Vv255 Applied Calculus III

Recitation VII

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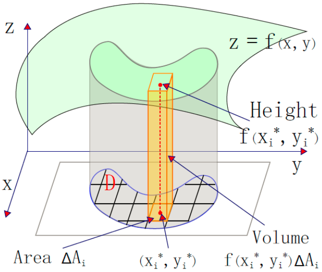
Contents

Double Integral



Net signed volume:

$$\iint_D f(x,y)dA = \lim_{n\to\infty}\sum_i^n f(x_i^*, y_i^*)\Delta A_i$$

A function $f(x, y)$ is called integrable if the limit actually exists and that its value does not depend on the choice of the partition.

Fubini's Theorem

Fubini's Theorem

Let R be the rectangle region defined by the inequalities

$$a \leq x \leq b, \qquad c \leq y \leq d$$

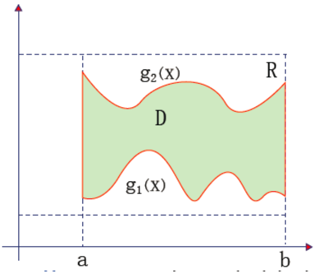
If $f(x,y)$ is continuous on this rectangle, then

$$\iint\limits_{R} f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

When $f(x,y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a simple form.

$$\iint\limits_{D} f(x,y)dA = \int_a^b g(x)dx \int_c^d h(y)dy$$

Two Basic Types of Double Integral



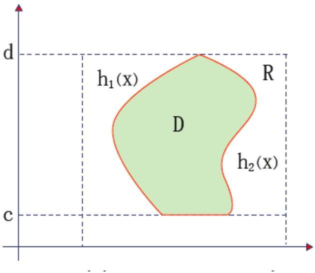


Figure: Type I Type II

$$\text{Type I:} \quad \iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$$\text{Type II:} \quad \iint_D f(x,y)\,dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$

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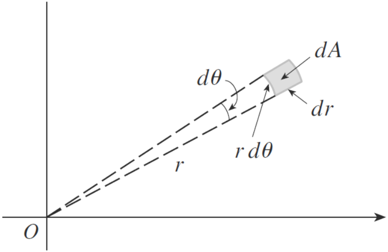
Double intergral in polar coordinates



$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ \mathbf{r} dr\ d\theta$$

r is known as the metric coefficient.

Applications of Double integral

1. Volume & area.
2. Average value

$$\text{Average value } = \frac{1}{A} \iint_D f(x,y)dA$$

3. Total mass of a lamina (inhomogeneous)

$$m = \iint_D \rho(x,y)dA$$

4. Center of mass/Center of gravity

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\rho(x,y)dA}{\iint_D \rho(x,y)dA} \qquad \bar{y} = \frac{M_x}{m} = \frac{\iint_D y\rho(x,y)dA}{\iint_D \rho(x,y)dA}$$

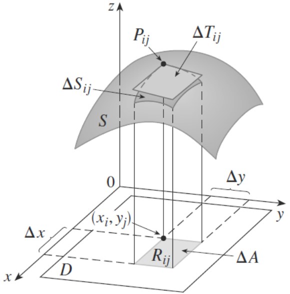
where M_y and M_x are the moments w.r.t. to the y and x axes.

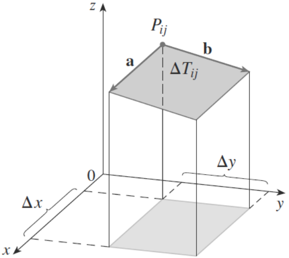
Applications of Double integral (cont.)

5. Centroid (the center of mass of a homogeneous lamina):

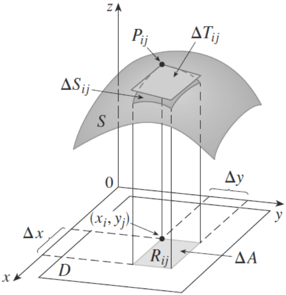
$$\bar{x} = \frac{\iint\limits_D x dA}{\iint\limits_D dA} = \frac{1}{\text{Area}} \iint\limits_D x dA \qquad \bar{y} = \frac{\iint\limits_D y dA}{\iint\limits_D dA} = \frac{1}{\text{Area}} \iint\limits_D y dA$$

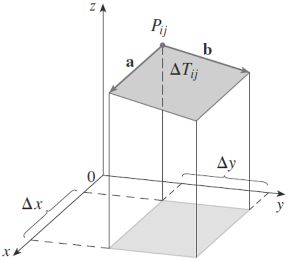
6. Surface area:





Applications of Double integral (cont.)



