# TIME EVOLUTION IN QUANTUM MECH.: THE SCHRÖDINGER EQN.

KEY IDEAS: (A.) QUANTUM POSTULATES SO FAR (LEC. 9, P.4-5)

1. STATE OF SYSTEM: VECTOR $|\psi\rangle$ IN A FINITE-DIM. LVS OR $\infty$-DIM. HILBERT SPACE
2. PHYSICAL OBS: HERMITIAN OP.S $\hat{\Omega} = \hat{\Omega}^{+}$
   $\hat{\Omega}|\omega_i\rangle = \omega_i|\omega_i\rangle$
3. MEASUREMENTS PRODUCE RANDOM RESULTS; PROBABILITY TO MEASURE $\omega_i$: $|\langle\omega_i|\psi\rangle|^2$

TOTAL PROBABILITY MUST SUM TO ONE:

$$\sum_i |\langle\omega_i|\psi\rangle|^2 = \sum_i \langle\psi|\omega_i\rangle\langle\omega_i|\psi\rangle = \langle\psi|\psi\rangle = 1$$

$\Longrightarrow$ TIME EVOLUTION CANNOT CHANGE THIS.

$\therefore$ $$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle; \quad \hat{U}^{+}(t,t_0)\,\hat{U}(t,t_0) = \hat{\mathbb{I}}$$

TIME EVOLUTION MUST BE A "ROTATION" INDUCED BY SOME UNITARY OPERATOR $\hat{U}(t,t_0)$, THE "PROPAGATOR"

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(B.) ENERGY-TIME UNCERTAINTY $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ LEC. 9, P.3

$\frac{\hbar}{\Delta t} \sim \Delta E$ FOR A TIME-EVOLVING QUANTUM STATE

- STATE WITH WELL-DEFINED ENERGY $\mathcal{E}_i$: MUST BE COMPLETELY NON-LOCAL IN TIME

IF $\hat{H}|\mathcal{E}_i\rangle = \mathcal{E}_i|\mathcal{E}_i\rangle$, THEN $|\psi_{\mathcal{E}_i}(t)\rangle = \hat{U}(t)|\mathcal{E}_i\rangle = e^{-i\frac{\mathcal{E}_i t}{\hbar}}|\mathcal{E}_i\rangle$

HAMILTONIAN, HERMITIAN OP. THAT MEASURES ENERGY — LEC. 9, P.5
LEC. 11, P.6

$\therefore$ ENERGY E'STATES ACQUIRE PURE PHASE UNDER TIME EVOLUTION

$\Longrightarrow$ SUGGESTS $$\hat{U}(t,t_0) = e^{-i\frac{\hat{H}(t-t_0)}{\hbar}}$$

LEC. 6, P.3: EXP. OF ANTIHERM. OP. IS UNITARY!

INFINITESIMAL VERSION: $|\psi(\Delta t)\rangle = e^{-i\frac{\hat{H}\Delta t}{\hbar}}|\psi(0)\rangle \simeq \left(\hat{\mathbb{I}} - \frac{i\Delta t}{\hbar}\hat{H} + O(\Delta t)^2\right)|\psi(0)\rangle$

$$\Rightarrow \frac{i\hbar}{\Delta t}\left[|\psi(\Delta t)\rangle - |\psi(0)\rangle\right] = \hat{H}|\psi(0)\rangle + O(\Delta t) = \hat{H}|\underbrace{\psi(\Delta t)}\rangle + O(\Delta t)$$

DIFFERS FROM $|\psi(0)\rangle$ BY A CHANGE THAT IS $\propto \Delta t$.

- TAKE $\Delta t \to 0$, WE GET A $5^{TH}$ POSTULATE OF Q.M.: (SEE LEC. $\frac{9}{???}$, P. 4-5 FOR POSTULATES ①–④)

## THE SCHRÖDINGER EQUATION:

(SOMETIMES CALLED THE TIME-DEPT. SCHRÖDINGER EQ. IN CONTRAST TO TIME-INDEPT. ONE $\hat{H}|E\rangle = E|E\rangle$)

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

## PROPERTIES:

① TIME-INDEPT. HAMILTONIAN $\hat{H}(t) = \hat{H}$ (CONSTANT)

- FORMAL SOLUTION: TIME-EVOLUTION OPERATOR $\hat{U}(t, t_0) = \hat{U}(t-t_0) = e^{-i\frac{\hat{H}(t-t_0)}{\hbar}}$

FINAL TIME ($t$), INITIAL TIME ($t_0$)

