2).

$$\begin{aligned}
&\underset{\theta_{0_1},\theta_{1_1},\ldots,\theta_{0_N},\theta_{1_N},W}{\text{minimize}}\ W && \text{(P2)}\\
&\text{subject to } P(\theta_{0_i} S_0(T_i) + \theta_{1_i} S_1(T_i) \le \bar{W}_i) \le \alpha_i\\
&\quad\text{for all } i = 1,\ldots,N\\
&\sum_{i=1}^{N} \theta_{0_i} S_0(0) + \theta_{1_i} S_1(0) \le W\\
&\theta_{0_1},\theta_{1_1},\ldots,\theta_{0_N},\theta_{1_N},W \ge 0
\end{aligned}$$

The investment problem (P2) aims to minimize the initial total investment W, while achieving the aspiration level $\bar{W}_i$ with the associated probability of success $1-\alpha_i$ at time T_i for every $i = 1, \ldots, N$. The first group of inequalities in (P2) states that requirement. In these inequalities, the term $\theta_{0_i} S_0(T_i) + \theta_{1_i} S_1(T_i)$ denotes the realized cash flow at time T_i. Moreover, the term θ_{0_i} expresses the units of the risk-free asset that the investor holds for the time T_i cash flow target and θ_{1_i} is the units of the risky asset held. The second inequality in (P2) is a budget constraint that states that (in a risk-neutral sense) the present value of what an investor receives in the future cannot exceed what she invests today. The last inequality is a no short-sale constraint. The following proposition, whose proof is in the Appendix, states that the multi-period problem (P2) can be solved by treating each cash flow requirement as an independent problem.

Proposition 2: *The optimal solution to the investment decision problem (P2) is the sum of the optimal solutions corresponding to the investment problems in which each cash flow requirement for each period is treated independently.*

Proposition 2 significantly simplifies the calculation for the initial wealth requirement for a multi-period goal. Suppose, for example, that an investor would like to receive \$100,000 annually over the 30 years after retirement. All we need to do is to decompose this goal into 30 cash flows, treat each cash flow independently, and apply Proposition 1 to obtain the optimal solutions for each of the 30 single-period goals. The optimal solution for this multi-period goal is simply the sum of the 30 solutions to the single-period goal problems.

Multi-Asset Case

Next, we use the single-asset result discussed previously to address the multi-asset case. As in Das, Markowitz, Sheid, and Statman [2009] and Nevins [2004], we assume that assets are exogenously given. The investor can select from among a range of funds with various asset allocations. One fund may have 30% in equity, 50% in bonds, and 20% in cash, and another 40% in equity, 45% in bonds, and 15% in cash. We further assume that the number of funds is so large that they form a continuum, so that any portfolio composed of these funds will itself behave like one of the funds. Alternatively, we can assume that the given funds are on the mean–variance efficient frontier.

Once the drifts and volatilities of the portfolios are given, an investor faces the problem of how to choose from among the funds to minimize the cost of fulfilling her goal. Since we can solve a multi-period goal problem by decomposing it into multiple single-goal problems, it suffices to discuss the single-period goal case. We can solve the single-period goal problem in a multi-asset market using the result we derived earlier. First compute the minimum required wealth for each given fund with the risk-free asset and then choose the portfolio with the minimum required wealth.

Regarding the multi-goal case, it bears emphasizing that our framework has the flexibility to accommodate a distinct aspiration level and probability of success for each required cash flow. Consequently, the different single-period goal problems that are components of a multi-period goal problem may lead to different portfolios.

Multi-Period, Multi-Goal, Multi-Asset Case

Finally, we consider the most general case of multiple goals with multi-period cash flow requirements in a multi-asset market. Following Shefrin and Statman [2000] and Barberis and Huang [2001], each goal is considered independently and, given the results of Proposition 2, the covariance between any pair of goal-specific portfolios is appropriately ignored. This makes it straightforward to solve this general multi-goal problem by decomposing it into different single-goal problems and then applying the results discussed above. Summing the solutions to the single-period problems gives the solution to the general multi-goal problem.

