Robot Dynamics HS 2019

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1 Parametrizations

1.1 Position and velocity

For every position parametrization, there is a linear mapping between linear velocities $\dot{\boldsymbol{r}}$ and derivatives of the representation $\dot{\boldsymbol{\chi}}$.

$$\boxed{\dot{\boldsymbol{r}} = \boldsymbol{E}_P(\boldsymbol{\chi}_P)\,\dot{\boldsymbol{\chi}}_P\ ,\ \dot{\boldsymbol{\chi}}_P = \boldsymbol{E}_P(\boldsymbol{\chi}_P)^{-1}\,\dot{\boldsymbol{r}}}$$

Cartesian Coordinates:
$\boldsymbol{E}_{P_c} = \mathbb{I}$
$\boldsymbol{\chi}_{P_c} = [x \quad y \quad z]^T$, ${}_{\mathcal{A}}\boldsymbol{r} = [x \quad y \quad z]^T$

Cylindrical coordinates:
$\boldsymbol{\chi}_{P_z} = [\rho \quad \theta \quad z]^T$,
${}_{\mathcal{A}}\boldsymbol{r} = [\rho\cos\theta \quad \rho\sin\theta \quad z]^T$

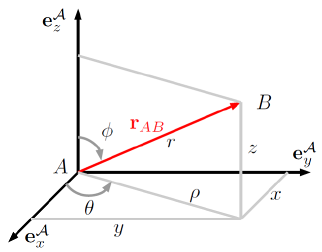
$$\boldsymbol{E}_{P_z} = \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & -\rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{E}_{P_z}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spherical coordinates:
$\boldsymbol{\chi}_{P_s} = [r \quad \theta \quad \phi]^T$,
${}_{\mathcal{A}}\boldsymbol{r} = [r\cos\theta\sin\phi \quad r\sin\theta\sin\phi \quad r\cos\phi]^T$

$$\boldsymbol{E}_{P_s} = \begin{bmatrix} \cos\theta\sin\phi & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\cos\theta\sin\phi & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}$$

$$\boldsymbol{E}_{P_s}^{-1} = \begin{bmatrix} \cos\theta\sin\phi & \sin\phi\sin\theta & \cos\phi \\ -\sin\theta/(r\sin\phi) & \cos\theta/(r\sin\phi) & 0 \\ (\cos\phi\cos\theta)/r & (\cos\phi\sin\theta)/r & -\sin\phi/r \end{bmatrix}$$



1.2 Rotation

${}_{\mathcal{A}}\boldsymbol{u} = \mathbf{C}_{\mathcal{AC}} \cdot {}_{\mathcal{C}}\boldsymbol{u} = \mathbf{C}_{\mathcal{AB}}\mathbf{C}_{\mathcal{BC}} \cdot {}_{\mathcal{C}}\boldsymbol{u}$
$\mathbf{C}_{\mathcal{BA}} = \mathbf{C}_{\mathcal{AB}}^{-1} = \mathbf{C}_{\mathcal{AB}}^T$
$\mathbf{C}_{\mathcal{AB}}\mathbf{C}_{\mathcal{AB}}^T = I_n$ (Orthogonality)

Elementary rotations:

$$\mathbf{C}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$\mathbf{C}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

$$\mathbf{C}_z = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler ZYZ (proper) angles:

$$\boldsymbol{\chi}_{R,ZYZ} = \begin{pmatrix} \mathrm{atan2}(c_{23}, c_{13}) \\ \mathrm{atan2}(\sqrt{c_{13}^2 + c_{23}^2}, c_{33}) \\ \mathrm{atan2}(c_{32}, -c_{31}) \end{pmatrix}$$

Euler ZXZ (proper) angles:

$$\boldsymbol{\chi}_{R,ZXZ} = \begin{pmatrix} \mathrm{atan2}(c_{13}, -c_{23}) \\ \mathrm{atan2}(\sqrt{c_{13}^2 + c_{23}^2}, c_{33}) \\ \mathrm{atan2}(c_{31}, c_{32}) \end{pmatrix}$$

Euler ZYX (Tait-Bryan) angles:

$$\boldsymbol{\chi}_{R,ZYX} = \begin{pmatrix} \mathrm{atan2}(c_{21}, c_{11}) \\ \mathrm{atan2}(-c_{31}, \sqrt{c_{32}^2 + c_{33}^2}) \\ \mathrm{atan2}(c_{32}, c_{33}) \end{pmatrix}$$

Euler XYZ (Cardan) angles:

$$\boldsymbol{\chi}_{R,XYZ} = \begin{pmatrix} \mathrm{atan2}(-c_{23}, c_{33}) \\ \mathrm{atan2}(c_{13}, \sqrt{c_{11}^2 + c_{12}^2}) \\ \mathrm{atan2}(c_{12}, -c_{11}) \end{pmatrix}$$

Angle-axis/Rotation-vector (non-minimal):

$$\boldsymbol{\chi}_{R,AA} = \begin{pmatrix} \theta \\ \boldsymbol{n} \end{pmatrix},\ \boldsymbol{n} = \frac{1}{2\sin(\theta)} \cdot \begin{pmatrix} c_{32} - c_{23} \\ c_{31} - c_{13} \\ c_{21} - c_{12} \end{pmatrix},$$

$$\theta = \mathrm{acos}(\frac{c_{11} + c_{22} + c_{33} - 1}{2}),\ \boldsymbol{\varphi} = \theta \cdot \mathbf{n}(nunit)$$

Unit Quaternions (non-minimal):
$\boldsymbol{\chi}_{R,quat} = \boldsymbol{\xi} = \begin{pmatrix} \xi \\ \check{\boldsymbol{\xi}} \end{pmatrix}$, $\boldsymbol{\xi}^{-1} = \begin{pmatrix} \xi \\ -\check{\boldsymbol{\xi}} \end{pmatrix}$
$\xi_0 = \cos(\theta/2), \quad \check{\boldsymbol{\xi}} = \boldsymbol{n} \cdot \sin(\theta/2)$

$$\boldsymbol{\chi}_{R,quat} = \frac{1}{2} \begin{pmatrix} \sqrt{c_{11} + c_{22} + c_{33} + 1} \\ \mathrm{sgn}(c_{32} - c_{23})\sqrt{c_{11} - c_{22} - c_{33} + 1} \\ \mathrm{sgn}(c_{13} - c_{31})\sqrt{c_{22} - c_{11} - c_{33} + 1} \\ \mathrm{sgn}(c_{21} - c_{12})\sqrt{c_{33} - c_{11} - c_{22} + 1} \end{pmatrix}$$

$$\boldsymbol{\xi}_{\mathcal{AB}} \otimes \boldsymbol{\xi}_{\mathcal{BC}} = \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}_{\mathcal{AB}} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}_{\mathcal{BC}}$$

$$\begin{pmatrix} 0 \\ {}_{\mathcal{A}}\boldsymbol{r} \end{pmatrix} = \boldsymbol{\xi}_{\mathcal{AB}} \otimes \begin{pmatrix} 0 \\ {}_{\mathcal{B}}\boldsymbol{r} \end{pmatrix} \otimes \boldsymbol{\xi}_{\mathcal{AB}}^{-1}$$

1.3 Angular Velocity

$$[{}_{\mathcal{A}}\boldsymbol{\omega}_{AB}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \dot{\mathbf{C}}_{\mathcal{AB}}\mathbf{C}_{\mathcal{AB}}^T$$

${}_{\mathcal{A}}\boldsymbol{\omega}_{AB} = \boldsymbol{E}_R(\boldsymbol{\chi}_R)\,\dot{\boldsymbol{\chi}}_R$ (see Script p.23-25)

