

Let X_1, \dots, X_n be a sequence of r.v.'s and X be another r.v. Let F_n be cdf of X_n , and F be cdf of X .

Recall: $X_n \xrightarrow{P} X$ if for every $\varepsilon > 0$,

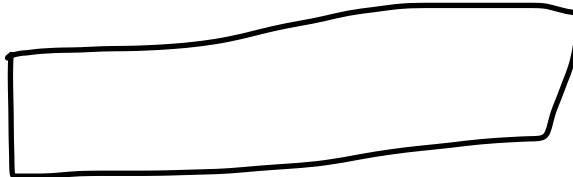
$$P(|X_n - X| > \varepsilon) \xrightarrow{n \rightarrow \infty} 0$$

$$X_n \xrightarrow{d} X \text{ if } \lim_{n \rightarrow \infty} F_n(x) = F(x)$$

at all x for which F is continuous.

$$P(\lim |X_n - X| < \varepsilon) = 1$$

or



For every $\omega \in [0, 1)$, $\omega^n \rightarrow 0$ as $n \rightarrow \infty$

$$X_n(\omega) \rightarrow \omega = X(\omega)$$

$X_n(1) = 2$ for every n , so $X_n(1) \not\rightarrow 1 = X(1)$

But $P([0, 1)) = 1 \Rightarrow X_n \xrightarrow{a.s.} X$ (but not pointwise)

$$\begin{aligned}
 (i) \quad X_n &\xrightarrow{q.m.} X \Rightarrow X_n \xrightarrow{P} X \\
 (ii) \quad X_n &\xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X \\
 (iii) \quad X_n &\xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c \\
 (iv) \quad X_n &\xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X
 \end{aligned}$$

$q.m. \downarrow$
 $a.s. \rightarrow P \rightarrow d$

Pf. (i) $X_n \xrightarrow{q.m.} X$. Fix $\varepsilon > 0$

$$P(|X_n - X| > \varepsilon) = P((X_n - X)^2 > \varepsilon^2) \stackrel{\text{Markov's}}{\leq} \frac{E[(X_n - X)^2]}{\varepsilon^2}$$

$$\xrightarrow{n \rightarrow \infty} 0 \Rightarrow X_n \xrightarrow{P} X$$

(ii), (iii) were proven before
 (iv) won't be proved here.

$$U \sim \text{Unif}(0,1), \quad X_n = \sqrt{n} \mathbb{I}_{(0, \frac{1}{n})}(U) = \begin{cases} \sqrt{n}, & U \in (0, \frac{1}{n}) \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 P(|X_n - 0| > \varepsilon) &= P(|X_n| > \varepsilon) = P(0 < U < \frac{1}{n}) \\
 &= \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0 \Rightarrow X_n \xrightarrow{P} 0.
 \end{aligned}$$

But $E[(X_n - 0)^2] = E(X_n^2) = \int_0^{1/n} (\sqrt{n})^2 du$

$$\begin{aligned}
 &= n \int_0^{1/n} du = n \cdot \frac{1}{n} = 1 \text{ for all } n \\
 &\Rightarrow X_n \not\xrightarrow{q.m.} 0.
 \end{aligned}$$

$$\Omega = [0, 1]$$

$$\text{Let } X_1(\omega) = \omega + \mathbb{I}_{[0, 1]}^{(\omega)} = \begin{cases} \omega + 1, & \omega \in [0, 1) \\ \omega, & \omega \notin [0, 1] \end{cases}$$

$$X_2(\omega) = \omega + \mathbb{I}_{[0, \frac{1}{2}]}^{(\omega)}, X_3^{(\omega)} = \omega + \mathbb{I}_{[\frac{1}{2}, 1]}^{(\omega)}$$

$$X_4(\omega) = \omega + \mathbb{I}_{[0, \frac{1}{3}]}^{(\omega)}, X_5(\omega) = \omega + \mathbb{I}_{[\frac{1}{3}, \frac{2}{3}]}^{(\omega)}, X_6(\omega) = \omega + \mathbb{I}_{[\frac{2}{3}, 1]}^{(\omega)}$$