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle \; ; \; \hat{U}(t_0,t_0) = \hat{\mathbb{I}} \quad \text{(IDENTITY OPERATOR)}$$

a) SATISFIES S.E.: $i\hbar\frac{d}{dt}\hat{U}(t,t_0) = i\hbar\frac{d}{dt}e^{-i\frac{\hat{H}(t-t_0)}{\hbar}} = \hat{H}\,\hat{U}(t,t_0)$

NO ORDERING ISSUES BECAUSE $[\hat{H}, \hat{U}(t,t_0)] = 0$ FOR ALL $(t-t_0)$

b) TIME TRANSLATION INVARIANCE:

$$\hat{U}(t_2,t_1)\,\hat{U}(t_1,t_0) = \hat{U}(t_2-t_1)\,\hat{U}(t_1-t_0) = \hat{U}(t_2-t_0)$$

c) ENERGY EIGENSTATE EXPANSION: $\hat{H}|E_i\rangle = E_i|E_i\rangle$ SOLVE SPECTRUM OF TIME-INDEPT. S.E.

$$\hat{U}(t-t_0) = e^{-i\frac{\hat{H}(t-t_0)}{\hbar}}\underbrace{\sum_i |E_i\rangle\langle E_i|}_{\text{ENERGY EIGENSTATE RESOLUTION OF IDENTITY}} = \sum_i e^{-i\frac{E_i t}{\hbar}}|E_i\rangle\langle E_i|$$

TIME EVOLUTION BECOMES PURE PHASE FACTORS IN ENERGY EIGEN BASIS

Energy Eigenstate Expansion Gives **General Solution*** To S.E. with a Time-Indept. Hamiltonian:

$$|\psi(t)\rangle = \sum_i \underbrace{|E_i\rangle}_{i^{\text{th}}\text{ Energy Eigenket}} \overbrace{\langle E_i|\psi(0)\rangle}^{\text{Proj. of Initial State}} \underbrace{e^{-i\frac{E_i t}{\hbar}}}_{\text{Time Evolution of } |E_i\rangle\text{: Pure Phase Factor}} \qquad (1)$$

(*) While Eq. (1) Always works (For a Time-Indept. $\hat{H}$), it is Not Always the Easiest Way to Get $|\psi(t)\rangle$. For some Initial Conditions and/or Hamiltonians, Might Be Easier to "Directly Compute" $\hat{U}(t)|\psi(0)\rangle$. $\Rightarrow$ See Spin-1 Example, Below.

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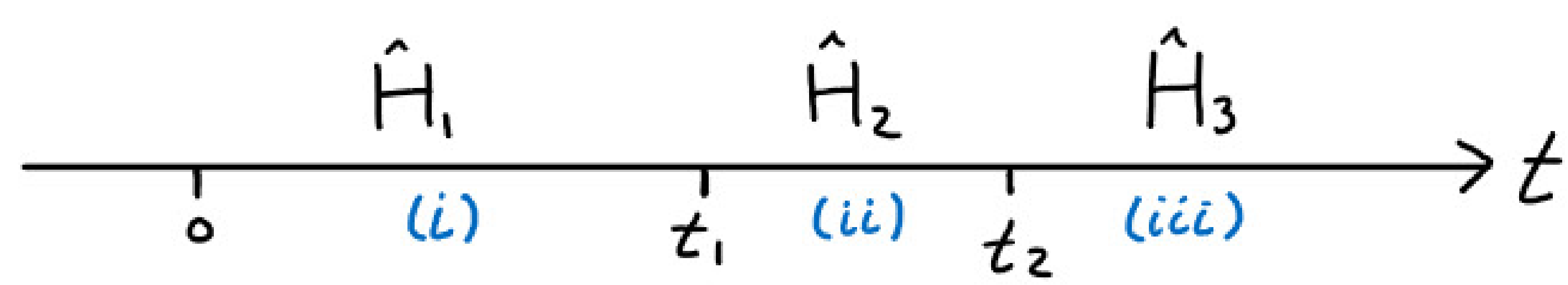
## (2.) Time-Dependent Hamiltonian $\hat{H} = \hat{H}(t)$

- Encodes, e.g., a Macroscopic (effectively Classical) Time-Varying Electromagnetic Field

### (A.) Piecewise Constant in Time

e.g., $\hat{H}(t) = \begin{cases} \hat{H}_1, & 0 \le t \le t_1, \\ \hat{H}_2, & t_1 \le t \le t_2, \\ \hat{H}_3, & t \ge t_2 \end{cases}$




$\Rightarrow$ (i) $0 \le t \le t_1$: $|\psi(t)\rangle = e^{-i\frac{\hat{H}_1 t}{\hbar}}|\psi(0)\rangle$