1.4 Transformations

$$\begin{pmatrix} {}_{\mathcal{A}}\boldsymbol{r}_{AP} \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} \mathbf{C}_{\mathcal{AB}} & {}_{\mathcal{A}}\boldsymbol{r}_{AB} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}}_{\mathbf{T}_{\mathcal{AB}}} \begin{pmatrix} {}_{\mathcal{B}}\boldsymbol{r}_{BP} \\ 1 \end{pmatrix}$$

$$\mathbf{T}_{\mathcal{AB}}^{-1} = \begin{bmatrix} \mathbf{C}_{\mathcal{AB}}^T & \overbrace{-\mathbf{C}_{\mathcal{AB}}^T\,{}_{\mathcal{A}}\boldsymbol{r}_{AB}}^{{}_{\mathcal{B}}\boldsymbol{r}_{BA}} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$

2 Kinematics

2.1 Velocity in rigid bodies

- $\boldsymbol{v}_P$: abs. velocity of P
- $\boldsymbol{a}_P$: abs. acceleration of P
- $\boldsymbol{\Omega}_{\mathcal{B}} = {}_{\mathcal{I}}\boldsymbol{\omega}_{\mathcal{B}}$: angular vel. of frame $\mathcal{B}$
- $\boldsymbol{\Psi}_{\mathcal{B}} = \dot{\boldsymbol{\Omega}}_{\mathcal{B}}$: angular accel. of frame $\mathcal{B}$

$${}_{\mathcal{A}}\boldsymbol{v}_{AP} = {}_{\mathcal{A}}(\dot{\boldsymbol{r}}_{AP}) = {}_{\mathcal{A}}\boldsymbol{v}_{AB} + {}_{\mathcal{A}}\boldsymbol{\omega}_{\mathcal{AB}} \times {}_{\mathcal{A}}\boldsymbol{r}_{BP}$$

In general, unless $\mathcal{C}$ is an inertial frame:
${}_{\mathcal{C}}\boldsymbol{v}_{AP} = {}_{\mathcal{C}}(\dot{\boldsymbol{r}}_{AP}) \neq \frac{\mathrm{d}}{\mathrm{d}t}({}_{\mathcal{C}}\boldsymbol{r}_{AP})$
In rigid body formulation:

$$\boldsymbol{v}_P = \boldsymbol{v}_B + \Omega \times \boldsymbol{r}_{BP}$$
$$\boldsymbol{a}_P = \boldsymbol{a}_B + \Psi \times \boldsymbol{r}_{BP} + \Omega \times (\Omega \times \boldsymbol{r}_{BP})$$

In a kinematic chain:

$${}_{\mathcal{I}}\boldsymbol{v}_{IE} = {}_{\mathcal{I}}\boldsymbol{\omega}_{I1} \times {}_{\mathcal{I}}\boldsymbol{r}_{12} + ... + {}_{\mathcal{I}}\boldsymbol{\omega}_{In} \times {}_{\mathcal{I}}\boldsymbol{r}_{nE}$$
$${}_{\mathcal{I}}\boldsymbol{\omega}_{IE} = {}_{\mathcal{I}}\boldsymbol{\omega}_{I1} + {}_{\mathcal{I}}\boldsymbol{\omega}_{12} + ... + {}_{\mathcal{I}}\boldsymbol{\omega}_{nE}$$

2.2 Forward kinematics

$$\mathbf{T}_{\mathcal{IE}}(\boldsymbol{q}) = \mathbf{T}_{\mathcal{I}0}\left(\prod_{k=1}^{n_j} \mathbf{T}_{k-1,k}(q_k)\right)\mathbf{T}_{n_j\mathcal{E}}$$

2.3 Analytical Jacobian

$$\dot{\chi}(\mathbf{q}) = \frac{\partial\boldsymbol{\chi}}{\partial\mathbf{q}}\dot{\mathbf{q}} = J_A(\mathbf{q}) \cdot \dot{\mathbf{q}} = \begin{bmatrix} \frac{\partial\boldsymbol{\chi}_{pos}}{\partial\mathbf{q}} \\ \frac{\partial\boldsymbol{\chi}_{rot}}{\partial\mathbf{q}} \end{bmatrix}\dot{\mathbf{q}}$$

2.4 Geometric / Basic Jacobian

$$\mathbf{w}_E = \begin{bmatrix} \mathbf{v}_E \\ \omega_E \end{bmatrix} = J_0(\mathbf{q})\dot{\mathbf{q}}$$

$$J_{0re}(\mathbf{q}) = \begin{bmatrix} J_{0,P} \\ J_{0,R} \end{bmatrix} = \begin{bmatrix} n_1 \times r_{1,E} & ... & n_n \times r_{n,E} \\ n_1 & ... & n_n \end{bmatrix}$$

$$J_{0pr}(\mathbf{q}) = \begin{bmatrix} J_{0,P} \\ J_{0,R} \end{bmatrix} = \begin{bmatrix} {}_{\mathcal{I}}\boldsymbol{n}_1 & ... & {}_{\mathcal{I}}\boldsymbol{n}_n \\ 0 & ... & 0 \end{bmatrix}$$

$${}_{\mathcal{I}}\boldsymbol{n}_i = \mathbf{C}_{\text{I i-1}}\,{}_{\text{i-1}}\boldsymbol{n}_i$$

$$J_0(q) = E_e(\chi)J_A(q)\ ;\ E_e(\chi) = \begin{bmatrix} E_p & 0 \\ 0 & E_R \end{bmatrix}$$

2.5 Inverse differential kinematics

$$\boxed{\mathbf{w}_E = J\dot{\mathbf{q}} \Rightarrow \dot{\mathbf{q}} = J^+\mathbf{w}_E}$$

where $J^+ = J^T(JJ^T)^{-1}$ (right Moore-Penrose). However we risk encountering **singular configurations** $\mathbf{q}_s$ where $rank(J(\mathbf{q}_s)) < m_0$, m_0 being the number of operational-space coordinates. Here J is badly conditioned.
We can mitigate this by using a redundant robot to carefully avoid singularities, and/or by damping the pseudo-inverse:

$$\dot{\mathbf{q}} = J^T(JJ^T + \lambda^2\mathbb{I})^{-1}\mathbf{w}_E$$

Now the pseudo-inverse minimizes $||\mathbf{w}_E^* - J\dot{\mathbf{q}}||^2 + \lambda^2||\dot{\mathbf{q}}||^2$ instead of just $||\mathbf{w}_E^* - J\dot{\mathbf{q}}||^2$, so convergence is slower but more stable for larger λ. In a **redundant configuration** q^* where $rank(J(\mathbf{q}^*)) < n$, the pseudoinverse minimizes $||\dot{\mathbf{q}}||^2$ while satisfying $\mathbf{w}_E^* = J\dot{\mathbf{q}}$ by using

$$\dot{\mathbf{q}} = J\mathbf{w}_E^* + N\dot{\mathbf{q}}_0$$
$$J(J^+\mathbf{w}_E^* + N\dot{\mathbf{q}}_0) = \mathbf{w}_E^* \quad \forall\dot{\mathbf{q}}_0$$

where $\boxed{N = \mathbb{I} - J^+J \longrightarrow JN = 0}$.

2.6 Multi-task IDK

Equal Priority

Given n_t tasks $\{J_i, \mathbf{w}_i^*\}$, we have:

$$\dot{\mathbf{q}} = \begin{bmatrix} J_1(q) \\ \vdots \\ J_{n_t}(q) \end{bmatrix}^+ \begin{pmatrix} \mathbf{w}_1^* \\ \vdots \\ \mathbf{w}_{n_t}^* \end{pmatrix}$$

In case the row-rank of the stacked Jacobian is greater that the column-rank, we are only minimizing $||\bar{\mathbf{w}} - \bar{J}\dot{\mathbf{q}}||^2$. We can weigh the tasks with

$$\bar{J}^{+W} = (\bar{J}^TW\bar{J})^{-1}\bar{J}^TW$$

where $W = diag(w_1, ..., w_m)$ and we minimize $||W^{1/2}(\bar{\mathbf{w}} - \bar{J}\dot{\mathbf{q}})||^2$.