CASE STUDY

Now we illustrate how to apply our results to the most general case discussed above. We begin by describing an investor's goals. We then prioritize the goals according to how much uncertainty in fulfilling them is acceptable to the investor. Finally, we apply the results developed in the previous section to compute the minimum initial wealth required to fulfill the goals.

Consider an investor with three goals: 1) buying a home, 2) funding the children's education, and 3) funding retirement. Exhibit 2 summarizes her goals. The investor plans to purchase a home five years from now (i.e., $T = 5$). The aspiration of this goal is classified in three levels. Since she would like to maintain her quality of life in terms of living environment, she would like to make a down payment of no less than \$75,000 to buy a house. Hence, she sets a goal with aspiration level \$75,000 ($\bar{W} = 75{,}000$) and an 80% probability of success ($(1 - \alpha) = 0.8$). Moreover, since she would prefer to move to a better neighborhood, she also hopes to save an additional \$75,000 to meet the first goal, with 60% probability. If possible, she would also prefer to have enough funds to build a home office. Therefore, she aims to accumulate yet another \$75,000 toward her home goal with a 40% probability of success. In summary, for the goal of buying a home, the investor sets three priorities: the first \$75,000 with probability 80%, an additional \$75,000 with probability 60%, and yet another \$75,000 with 40% chance of success.

Exhibit 2
Summary of Investor's Goals

Goals	Amount ($ thousands)	Probability of Success
Home Purchase	75	80%
(Year 5)	75	60%
	75	40%
Children's Education*	25	90%
(Years 11–14)	25	80%
	25	50%
Retirement**	50	95%
(Years 21–50)	50	90%
	50	50%

*Notes: * The aspiration amount increases 4% annually to offset inflation.*
*** The aspiration amount increases 2% annually to offset inflation.*
Both inflation offsets will begin in Year 0.

Note that a higher probability of success reflects a stronger determination to fulfill a goal. The investor designates the initial $75,000 goal as high priority by assigning a relatively high success probability to it. For the additional increments of $75,000, she assigns lower probabilities. This reflects her willingness to tolerate greater risk while pursuing those more ambitious goals. Although the dollar increments in this example are equal, the analysis works just as well for unequal increments.

The investor also sets a goal for her children's college education. Unlike the goal of a home purchase down payment, which is for one period, this goal is multi-period, beginning 11 years from now and lasting for 4 years ($T = 11, \ldots, 14$). The investor prioritizes this goal according to three categories: funding $25,000 annually with a 90% probability of success, an additional $25,000 annually with an 80% probability of success, and yet another $25,000 annually with a 50% chance of success. To offset anticipated tuition inflation, these target amounts increase 4% annually.

Finally, the investor sets a third goal, funding retirement. She expects to retire 20 years from now and to receive a stream of cash flows over the next 30 years ($T = 21, 22, \ldots, 50$). To ensure that she maintains her current living standard, she sets as her highest priority receiving $50,000 annually with a 95% probability of success. In addition, she aims to receive an additional $50,000 per year in retirement with a 90% probability of success and yet another $50,000 annually with probability 50%. To offset inflation, each of these target cash flows increases 2% annually.

The investor is saving to attain her three goals. Her current wealth is $1,000,000 and she plans to save $25,000 this year and to boost the amount she saves by 2% annually over the next 20 years. To compute the net present value of her saving plan, we use a discount rate of 8%.[6] Exhibit 3 summarizes the investor's saving plan.

The last piece of information needed is the set of funds from which the investor can choose her investments. We choose five funds from the efficient frontier. Exhibit 4 summarizes the risk and return of the funds, which are plotted along the efficient frontier in Exhibit 5.[7] As these exhibits show, fund 1 is the least risky and fund 5 is the riskiest. We expect that in striving to meet a goal associated with a high probability of success, the investor will choose the lower risk funds, consistent with her high risk aversion with respect to the goal. Conversely, to meet more aggressive goals for which she tolerates a lower probability of success, the investor will invest in higher risk, higher return funds.

The savings plan and goals of the investor can be depicted as a series of cash flows (Exhibit 6). Can the investor's planned savings fund her goals? If so, we would like to know the minimum wealth required to fund the goals. Furthermore, for each goal, we need to know which fund to invest in to attain it at minimum cost, given its aspiration level, time horizon, and success probability.

