Vv255 Applied Calculus III

Recitation X

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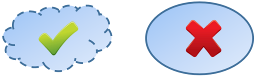
Contents

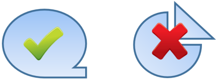
Region of Interest D

There are three common adjectives in front of the region of interest D in the field of line integral.

1. Open: the points on all the boundaries does NOT count!



2. Connected: any two points in D can actually be connected by a path that lies entirely within D.



3. Simply connected: one piece + NO “holes”. Or say, any closed path in the region could be shrunk to a point.

The Fundamental theorem for line integrals

Suppose that

$$\mathbf{F}(x,y) = P(x,y)\mathbf{e}_x + Q(x,y)\mathbf{e}_y,$$

is a conservative vector field, that is, $\boxed{\mathbf{F} = \nabla f}$ in some open region D containing the points A and B and that

$$P(x,y) \text{ and } Q(x,y) \text{ are continuous in this region } D.$$

If C be a piecewise smooth parametric curve given by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y, \qquad \text{for} \quad a \le t \le b$$

starts at $A = \mathbf{r}(a)$ and ends at $B = \mathbf{r}(b)$, and lies entirely in the region D, then

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))}$$

The Fundamental theorem for line integrals (cont.)

Independent of path:

- Conditions:
  - $\mathbf{F}$ being conservative: $\mathbf{F} = \nabla f$
  - Region D being open.
  - $P(x, y)$ and $Q(x, y)$ being continuous.
  - C being piecewise smooth.
  - A and B are in region D.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- The FTL states the value of the integral depends on the endpoints but not on the actual path C, it is said to be independent of the path.
- The FTL can be easily extended to $\mathbb{R}^3$.

$$\text{Conservative} \xrightarrow{D \text{ being Open}} \text{Independent of path}$$

The Fundamental theorem for line integrals

Conservative:
If the line integral of a vector field F is independent of path within D, then F is a conservative vector field on D.
Proof see lecture.

$$\text{Independent of path} \xrightarrow{D \text{ being Open \& Connected}} \text{Conservative}$$

Therefore, we have the following iff statement:

On an open connected region D, a continuous vector field $\mathbf{F}$ is

a conservative vector field if and only if its line integral is independent of path.

Conservative Field Test

Suppose $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$ is a vector field on an open simply connected region D, and if P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D \text{, then } \mathbf{F} \text{ is conservative.}$$

Similarly, for $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y + R\mathbf{e}_z$ defined on an open simply connected region E, and if P, Q and R have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \quad \text{throughout } E \text{, then } \mathbf{F} \text{ is conservative.}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \xrightarrow{D \text{ being Open \& Simply Connected}} \text{Conservative}$$

For $\mathbb{R}^3$, we can mimic the way introduced by your Physics professor M.K:

1. Irrotational? $(\nabla \times \mathbf{F} = 0)$
2. Open simply connected?

Actually consistent with the previous result!

The general technique of finding potential function

-It remains to find the function f such that $\nabla f = \mathbf{F}$ for a given vector field $\mathbf{F}$ that is known to be conservative before we can apply FTL.

For a conservative vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{e}_x + Q(x, y)\mathbf{e}_y$,

1. Integrate $P(x, y)$ w.r.t x to obtain

$$f(x, y) = f_1(x, y) + g(y), \quad \text{where } f_1(x, y) = \int P(x, y)\, dx, \text{ and } g(y)$$

   is an unknown function that plays the role of the constant of integration.

2. Differentiate $f = f_1 + g$ w.r.t y to obtain

$$\frac{\partial}{\partial y}\left[f_1(x, y)\right] + g'(y) = Q(x, y), \quad \text{and solve for } g'(y).$$

3. Integrate $g'(y)$ w.r.t y to complete the definition of f, up to a constant.

A similar procedure can be used for a vector field defined on $\mathbb{R}^3$.