Task Prioritization

$$\dot{\mathbf{q}} = J_1^+\mathbf{w}_1^* + N_1\dot{\mathbf{q}}_0$$
$$\mathbf{w}_2^* = J_2\dot{\mathbf{q}} = J_2(J_1^+\mathbf{w}_1^* + N_1\dot{\mathbf{q}}_0)$$
$$\Rightarrow \dot{\mathbf{q}}_0 = (J_2N_1)^+(\mathbf{w}_2^* - J_2J_1^+\mathbf{w}_1^*)$$
$$\Rightarrow \dot{\mathbf{q}} = J_1^+\mathbf{w}_1^* + N_1(J_2N_1)^+(\mathbf{w}_2^* - J_2J_1^+\mathbf{w}_1^*)$$

In general:

$$\dot{\mathbf{q}} = \sum_{i=1}^{n_t} \bar{N}_i\dot{\mathbf{q}}_i$$

$$\dot{\mathbf{q}}_i = (J_i\bar{N}_i)^+\left(\mathbf{w}_i^* - J_i\sum_{k=1}^{i-1}\bar{N}_k\dot{\mathbf{q}}_k\right)$$

whereby $\bar{N}_i$ is the Nullspace of the stacked Jacobian $\bar{J}_i = [J_1^T \dots J_{i-1}^T]$. With 2 tasks we first $\min||\dot{q}||^2$ and then $\min||J_2\dot{q} - \mathbf{w_2}||^2$ s.t. $J_1\dot{q} - \mathbf{w_1} = 0$.

2.7 Inverse Kinematics

Genernal goal: $q = q(\chi^*)$

1. $\mathbf{q} \leftarrow \mathbf{q}^0$
2. While $||\boldsymbol{\chi}_e^* \boxminus \boldsymbol{\chi}_e(\mathbf{q})|| > tol$ do
3. $\quad J_A \leftarrow J_A(\mathbf{q}) = \frac{\partial\boldsymbol{\chi}_e}{\partial\mathbf{q}}(\mathbf{q})$
4. $\quad J_A^+ \leftarrow (J_A)^+$
5. $\quad \Delta\boldsymbol{\chi}_e \leftarrow \boldsymbol{\chi}_e^* \boxminus \boldsymbol{\chi}_e(\mathbf{q})$
6. $\quad \mathbf{q} \leftarrow \mathbf{q} + J_A^+\Delta\boldsymbol{\chi}_e$

One issue is that **for very large errors** $\Delta\boldsymbol{\chi}_e$, we get too imprecise. We can avoid this by scaling the update with a factor $0 < k < 1$: $\mathbf{q} \leftarrow \mathbf{q} + kJ_A^+\Delta\boldsymbol{\chi}_e$. But we still have issues inverting J_A in **singular configurations**. An alternative is $\mathbf{q} \leftarrow \mathbf{q} + \alpha J_A^T\Delta\boldsymbol{\chi}_e$, which converges for small α. We must also appropriately compute the difference $\boldsymbol{\chi}_e^* \boxminus \boldsymbol{\chi}_e(\mathbf{q})$ depending on the parametrization. For cartiesian coordinates, this is regular vector subtraction. Also note that with cartesian coordinates $J_{0,P} = J_{A,P}$. For rotational difference we can extract the rotation vector $\Delta\boldsymbol{\varphi}$ from the "rotation difference", and use that for the update:

$$\mathbf{C}_{\mathcal{GS}}(\Delta\boldsymbol{\varphi}) = \mathbf{C}_{\mathcal{GI}}(\boldsymbol{\varphi}^*)\mathbf{C}_{\mathcal{SI}}(\boldsymbol{\varphi}^t)^T$$
$$\mathbf{q} \leftarrow \mathbf{q} + k_{pR} J_{0,R}^{+} \Delta\boldsymbol{\varphi}$$

2.8 Trajectory control

Position: with $\Delta\mathbf{r}_e^t = \mathbf{r}_e^*(t) - \mathbf{r}_e(\mathbf{q}^t)$

$$\dot{\mathbf{q}}^* = J_{e0P}^{+}(\mathbf{q}^t)(\dot{\mathbf{r}}_e^*(t) + k_{pP}\Delta\mathbf{r}_e^t)$$

Orientation: with $\Delta\boldsymbol{\varphi}$ as above,

$$\dot{\mathbf{q}}^* = J_{e0R}^{+}(\mathbf{q}^t)(\boldsymbol{\omega}_e^*(t) + k_{pR}\Delta\boldsymbol{\varphi})$$

3 Dynamics

$$\boxed{\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau} + \mathbf{J}_c(\mathbf{q})^T\mathbf{F}_c}$$

- $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n_q \times n_q}$ Mass matrix ($\perp$).
- $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \in \mathbb{R}^{n_q}$ Gen. pos., vel., accel.
- $\mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n_q}$ Coriolis and centrifugal terms
- $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^{n_q}$ Gravity terms
- $\boldsymbol{\tau} \in \mathbb{R}^{n_q}$ External generalized forces
- $\mathbf{F}_c \in \mathbb{R}^{3\times n_c}$ External cartesian forces
- $\mathbf{J}_c(\mathbf{q}) \in \mathbb{R}^{n_c \times n_q}$ Geometric Jacobian of location where external forces apply

$$\begin{pmatrix}\mathbf{v}_s\\ \boldsymbol{\Omega}\end{pmatrix} = \begin{bmatrix}J_P\\ J_R\end{bmatrix}\dot{\mathbf{q}}$$
$$\begin{pmatrix}\mathbf{a}_s\\ \boldsymbol{\Psi}\end{pmatrix} = \begin{pmatrix}\dot{\mathbf{v}}_s\\ \dot{\boldsymbol{\Omega}}\end{pmatrix} = \begin{bmatrix}J_P\\ J_R\end{bmatrix}\ddot{\mathbf{q}} + \begin{bmatrix}\dot{J}_P\\ \dot{J}_R\end{bmatrix}\dot{\mathbf{q}}$$

3.1 Newton-Euler method

- m body mass
- $\boldsymbol{\Theta}_S$ inertia matrix around CoG
- $\mathbf{p}_S = m\mathbf{v}_S$ linear momentum
- $\mathbf{N}_S = \boldsymbol{\Theta}_S \cdot \boldsymbol{\Omega}$ angular momentum around CoG
- $\dot{\mathbf{p}} = m\mathbf{a}_S$ change in linear momentum
- $\dot{\mathbf{N}}_S = \boldsymbol{\Theta}_S \cdot \boldsymbol{\Psi} + \boldsymbol{\Omega} \times \boldsymbol{\Theta}_S \cdot \Omega$ change in angular momentum

Cut each link free as a single rigid body, and introduce constraint forces $\mathbf{F}_i$ acting on the body at the joint. Then apply conservation of linear and angular momentum in all DoFs subject to all external forces (*including* contraints $\mathbf{F}_i$):

$$\dot{\mathbf{p}}_S = \mathbf{F}_{ext,S}$$
$$\dot{\mathbf{N}}_S = \mathbf{T}_{ext}$$

For calculations all quantities must be in the same coordinate system.

3.2 Lagrange method

Define the *Lagrangian function*:

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$

Where $\mathcal{T}$ is the kinetic energy and $\mathcal{U}$ the potential energy. Then the *Euler-Lagrange equation of the second kind* holds for the total external generalized forces $\boldsymbol{\tau}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\right) - \left(\frac{\partial\mathcal{L}}{\partial\mathbf{q}}\right) = \boldsymbol{\tau}$$

The kinetic energy for a system of n_b bodies is defined as:

$$\mathcal{T} := \sum_{i=1}^{n_b}\left(\tfrac{1}{2}m_i\,{}_{\mathcal{A}}\dot{\boldsymbol{r}}_{S_i}^T\,{}_{\mathcal{A}}\dot{\boldsymbol{r}}_{S_i} + \tfrac{1}{2}\,{}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i}^T \cdot {}_{\mathcal{B}}\boldsymbol{\Theta}_{S_i} \cdot {}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i}\right)$$
$$= \frac{1}{2}\dot{\mathbf{q}}^T\underbrace{\left(\sum_{i=1}^{n_b}(J_{S_i}^T m J_{S_i} + J_{R_i}^T\boldsymbol{\Theta}_{S_i}J_{R_i})\right)}_{\mathbf{M}(\mathbf{q})}\dot{\mathbf{q}}$$
$$= \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$

The potential energy is typically in the form of gravitational and elastic terms:

$$\mathcal{U} = \underbrace{-\sum_{i=1}^{n_b}\boldsymbol{r}_{S_i}^T(m_i g \cdot \mathbf{e}_g)}_{\text{gravitational}} + \underbrace{\sum_{j=1}^{n_E}\frac{1}{2}k_j(d(\mathbf{q}) - d_{0,j})^2}_{\text{elastic}}$$

Here we have n_E elastic components with coefficients k_j and rest configuration $d_{0,j}$.