(ii) $t_1 \le t \le t_2$: $|\psi(t)\rangle = e^{-i\frac{\hat{H}_2(t-t_1)}{\hbar}} \cdot e^{-i\frac{\hat{H}_1 t_1}{\hbar}}|\psi(0)\rangle \neq e^{\frac{-i\hat{H}_2(t-t_1) - i\hat{H}_1 t_1}{\hbar}}|\psi(0)\rangle$

Unless: $[\hat{H}_1, \hat{H}_2] = 0$ Lec. 6, p. 1

(iii) $t \ge t_2$: $|\psi(t)\rangle = e^{-i\hat{H}_3\frac{(t-t_2)}{\hbar}} \cdot e^{-i\hat{H}_2\frac{(t_2-t_1)}{\hbar}} \cdot e^{-i\hat{H}_1\frac{t_1}{\hbar}}|\psi(0)\rangle$

For Piecewise Constant-in-Time $\hat{H}(t)$, Time Evolution is Product of "Chunks" Associated to Each Constant-$\hat{H}$ Time Interval.

## (B.) General Time-Dept. $\hat{H}(t)$

- Evolution in infinitesimal time step $\Delta t$: $|\psi(t+\Delta t)\rangle \simeq (\hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}(t)\Delta t + \mathcal{O}(\Delta t)^2)|\psi(t)\rangle \simeq e^{-i\frac{\hat{H}(t)\Delta t}{\hbar}}|\psi(t)\rangle$

- Total Evolution: Cascade ("Time-Ordered Product") of infinitesimal slices

$$|\psi(t)\rangle = e^{-i\frac{\Delta t\,\hat{H}(t-\Delta t)}{\hbar}} \times e^{-i\frac{\Delta t\,\hat{H}(t-2\Delta t)}{\hbar}} \times \dots \times e^{-i\frac{\Delta t\,\hat{H}(2\Delta t)}{\hbar}} \times e^{-i\frac{\Delta t\,\hat{H}(\Delta t)}{\hbar}} |\psi(0)\rangle$$

$$\neq e^{-\frac{i}{\hbar}\int_0^t dt'\,\hat{H}(t')}|\psi(0)\rangle \quad \text{unless } [\hat{H}(t_1),\hat{H}(t_2)] = 0 \text{ for generic } 0 \le t_{1,2} \le t.$$

How to Solve?

① Directly, e.g. S.E. as a Partial Diff. Eq.:

$$i\hbar\,\partial_t \psi(t,x) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi(t,x)$$

② Path Integral (... later ...)

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# Spin-1 Q.M.: Dynamics

$$\hat{H}(t) = -\gamma\,\hat{\vec{S}}\cdot\vec{B}(t)$$

External magnetic field

## (A.) Static $\vec{B}$: $\vec{B} = B\,\vec{n}_z$ WLoG

$$\hat{H}|m_z\rangle = E_{m_z}|m_z\rangle \; ; \; E_{m_z} = -\gamma B\hbar\, m_z \equiv -\hbar\omega_L\, m_z$$

$$\hat{S}_z|m_z\rangle = \hbar m_z|m_z\rangle$$

$\omega_L \equiv \gamma B$

"Larmor" Freq.

- Note: sign of $\omega_L$ determined by sign of $\gamma$.

⇒ $\gamma > 0$ for $q > 0$;

⇒ $\gamma < 0$ for $q < 0$ [e.g., spin-$\frac{1}{2}$ electron]

### ① Eigenstate Expansion

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}\sum_{m_z=-1}^{1}|m_z\rangle\langle m_z|\psi(0)\rangle$$

$$= \sum_{m_z=-1}^{1}|m_z\rangle\langle m_z|\psi(0)\rangle\, e^{-i\frac{E_{m_z}t}{\hbar}} = \sum_{m_z=-1}^{1}|m_z\rangle\langle m_z|\psi(0)\rangle\, e^{i m_z\omega_L t}$$

Observables:

ⓐ Prob. to measure $S_z = m_z\hbar$: $|\langle m_z|\psi(t)\rangle|^2 = |\langle m_z|\psi(0)\rangle|^2$, Indept. of time

⇑ Measurement of $\hat{\vec{S}}$ parallel to $\vec{B}$.