Exhibit 3
Investor's Saving Plan

Saving Plan	Amount ($ thousands)	
Savings (Year 1–20)	25	(Annual Saving)
Initial Wealth	1000	

Note: The savings will increase by 2% annually.

Exhibit 4
Statistics of Available Funds

	Fund 1	Fund 2	Fund 3	Fund 4	Fund 5
Expected Annual Return μ	5.57%	6.76%	7.77%	8.19%	8.61%
Annual Volatility σ	5.63%	8.54%	11.67%	13.27%	14.93%

EXHIBIT 5
Efficient Frontier and Available Funds

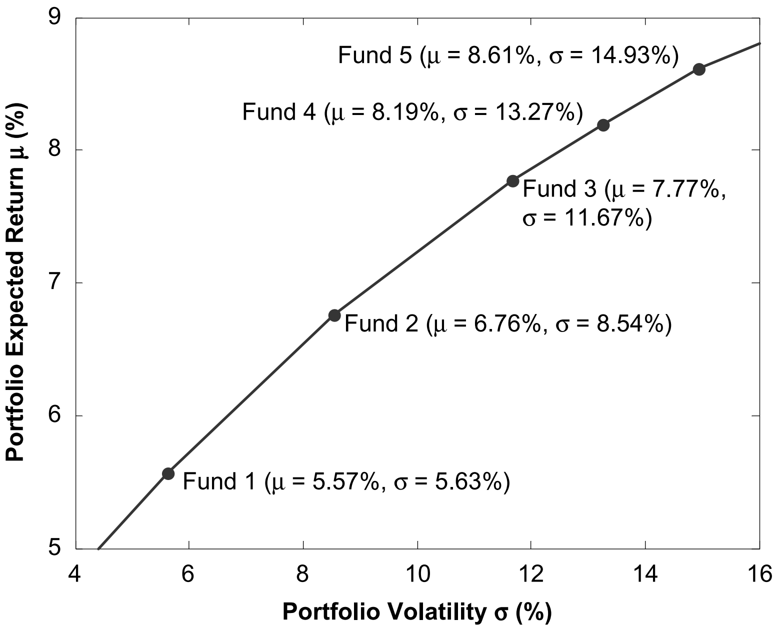
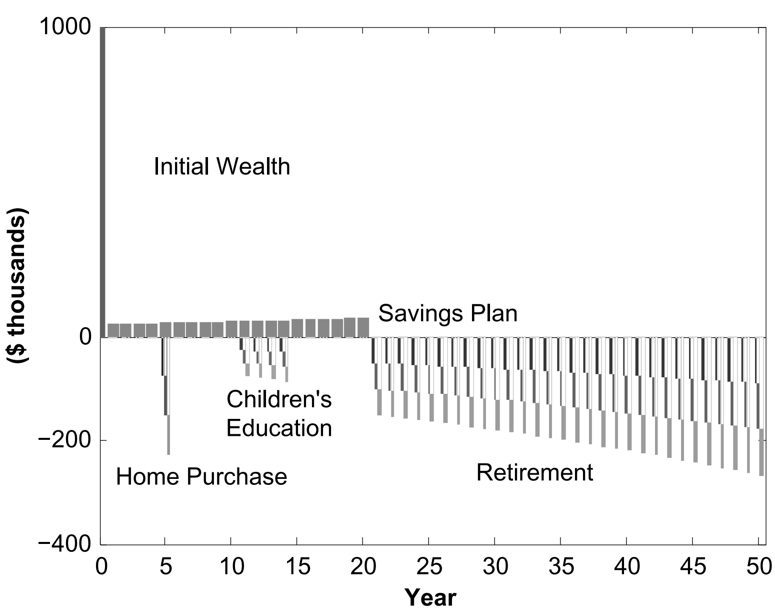


EXHIBIT 6
Goals Specification in Cash Flow View



If, on the other hand, the savings are insufficient to cover the goals, we would like to know how we could modify the goals to make the overall plan attainable. For example, delaying retirement could help in this situation.

