Vv255 Applied Calculus III

Recitation IX

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Contents

Lecture 20: Vector Fields

Lecture 21: Line Integrals

Vector Field

- A vector field in $\mathbb{R}^2$ is a function that associates with each point (x, y) in the plane a unique vector $\mathbf{F}(x, y)$ parallel to the plane.

$$\mathbf{F}(x, y) = \begin{bmatrix} P \\ Q \end{bmatrix}, \text{where components } P \text{ and } Q \text{ are functions of } x \text{ and } y.$$

- A vector field in $\mathbb{R}^3$ is a function that associates with each point (x, y, z) in the 3-space a unique vector $\mathbf{F}(x, y, z)$ in 3-space.

$$\mathbf{F}(x, y, z) = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \text{ where components } P, Q \text{ and } R \text{ are functions of } x, y \text{ and } z.$$

- Functions P, Q and R are known as component functions.
- Sometimes written compactly as $\mathbf{F}(\mathbf{r})$, where $\mathbf{r}$ is the position vector $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$

Gradient Field

A vector field **F** in $\mathbb{R}^2$or $\mathbb{R}^3$ is said to be conservative in a region if it is the gradient field for some function f in that region, that is, if

$$\boxed{\mathbf{F} = \nabla f}$$

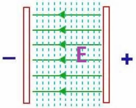
The function f is called a potential function for **F** in the region.

Examples:

1. Static electric potential V is the negative gradient of the electric field intensity **E**.

$$\mathbf{E} = -\nabla V$$

meaning, **E** field always points in the direction of maximum rate of decrease of V.



About Gradient Operator: In Cartesian Coordinates

We follow the approach in rc_05. First, we found that

$$dV = (\nabla V) \cdot d\mathbf{l}$$

where $dl = \mathbf{a}_l dl$. This essentially says that the total differential of a scalar function V could be expressed as the dot product of its gradient and the directional line vector $d\mathbf{l}$.
Simultaneously, we could express dV, the total differential of a scalar function V in Cartesian Coordinates as:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz.$$

Note that in Cartesian Coordinates

$$d\mathbf{l} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz.$$

About Gradient Operator: In Cartesian Coordinates (cont.)

Then, expressing dV as the dot product of two vectors, we have:

$$\begin{aligned} dV &= \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right) \cdot (\mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz) \\ &= \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right) \cdot d\mathbf{l} \end{aligned}$$

With a little comparison, we could get that

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

or

$$\nabla V = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) V$$

About Gradient Operator: In Cartesian Coordinates (cont.)

In view of the equation above, it is convenient to consider V in Cartesian coordinates as a vector differential **operator**

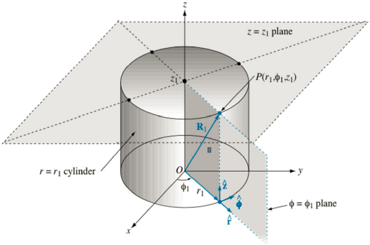
$$\nabla \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}$$

Remember,

$$dV = (\nabla V) \cdot d\mathbf{l}$$

is valid in all orthogonal curvilinear coordinate systems, not just Cartesian, but also Cylindrical and Spherical.
Now, with $dV = (\nabla V) \cdot d\mathbf{l}$ in mind, we could derive the formula for the gradient in cylindrical coordinate systems in the exact same way:

About Gradient Operator: In Cylindrical Coordinates



Note it is true in cylindrical coordinates that

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial \phi}d\phi + \frac{\partial V}{\partial z}dz$$

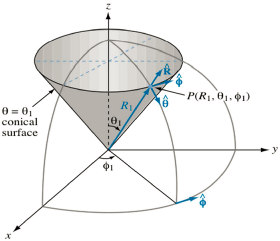
and

$$d\mathbf{l} = \mathbf{a}_r dr + \mathbf{a}_\phi r d\phi + \mathbf{a}_z dz$$

Therefore:

$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

About Gradient Operator: In Spherical Coordinates



Note it is true in spherical coordinates that

$$dV = \frac{\partial V}{\partial R} dR + \frac{\partial V}{\partial \theta} d\theta + \frac{\partial V}{\partial \phi} d\phi$$

and

$$d\mathbf{l} = \mathbf{a}_R dR + \mathbf{a}_\theta R d\theta + \mathbf{a}_\phi R \sin\theta d\phi$$

Therefore:

$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R} \frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R \sin\theta} \frac{\partial V}{\partial \phi}$$

About Gradient Operator: In Spherical Coordinates (cont.)

NOTE THAT my notation is different from the one in your lecture slides! But as long as it indicates the correct directional vector, it's OK!

Notation is just a mask, you have to see through what's behind it.

Dr. Jing's version should be:

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z$$

$$\nabla = \frac{\partial}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho \sin\phi}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial}{\partial \phi}\mathbf{e}_\phi$$

Algebra Rules for Gradients

Algebra Rules for Gradients

Sum:

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

Constant Multiple:

$$\nabla(\alpha f) = \alpha \nabla f \qquad \text{for any real number } \alpha.$$

Product:

$$\nabla(fg) = f\nabla g + g\nabla f$$

Quotient:

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

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Lecture 20: Vector Fields

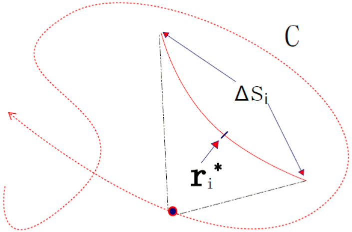
Lecture 21: Line Integrals

Line Integral

- If a constant force F is applied to an object to move it along a straight line from $x = a$ to $x = b$, then the amount of work done is the force times the distance,
$$W = F(b - a)$$
- More generally, if the force is not constant, but is instead dependent on x so that the amount of force applied when the object is at the point x is given by $F(x)$, then the work done is given by the integral
$$W = \int_a^b F(x)dx$$
This definition is based on applying the "basic" formula inside each subinterval
$$\sum_{i=1}^{n} F(x^*)\Delta x_i$$
and take the limit of the sum.

Line Integral (cont.)

- Now, suppose that a force is applied to an object to move it along a curve C defined by a smooth parametrization $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y$, instead of along a straight line.



If the force on the object in the direction of motion at (x, y) is given by

$$F(x, y) = F(\mathbf{r})$$

Line Integral (cont.)

Then it is reasonable to expect the following definition,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(\mathbf{r}^*)\Delta s_i = \int_C F(\mathbf{r})ds$$

where s is the arc length parameter.

Line Integral

- To actually evaluate a line integral, we express the curve C parametrically,

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad \text{or} \qquad \mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \qquad \text{where } a \le t \le b.$$

- Since the arc length function and its derivative is given by

$$s(t) = \int_\alpha^t |\mathbf{r}'(u)|\,du \qquad \text{and} \qquad \frac{ds}{dt} = |\mathbf{r}'(t)|$$

- Thus the differentials are connected by the following relationship,

$$ds = |\mathbf{r}'(t)|\,dt$$

- Therefore,

$$\int_C f(\mathbf{r})\,ds = \int_a^b f(\mathbf{r})|\mathbf{r}'(t)|\,dt$$

Line Integral

INDEED, you only need ONE formula to evaluate line integral, which is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

or you can first find

$$dW = \mathbf{F} \cdot d\mathbf{r}$$

then do the integration

$$W = \int_C dW$$

Application

1. Finding the total work done by a mechanical force acting on an object.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

2. Ampere's circuital law:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$$

   which states that the circulation of the magnetic flux density in free space around any closed path is equal to μ_0 times the total current flowing through the surface bounded by the path.

3. Stoke's theorem (will cover latter in this semester)

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$$

   which states that the surface integral of the curl of a vector filed over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.