Vv255 Applied Calculus III

Recitation I

LIU Xieyang

Teaching Assistant

University of Michigan - Shanghai Jiaotong University
Joint Institute

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Multivariable calculus

Multivariable calculus (A.K.A. multivariate calculus) is the extension of calculus in one variable to calculus in more than one variable.

- Pay attention to the figures on Page 2, Lecture 1

1. Different dimensions will lead to different meanings/interpretations of the same equation.
   - For example $x^2 + y^2 = 1$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.
2. In 2D analytical geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$.
   In 3D analytical geometry, an equation in x, y, and z represents a surface in $\mathbb{R}^3$.
3. Distance formula in $\mathbb{R}^3$:
   $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Vector

1. Definition:
   The term vector is used to indicate a quantity that has both magnitude and direction .
   - Equal/Equivalence: same magnitude & direction.
   - **0** is the only vector that has no specific direction.
2. Manipulation:
   - Addition
   - Scalar Multiplication (eg. scalar multiple $\Rightarrow$ parallel)
   - length or magnitude
3. 8 defining properties (axioms) of addition and scalar multiplication: see Page 7, Lecture 1.
4. Vector Space: A set V satisfies the following two additional axioms as well as the eight axioms above is known as a vector space.
   - If $\mathbf{u}$ and $\mathbf{v}$ are in V, then the sum of $\mathbf{u}$ and $\mathbf{v}$ is in V.
   - If $\mathbf{u}$ is in V, then the scalar multiple of $\mathbf{u}$ by α is in V.

Matrices

1. Definition:
   - Row-major propertity: By an $m \times n$ matrix (read m by n matrix) we mean a matrix with m rows and n columns.
2. Manipulation:
   - Addition of Matrices: same size!
   - Scalar Multiplication
3. 8 properties of Matrices addition and scalar multiplication
   - Here $\mathbf{0}$ denote the zero matrix of the right size.
4. Multiplication of a Matrix by a Matrix:
   - The no. of columns of the first matrix = the no. of rows of the second matrix (size matches!)
   - Properties: associativity, left & right distributivity
   - Identity matrix
5. Three ways of looking at a linear system:
   - Row (Intersection) Column (Combination) Matrix (Inverse image)

Dot (inner) product

1. Dot product is a scalar quantity!
2. Properties of the dot product
   - Magnitude: $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
   - Normalizing: $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{\mathbf{v}}{\sqrt{\mathbf{v} \cdot \mathbf{v}}}$
3. Geometric definition: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cos\theta$
   - Cauchy-Schwarz inequality
   - Triangle Inequality
   - Parallelogram Law
4. Orthogonal & Orthonormal
   - The dot product measures the extent to which two vectors point in the same general direction.
5. Scalar and Vector projection
   - Parallel and perpendicular components

Bases

1. Basis in $\mathbb{R}^3$ must be non-coplanar.
2. *Linear independence: The vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_k$ for some integer k are linearly independent if the only way to satisfy the following equation
$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_k\mathbf{a}_k = \mathbf{0}$$
is for all α's to be zero.
3. *Dimension: The dimension of a space is the largest number of linearly independent vectors, n, in that space. A basis for that space consists of n linearly independent vectors. A vector $\mathbf{v}$ in that space has n components (some of them possibly zero) with respect to any basis in that space.
4. Orthonormal basis in $\mathbb{R}^3$
5. Basis Transformation: see worksheet

Cross Product

1. Definition of cross product:
   - Magnitude: the area of the parallelogram ($|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$)
   - Direction: right-hand rule
   - The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, actually it's perpendicular to all the vectors in the plane formed by $\mathbf{u}$ and $\mathbf{v}$.
2. Determination of parallel & perpendicular:
   - Parallel: $\mathbf{u} \times \mathbf{v} = \mathbf{0}$
   - Perpendicular: $\mathbf{u} \cdot \mathbf{v} = 0$
3. Properties: see page 7, lecture 3.
   - non-commutative
   - non-associative
   - left & right distributive
4. Scalar triple product $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$: invariant under circular shift!
5. Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
   - "Back-cab rule": $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

Determinant

1. Only square matrix has determinant.
2. A third-order can be defined in terms of second-order determinants.

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

3. Physical meaning:
   - 2D: Theorem & its proof on page 10, lecture 3:

     The area of the parallelogram determined by two vectors **u** and **v** in $\mathbb{R}^2$ is

     $$\text{Area} = \left| \det \left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right) \right|, \quad \text{where} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

   - 3D: Theorem & its proof on page 11, lecture 3:

Determinant (cont.)

Theorem

The volume of the parallelepiped determined by the vectors **u**, **v**, and **w** is equal to the absolute value of the determinant of the corresponding matrix,

$$\text{Volume} = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})| = \left| \det \left(\begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \right) \right|$$

4. *What does it mean to have a determinant equal to zero?
   - zero area $\Rightarrow$ collinear
   - zero volume $\Rightarrow$ collinear / coplanar
   - $\det(A) = \det(A^T)$ (proof?)
   - *If at least one of the row matrix in the original matrix can be represented by a linear combination of the other row matrices, the determinant of the original matrix must be 0; in other words, the rows of the original matrix are not linearly independent.

Eigenvalue & Eignevector

1. Definition:

   The scalar λ is called the eigenvalue of $\mathbf{A}$ if there is a nonzero vector $\mathbf{x}$ of

   $$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

   Such nonzero vector $\mathbf{x}$ is known as the eigenvector corresponding to λ.

2. A common question: page 16, lecture 3:

Eigenvalue & Eignevector (cont.)

- The equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

states that $\mathbf{x}$ is a vector that is perpendicular to all rows of

$$\text{the matrix} \quad (\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{bmatrix}$$

where $\mathbf{r}_1, \ldots \mathbf{r}_n$ are row vectors.

E3.6 When will we have such a vector $\mathbf{x}$?

linearly dependent rows

collinear row vectors in $\mathbb{R}^2$

coplanar or collinear row vectors in $\mathbb{R}^3$

Think about the definition of linear independence, and assume that if $\mathbf{r}_1$ to $\mathbf{r}_n$ are linear independent, what would happen (contrary to the definition of linear independence) ?

Eigenvalue & Eignevector (cont.)

3. Calculating eigenvalues and eigenvectors:
   - Practice!