



$$\begin{aligned}
 P(A \cup B \cup C) &= P(A \cup B) + P(C) - P((A \cup B) \cap C) \\
 &\leq P(A \cup B) + P(C) \\
 &= P(A) + P(B) - P(A \cap B) + P(C) \\
 &\leq P(A) + P(B) + P(C)
 \end{aligned}$$

$$A \subset B$$

$$P(A \cap B) = P(A)$$

If  $A, B$  are indep,

$$P(A \cap B) = P(A) P(B)$$

→ possible if  $P(B) = 1$  or  $P(A) = 0$

$$y = \frac{1}{x} = h(x) ; h^{-1}(y) = \frac{1}{y} , \frac{d}{dy} h^{-1}(y) = -\frac{1}{y^2}$$

$$y \geq 1$$

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right|$$

$$= 2 \left(1 - \frac{1}{y}\right) \cdot \left| -\frac{1}{y^2} \right| = \frac{2}{y^2} \left(1 - \frac{1}{y}\right)$$

$$f_Y(y) = \begin{cases} \frac{2}{y^2} \left(1 - \frac{1}{y}\right), & y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$0 \leq Y \leq 1$$

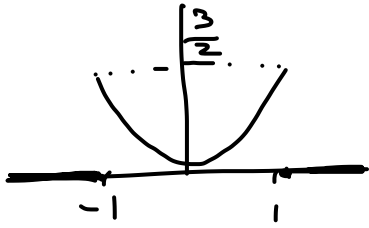
$$F_Y(y) = P(Y \leq y) = P(F(X) \leq y)$$

$$= P(F^{-1}(F(X)) \leq F^{-1}(y))$$

$$= P(X \leq F^{-1}(y))$$

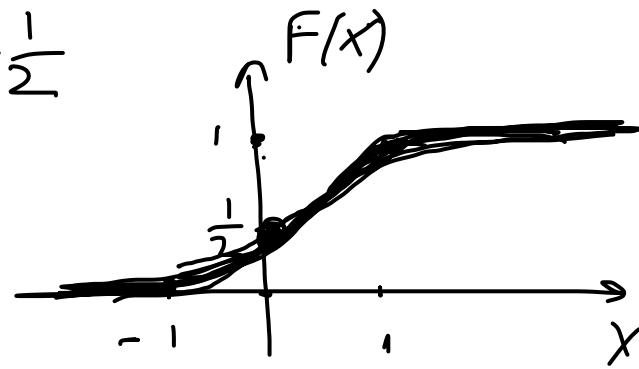
$$= F(F^{-1}(y)) = y$$

$$\Rightarrow Y \sim \text{Unif}(0, 1)$$



$$F(x) = \int_{-1}^x \frac{3}{2} t^2 dt = \frac{x^3}{2} + \frac{1}{2}$$

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x^3}{2} + \frac{1}{2}, & -1 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$



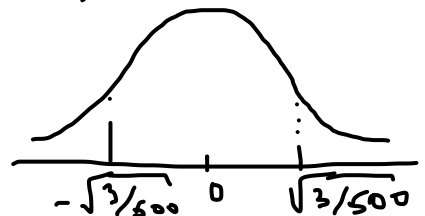
$$S = \frac{X_1 + \dots + X_{100}}{100}$$

$$E(S) = \frac{1}{100} \sum E(X_i) = 0, \text{Var}(S) = \frac{1}{100^2} \cdot \frac{3}{5} \cdot 100$$

$$E(X_i) = \int_{-1}^1 \frac{3}{2} x^3 dx = 0 \quad = \frac{3}{500}$$

$$E(X_i^2) = \int_{-1}^1 \frac{3}{2} x^4 dx = \frac{3}{5} \Rightarrow \text{Var}(X_i) = \frac{3}{5}$$