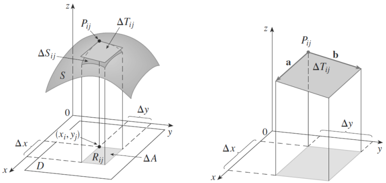


$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta \ |\mathbf{a} \times \mathbf{b}|$$

$$\mathbf{a} = \Delta x\, \mathbf{i} + f_x(x_i, y_j)\, \Delta x\, \mathbf{k}$$

$$\mathbf{b} = \Delta y\, \mathbf{j} + f_y(x_i, y_j)\, \Delta y\, \mathbf{k}$$

Applications of Double integral (cont.)



$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\,\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\,\Delta y \end{vmatrix}$$

$$= -f_x(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{i} - f_y(x_i, y_j)\,\Delta x\,\Delta y\,\mathbf{j} + \Delta x\,\Delta y\,\mathbf{k}$$

$$= [-f_x(x_i, y_j)\mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}]\,\Delta A$$

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,\Delta A$$

Therefore, $$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Applications of Double integral (cont.)

More advanced:

1. Electric potential V

$$V = \frac{1}{4\pi\epsilon_0} \iint\limits_{s} \frac{\rho_s}{R} ds$$

2. $\cdots$

Contents

Triple integral & Fubini's Theorem

Triple integral:

$$\iiint\limits_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

Find the boundary expression!

Fubini's theorem

If f is continuous on the rectangle $B = [a,b] \times [c,d] \times [r,s]$, then

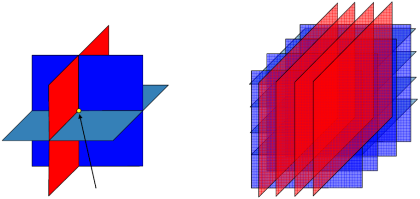
$$\iiint\limits_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

There are five other possible orders, all of which give the same value.

Contents

Orthogonal Coordinate Systems

- A tool to ease the solving process for problems with certain geometry;
- A point in 3D space can be determined by the intersection of three surfaces.
- If these three surfaces are mutually orthogonal to each other, this sets up an orthogonal coordinate system.

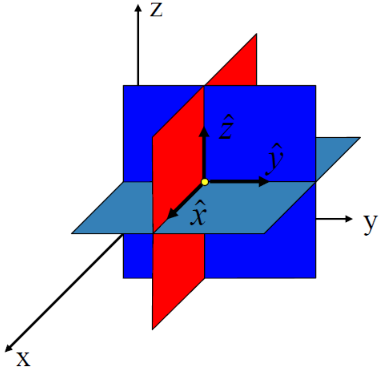


Point in space is an intersection of coordinate surfaces.

To describe location of each point in space we need 3 sets of surfaces.

- Unit vectors are defined as the normal direction of each coordinate surface.

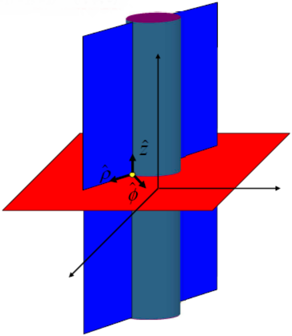
Cartesian Coordinates



$$(x, y, z)$$

Coordinate system where all coordinate surfaces are planar is called Cartesian. Note that in this system, the direction of coordinate vectors is the same at every point in space