(b) SUPPOSE WE MEASURE SPIN ALONG A DIFFERENT AXIS. LET $\hat{S}_a |m_a\rangle = \hbar m_a |m_a\rangle$, $a \in \{x, y, z\}$, $m_a \in \{-1, 0, 1\}$

PROBABILITY THAT WE MEASURE $S_a = m_a \hbar$: $|\langle m_a | \psi(t)\rangle|^2$

**FOR $a \neq z$: MEASUREMENT $\perp$ TO THE STATIC $\vec{B}$**

- $\langle m_a | \psi(t)\rangle = \sum_{m_z=-1}^{1} \langle m_a | m_z\rangle\langle m_z | \psi_0\rangle e^{-i\frac{E_{m_z} t}{\hbar}} = \sum_{m_z=-1}^{1} \langle m_a | m_z\rangle\langle m_z|\psi_0\rangle e^{i m_z \omega_L t}$

ex:) $|m_x = +1\rangle$: NEED TO FIND E'STATES OF $\hat{S}_x$. MUST CHOOSE A BASIS AND STICK WITH IT!

- TO EVALUATE $\langle m_x = +1 | \psi(t)\rangle$, NEED ALL 3 STATES $\{|-1\rangle_z, |0\rangle_z, |1\rangle_z\}$, WHERE $|m_z = 1\rangle \equiv |1\rangle_z$

$\therefore$ WORK IN $\{|m_z\rangle_z\}$ (ENERGY EIGEN-) BASIS.

- THEN:

$$\hat{S}_x \Rightarrow \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad ; \quad \text{SOLVE } (\hat{S}_x - \hbar \hat{\mathbb{I}})|1\rangle_x = 0 \Rightarrow |1\rangle_x \Rightarrow \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

OR: $|1\rangle_x = \frac{1}{2}(|1\rangle_z + \sqrt{2}|0\rangle_z + |-1\rangle_z)$

$$\therefore {}_x\langle 1 | \psi(t)\rangle = \frac{1}{2}\left[{}_z\langle 1| + \sqrt{2}\,{}_z\langle 0| + {}_z\langle -1|\right]\left[|1\rangle_z\,{}_z\langle 1|\psi_0\rangle e^{i\omega_L t} + |0\rangle_z\,{}_z\langle 0|\psi_0\rangle + |-1\rangle_z\,{}_z\langle -1|\psi_0\rangle e^{-i\omega_L t}\right]$$

$$= \frac{1}{2}\,{}_z\langle 1|\psi_0\rangle e^{i\omega_L t} + \frac{1}{\sqrt{2}}\,{}_z\langle 0|\psi_0\rangle + \frac{1}{2}\,{}_z\langle -1|\psi_0\rangle e^{-i\omega_L t}$$

$$\equiv \alpha e^{i\omega_L t} + \beta + \gamma e^{-i\omega_L t}$$

$\therefore$ PROBABILITY TO FIND $S_x = \hbar$: $|{}_x\langle 1|\psi(t)\rangle|^2$ OSCILLATES IN TIME FOR GENERIC [NON-ENERGY-EIGENSTATE] INITIAL CONDITION $|\psi_0\rangle$.

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## (2) SPIN DYNAMICS AS A ROTATION

$\hat{H} = -\gamma \hat{\vec{S}} \cdot \vec{B}$ ; LET $\vec{B} = \underset{\text{MODULUS}}{B}\ \underset{\text{DIRECTION}}{\vec{n}_B}$, $\vec{n}_B \cdot \vec{n}_B = 1$

$$\Rightarrow \hat{U}(t) = e^{-i\frac{\hat{H}t}{\hbar}} = e^{-i\frac{(-\gamma B \vec{n}_B t)\cdot\hat{\vec{S}}}{\hbar}} \equiv e^{-i\frac{\vec{\Theta}(t)\cdot\hat{\vec{S}}}{\hbar}}$$

JUST A ROTATION MATRIX!

HERE $\vec{\Theta} = -\omega_L t \cdot \vec{n}_B$ ; $\omega_L = \gamma B$ (LARMOR FREQ., POSITIVE OR NEGATIVE, DEPENDING ON $\gamma$)

e.g., $\vec{B} = B\vec{n}_z$:

$$\hat{U}(t) = e^{-i\frac{(-\omega_L t)\hat{S}_z}{\hbar}} \Rightarrow \begin{bmatrix} e^{i\omega_L t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\omega_L t} \end{bmatrix}$$

Z-AXIS ROTATION BY ANGLE $\Theta = -\omega_L t$

IN $\{|1\rangle_z, |0\rangle_z, |-1\rangle_z\}$ BASIS

- Rotation Around Another Axis

$\vec{B} = B\vec{n}_y$ (e.g.) $\Rightarrow$ $\hat{U}(t) = e^{\frac{-i(-\omega_L t)\hat{S}_y}{\hbar}} = \hat{R}_y(-\omega_L t)$ Rotation by $\theta = -\omega_L t$ around y-axis

- Basis Note: If we want to compute $|\psi(t)\rangle \equiv \hat{R}_y(-\omega_L t)|m_z\rangle_z$ (e.g.)