3.3 Proj. Newton-Euler Method

$$\mathbf{M} = \sum_{i=1}^{n_b}({}_{\mathcal{A}}\boldsymbol{J}_{S_i}^T m\,{}_{\mathcal{A}}\boldsymbol{J}_{S_i} + {}_{\mathcal{B}}\boldsymbol{J}_{R_i}^T\,{}_{\mathcal{B}}\boldsymbol{\Theta}_{S_i}\,{}_{\mathcal{B}}\boldsymbol{J}_{R_i})$$
$${}_{\mathcal{B}}\boldsymbol{\Theta} = \mathbf{C}_{\mathcal{BA}} \cdot {}_{\mathcal{A}}\boldsymbol{\Theta} \cdot \mathbf{C}_{\mathcal{BA}}^T$$
$$\mathbf{b} = \sum_{i=1}^{n_b}({}_{\mathcal{A}}\boldsymbol{J}_{S_i}^T m\,{}_{\mathcal{A}}\dot{\boldsymbol{J}}_{S_i}\dot{\mathbf{q}} + {}_{\mathcal{B}}\boldsymbol{J}_{R_i}^T({}_{\mathcal{B}}\boldsymbol{\Theta}_{S_i}\,{}_{\mathcal{B}}\dot{\boldsymbol{J}}_{R_i}\dot{\mathbf{q}}$$
$$+ {}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i} \times {}_{\mathcal{B}}\boldsymbol{\Theta}_{S_i}\,{}_{\mathcal{B}}\boldsymbol{\Omega}_{S_i}))$$
$$\mathbf{g} = \sum_{i=1}^{n_b}(-\,{}_{\mathcal{A}}\boldsymbol{J}_{S_i}^T\,{}_{\mathcal{A}}\boldsymbol{F}_{g,i})$$
$$\tau_{F,ext} = \sum_{j=1}^{n_{f,ext}} J_{P,j}^T F_j\,;\, \tau_{T,ext} = \sum_{k=1}^{n_{m,ext}} J_{R,k}^T T_{ext,k}$$

4 Floating-base dynamics

Generalized coordinates are now $\mathbf{q} = [\mathbf{q}_b^T\ \mathbf{q}_j^T]^T$, where $\mathbf{q}_b$ are the generalized coordinates of the base (position and orientation). The generalized velocities are therefore no longer $\dot{\mathbf{q}}$, but are denoted $\mathbf{u} = [{}_{\mathcal{I}}\boldsymbol{v}_B^T\ {}_{\mathcal{B}}\boldsymbol{\omega}_{IB}^T\ \dot{\mathbf{q}}_j^T]^T$.

$$\boxed{\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} + \mathbf{b}(\mathbf{q},\mathbf{u}) + \mathbf{g}(\mathbf{q}) = \mathbf{S}^T\boldsymbol{\tau} + \mathbf{J}_{ext}^T\mathbf{F}_{ext}}$$

- $\mathbf{u}, \dot{\mathbf{u}} \in \mathbb{R}^{n_u}$ Gen. vel., accel.
- $\mathbf{S}$ selection matrix of actuated joints; $u_j = Su = [0_{6x6}\quad \mathbb{I}_{6xn_j}](u_b u_j)^T$
- $\mathbf{F}_{ext} \in \mathbb{R}^{3\times n_c}$ External cartesian forces acting on robot
- $\mathbf{J}_{ext}(\mathbf{q}) \in \mathbb{R}^{n_c \times n_u}$ Geometric Jacobian of location where external forces apply

Position and velocity of a point Q on the robot:

$${}_{\mathcal{I}}\boldsymbol{r}_{IQ}(\mathbf{q}) = {}_{\mathcal{I}}\boldsymbol{r}_{IB}(\mathbf{q}) + \mathbf{C}_{\mathcal{IB}}(\mathbf{q}) \cdot {}_{\mathcal{B}}\boldsymbol{r}_{BQ}(\mathbf{q})$$
$${}_{\mathcal{I}}\boldsymbol{v}_Q = \underbrace{\left[\mathbb{I}_{3\times3}\ \ -\mathbf{C}_{\mathcal{IB}}\cdot[{}_{\mathcal{B}}\boldsymbol{r}_{BQ}]_\times\ \ \mathbf{C}_{\mathcal{IB}}\cdot{}_{\mathcal{B}}\boldsymbol{J}_{P_{q_j}}(\mathbf{q}_j)\right]}_{={}_{\mathcal{I}}\boldsymbol{J}_Q(\mathbf{q})}\cdot\mathbf{u}$$

4.1 Contact kinematics

The point of contant C is not allowed to move: $\mathbf{r}_C = const.$ and $\dot{\mathbf{r}}_C = \ddot{\mathbf{r}}_C = \mathbf{0}$. Written in generalized coordinates these are:

$${}_{\mathcal{I}}\boldsymbol{J}_{C_i}\mathbf{u} = \mathbf{0}, \quad {}_{\mathcal{I}}\boldsymbol{J}_{C_i}\dot{\mathbf{u}} + {}_{\mathcal{I}}\dot{\boldsymbol{J}}_{C_i}\mathbf{u} = \mathbf{0}$$

We can therefore stack the constraint Jacobians:

$$\mathbf{J}_c = \begin{bmatrix}{}_{\mathcal{I}}\boldsymbol{J}_{C_1}\\ \vdots\\ {}_{\mathcal{I}}\boldsymbol{J}_{C_{n_c}}\end{bmatrix} \in \mathbb{R}^{3n_c\times(n_b+n_j)}$$

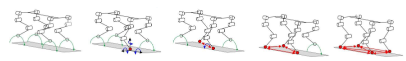
By using the nullspace projection $\mathbf{N}_c$ of $\mathbf{J}_c$ we can still move the system:

$$\mathbf{0} = \dot{\mathbf{r}} = \mathbf{J}_c\dot{\mathbf{q}} \quad\Rightarrow\quad \dot{\mathbf{q}} = \mathbf{J}_c^{+}\mathbf{0} + \mathbf{N}_c\dot{\mathbf{q}}_0$$
$$\mathbf{0} = \ddot{\mathbf{r}} = \mathbf{J}_c\ddot{\mathbf{q}} + \dot{\mathbf{J}}_c\dot{\mathbf{q}} \quad\Rightarrow\quad \ddot{\mathbf{q}} = \mathbf{J}_c^{+}(-\dot{\mathbf{J}}_c\dot{\mathbf{q}}) + \mathbf{N}_c\ddot{\mathbf{q}}_0$$

The contact Jacobian tells us how the system can move. If we partition it into the part relating to the base and the part relating to the joints:

- $\mathbf{J}_c = [\mathbf{J}_{c,b}\ \mathbf{J}_{c,j}]$
- $rank(\mathbf{J}_{c,b})$ is the number of constraints on the base $\rightarrow$ the number of controllable base DoFs.
- $rank(\mathbf{J}_c) - rank(\mathbf{J}_{c,b})$ is the number of contraints on the actuators.

Quadruped (18 DoF; 6 for base, 12 actuators):



Total constraints $rank(\mathbf{J}_c)$	0	3	6	9	12
Base constraints $rank(\mathbf{J}_{c,b})$	0	3	5	6	6
Internal constraints $rank(\mathbf{J}_c) - rank(\mathbf{J}_{c,b})$	0	0	1	3	6
Uncontrollable DoFs $6 - rank(\mathbf{J}_{c,b})$	6	3	1	0	0

4.2 Support-consistent dynamics

If we use **soft contacts** to model the contact, we simply introduce an external force acting on the robot:

$$\mathbf{F}_c = k_p(\mathbf{r}_c - \mathbf{r}_{c0}) + k_d\dot{\mathbf{r}}_c$$

However such problems are hard to accurately solve numerically (slow system dynamics, fast contact dynamics).