⋮

$$\text{Let } X(\omega) = \omega$$

As $n \rightarrow \infty$, $P(|X_n - X| \geq \varepsilon) = P(\text{an interval of } \omega \text{ values whose length is going to } 0)$

$$\rightarrow 0 \Rightarrow X_n \xrightarrow{P} X$$

For every ω , $X_n(\omega)$ alternates between ω and $\omega + 1$ infinitely often.

$$\text{If } \omega = \frac{3}{8}, X_1(\omega) = \frac{3}{8} + 1, X_2(\omega) = \frac{3}{8} + 1,$$

$$X_3(\omega) = \frac{3}{8}, X_4(\omega) = \frac{3}{8}, X_5(\omega) = \frac{3}{8} + 1,$$

$$X_6(\omega) = \frac{3}{8}, \text{ etc}$$

$$\Rightarrow X_n \not\xrightarrow{P} X \text{ a.s.}$$

$$P(X_n = n^2) = \frac{1}{n}, \quad \underline{P(X_n = 0) = 1 - \frac{1}{n}}$$

$$P(|X_n - 0| < \varepsilon) = P(X_n = 0) = 1 - \frac{1}{n} \xrightarrow{n \rightarrow \infty} 1$$

$$P(|X_n - 0| \geq \varepsilon) \rightarrow 0 \Rightarrow X_n \xrightarrow{P} 0$$

$$\text{But } E(X_n) = n^2 \cdot \frac{1}{n} + 0 \left(1 - \frac{1}{n}\right) = n$$

$$E(X_n) \xrightarrow{n \rightarrow \infty} \infty$$

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$$P(|X_{(n)} - 1| \geq \varepsilon) = P(X_{(n)} \geq 1 + \varepsilon \text{ or } X_{(n)} \leq 1 - \varepsilon)$$

$$= P(X_{(n)} \leq 1 - \varepsilon)$$

$$= P(X_1 \leq 1 - \varepsilon, \dots, X_n \leq 1 - \varepsilon)$$

$$= \prod_{i=1}^n P(X_i \leq 1 - \varepsilon) = (1 - \varepsilon)^n \xrightarrow{n \rightarrow \infty} 0$$

$$\Rightarrow X_{(n)} \xrightarrow{P} 1$$

$$n(1 - X_{(n)}) \xrightarrow{d} \xi_{\text{Exp}(1)}$$

$$P(n(1 - X_{(n)}) \leq t) = P(X_{(n)} \geq 1 - \frac{t}{n})$$

$$= 1 - P(X_{(n)} \leq 1 - \frac{t}{n}) = 1 - \left(1 - \frac{t}{n}\right)^n \xrightarrow{n \rightarrow \infty} 1 - e^{-t}$$

$$\text{Let } X_n \sim \mathcal{N}(0, \frac{1}{n})$$

Let F be a dist. fn for $c=0$

(dist'n function for a point mass at 0)

$$\sqrt{n} X_n \sim \mathcal{N}(0, 1)$$

$$F_n(t) = P(X_n \leq t) = P(\sqrt{n} X_n \leq \sqrt{n} t)$$

$$= P(Z \leq \sqrt{n} t), \quad Z \sim \mathcal{N}(0, 1)$$

$$F_n(t) \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{if } t < 0 \quad (\sqrt{n} t \rightarrow -\infty)$$

$$F_n(t) \xrightarrow[n \rightarrow \infty]{} 1 \quad \text{if } t > 0 \quad (\sqrt{n} t \rightarrow \infty)$$

$$F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

$$F_n(t) \rightarrow F(t) \Rightarrow X_n \xrightarrow{d} 0$$

$$P(|X_n - 0| > \varepsilon) = P(|X_n|^2 > \varepsilon^2) \leq \frac{E(X_n^2)}{\varepsilon^2}$$

$$= \frac{1/n}{\varepsilon^2} = \frac{1}{n \varepsilon^2} \xrightarrow[n \rightarrow \infty]{} 0$$

$$\Rightarrow X_n \xrightarrow{P} 0$$