Must choose a basis and stick with it.

If $|m_z=1\rangle_z \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then $\hat{R}_y(\theta) = e^{-i\frac{\hat{S}_y}{\hbar}\theta}$, $\hat{S}_y \Rightarrow \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$

$\{|1\rangle_z, |0\rangle_z, |-1\rangle_z\}$ basis

See Homework 6

In particular, $\hat{R}_y(\theta) \not\Rightarrow \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ Wrong basis! (Lec. 10)

In this basis $(\{|\vec{n}_x\rangle, |\vec{n}_y\rangle, |\vec{n}_z\rangle\})$, $|1\rangle_z \Rightarrow \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$

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## (B.) Time-Dept. $\vec{B} = \vec{B}(t)$:

- For some initial state $|\psi_0\rangle$, solve $i\hbar \frac{d}{dt}|\psi(t)\rangle = -\gamma\vec{B}(t)\cdot\hat{\vec{S}}\,|\psi(t)\rangle$

- Alternative: Operator Equation of Motion

Consider the time evolution of the spin vector operator expectation value:

$$\langle\hat{\vec{S}}\rangle(t) \equiv \langle\psi(t)|\hat{\vec{S}}|\psi(t)\rangle, \quad \hat{\vec{S}} = \hat{S}_x\vec{n}_x + \hat{S}_y\vec{n}_y + \hat{S}_z\vec{n}_z$$

In components:
$$\frac{d}{dt}\langle\psi(t)|\hat{S}_a|\psi(t)\rangle = \langle\dot{\psi}(t)|\hat{S}_a|\psi(t)\rangle + \langle\psi(t)|\hat{S}_a|\dot{\psi}(t)\rangle$$

From the S.E.:
$$|\dot{\psi}\rangle = -\frac{i}{\hbar}\hat{H}(t)|\psi\rangle = \left(\cancel{-}\frac{i}{\hbar}\right)\left(\cancel{-}\gamma B_b(t)\hat{S}_b\right)|\psi\rangle$$
Einstein sum on $b \in \{x,y,z\}$
$$\langle\dot{\psi}| = \langle\psi|\left(\frac{-i}{\hbar}\right)\left(\gamma B_b(t)\hat{S}_b\right)$$

$$\therefore \frac{d\langle\hat{S}_a\rangle(t)}{dt} = \left(\frac{-i}{\hbar}\right)\gamma B_b(t)\langle\psi(t)|\hat{S}_b\hat{S}_a - \hat{S}_a\hat{S}_b|\psi(t)\rangle$$
$$= \left(\frac{\cancel{-i}}{\cancel{\hbar}}\right)\gamma B_b(t)\,\cancel{i\hbar}\,\epsilon_{bac}\langle\psi(t)|\hat{S}_c|\psi(t)\rangle$$

$$\frac{d}{dt}\langle \hat S_a\rangle = \frac{-i}{\hbar}\gamma B_b(t)\langle[\hat S_b,\hat S_a]\rangle = \frac{\cancel{i\hbar}}{\cancel{\hbar}}\gamma B_b(t)\,\cancel{i}\,\epsilon_{bac}\,\langle \hat S_c\rangle = \gamma\,\epsilon_{abc}\,\langle \hat S_b\rangle(t)\,B_c(t)$$

QUANTUM MAG. MOMENT OPERATOR: $\hat{\vec\mu} = \gamma\,\hat{\vec S}$

$$\boxed{\therefore\quad \frac{d}{dt}\langle\hat{\vec\mu}\rangle(t) = \gamma\,\langle\hat{\vec\mu}\rangle(t)\times\vec B(t)}$$

E.O.M. FOR EXPECTATION VALUE OF MAGNETIC MOMENT OPERATOR.

STATIC $\vec B$: LET $\langle\hat{\vec\mu}\rangle(t)\equiv\vec\mu(t)$ ; $\dot{\vec\mu} = \gamma\vec\mu\times\vec B$

$\Rightarrow \vec\mu(t+\Delta t) = \vec\mu(t) + \vec{\Delta\theta}\times\vec\mu(t)$ (1) , AN INFINITESIMAL ROTATION !

$\vec{\Delta\theta} = -\gamma\vec B t = -\omega_L t\,\vec n_B$ , $\omega_L = \gamma B$ LARMOR FREQ.

- TO BE PRECISE: EQ. (1) IS AN INFINITESIMAL ROTATION OF A VECTOR $\vec\mu$, EXPRESSED IN COMPONENTS IN THE USUAL $\{\vec n_x, \vec n_y, \vec n_z\}$ BASIS
  — LEC. 10, P. 3-4.