By CLT,  $S \sim N(0, \frac{3}{500})$



$$\begin{aligned}
\sum_{y=0}^{\infty} \sum_{x=0}^{\infty} k q^2 p^{x+y} &= \sum_{y=0}^{\infty} k q^2 p^y \sum_{x=0}^{\infty} p^x \\
&= \sum k q^2 p^y \cdot \frac{1}{1-p} \\
&= k q \sum_{y=0}^{\infty} p^y = k q \cdot \frac{1}{1-p} = \boxed{k=1}
\end{aligned}$$

Yes

$$P(X, Y) = P(X)P(Y), \quad \begin{aligned} P(X) &= q p^x \\ P(Y) &= q p^y \end{aligned}$$

$$T = X + Y$$

$$\begin{aligned}
P(T=t) &= P(X+Y=t) = \sum_{x=0}^t q^2 p^x p^{t-x} \\
&= \sum_{x=0}^t q^2 p^t = q^2 p^t \sum_{x=0}^t 1 \\
&= (t+1) q^2 p^t, \quad t = 0, 1, 2, \dots
\end{aligned}$$

$X = \#$  of heads in the 1<sup>st</sup> 10 flips

$Y = \#$  of heads in the 2<sup>nd</sup> 10 flips

$$P(X=x | X+Y=12) = \frac{P(X=x \text{ and } Y=12-x)}{P(X+Y=12)}$$

$$= \frac{\binom{10}{x} \left(\frac{1}{2}\right)^{10} \cdot \binom{10}{12-x} \left(\frac{1}{2}\right)^{10}}{\binom{20}{12} \left(\frac{1}{2}\right)^{20}} = \frac{\binom{10}{x} \binom{10}{12-x}}{\binom{20}{12}}, x=2,3,\dots$$

$$\lambda \sim \exp(1)$$

$$P(X=x) = \int_0^{\infty} P_{X,\lambda} \lambda \lambda$$

$$= \int_0^{\infty} P(X=x | \lambda) f(\lambda) d\lambda$$

$$= \int_0^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} e^{-\lambda} d\lambda$$

$$= \frac{1}{x!} \int_0^{\infty} \lambda^x e^{-2\lambda} d\lambda = \left( \begin{array}{l} u = 2\lambda \\ du = 2 d\lambda \end{array} \right)$$

$$= \frac{1}{x!} \frac{1}{2} \int_0^{\infty} \frac{u^x}{2^x} e^{-u} du = \frac{\Gamma(x+1)}{x! 2^{x+1}} \underbrace{\int_0^{\infty} \frac{u^x e^{-u}}{\Gamma(x+1)} du}_{=1}$$

$$= \frac{x!}{x!} \frac{1}{2^{x+1}} = \left(\frac{1}{2}\right)^{x+1}$$

$$\begin{aligned}
\text{Cov}(X, XY) &= E(X^2Y) - E(X)E(XY) \\
&= E(X^2)E(Y) - E(X)^2E(Y) \\
&= E(Y)[E(X^2) - E(X)^2] \\
&= E(Y)\text{Var}(X)
\end{aligned}$$

$$\begin{aligned}
\rho(X+Y, X-Y) &= \frac{\text{Cov}(X+Y, X-Y)}{\sqrt{\text{Var}(X+Y)}\sqrt{\text{Var}(X-Y)}} \\
&= \frac{\text{Cov}(X, X) - \cancel{\text{Cov}(X, Y)} + \cancel{\text{Cov}(X, Y)} - \text{Cov}(Y, Y)}{\sqrt{\text{Var}(X) + \text{Var}(Y)}\sqrt{\text{Var}(X) + \text{Var}(Y)}} \\
&= \frac{\text{Var}(X) - \text{Var}(Y)}{\text{Var}(X) + \text{Var}(Y)}
\end{aligned}$$

$$P(X \geq 11) \leq \frac{E(X)}{11} = \frac{5}{11}$$

$$\underbrace{P(|X - \mu| \geq a)}_{P(X \geq 11)} \leq \frac{\text{Var } X}{a^2}$$

$$P(|X - 5| \geq 6) = P\left(\begin{array}{l} X - 5 \geq 6 \text{ or} \\ X - 5 \leq -6 \end{array}\right)$$