Instead it works better to use **hard contacts**. We impose the kinematic constraint ${}_{\mathcal{I}}\boldsymbol{J}_{C_i}\dot{\mathbf{u}} + {}_{\mathcal{I}}\dot{\boldsymbol{J}}_{C_i}\mathbf{u} = \mathbf{0}$ from above and calculate the resulting force and null-space matrix:

$$\mathbf{F}_c = (\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^T)^{-1}(\mathbf{J}_c\mathbf{M}^{-1}(\mathbf{S}^T\tau - \mathbf{b} - \mathbf{g}) + \dot{\mathbf{J}}_c\mathbf{u})$$
$$\mathbf{N}_c = \mathbb{I} - \mathbf{M}^{-1}\mathbf{J}_c^T(\mathbf{J}_c\mathbf{M}^{-1}\mathbf{J}_c^T)^{-1}\mathbf{J}_c$$
$$\Rightarrow \boxed{\mathbf{N}_c^T(\mathbf{M}\dot{\mathbf{u}} + \mathbf{b} + \mathbf{g}) = \mathbf{N}_c^T\mathbf{S}^T\boldsymbol{\tau}, \quad \mathbf{J}_c\mathbf{N}_c = \mathbf{0}}$$

By defining the *end-effector inertia* $\boldsymbol{\Lambda}_c = (\mathbf{J}_c\mathbf{M}_c^{-1}\mathbf{J}_c^T)^{-1}$ we can write the kinetic energy loss on impact:

$$\mathbf{u}^+ = \mathbf{N}_c\mathbf{u}^-$$
$$E_{loss} = \Delta E_{kin} = -\frac{1}{2}\Delta\mathbf{u}^T\mathbf{M}\Delta\mathbf{u} = -\frac{1}{2}\dot{\mathbf{r}}^{-T}\mathbf{M}\dot{\mathbf{r}}^-$$

5 Dynamic control

5.1 Joint-space Dynamic Control

$$\boxed{\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}}$$

Torque as a function of position and velocity error:

$$\boldsymbol{\tau}^* = k_p(\mathbf{q}^* - \mathbf{q}) + k_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$$

Compensate for gravity by adding an estimated gravity term:

$$\boldsymbol{\tau}^* = k_p(\mathbf{q}^* - \mathbf{q}) + k_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$$

Compensate for **system dynamics**:

$$\boldsymbol{\tau} = \hat{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}}^* + \hat{\mathbf{b}}(\mathbf{q},\dot{\mathbf{q}}) + \hat{\mathbf{g}}(\mathbf{q})$$

If the model is exact, we have $\mathbb{I}\ddot{\mathbf{q}} = \ddot{\mathbf{q}}^*$ (decoupled control), meaning we can perfectly control system dynamics. We could apply a PD-control law, making each joint behave like a mass-spring-damper with unitary mass:

$$\ddot{\mathbf{q}}^* = k_p(\mathbf{q}^* - \mathbf{q}) + k_d(\dot{\mathbf{q}}^* - \dot{\mathbf{q}})$$

$$\omega = \sqrt{k_p}, \quad D = \frac{k_d}{2\sqrt{k_p}}$$

5.2 Task-space Dynamic Control

$$\boxed{\dot{\mathbf{w}}_{\mathbf{e}} = J_e\ddot{q} + \dot{J}_e\dot{q} = J_eM^{-1}(\tau - b - g) + \dot{J}_e\dot{q}}$$

$$\tau = J_e^T F_e \quad , \quad \ddot{q}^* = J_e^{+}(\dot{\mathbf{w}}_{\mathbf{e}}{}^* - \dot{J}_e\dot{q})$$

End-Effector Motion Control

Generalized framework to control motion and force. **End-effector dynamics**:

$$\mathbf{\Lambda}\dot{\mathbf{w}}_e + \boldsymbol{\mu} + \mathbf{p} = \mathbf{F}_e$$
$$\mathbf{\Lambda} = (\mathbf{J}_e\mathbf{M}^{-1}\mathbf{J}_e^T)^{-1}$$
$$\boldsymbol{\mu} = \mathbf{\Lambda}\mathbf{J}_e\mathbf{M}^{-1}\mathbf{b} - \mathbf{\Lambda}\dot{\mathbf{J}}_e\dot{\boldsymbol{q}}$$
$$\mathbf{p} = \mathbf{\Lambda}\mathbf{J}_e\mathbf{M}^{-1}\mathbf{g}$$

represent the end-effector inertia, centrifugal/coriolis and gravitational terms in task space. Following from the dynamics the **end-effector control** can be found:

$$\boxed{\boldsymbol{\tau}^* = \hat{\mathbf{J}}^T(\hat{\mathbf{\Lambda}}\dot{\mathbf{w}}_e^* + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}})}$$

$$\mathbf{w}_\mathbf{e}^* = \mathbf{k}_\mathbf{p}\begin{pmatrix} r_e^* - r_e \\ \Delta\phi_e \end{pmatrix} + \mathbf{k}_\mathbf{d}(\mathbf{w}_\mathbf{e}^* - \mathbf{w}_\mathbf{e})$$

$$\Rightarrow \begin{pmatrix} r_e^* - r_e \\ \Delta\phi_e \end{pmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & E_R \end{bmatrix}$$

$$\dot{\mathbf{w}}_e^* = k_p\mathbf{E}(\boldsymbol{\chi}_e^* \boxminus \boldsymbol{\chi}_e) + k_d(\mathbf{w}_e^* - \mathbf{w}_e) \quad \underbrace{+\dot{\mathbf{w}}_e^*(t)}_{\text{Trajectory Control}}$$

Operational Space Control

$$\boxed{\underline{\mathbf{F}_c^*} + \mathbf{\Lambda}\underline{\dot{\mathbf{w}}_e^*} + \boldsymbol{\mu} + \mathbf{p} = \mathbf{F}_e}$$

$$\boldsymbol{\tau}^* = \hat{\mathbf{J}}^T(\hat{\mathbf{\Lambda}}\mathbf{S}_M\underline{\dot{\mathbf{w}}_e^*} + \mathbf{S}_F\underline{\mathbf{F}_c^*} + \hat{\boldsymbol{\mu}} + \hat{\mathbf{p}})$$

with S_M and S_F being the selection matrices for Motion and Force. Let $\mathbf{C}$ represent the rotation from the inertial frame to the contact force frame. The selection matrices can be calculated as (with $\sigma_i \in \{0, 1\}$) :

$$\mathbf{\Sigma}_p = \begin{bmatrix} \sigma_{px} & 0 & 0 \\ 0 & \sigma_{py} & 0 \\ 0 & 0 & \sigma_{pz} \end{bmatrix}, \; \mathbf{\Sigma}_r = \begin{bmatrix} \sigma_{rx} & 0 & 0 \\ 0 & \sigma_{ry} & 0 \\ 0 & 0 & \sigma_{rz} \end{bmatrix}$$

$$\mathbf{S}_M = \begin{bmatrix} \mathbf{C}^T\mathbf{\Sigma}_p\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^T\mathbf{\Sigma}_r\mathbf{C} \end{bmatrix}$$

$$\mathbf{S}_F = \begin{bmatrix} \mathbf{C}^T(\mathbb{I}-\mathbf{\Sigma}_p)\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^T(\mathbb{I}-\mathbf{\Sigma}_r)\mathbf{C} \end{bmatrix}$$

OSC with multiple objectives

Example: Quadruped with three stationary legs and one in swing.

