Vv255 Applied Calculus III

Recitation II

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Contents

Line in $\mathbb{R}^3$

- Vector equation:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

- Parametric equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

- Symmetric equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- Line segments from $\mathbf{r}_1$ to $\mathbf{r}_2$ is given by:

$$\mathbf{r} = (1 - t)\mathbf{r}_1 + t\mathbf{r}_2 \qquad 0 \leq t \leq 1$$

Plane in $\mathbb{R}^3$

- Vector equation:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

- Scalar equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- Linear equation:

$$ax + by + cz + d = 0 \quad \text{where} \quad d = -ax_0 - by_0 - cz_0$$

- Distance D from a Point to a Line in Space:

$$D = \frac{\left|\overrightarrow{P_0P_1} \times \mathbf{v}\right|}{|\mathbf{v}|}$$

- Distance D from a Point to a Plane:

$$D = \frac{\left|\overrightarrow{P_0P_1} \cdot \mathbf{n}\right|}{|\mathbf{n}|}$$

Contents

Limit, Continuity, Derivative, Integral

- Limit (A practical way to calculate)

$$\lim_{t\to a}\begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} = \begin{bmatrix} \lim_{t\to a} f(t) \\ \lim_{t\to a} g(t) \\ \lim_{t\to a} h(t) \end{bmatrix}$$

- Continuity:
  A vector function $\mathbf{r}(t)$ is continuous at a point $t = a$ in its domain if

$$\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a)$$

  The function is continuous if it is continuous at every point in its domain.
- Derivative:

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta\to 0}\frac{\Delta\mathbf{r}}{\Delta t} = \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix}$$

Limit, Continuity, Derivative, Integral (cont.)

$\mathbf{r}'(t)$ is known as the tangent vector.

- Differentiation rules for vector-valued function.

  Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions of t.

  1. Addition
  $$\frac{d}{dt}[\mathbf{u}+\mathbf{v}] = \mathbf{u}' + \mathbf{v}'$$
  2. Scalar multiplication
  $$\frac{d}{dt}[f\mathbf{u}] = f'\mathbf{u} + f\mathbf{u}', \quad \text{where } f \text{ is a real-valued function of } t.$$
  3. Dot product
  $$\frac{d}{dt}[\mathbf{u}\cdot\mathbf{v}] = \mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}'$$
  4. Cross product
  $$\frac{d}{dt}[\mathbf{u}\times\mathbf{v}] = \mathbf{u}'\times\mathbf{v} + \mathbf{u}\times\mathbf{v}'$$
  5. Chain rule
  $$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)), \quad \text{where } f \text{ is a real-valued function of } t.$$

Limit, Continuity, Derivative, Integral (cont.)

- Indefinite integral

$$\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C}$$

- Definite integral

$$\int_a^b \mathbf{r}(t)\,dt = \begin{bmatrix} \int_a^b f(t)\,dt \\ \int_a^b g(t)\,dt \\ \int_a^b h(t)\,dt \end{bmatrix}$$

- The Fundamental Theorem of Calculus

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

Contents

$\mathbf{T}, \mathbf{B}, \mathbf{N}$, and curvature κ

- $\mathbf{T}$: unit tangent vector

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}$$

- κ: curvature

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{|f''(x)|}{(1+[f'(x)]^2)^{\frac{3}{2}}} = \frac{|x''y' - y''x'|}{((x')^2 + (y')^2)^{\frac{3}{2}}}$$

  If κ is large, $\mathbf{T}$ turns sharply, otherwise $\mathbf{T}$ turns slowly.

- $\mathbf{N}$: principal unit normal vector

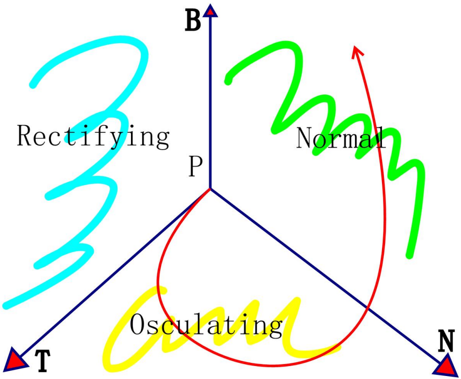
$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

- $\mathbf{B}$: binormal vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

T, **B**, **N**, and curvature κ (cont.)

- Three planes:



The curvature can be thought of as the rate at which the normal plane turns as P moves along its path.

Contents

Kepler's Law

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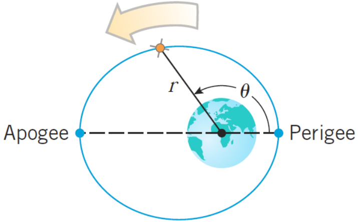
1. First law (Law of Orbits).

   The orbit of a planet is an ellipse with the Sun at one of the two foci.

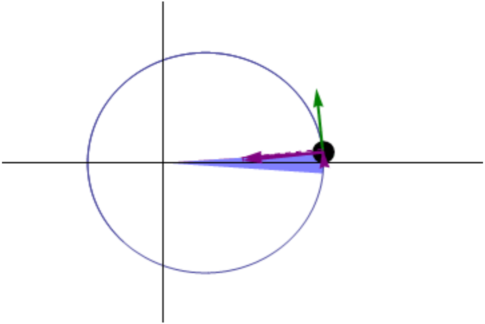
   E.g. Satellite motion around the Earth:



Kepler's Law (cont.)

2. Second law (Law of Areas).

   A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.



Kepler's Law (cont.)

3. Third law (Law of Periods).

The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

$$T^2 = \frac{4\pi^2}{GM} a^3 \quad \Rightarrow \quad T = \frac{2\pi}{\sqrt{GM}} a^{\frac{3}{2}}$$