Vv255 Applied Calculus III

Recitation IV

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Contents

Functions of several variables

Definition:

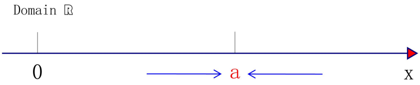
- A function of two variables, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy-plane.
- A function of three variables, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D in three-dimensional space.
- Natural Domain: the domain of independent variables consisting of all points for which the function formula yields a real value for the dependent variable.

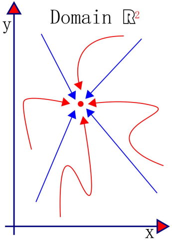
Question:

Find and sketch the domain of the function

$$f(x, y) = \ln(9 - x^2 - 9y^2)$$

Limit





The first thing we can do and need to do in 3D is to specify the curve along which we have consider the limit as (x, y) approaches (x_0, y_0).

Limit (cont.)

Limit along a curve:

Definition

If C is a smooth parametric curve that is represented by the equations

$$x = x(t), \quad y = y(t)$$

and if $x_0 = x(t_0)$ and $y_0 = y(t_0)$, then the limit of $f(x, y)$

as (x, y) approaches a point (x_0, y_0) along a curve C

is defined to be

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ \text{along } C}} f(x, y) = \lim_{t\to t_0} f(x(t), y(t))$$

Limit (cont.)

General definition of limit:

Definition

Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (a, b), except possibly at (a, b).

Then we say that the limit of f as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (a, b) satisfies

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

Limit (cont.)

The general definition of limit says nothing about along which curve should we approach a point, and we should expect that the limits along any smooth curve C are equal.

The relationship between the limit along a specific curve and the general definition of limit are stated as follows:

1. If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, then $\lim\limits_{\substack{(x,y)\to(a,b)\\ \text{along } C}} f(x,y) = L$ for any smooth curve C.
2. If $\lim\limits_{\substack{(x,y)\to(a,b)\\ \text{along } C}} f(x,y)$ fails to exist, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist .
3. If $\lim\limits_{\substack{(x,y)\to(a,b)\\ \text{along } C_1}} f(x,y) \neq \lim\limits_{\substack{(x,y)\to(a,b)\\ \text{along } C_2}} f(x,y)$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist .

Limit (cont.)

Question:
Find out the limit if it exists, or explain why it doesn't exist.

$$\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+y^4}$$

Continuity

Definition:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Basically, we require that the limit of the function and the value of the function to be the same at the point.

Theorem:

1. If $g(t)$ and $h(t)$ are continuous at d with $g(d) = a$ and $h(d) = b$, and $f(x,y)$ is continuous at (a,b), then the composition $f(g(t),h(t))$ is continuous at d.
2. A sum, difference, or product of continuous functions is continuous.
3. A quotient of continuous functions is continuous when the denominator is not 0.
4. If $h(x,y)$ is continuous at (a,b) and $g(u)$ is continuous at $u = h(a,b)$, then the composition $f(x,y) = g(h(x,y))$ is continuous at (a,b).

Continuity (cont.)

Question:
Find out the limit if it exists, or explain why it doesn't exist.

1. $\lim_{(x,y)\to(6,3)} xy\cos(x-2y)$

2. $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ Hint: *Use SQUEEZE THEOREM*

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Partial Derivative

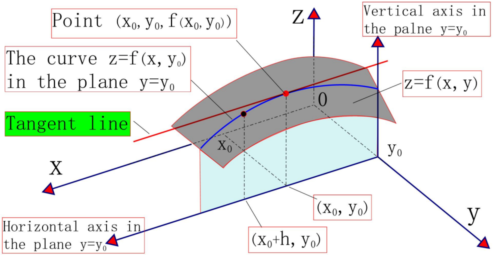

Figure: Holding the y-value constant.

Definition:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \left.\frac{d}{dx}f(x,y_0)\right|_{x=x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} = \left.\frac{d}{dy}f(x_0,y)\right|_{y=y_0}$$

Partial Derivative (cont.)

For functions of more than two independent variables, the definitions of partial derivatives are very similar. They are ordinary derivatives with respect to one independent variable taken while the other independent variables are held constant.

Question:
Find the indicated partial derivatives. $f(x, y) = \sqrt{x^2 + y^2}$. What's $f_x(3, 4)$?

Implicit Differentiation

Assume that the following equation

$$f(x, y, z) = 0$$

defines z as a function of the two independent variables x and y and the partial derivatives exists.
We find $\dfrac{\partial z}{\partial x}$ by differentiate both sides of the equation w.r.t x while holding y constant and treating z as a differentiable function of x, say $z = g(x)$

Question:
Implicitly differentiate on $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$x^2 + y^2 + z^2 = 3xyz$$

Higher-order Partial Derivatives

Second-order Partial Derivatives:

- If we differentiate a function f (x; y) twice, we obtain its second-order partial derivatives.
- For example:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = (f_x)_x \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} = (f_y)_x$$

Clairaut's Theorem:

- If $f(x, y)$ and its partial derivatives f_x , f_y , f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Higher-order Partial Derivatives:

- Basically the same. Mind the order!