- Leg swing: $\ddot{\mathbf{r}}_{OF} = \mathbf{J}_F\ddot{\mathbf{q}}_F + \dot{\mathbf{J}}_F\dot{\mathbf{q}}_F = \ddot{\mathbf{r}}_{OF,des}(t) = k_p(\mathbf{qr}^* - \mathbf{r}) + k_d(\dot{\mathbf{r}}^* - \dot{\mathbf{r}}) + \ddot{\mathbf{r}}^*$
- Body movement (translation and orientation): $\dot{\mathbf{w}}_B = \mathbf{J}_B\ddot{\mathbf{q}}_B + \dot{\mathbf{J}}_B\dot{\mathbf{q}}_B = \dot{\mathbf{w}}_{OB,des}(t) = k_p\begin{pmatrix} \mathbf{r}^* - \mathbf{r} \\ \boldsymbol{\varphi}^* \boxminus \boldsymbol{\varphi} \end{pmatrix} + k_d(\mathbf{w}^* - \mathbf{w}) + \dot{\mathbf{w}}^*$
- Enforce contact constraints: $\ddot{\mathbf{r}}_c = \mathbf{J}_c\ddot{\mathbf{q}}_c + \dot{\mathbf{J}}_c\dot{\mathbf{q}}_c = \mathbf{0}$

Solve for generalized acceleration and torque giving each task **equal priority**:

$$\ddot{\mathbf{q}}^* = \begin{bmatrix} \mathbf{J}_F \\ \mathbf{J}_B \\ \mathbf{J}_c \end{bmatrix}^+ \left(\begin{pmatrix} \ddot{\mathbf{r}}_{OF,des}(t) \\ \dot{\mathbf{w}}_{B,des}(t) \\ \mathbf{0} \end{pmatrix} - \begin{bmatrix} \dot{\mathbf{J}}_F \\ \dot{\mathbf{J}}_B \\ \dot{\mathbf{J}}_c \end{bmatrix} \dot{\mathbf{q}} \right)$$

Solve **with prioritization**:

$$\ddot{\mathbf{q}}^* = \sum_{i=1}^{n_t} \mathbf{N}_i\ddot{\mathbf{q}}_i,$$

$$\ddot{\mathbf{q}}_i := (\mathbf{J}_j\mathbf{N}_i)^+ \left(\mathbf{w}_i^* - \dot{\mathbf{J}}_i\dot{\mathbf{q}} - \mathbf{J}\sum_{k=1}^{i-1}\mathbf{N}_k\dot{\mathbf{q}}_k \right)$$

Where $\mathbf{N}_i$ is the nullspace projection of $\mathbf{J}_i := [\mathbf{J}_1^T \ldots \mathbf{J}_i^T]^T$.

5.3 Inv. Dynamics Floating-Base

Given a desired acceleration $\dot{\mathbf{u}}^*$ from the support-consistent dynamics follows:

$$\boldsymbol{\tau}^* = (\mathbf{N}_c^T\mathbf{S}^T)^+\mathbf{N}_c^T(\mathbf{M}\dot{\mathbf{u}}^* + \mathbf{b} + \mathbf{g}) \underbrace{+\mathcal{N}(\mathbf{N}_c^T\mathbf{S}^T)\boldsymbol{\tau}_0^*}_{\text{multiple solutions}}$$

Task-Space Control as QP

The behaviour of a robotic system can be described as multi-task control problem with the optimization variable x as follows:

$$x_{fixedB} = \begin{pmatrix} \ddot{q} \\ F_c \\ \tau \end{pmatrix} \quad \text{or} \quad x_{floatingB} = \begin{pmatrix} \dot{u} \\ F_c \\ \tau \end{pmatrix}$$

Using the optimization variable x the EoM $M\ddot{q} + b + g + J_c^T F_c = S^T\tau$ can be formulated as least square problem $Ax - b = 0$:

$$A = \begin{bmatrix} \hat{M} & \hat{J}_c^T & -S^T \end{bmatrix} \qquad b = -\hat{b} - \hat{g}$$

To achieve a desired acceleration in the **joint space** $\ddot{q}$ or at a point of interest in the **task space** $J\ddot{q} + \dot{J}\dot{q} = \dot{\mathbf{w}}_\mathbf{e}$:

$$A = \begin{bmatrix} \mathbb{I} \text{ or } \hat{J}_i & 0 & 0 \end{bmatrix} \qquad b = \ddot{q} \text{ or } \dot{\mathbf{w}}_\mathbf{e}^* - \dot{\hat{J}}_i\dot{q}$$

Pushing with a certain force $F_i = F_i^*$:

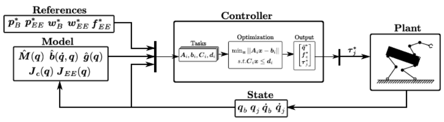
$$A = \begin{bmatrix} 0 & \mathbb{I} & 0 \end{bmatrix} \qquad b = F_i^*$$

6 Legged Robots

6.1 Hierarchical Optimization

Formulating a Hierarchical Optimization (HO) problem as a QP:

$$\min||A_i x - b_i|| \quad , \quad C_i x \le d_i$$



The HO variable x and EoM are defined as:

$$M(q)\ddot{q} + b(q, \dot{q}) + g(q) = S^T\tau_j + J_c^T(q)f_c$$
$$x = [\ddot{q}^T \quad f_c^T \quad \tau_j^T]^T$$

Task 1: Fulfill equation of Motion:

$$A_1 = [M(q) - J_c^T - S^T] \quad , \quad b_1 = -b - g$$

Task 2: Ensure feet stationary on ground:

$$A_2 = [J_{c,lin} \quad 0 \quad 0] \quad , \quad b_2 = \dot{\mathbf{w}}_\mathbf{c}^* - \dot{J}_{c,lin}\dot{q}$$

Task 3: Move body accord ref. trajectory:

$$\dot{w}_B^* = k_p(p_B^* - p_B) + k_d(w_B^* - w_B)$$
$$A_3 = [J_B \quad 0 \quad 0] \quad , \quad b_3 = \dot{w}_B^* - \dot{J}_B\dot{q}$$

Task 4: Torque min $(min(||\tau||_2)$:

$$A_4 = [0 \quad 0 \quad \mathbb{I}] \quad , \quad b_4 = 0$$

7 Rotorcraft

Propeller thrust and drag proportional to squared rotational speed (b: thrust constant; d: drag constant):

$$T_i = b\omega_{p,i}^2, \quad Q_i = d\omega_{p,i}^2$$

7.1 Kinematics

Use Tait-Bryan angles, consisting of yaw ψ (Z-axis), pitch θ (Y-axis) and roll ϕ (X-axis).

$$\mathbf{C}_{EB} = \mathbf{C}_{E1}(\mathbf{z}, \psi) \cdot \mathbf{C}_{12}(\mathbf{y}, \theta) \cdot \mathbf{C}_{2B}(\mathbf{x}, \phi)$$

Angular velocity:

$$_\mathcal{B}\boldsymbol{\omega} = {}_\mathcal{B}\boldsymbol{\omega}_{\text{roll}} + {}_\mathcal{B}\boldsymbol{\omega}_{\text{pitch}} + {}_\mathcal{B}\boldsymbol{\omega}_{\text{yaw}}$$
$$_\mathcal{B}\boldsymbol{\omega}_{\text{roll}} = (\dot{\psi}, 0, 0)^T$$
$$_\mathcal{B}\boldsymbol{\omega}_{\text{pitch}} = \mathbf{C}_{2B}^T(0, \dot{\theta}, 0)^T$$
$$_\mathcal{B}\boldsymbol{\omega}_{\text{yaw}} = [\mathbf{C}_{12} \cdot \mathbf{C}_{2E}]^T(0, 0, \dot{\phi})^T$$

$$_\mathcal{B}\boldsymbol{\omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} = J_r\dot{\boldsymbol{\chi}}_r = J_r\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

$$J_r = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \overset{\theta=\phi=0}{=} \mathbb{I}_{3x3}$$

NB: singularity for $\theta = \pm 90°$ (Gimbal lock).