Applying Proposition 2 and the discussion following it, we decompose the goals in this more general case into their simplest components: single-period goals. In this case study, the three goals constitute 105 simple, single-period goals. Then, by Proposition 1, we compute the minimum wealth necessary for each simple goal, and finance each of the simple goals by investing in the fund requiring the smallest initial wealth to achieve it. The closed-form solution in Proposition 1 facilitates the computation. By solving for each simple goal, we know how much is needed to fund the overall plan and which funds to invest in to meet the goal.

From Exhibit 7, which summarizes the results, we observe that the goals with a higher probability of success are supported by investments in the less risky funds. For example, in the retirement plan, the investor's highest priority is to receive \$50,000 annually with a success probability of 95%. The investor supports this goal by investing in the least risky funds: fund 2 for years 21 to 32, and fund 1 for years 33 to 50. Conversely, for a goal where the investor is willing to tolerate more uncertainty, there will be a greater allocation to riskier funds. For example, the goal of home purchase requiring an incremental \$75,000 in year 5 with a 40% success probability will be supported with an investment in the most risky fund.

The sum of the investor's current wealth and the net present value of her saving plan is approximately \$1,289,000. But from Exhibit 7 we know that not all goals can be funded with these resources. Our approach identifies when goals cannot be fulfilled with the wealth available, requiring the investor to modify her plan. The investor can then adjust some goals not funded by the current plan, or accept increased uncertainty regarding the goals that are funded.

CONCLUSION

This article proposes a framework for goals-based wealth management. We start by investigating a simple case where there is one single-period goal and two assets and derive a closed-form solution

EXHIBIT 7

Final Result of Goals-Based Wealth Management Case Study

Goals	Probability of Success	Beginning Year	Final Year	Cost of Using Fund to Meet Goals: Fund 1	Fund 2	Fund 3	Fund 4	Fund 5	Cumulative Funds Required
Retirement Plan	95%	21	32	$284,589	$274,575	$310,800	$351,580	$408,285	$274,575
Retirement Plan	95%	33	50	$270,374	$276,181	$313,331	$338,227	$349,578	$544,949
Retirement Plan	90%	21	50	$496,408	$466,381	$498,659	$541,937	$601,471	$1,011,331
Children's Education	90%	11	14	$71,094	$72,816	$77,535	$78,092	$78,092	$1,082,424
Children's Education	80%	11	14	$65,441	$64,317	$66,357	$68,678	$71,646	$1,146,741
Home Purchase	80%	5	5	$64,154	$64,894	$66,353	$66,353	$66,353	$1,210,895
Home Purchase	60%	5	5	$59,808	$58,419	$58,130	$58,415	$58,874	$1,269,025
Retirement Plan	50%	21	50	$335,519	$260,345	$227,359	$222,765	$222,121	$1,491,145
Children's Education	50%	11	14	$55,851	$50,727	$48,156	$47,778	$47,724	$1,538,870
Home Purchase	40%	5	5	$56,301	$53,362	$51,437	$50,864	$50,422	$1,589,291

Note: For each goal, the shaded box indicates the fund for which the required initial wealth is minimized.

to the problem. Through the closed-form solution, we establish a relationship between the goals-based optimal portfolio and the portfolios on the mean–variance efficient frontier. We further extend our result to the more general case of multi-period goals. Applying Sharpe's lockbox separation concept and the martingale methodology, we derive a solution for a multi-period goal by decomposing it into single-period goals. Finally, we show how to apply our result to the general case of multi-period goals and a market with many assets.

Although this framework facilitates goals-based wealth management, much work remains. First, our approach implicitly assumes that once the investor allocates assets to certain funds, she will continue to hold the portfolio regardless of market dynamics. In reality, we expect that a dynamic rebalancing strategy contingent on market conditions would be more beneficial for the investor. Second, as in Das, Markowitz, Sheid, and Statman [2009], we assume that the funds are given exogenously. It would be more practical if the investor could assemble her portfolio by investing in equities, bonds, and other assets directly. One technique that can deal with this type of problem is stochastic dynamic programming, with which one can simulate numerous scenarios and compute optimal rebalancing strategies. Third, this study focuses on achieving goals without considering in detail the uncertainties surrounding the savings that an investor plans to accumulate. Our framework can be broadened to more carefully account for the uncertainty of those cash flows due to potential life events, such as unemployment or illness. This too can be studied in a stochastic dynamic programming framework. Such an approach would, however, significantly complicate the model. Finally, an investor must adjust her goals or savings if those goals are not attainable with the wealth available to her. Those adjustments are feasible because of the simplicity of our framework but are not a focus of this article. We leave these extensions for future research.