Conclusion

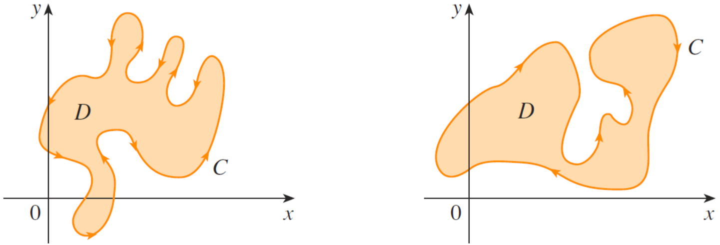
Suppose $P(x,y)$ and $Q(x,y)$ are continuous on some open simply connected region D, then then the following statements are equivalent :

1. $\mathbf{F} = P\mathbf{e}_x + Q\mathbf{e}_y$ is a conservative vector field on the region D.
2. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ at every point in D.
3. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth *closed* curve C in D.
4. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from any point A in D to any point B in D for every piecewise smooth curve C in D.

Contents

Green Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C . In stating Green's Theorem we use the convention that the positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C. Thus, if C is given by the vector function $\mathbf{r}(t), a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C.



(a) Positive orientation (b) Negative orientation

Green Theorem (cont.)

Green's Theorem

If C is a positively oriented, piecewise smooth, simple closed curve that encloses a region D, and $P(x, y)$ and $Q(x, y)$ are functions that have continuous first partial derivatives on some open set containing D, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where

$$\mathbf{F} = P\mathbf{a}_x + Q\mathbf{a}_y \quad \text{and} \quad d\mathbf{r} = dx\mathbf{a}_x + dy\mathbf{a}_y$$

Proof see lecture.

Applications of Green Theorem

- Double integral is easier to evaluate than line integral, use Green's theorem in the positive direction.
- Line integral is easier to evaluate than double integral, use Green's theorem in the reverse direction.
- Computing areas.
  Note that the area of a region D is $\iint_D 1dA$, we wish to choose P and Q so that
  $$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$
  There are several possibilities:
  $$\begin{cases} P(x,y) = 0 \\ Q(x,y) = x \end{cases} \quad \begin{cases} P(x,y) = -y \\ Q(x,y) = 0 \end{cases} \quad \begin{cases} P(x,y) = -\dfrac{1}{2}y \\ Q(x,y) = \dfrac{1}{2}x \end{cases}$$

Applications of Green Theorem (cont.)

Then Green's Theorem gives the following formulas for the area of D :

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx$$

For example, the area of a ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be computed in the following way:
The ellipse has parametric equations: $x = a\cos t, y = b\sin t$, where $0 \le t \le 2\pi$. Thus:

$$\begin{aligned} A &= \frac{1}{2}\int_C xdy - ydx \\ &= \int_0^{2\pi} (a\cos t)(b\cos t)dt \\ &\quad - (b\sin t)(-a\sin t)dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab \end{aligned}$$

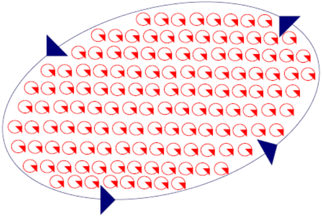
or

$$\begin{aligned} A &= \oint_C xdy \\ &= \int_0^{2\pi} (a\cos t)(b\cos t)dt \\ &= ab\int_0^{2\pi} \cos^2 t dt \\ &= \pi ab \end{aligned}$$

Applications of Green Theorem (cont.)

- Help you understand Stoke's Theorem! (Will be covered later!)
- Physical meaning:

$$\underbrace{\oint_C P\,dx + Q\,dy}_{\text{Macroscopic circulation}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{\text{Microscopic circulation}}$$



Green's theorem says that if you add up all the microscopic circulation inside C, then the sum is exactly the same as the macroscopic circulation around C.

Applications of Green Theorem (cont.)

- Normal & Tangential form

$$\text{Normal form} \qquad \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

$$\text{Tangential form} \qquad \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

where **n** is the unit outward normal, and the **T** is the unit tangent vector.