Vv255 Applied Calculus III

Recitation VI

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Contents

Linearization & Differential

Linearization:
The linearization of a function $f(x, y)$ at a point (x_0, y_0), where f is differentiable,

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Differential: If we move from(x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

in the linearization of f is called the total differential of f.

Linearization & Differential (cont.)

- The plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point (x_0, y_0) .
- The linearization of a function of two variables is a tangent-plane approximation in the same way that the linearization of a function of a single variable is a tangent line approximation.

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Extreme Values for function of several variables

Let (a, b) be a point in the domain D of a function $f(x, y)$, then $f(a, b)$ is

- Relative maximum value of f if

$$f(a, b) \geq f(x, y) \qquad \text{for some disk with center } (a, b).$$

- Relative minimum value of f if

$$f(a, b) \leq f(x, y) \qquad \text{for some disk with center } (a, b).$$

- *Absolute* maximum value of f for an interval I if

$$f(a, b) \geq f(x, y) \qquad \text{for all points } (x, y) \text{ in } R.$$

- *Absolute* minimum value of f for an interval I if

$$f(a, b) \leq f(x, y) \qquad \text{for all points } (x, y) \text{ in } R.$$

Finding Local Extrema

Procedures for finding local/relative extrema:
Step 1: Gradient Test: Finding Critical Points

$$\text{critical points} \begin{cases} 1. & \nabla f \text{ does not exist.} \\ 2. & \nabla f = 0. \end{cases}$$

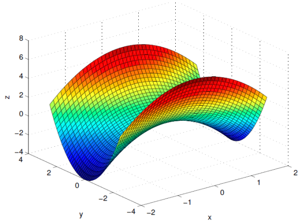

Figure: Saddle point: In general, we will say that a function has a saddle point P if there are two distinct vertical planes through P such that P in one of the planes is a local maximum and P in the other is a local minimum.

Finding Local Extrema (cont.)

Step 2: Second derivative Test: Hessian Matrix

$$\mathbf{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

Suppose f is differentiable and $\nabla f = 0$ at a point P_0, and if

$$\begin{aligned} \text{all the eigenvalues of } \mathbf{H} \text{ at } P_0 \text{ are positive} &\implies \text{local minimum} \\ \text{all the eigenvalues of } \mathbf{H} \text{ at } P_0 \text{ are negative} &\implies \text{local maximum} \\ \mathbf{H} \text{ at } P_0 \text{ has both positive and negative eigenvalues} &\implies \text{saddle point} \\ \text{One of the eigenvalues of } \mathbf{H} \text{ is zero} &\implies \text{inconclusive} \end{aligned}$$

Finding Global Extrema

Procedures for finding global/Absolute extrema:

1. Find the local extreme values of f in the domain D.
2. Find the local extreme values of f on boundary of the domain D.
   - Direct examination;
   - Lagrange multiplier (will be talked about later)
3. Compare values in step 1. and step 2., the largest of them is the global maximum, the smallest is the global minimum.

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Geometry Basis of Lagrange Multipliers

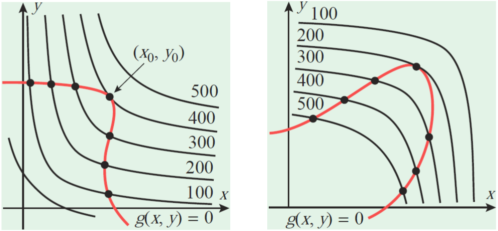
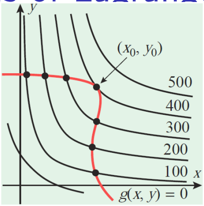


Figure: Maximum of f is 400; minimum of f is 200

To motivate the method of Lagrange multipliers, suppose that we are trying to maximize a function $f(x, y)$ subject to the constraint $g(x, y) = 0$. Geometrically, this means that we are looking for a point (x_0, y_0) on the graph of the constraint curve at which $f(x, y)$ is as large as possible.

Geometry Basis of Lagrange Multipliers (cont.)



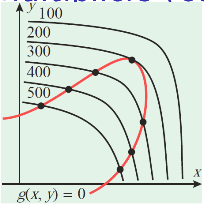


Figure: Maximum of f is 400; minimum of f is 200

To help locate such a point, let us construct a contour plot of $f(x, y)$ in the same coordinate system as the graph of $g(x, y) = 0$. For example, the graph on the left shows some typical level curves of $f(x, y) = c$, which we have labeled $c = 100, 200, 300, 400, 500$ for purpose of illustration. In this figure, each point of intersection of $g(x, y) = 0$ with a level curve is a candidate for a solution, since these points lie on the constraint curve. Among the seven such intersections shown in the figure, the maximum value of $f(x, y)$ occurs at the intersection (x_0, y_0) where $f(x, y)$ has a value of 400.

Geometry Basis of Lagrange Multipliers (cont.)

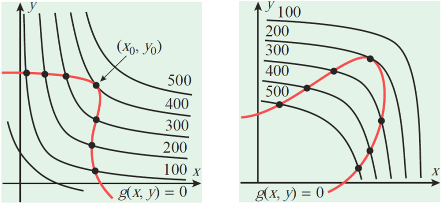


Figure: Maximum of f is 400; minimum of f is 200

Note that at (x_0, y_0) the constraint curve and the level curve just touch and thus have a common tangent line at this point.
Since $\nabla f(x_0, y_0)$ is normal to the level curve $f(x, y) = 400$ at (x_0, y_0), and since $\nabla g(x0, y0)$ is normal to the constraint curve $g(x, y) = 0$ at (x_0, y_0), we conclude that the vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ must be parallel. That is

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \qquad \text{for some scalar} \quad \lambda$$

Lagrange Multipliers

Ultimate Goal:

Minimize/Maximize f subject to the constraint(s) $g_1 = k_1$, $g_2 = k_2$, $\cdots$, $g_n = k_n$,
where f and g_1, $\cdots$, g_n are functions of several variables.

Method:
Step 1: Solve the simultaneous equations:

$$\begin{gathered}\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 + \cdots \\ g_1 = k_1 \\ g_2 = k_2 \\ \cdots\end{gathered}$$

Step 2:
Evaluate f at all the points $(x, y, z, \cdots)$ that result from step 1. The largest of these values is the maximum value of f ; the smallest is the minimum value of f.