APPENDIX

Proof of Proposition 1: We first show that the investor's decision problem (P1) is equivalent to the following linear program problem.

$$\begin{aligned}
&\underset{\theta_0,\theta_1,W}{\text{minimize}}\ W\\
&\text{subject to}\quad -\theta_0 S_0(0)\exp(rT) - \theta_1 S_1(0)\\
&\qquad \times \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \Phi^{-1}(\alpha)\sigma\sqrt{T}\right) \le -\bar{W}\\
&\qquad \theta_0 S_0(0) + \theta_1 S_1(0) \le W\\
&\qquad \theta_0, \theta_1, W(t) \ge 0, \quad \forall t \in [0.T]
\end{aligned}$$

where $\Phi^{-1}(\cdot)$ denotes the inverse cumulative distribution function of the standard normal distribution.

It suffices to show that the first inequalities in both problems are equivalent. Observe that

$$P(W(T) \le \bar{W}) \le \alpha \Leftrightarrow P\Bigg(\theta_0 S_0(0)\exp(rT) + \theta_1 S_1(0)$$

$$\times \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\varepsilon\right) \le \bar{W}\Bigg) \le \alpha$$

$$\Leftrightarrow P\Bigg(\log(\theta_1 S_1(0)) + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\varepsilon$$

$$\le \log(\bar{W} - \theta_0 S_0(0)\exp(rT))\Bigg) \le \alpha$$

$$\Leftrightarrow \log\left(\frac{\bar{W} - \theta_0 S_0(0)\exp(rT)}{\theta_1 S_1(0)}\right) \le \left(\mu - \frac{1}{2}\sigma^2\right)T + \Phi^{-1}(\alpha)\sigma\sqrt{T}$$

$$\Leftrightarrow \bar{W} - \theta_0 S_0(0)\exp(rT) \le \theta_1 S_1(0)\exp\Bigg(\left(\mu - \frac{1}{2}\sigma^2\right)$$

$$\times T + \Phi^{-1}(\alpha)\sigma\sqrt{T}\Bigg)$$

$$\Leftrightarrow -\theta_0 S_0(0)\exp(rT) - \theta_1 S_1(0)\exp\Bigg(\left(\mu - \frac{1}{2}\sigma^2\right)$$

$$\times T + \Phi^{-1}(\alpha)\sigma\sqrt{T}\Bigg) \le -\bar{W}$$

where ε denotes a stand normally distributed random variable. Hence, the linear programming problem above is equivalent to investment problem (P1).

It is clear that the optimal solution exists, because the constraint set is a nonempty polyhedral, and that the second inequality will become binding at optimality since the variable W only relates to the term $\theta_0 S_0(0) + \theta_1 S_1(0)$ in the constraint set. Since the optimality of this linear programming problem gives the minimum sum of the terms $\theta_0 S_0(0)$ and $\theta_1 S_1(0)$, the first inequality becomes binding. Therefore, the term with a larger coefficient in the first inequality will have a smaller value and will be positive, while the other term will become zero.

Proof of Corollary 1: Suppose that a portfolio ω is goals-based efficient but not mean–variance efficient. Then there must exist a portfolio υ, such that $\mu_w \le \mu_v$ and $\sigma_\omega \ge \sigma_v$, where at least one of the inequalities is strict. From the result of Proposition 1 and the condition $\alpha < 0.5$, we have $W_\omega^* \ge W_V^*$, violating the assumption that portfolio ω is goals-based efficient. We next prove that mean–variance efficiency implies goals-based efficiency. Suppose that a portfolio ω is mean–variance efficient, but not goals-based efficient. Then there exists a portfolio υ such that $W_\omega^* \ge W_V^*$, $\mu_\omega \le \mu_v$, and $\sigma_w \ge \sigma_v$, where at least one of the inequalities is strict. Since the portfolio ω is mean–variance efficient, then the last two inequalities will become binding, then the first inequality $W_\omega^* \ge W_v^*$ would not be true under the assumption of $\alpha < 0.5$. Similarly, any other combination of binding inequalities violates the assumption that portfolio ω is mean–variance efficient.