I.e., $\hat{\vec\mu} = \gamma\hat{\vec S} = \gamma\left[\hat S_x\vec n_x + \hat S_y\vec n_y + \hat S_z\vec n_z\right]$ ; WE WILL ALWAYS USE THE $\{\vec n_x, \vec n_y, \vec n_z\}$ BASIS FOR **VECTOR OPERATORS.**

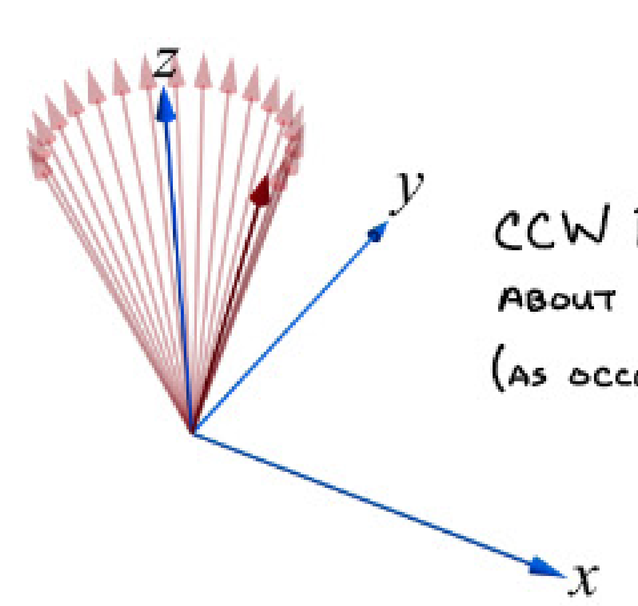
- BY COMPARISON, WE USUALLY (THOUGH NOT ALWAYS) WORK IN THE $\{|M_z\rangle_z\}$ BASIS FOR **STATES** (VS. **VECTOR OPERATORS**)

## SOLUTION: LARMOR PRECESSION OF $\vec\mu$ AROUND $\vec B$

LET $\vec B = B\,\vec n_z$

- $\mu_z(t) = \mu_z(0)$, CONSTANT.
- $\dot\mu_x = \omega_L\mu_y$ , $\dot\mu_y = -\omega_L\mu_x$

$$\Rightarrow \mu_x(t) = \mu_x(0)\cos(\omega_L t) + \mu_y(0)\sin(\omega_L t)$$
$$\mu_y(t) = -\mu_x(0)\sin(\omega_L t) + \mu_y(0)\cos(\omega_L t).$$




CCW PRECESSION ABOUT $\vec n_z$ (AS OCCURS FOR $\omega_L<0$)

CW (CCW) PRECESSION FOR $\omega_L>0$ ($\omega_L<0$)

✓ CONSISTENT WITH $\hat U(t)$ ON P. 5 (BOTTOM)

① CLASSICAL E.O.M. FOR IDEAL CURRENT LOOP WITH MAG. MOMENT $\vec{\mu} = \left(\frac{IA}{c}\right)\hat{\mu}$ IN EXT. FIELD $\vec{B}(t)$:

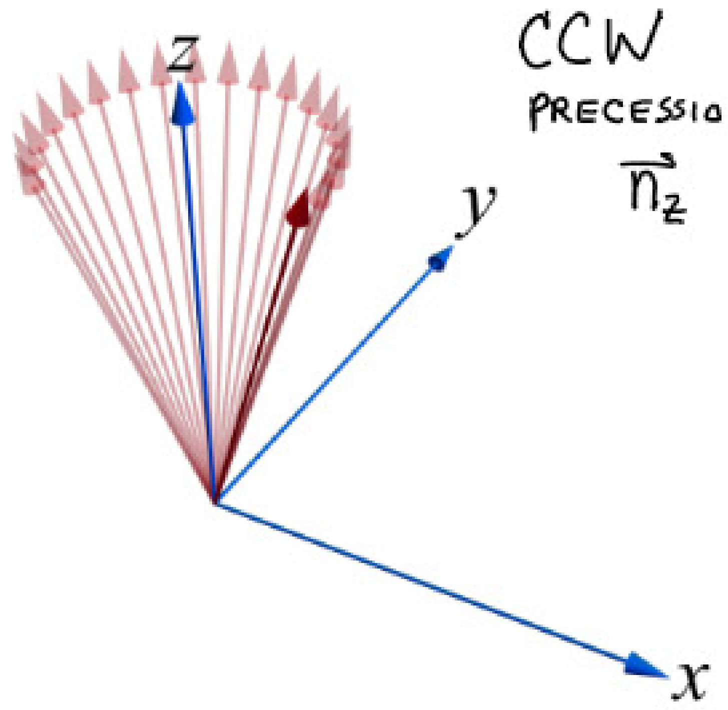
$$\frac{d\vec{\mu}}{dt} = \gamma\, \vec{\mu} \times \vec{B}(t)$$ — SEE p.9 FOR A DERIVATION.