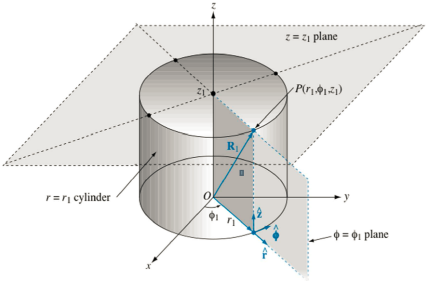
Cylindrical Coordinates

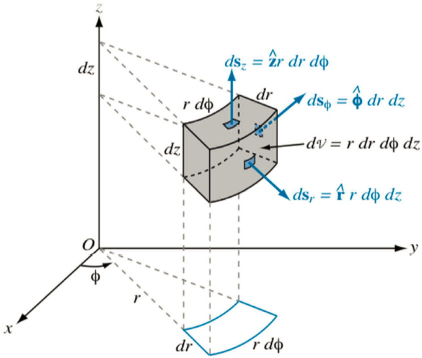


$$(\rho \text{ (or } r), \phi \text{ (or } \theta), z)$$

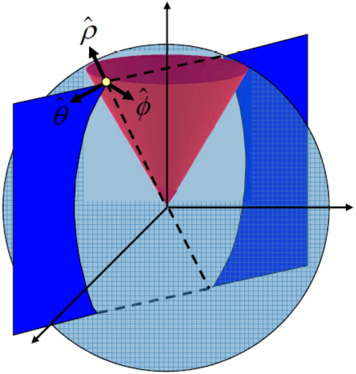
In a cylindrical coordinate system, a cylindrical surface and two planar surfaces all orthogonal to each other define location of a point in space.
The coordinate vectors are $\hat{\rho}$ (or $\hat{r}$), $\hat{\phi}$ (or $\hat{\theta}$), $\hat{z}$. Note, in this system the direction of coordinate vectors changes from point to point.

Cylindrical Coordinates (cont.)





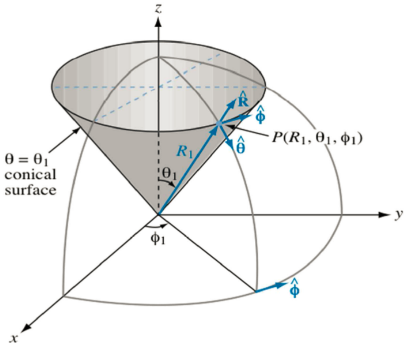
Spherical Coordinates

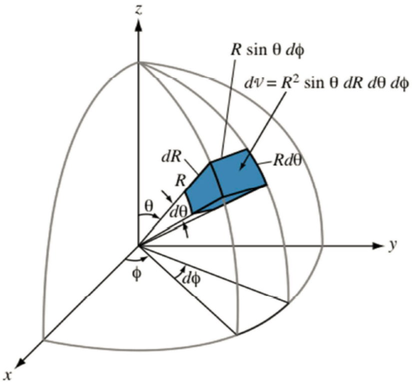


$$(R \text{ (or } \rho), \theta \text{ (or } \phi), \phi \text{ (or } \theta))$$

In a spherical coordinate system the surfaces of spheres, cones and planes are coordinate surfaces. The corresponding coordinate vectors $(\hat{R}$ (or $\hat{\rho}), \hat{\theta}$ (or $\hat{\phi}), \hat{\phi}$ (or $\hat{\theta}))$ also change their direction from point to point.

Spherical Coordinates (cont.)





Triple integral in Cylindrical and Spherical coordinates

Cylindrical:

$$\iiint_E f(r,\theta,z)\ dr\ rd\theta\ dz = \iiint_E f(r,\theta,z)\underline{r}drd\theta dz$$

Spherical:

$$\iiint_E f(\rho,\phi,\theta)\ d\rho\ \rho d\phi\ \rho\sin\phi d\theta = \iiint_E f(\rho,\phi,\theta)\underline{\rho^2\sin\phi}d\rho d\theta d\phi$$

The additional r and $\rho^2\sin\phi$ can also be interpreted as Jacobian.

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Jacobian

Definition:

The Jacobian of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

It gives how much the transformation is expanding or contracting the area around a point in uv-plane as the point is transformed into xy-plane.

Theorem:

If $g(u, v)$, $h(u, v)$ and $f(x, y)$ have continuous partial derivatives and $J(u, v)$ is zero only at isolated points, if at all, then

$$\iint_R f(x, y)\, dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Jacobian

Definition:

For an one-to-one transformation that maps a region in $\mathbb{R}^3$ onto a region in $\mathbb{R}^3$,

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

the Jacobian is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

This determinant measures how much the volume near a point is being expanded or contracted by the transformation from (u, y, w) to (x, y, z) coordinates.