Proof of Proposition 2: We restate the investment decision problem (P2) with a group of dummy variables W_i, $i = 1, \ldots, N$, in the following problem (P2′).

$$\begin{aligned} &\underset{\theta_{0_1}, \theta_{1_1}, \ldots, \theta_{0_N}, \theta_{1N}, W}{\text{minimize}} \; W && \text{(P2′)} \\ &\text{subject to } P(\theta_{0_i} S_0(T_i) + \theta_{1_i} S_1(T_i) \le \bar{W}_i) \le \alpha_I, \text{ for all } i = 1, \ldots, N \\ &\theta_{0_i} S_0(0) + \theta_{1_i} S_1(0) \le W_i, \text{ for all } i = 1, \ldots, N \\ &\sum_{i=1}^{N} W_i \le W \\ &\theta_{0_1}, \theta_{1_1}, \ldots, \theta_{0_N}, \theta_{1_N}, W \ge 0 \end{aligned}$$

Problems (P2) and (P2′) are equivalent, because the only difference between them is that in the second inequality of (P2), the sum of the terms

$$\theta_{0_i} S_0(0) + \theta_{1_i} S_1(0)$$

is replaced by the dummy variable W_i for all possible i in problem (P2′). At optimality, the inequality would become binding and therefore the equivalence holds. Next, we consider a problem associated with a single period of the multiperiod goal:

$$\begin{aligned} &\underset{\theta_{0_i}, \theta_{1_i}, W_i}{\text{minimize}} \; W_i && (\mathrm{P_i}) \\ &\text{subject to } P(\theta_{0_i} S_0(T_i) + \theta_{1_i} S_1(T_i) \le \bar{W}_i) \le \alpha_I \\ &\theta_{0_i} S_0(0) + \theta_{1_i} S_1(0) \le W_i \\ &\theta_{0_i}, \theta_{1_i}, W_i \ge 0 \end{aligned}$$

Let W^* denote the optimal solution to problem (P2′) and W_i^* for problem ($\mathrm{P_i}$) for all $i = 1, \ldots, N$. Then it suffices to show that $W^* = \Sigma_{t=1}^N W_i^*$. It is clear that $W^* \le \Sigma_{t=1}^N W_i^*$, because what satisfies the constraint set in (P2) would naturally satisfy those in ($\mathrm{P_i}$). It remains to be proved that $W^* \ge \Sigma_{t=1}^N W_i^*$. By construction, the dummy variable W_i in (P2′) satisfies $W_i \ge W_i^*$, because W_i^* is the minimum wealth needed to meet the requirement related to the first inequality in (P2) and the corresponding constraint in (P2′). Since $\Sigma_{t=1}^N W_i \le W$, we have $W^* \ge \Sigma_{t=1}^N W_i^*$.

ENDNOTES

The views expressed here are those of the authors and not necessarily those of any entity within and including Bank of America.

[1]See CFP Board, *Standards of Professional Conduct*, "Financial Planning Practice Standard 200–1: Determining a Client's Personal and Financial Goals, Needs and Priorities"; and ISO Standard 22222:2005, "Personal Financial Planning—Requirements for Personal Financial Planners."

[2]The subportfolio can be interpreted as a mental account. We use the terms interchangeably.

[3]This setup is consistent with that in Shefrin and Statman [2000] and Das, Markowitz, Sheid, and Statman [2009]. Moreover, the latter relates the success probability to value at risk.

[4]Although this depends upon r, μ, σ, and T among other variables, this dependency is negligible for sufficiently small values of α.

[5]Note that the structure of the market we described earlier implies that the market is complete (Shreve [2004]).

[6]This discount rate reflects uncertainties surrounding future savings, such as the risks of unemployment and disability.

[7]The figures given here are the simple expected rate of return and associated volatilities. To conduct this analysis, we first convert these into logarithmic rates of return and volatilities.

REFERENCES

Barber, B.M., and T. Odean. "Trading Is Hazardous to Your Wealth: The Common Stock Investment Performance of Individual Investors." *Journal of Finance*, Vol. 55, No. 2 (2000), pp. 773-806.