② E.O.M. FOR EXPECTATION VALUE OF QUANTUM SPIN-1 MAG. MOMENT OPERATOR $\hat{\mu} = \gamma \hat{S}$:

$$\frac{d\langle\hat{\mu}\rangle}{dt} = \gamma \langle\hat{\mu}\rangle \times \vec{B}(t)\,, \quad \langle\vec{\mu}(t)\rangle \equiv \langle\psi(t)|\hat{\mu}|\psi(t)\rangle\,; \quad i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

③ SOLUTION FOR STATIC $\vec{B} = B\hat{z}$: LARMOR PRECESSION

$$\mu_z(t) = \mu_z(0)$$
$$\dot{\mu}_x(t) = \omega_L\, \mu_y(t)$$
$$\dot{\mu}_y(t) = -\omega_L\, \mu_x(t)$$




$$\mu_x(t) = \mu_x(0)\cos(\omega_L t)\,; \quad \omega_L = \gamma B$$
$$\mu_y(t) = -\mu_x(0)\sin(\omega_L t)$$

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? SO ARE CLASSICAL, QUANTUM RESPONSES IDENTICAL? NO

① CLASSICAL: $\gamma_{cl} = \left(\frac{q}{2mc}\right)$ ; $\gamma_Q = g \times \left(\frac{q}{2mc}\right)$, $g$: g-FACTOR

② $\frac{d}{dt}\langle\hat{\mu}\rangle = \gamma_Q \langle\hat{\mu}\rangle \times \vec{B}(t)$ IS A STATEMENT ABOUT THE TIME-EVOLUTION OF EXPECTATION VALUES

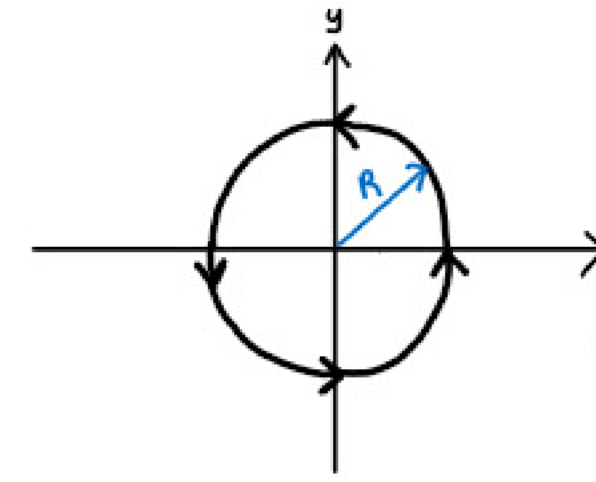
⇒ TO OBSERVE IN QUANTUM EXPERIMENTS, MUST

(a) INITIALIZE $|\psi(0)\rangle$ SAME WAY IN $N \gg 1$ EXPERIMENTS

(b) MEASURE NON-COMMUTING $\{\hat{\mu}_x, \hat{\mu}_y, \hat{\mu}_z\}$, EACH IN MANY SEP. EXPTS.

- CANNOT SIMULTANEOUSLY MEASURE NON-COMMUTING OBS.
  - UNCERTAINTY
  - "COLLAPSE OF THE STATE VECTOR" POST MEASUREMENT
- RESULT FOR ANY OBS. $\hat{\mu}_a$ IS RANDOM IN ANY GIVEN EXPT. MUST AVERAGE RESULTS OVER MANY EXPERIMENTS TO DETERMINE $\langle\hat{\mu}_a(t)\rangle$ ⇐ FOR EACH FIXED MEASUREMENT TIME, NEED MANY EXPERIMENTS!!