7.2 Dynamics

$$\boxed{M(\varphi)\ddot{\varphi} + b(\dot{\varphi}, \varphi) + g(\varphi) + J_{ext}^T F_{ext} = S^T\tau_{act}}$$

Change of momentum and spin in the body frame ($\mathbf{M}$ = total moment/torque):

$$\begin{bmatrix} m\mathbb{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} _\mathcal{B}\dot{\boldsymbol{v}} \\ _\mathcal{B}\dot{\boldsymbol{\omega}} \end{bmatrix} + \begin{bmatrix} _\mathcal{B}\boldsymbol{\omega} \times m\, _\mathcal{B}\boldsymbol{v} \\ _\mathcal{B}\boldsymbol{\omega} \times \mathbf{I}\, _\mathcal{B}\boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} _\mathcal{B}\boldsymbol{F} \\ _\mathcal{B}\boldsymbol{M} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = C_{EB}\, _\mathcal{B}\boldsymbol{v} = C_{EB}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

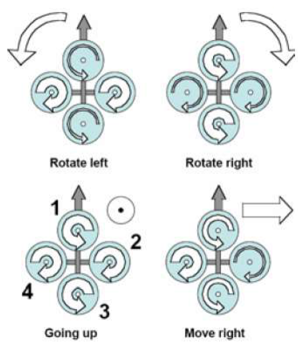
Forces and moments come from gravity and aerodynamics:

$$_\mathcal{B}\boldsymbol{F} = {}_\mathcal{B}\boldsymbol{F}_G + {}_\mathcal{B}\boldsymbol{F}_{Aero}$$
$$_\mathcal{B}\boldsymbol{M} = {}_\mathcal{B}\boldsymbol{M}_{Aero}$$
$$_\mathcal{B}\boldsymbol{F}_G = \mathbf{C}_{EB}^T\begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$$
$$_\mathcal{B}\boldsymbol{F}_{Aero} = \sum_{i=1}^{4}\begin{bmatrix} 0 \\ 0 \\ -T_i = -b\omega_{p,i}^2 \end{bmatrix}$$
$$_\mathcal{B}\boldsymbol{M}_{Aero} = {}_\mathcal{B}\boldsymbol{M}_T + {}_\mathcal{B}\boldsymbol{Q} = \begin{bmatrix} l(T_4 - T_2) \\ l(T_1 - T_3) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^4 Q_i(-1)^{(i-1)} \end{bmatrix}$$

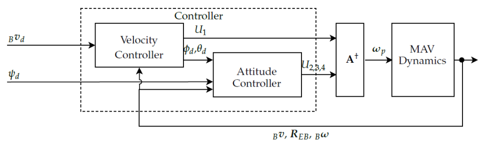
Full control over all rotational speeds, independently of the current position state. **Only directly control of vertical cartesian velocity - attitude control must be used for full position control.**

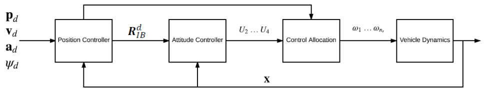
7.3 Control

Movement directions with four propellers:



Possible Control Structures:





To formulate the control architecture, a virtual control input U is used:

$$\begin{pmatrix}U_1\\U_2\\U_3\\U_4\end{pmatrix} = A\begin{pmatrix}\omega_1^2\\ \vdots\\ \omega_i^2\end{pmatrix} \quad , \quad A^\dagger = A^T(AA^T)^{-1}$$

Hence the translational and rotational dynamics are stated as follows for $\dot{X}$ for $X = (p\ v\ R_{EB}\ \omega)^T$:

$$\dot{p} = R_{EB\ \mathcal{B}}\boldsymbol{v}$$

$$_{\mathcal{B}}\dot{\boldsymbol{v}} = -\omega \times {}_{\mathcal{B}}\dot{\boldsymbol{v}} + \begin{pmatrix}0\\0\\\frac{U_1}{m}\end{pmatrix} + R_{EB}^T g$$

$$\dot{R}_{EB} = R_{EB}\omega$$

$$\dot{\omega} = I^{-1}(-\omega \times I\omega + \begin{pmatrix}U_2\\U_3\\U_4\end{pmatrix})$$

Equilibrium Point:
$\phi = \theta = p = q = r = 0$; $U_2 = U_3 = U_4 = 0$
$U_1 = mg$; $sin(x) \approx x$, $cos(x) \approx 1$

$$I_{xx}\dot{p} = qr(I_{yy} - I_{zz}) + U_2 \rightarrow \ddot{\phi} = \frac{1}{I_{xx}}U_2$$

$$I_{yy}\dot{q} = pr(I_{zz} - I_{xx}) + U_3 \rightarrow \ddot{\theta} = \frac{1}{I_{yy}}U_3$$

$$I_{zz}\dot{r} = U_4 \rightarrow \ddot{\psi} = \frac{1}{I_{zz}}U_4$$

This results in following PD-Control Inputs:

$$U_1 = T_{des}$$

$$U_2 = (\phi_{des} - \phi)k_{pRoll} - \dot{\phi}k_{dRoll}$$

$$U_3 = (\theta_{des} - \theta)k_{pPitch} - \dot{\theta}k_{dPitch}$$

$$U_3 = (\psi_{des} - \psi)k_{pYaw} - \dot{\psi}k_{dYaw}$$

7.4 Propeller aerodynamics

Propeller in hover:

- Thrust force T normal to prop. plane, $|T| = \frac{\rho}{2}A_P C_T(\omega_P R_p)^2$
- Drag torque Q, around rotor plane $|Q| = \frac{\rho}{2}A_P C_Q(\omega_P R_p)^2 R_p$
- C_T and C_Q depend on blade pitch angle (prop geometry), Reynolds number (prop speed, velocity, rotational speed).

Propeller in forward flight: additional forces due to force unbalance between forward- and backward-moving props.

- Hub force H (orthogonal to T, opposite to horizontal flight direction V_H), $|H| = \frac{\rho}{2}A_P C_H(\omega_P R_p)^2 R_p$
- Rolling torque R around flight direction $|R| = \frac{\rho}{2}A_P C_R(\omega_P R_p)^2 R_p$
- C_R and C_H depend on advance ratio $\mu = \frac{V}{\omega_P R_P}$

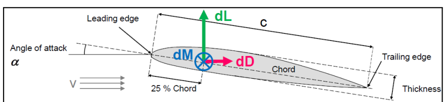
Ideal power consumption at hover: $P = \frac{F_{Thrust}^{3/2}}{\sqrt{2\rho A_R}} = \frac{(mg)^{3/2}}{\sqrt{2\rho A_R}}$. The prop efficiency is measured with the Figure of Merit FM:

$$FM = \frac{\text{Ideal power to hover}}{\text{Actual power to hover}} < 1$$

Blade Elemental and Momentum Theory (BEMT): blade shape determines drag and lift coefficients c_D, c_L.

8 Fixed-Wing

8.1 Aerodynamic Basics



Pressure distribution can be reduced to two forces and one moment ***per unit length***:

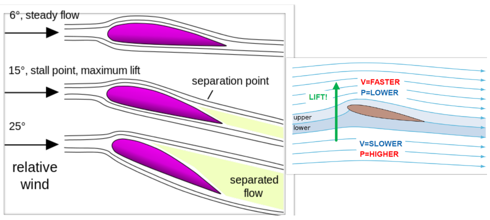
Lift force $dL = C_l \frac{\rho}{2} c \cdot dy \cdot V^2$

Drag force $dD = C_d \frac{\rho}{2} c \cdot dy \cdot V^2$

Moment $dM = C_m \frac{\rho}{2} c^2 \cdot dy \cdot V^2$

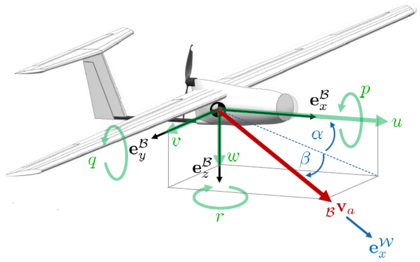
ρ : Density of fluid (air) [kg/m³]
c : Chord length [m]
V : Flight speed (w.r.t. air) [m/s]
C_l : Airfoil lift coefficient [-]
C_d : Airfoil drag coefficient [-]
C_m : Airfoil moment coefficient [-]