$$= P(X - 5 \geq 6) = P(X \geq 11)$$

$$P(X \geq 11) = P(|X - 5| \geq 6) \leq \frac{\text{Var } X}{36}$$

$$= \frac{42 - 25}{36} = \frac{17}{36}$$

$$x = uv, \quad y = u - uv$$

$$J = \det \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix} = -uv - u + uv = -u$$

$$f_{X,Y}(x,y) = \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1 - 1} y^{\alpha_2 - 1} e^{-\lambda(x+y)}, \quad x, y > 0$$

$$u > 0, \quad 0 < v < 1$$

$$f_{u,v}(u,v) = \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (uv)^{\alpha_1 - 1} (u - uv)^{\alpha_2 - 1} e^{-\lambda u} |1 - u|$$

$$= \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1 + \alpha_2 - 1} v^{\alpha_1 - 1} (1 - v)^{\alpha_2 - 1} e^{-\lambda u}$$

$$f_u(u) = \frac{\lambda^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2)} u^{\alpha_1 + \alpha_2 - 1} e^{-\lambda u}, \quad u > 0$$

$$U \sim \text{Gamma}(\alpha_1 + \alpha_2, \lambda)$$

$$f_v(v) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1 - 1} (1 - v)^{\alpha_2 - 1}, \quad 0 < v < 1$$

$$V \sim \text{Beta}(\alpha_1, \alpha_2)$$

$$= \sum_{i=1}^{\infty} p_i = 1$$

$$\begin{aligned} \pi_X(t) &= \sum_{i=0}^{\infty} p_i t^i \\ &= p_0 + p_1 t + p_2 t^2 + \dots \end{aligned}$$

$$= P(X=0) = p_0$$

$$\begin{aligned} &= \frac{1}{2} \left[ \sum_{i=0}^{\infty} p_i (1 + (-1)^i) \right] = \frac{1}{2} [2p_0 + 2p_2 + 2p_4 + \dots] \\ &= p_0 + p_2 + p_4 + \dots = P(X=0) + P(X=2) + P(X=4) + \dots \\ &= P(X \text{ is even}) \end{aligned}$$

$$\pi(t) = \frac{\alpha}{1+qt} \text{ is not pgf}$$

$$= \alpha \left( \underbrace{1 - q}_{< 0} t + q^2 t^2 + \dots \right)$$

$$m_X(t) = \frac{1}{81} (e^{4t} + 8e^{3t} + 24e^{2t} + 32e^t + 16)$$

$$P(X \leq 2) = p_0 + p_1 + p_2 = \frac{16}{81} + \frac{32}{81} + \frac{24}{81} \\ = \frac{72}{81} = \frac{8}{9}$$

$$m'_X(t) = \frac{1}{81} \cdot 4(e^t + 2)^3(e^t)$$

$$m'_X(0) = \frac{4}{81} \cdot 27 = \frac{108}{81}$$

$$P(X_{(4)} \geq 3\lambda) = 1 - P(X_{(4)} < 3\lambda) \\ = 1 - (1 - e^{-3\lambda^2})^4$$

$$E(X+2Y) = 3 + 2 \cdot 1 = 5$$

$$\text{Var}(X+2Y) = 9 + 4 \cdot 4 = 25$$

$$X+2Y \sim N(5, 25) \Rightarrow \frac{X+2Y-5}{5} \sim N(0,1)$$

$$P(X+2Y \leq 6) = P\left(Z \leq \frac{6-5}{5}\right) = P\left(Z \leq \frac{1}{5}\right) \\ = \Phi\left(\frac{1}{5}\right)$$

$$\frac{X-3}{3} \sim N(0,1), \quad \frac{Y-1}{2} \sim N(0,1)$$

$$Z = \underbrace{\left(\frac{X-3}{3}\right)^2}_{\sim \chi_1^2} + \underbrace{\left(\frac{Y-1}{2}\right)^2}_{\sim \chi_1^2} \sim \chi_{(2)}^2$$

