Vv255 Applied Calculus III

Recitation V

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Lecture 10: The Chain Rule

Lecture 11: Directional Derivatives and Gradient Vector

Chain Rule

Version 1:

1. Suppose $f(x, y)$ is differentiable, where
2. $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then the composite function $f(x(t), y(t))$ is a differentiable function of t and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Version 2:

1. Suppose $z = f(x, y)$ and $y = g(x)$, then x is the only independent variable

$$\begin{aligned}\frac{dz}{dx} &= \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}\end{aligned}$$

Chain Rule (cont.)

Version 3:

1. If $f(x, y, z)$, $x(r, s)$, $y(r, s)$, and $z(r, s)$ are differentiable, then f has partial derivatives with respect to r and s, given by

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Implicit Differentiation

Version 1:
Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Version 2:
Suppose F is a function of x, y and z, if the following conditions are satisfied

1. The partial derivatives F_x , F_y , and F_z are continuous throughout an open region R in space containing the point (x_0, y_0, z_0).
2. For some constant c, $F(x_0, y_0, z_0) = c$, $F_z(x_0, y_0, z_0) \neq 0$, then the equation $F(x, y, z) = c$ defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

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Directional Derivatives & Gradient Vector

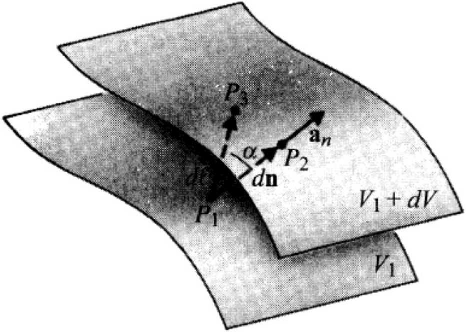
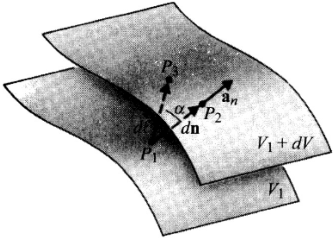


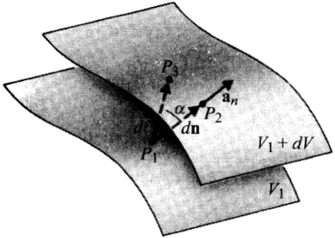
Figure: Directional derivatives & Gradient Vector

Directional Derivatives



Let us consider a scalar function of space coordinates $V(x, y, z)$, which may represent, say, the temperature distribution in a building, the altitude of a mountainous terrain, or the electric potential in a region. The magnitude of V, in general, depends on the position of the point in space, but it may be constant along certain lines or surfaces.
The figure above shows two surfaces on which the magnitude of V is constant and has the values V_1 and $V_1 + dV$, respectively, where dV indicates a small change in V.

Directional Derivatives (cont.)



Point P_1 is on surface V_1; P_2 is the corresponding point on surface $V_1 + dV$ along the normal vector $d\mathbf{n}$; and P_3 is a point close to P_2 along another vector $d\mathbf{l} \neq d\mathbf{n}$.

For the same change dV in V, the space rate of change, dV/dl, is obviously greatest along $d\mathbf{n}$ because dn is the shortest distance between the two surfaces.

Since the magnitude of dV/dl depends on the direction of $d\mathbf{l}$, dV/dl is a directional derivative.

Directional Derivatives (cont.)

We define the vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar as the gradient of that scalar. We write:

$$\mathbf{grad}V \equiv \mathbf{a}_n \frac{dV}{dn}$$

For brevity it is customary to employ the operator *del*, represented by the symbol ∇ and write ∇V in place of $\mathbf{grad}V$. Thus,

$$\nabla V \equiv \mathbf{a}_n \frac{dV}{dn}$$

We have assumed that dV is positive (an increase in V); if dV is negative (a decrease in V from P_1 to P_2), ∇V will be negative in the $\mathbf{a}_n$ direction.

Directional Derivatives (cont.)

The directional derivative along $d\mathbf{l}$ is

$$\begin{aligned} \frac{dV}{dl} &= \frac{dV}{dn}\frac{dn}{dl} \\ &= \frac{dV}{dn}\cos\alpha \\ &= \frac{dV}{dn}\mathbf{a}_n \cdot \mathbf{a}_l = (\nabla V)\cdot \mathbf{a}_l \end{aligned}$$

The above equation states that the space rate of increase of V in the $\mathbf{a}_l$, direction is equal to the projection (the component) of the gradient of V in that direction.

Gradient Vector

$$\frac{dV}{dl} = \frac{dV}{dn}\mathbf{a}_n \cdot \mathbf{a}_l = (\nabla V) \cdot \mathbf{a}_l$$

We can also write the above equation as:

$$dV = (\nabla V) \cdot d\mathbf{l}$$

where $dl = \mathbf{a}_l dl$.
Now, note that dV is the total differential of V as a result of a change in position; it can be expressed in terms of the differential changes in coordinates:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Note that

$$d\mathbf{l} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz$$

Gradient Vector (cont.)

Then, expressing dV as the dot product of two vectors, we have:

$$dV = \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right) \cdot (\mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz)$$
$$= \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right) \cdot d\mathbf{l}$$

With a little comparison, we could get that

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

or

$$\nabla V = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) V$$

In view of the equation above, it is convenient to consider V in Cartesian coordinates as a vector differential **operator**

$$\nabla \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}$$

Algebra Rules for Gradients

Algebra Rules for Gradients

Sum:

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

Constant Multiple:

$$\nabla(\alpha f) = \alpha \nabla f \qquad \text{for any real number } \alpha.$$

Product:

$$\nabla(fg) = f\nabla g + g\nabla f$$

Quotient:

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

More about ∇ Operator (Will be covered later in this semester)

∇ Operator:

$$\nabla \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}$$

Gradient:

$$\nabla V = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \right) V$$

Divergence:

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Curl (Rotation):

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$