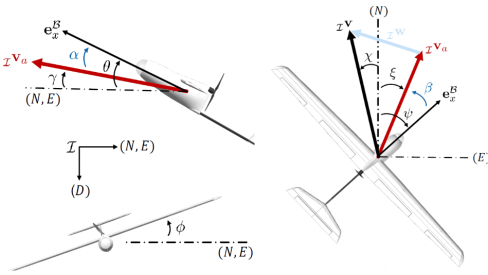
Stall does highly depend on fluid, foil and Reynolds number:



8.2 Kinematics

Body-axis $\mathcal{B}$
Body velocity: $_{\mathcal{B}}\boldsymbol{v}_a = (u, v, \omega)^T$
Body rates: $_{\mathcal{B}}\boldsymbol{\omega} = (p, q, r)^T$
Air-mass rel. s. (Airspeed): $V = \sqrt{u^2 + v^2 + \omega^2}$
Wind-axis $\mathcal{W}$
Angle of attack: $\alpha = tan^{-}1(\omega/u)$
Sideslip angle: $\beta = sin^{-}1(v/V)$



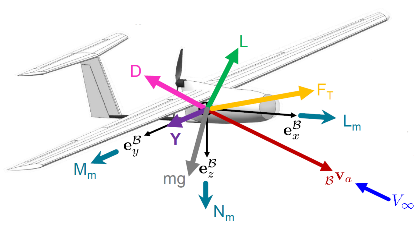


γ: Flight path angle from horizon
θ: Pitch anlge from horixon to x
ϕ: Roll angle, rotation about x
ξ: Heading angle, from North
ψ: Yaw angle, from North
χ: Course angle from North
$_{\mathcal{I}}\boldsymbol{v}$: Ground based internal velocity / ground speed)

$$_{\mathcal{I}}\boldsymbol{v}_a = C_{\mathcal{IB}\ \mathcal{B}}\boldsymbol{v}_a$$

$$_{\mathcal{I}}\boldsymbol{v} = {}_{\mathcal{I}}\boldsymbol{v}_a + {}_{\mathcal{I}}\boldsymbol{w} = {}_{\mathcal{I}}\dot{\boldsymbol{r}} = \begin{bmatrix}V\cos\gamma\cos\xi + \omega_N\\ V\cos\gamma\sin\xi + \omega_E\\ -V\sin\gamma + \omega_D\end{bmatrix}$$

8.3 Dynamics



$$\text{Lift}L = \frac{1}{2}\rho V^2 S c_L$$

$$\text{Drag}D = \frac{1}{2}\rho V^2 S c_D$$

$$\text{Rolling Moment}L_m = \frac{1}{2}\rho V^2 S b c_l$$

$$\text{Pitching Moment}M_m = \frac{1}{2}\rho V^2 S \bar{c} c_m$$

$$\text{Yawing Moment}N_m = \frac{1}{2}\rho V^2 S b c_n$$

$\Rightarrow$Y only composed of aerodynanmic Forces

EoM Translation

$$\dot{u} = rv - qw + \frac{1}{2}(F_T\cos\epsilon - D\cos\alpha + L\sin\alpha) - g\sin\theta$$

$$\dot{v} = pw - ru + \frac{1}{m}Y + g\sin\phi\cos\theta$$

$$\dot{w} = qu - pv + \frac{1}{m}(F_T\sin\epsilon - D\sin\alpha - L\cos\alpha) + g\cos\phi\cos\theta$$

$$\begin{bmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{bmatrix} = C_{IB}\begin{bmatrix}u\\v\\w\end{bmatrix} + {}_{\mathcal{I}}\boldsymbol{w}$$

EoM Rotation (Assumed $I_{xz} \approx 0$)

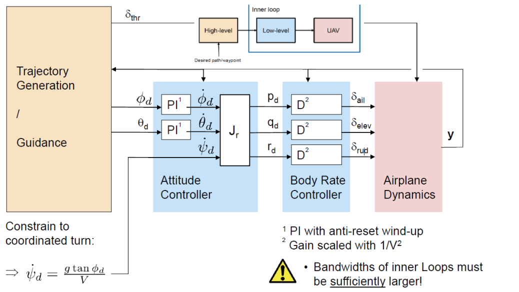
$$\dot{p} = \frac{1}{I_{xx}}(L_m + L_{m_T} - qr(I_{zz} - I_{yy}))$$

$$\dot{q} = \frac{1}{I_{yy}}(M_n + M_{m_T} - pr(I_{xx} - I_{zz}))$$

$$\dot{r} = \frac{1}{I_{zz}}(N_m + N_{m_T} - pq(I_{yy} - I_{xx}))$$

$$\begin{bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{bmatrix} = J_r^{-1}\begin{bmatrix}p\\q\\r\end{bmatrix} = \begin{bmatrix}p + q\tanh\theta\sin\phi + r\tan\theta\cos\phi\\ q\cos\phi - r\sin\phi\\ q\frac{\sin\phi}{\cos\theta} + r\frac{\cos\phi}{\cos\theta}\end{bmatrix}$$

8.4 Control



8.5 Steady level Turning Flight

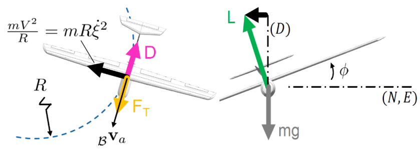
$_{\mathcal{B}}\dot{\boldsymbol{v}}_a = {}_{\mathcal{B}}\dot{\boldsymbol{\omega}} = 0 \rightarrow$ Steady (unaccelerated)
$\theta = \alpha \rightarrow \gamma = 0$Level
$\phi =$ const. $\neq 0 \rightarrow$ Turning
$\xi = \psi$ / $\dot{\xi} = \dot{\psi} \rightarrow$ No Sideslip
$Y = 0 \rightarrow$ Coordinated turn
L increases with $\frac{1}{\cos\phi}$
V_{min} increases with $\sqrt{\frac{1}{\cos\phi}}$

Force Balance:

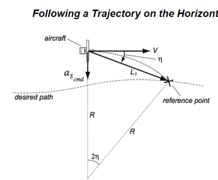
$$\frac{L\sin\phi}{L\cos\phi} = \frac{m\frac{V^2}{R}}{mg}$$

$$\tan\phi = \frac{V\dot{\xi}}{g}$$

$$\dot{\xi} = \dot{\psi} = g\tan\phi/V$$



8.6 $\mathcal{L}_1$-Guidance



$$\sin\eta = \frac{L_1}{2R} \Rightarrow R = \frac{L_1}{2\sin\eta}$$

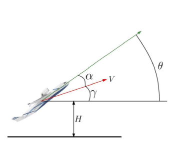
$$a_{s_{cmd}} = \frac{V^2}{R} = 2\frac{V^2\sin\eta}{L_1} = \dot{\xi}_d V$$

$$\Rightarrow \dot{\xi}_d = \frac{a_{s_{cmd}}}{V}$$

$$\dot{\xi}_d = \frac{g\tan\phi_d}{V} \Rightarrow \phi_d = \tan^{-1}\left(\frac{a_{s_{cmd}}}{g}\right)$$

8.7 TECS - Total Energy Control System

Control Altitude and Airspeed



$$E_{tot} = E_{kin} + E_{pot} = \frac{1}{2}mV^2 + mgH$$

$$\frac{\dot{E}_{tot}}{mg} = \frac{mV\dot{V}}{mg} + \frac{\dot{H}mg}{mg} = \frac{V\dot{V}}{g} + \dot{H}$$

$$\dot{E}_{spec} = \frac{\dot{E}_{tot}}{mgV} = \frac{\dot{V}}{g} + \frac{\dot{H}}{V} = \frac{\dot{V}}{g} + \sin\gamma$$

$$\dot{E}_{spec} = \frac{\dot{V}}{g} + sin\gamma \approx \frac{\dot{V}}{g} + \gamma$$

$$\dot{E}_{dist} = \gamma - \frac{\dot{V}}{g}$$