Barberis, N., and M. Huang. "Mental Accounting, Loss Aversion and Individual Stock Returns." *Journal of Finance*, Vol. 56, No. 4 (2001), pp. 1247-1292.

Berkelaar, A.B., R. Kouwenberg, and T. Post. "Optimal Portfolio Choice under Loss Aversion." *Review of Economics and Statistics*, Vol. 86, No. 4 (2004), pp. 973-987.

Brunel, J.L. "Revisiting the Asset Allocation Challenge through a Behavioral Finance Lens." *The Journal of Wealth Management*, Vol. 55, No. 1 (Fall 2003), pp. 10-20.

——. "A Behavioral Finance Approach to Strategic Asset Allocation–A Case Study." *Journal of Investing Consulting*, Vol. 7, No. 3 (2006), pp. 61-69.

——. "Editor's Letter." *The Journal of Wealth Management*, Vol. 13, No. 3 (Winter 2010), pp. 1-4.

Chhabra, A.B. "Beyond Markowitz: A Comprehensive Wealth Allocation Framework for Individuals." *The Journal of Wealth Management*, Vol. 7, No. 4 (2005), pp. 8-34.

Chhabra, A.B., Ravindra Koneru, Lex Zaharoff. "Creating a Goals-Based Wealth Allocation Process." *The Journal of Wealth Management*, Vol. 11, No. 3 (Winter 2008), pp. 48-55.

Cox, J., and C. Huang. "Optimum Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process." *Journal of Economic Theory*, Vol. 49 (1989), pp. 33-83.

Das, Sanjiv, Harry Markowitz, Jonathan Scheid, and Meir Statman. "Portfolio Optimization with Mental Accounts." *Journal of Financial and Quantitative Analysis*, Vol. 45, No. 2 (April 2010), pp. 311-334.

De Giorgi, E.G. "Goals-Based Investing with Cumulative Prospect Theory and Satisficing Behavior." Working paper, August 2009.

Duffie, D. *Dynamic Asset Pricing*, 3rd ed. Princeton, NJ: Princeton University Press, 2001.

Ebbinghaus, H. *Memory: A Contribution to Experimental Psychology.* Original 1885; translation, Teachers College, 1913.

Joshi, M.S. *The Concepts and Practice of Mathematical Finance*, 2nd ed. Cambridge, UK: Cambridge University Press, 2008.

Kahneman, D., and A. Tversky. "On the Psychology of Prediction." *Psychological Review*, No. 80 (1973), pp. 237-251.

——. "Prospect Theory: An Analysis of Decision under Risk." *Econometrica*, No. 47 (1979), pp. 263-290.

Kim, T.S., and E. Omberg. "Dynamic Nonmyopic Portfolio Behavior." *Review of Financial Studies*, Vol. 9, No. 1 (1996), pp. 141-161.

Markowitz, H. "Portfolio Selection." *Journal of Finance*, Vol. 7 (1952), pp. 77-91.

——. *Portfolio Selection: Efficient Diversification of Investments*. New Haven, CT: Yale University Press, 1959.

Merton, Robert C. "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case." *Review of Economics and Statistics*, Vol. 51, No. 3 (1969), pp. 247-251.

Nevin, D. "Goals-Based Investing: Integrating Traditional and Behavioral Finance." *The Journal of Wealth Management*, Vol. 6, No. 4 (Spring 2004), pp. 8-23.

Odean, T. "Do Investors Trade Too Much?" *American Economic Review*, Vol. 89 (1999), pp. 1279-1298.

Sharpe, William F. "Lockbox Separation." Working paper, June 2007.

Shefrin, H., and M. Statman. "Behavioral Portfolio Theory." *Journal of Financial and Quantitative Analysis*, No. 35 (2000), pp. 127-151.

Shreve, S.E. *Stochastic Calculus for Finance II: Continuous-Time Models*. New York, NY: Springer Finance, June 2004.

Thaler, R.H. "Toward a Positive Theory of Consumer Choice." *Journal of Economic Behavior and Organization*, Vol. 1 (1980), pp. 39-60.

——. "Mental Accounting and Consumer Choice." *Marketing Science*, Vol. 4, No. 3 (Summer 1985), pp. 199-214.