# Classical Mag. Moment Evolution in an External $\vec{B}$-Field

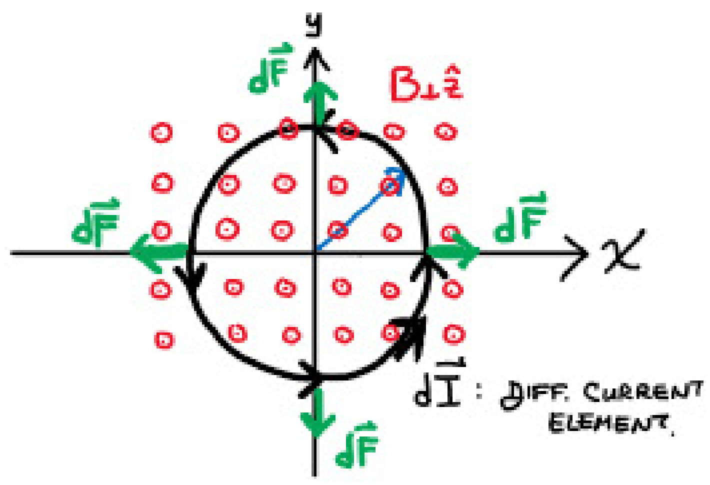



Total Current $I$ Flowing Through Loop

**Torque on a Current Loop:** Take loop to lie in the xy plane.

- Break ext. mag. field $\vec{B}$ into in-plane, out-of-plane components: $\vec{B} = B_\perp \hat{z} + \underbrace{\vec{B}_{\parallel}}_{\text{in x-y plane}}$

(1) Torque due to $B_\perp \hat{z}$




Charge is distributed uniformly through loop (no charge accumulation, $\vec{E}$-fields)

- $dq = \underset{\substack{\Uparrow \\ \text{charge density}}}{\rho_q} \underbrace{R\,d\phi}_{\text{diff. arc length}} \implies q = \int_0^{2\pi} \rho_q R\,d\phi = 2\pi R \rho_q \quad \therefore \rho_q = \frac{q}{2\pi R}$

- $d\vec{I} = dq \left(\frac{v}{2\pi R}\right)\hat{\phi}$, where $\hat{\phi} = \frac{d\hat{r}}{d\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$ is the unit vector pointing in the direction of increasing $\hat{\phi}$ ($\hat{r}, \hat{\phi}$: polar unit vectors – Vec. Calculus)

  Lin. velocity of circ. charges ($v$)

- Lorentz Force (Gaussian):

$$d\vec{F} = dq\,\frac{\vec{v}}{c}\times\vec{B}_\perp = \left(\frac{2\pi R}{\cancel{v}}\right)\frac{\cancel{v}}{c}\,d\vec{I}\times\vec{B}_\perp = \left(\frac{2\pi R}{c}\right)dI\;\hat{\phi}\times\hat{z} = \left(\frac{2\pi R}{c}\right)dI\;\hat{r}(\phi)$$

- Torque: $d\vec{N} = R\hat{r}\times d\vec{F} = 0$

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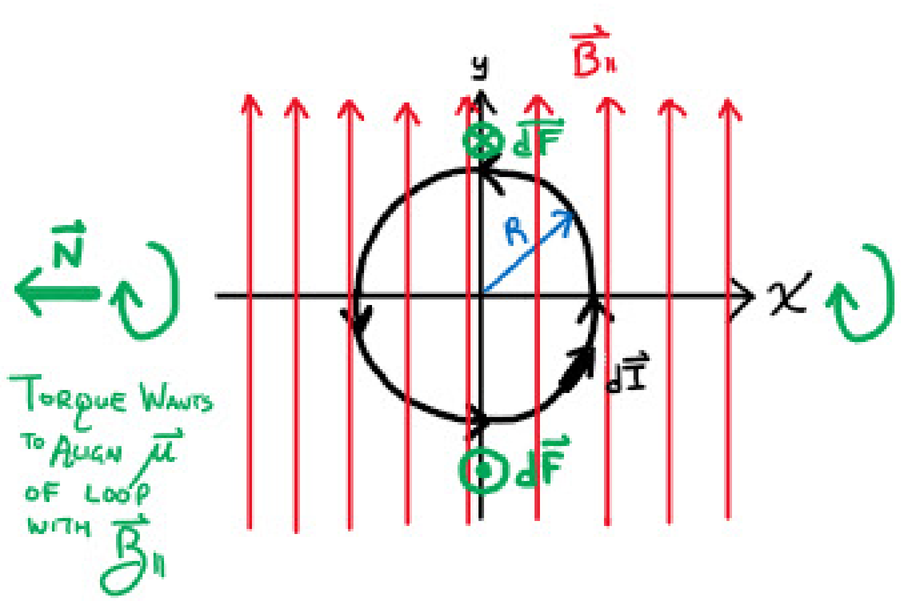
(2) Torque due to $\vec{B}_{\parallel}$. WLoG take $\vec{B}_{\parallel} = B_{\parallel}\hat{y}$

- $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$
- $dI = \frac{I}{2\pi}\,d\phi \iff I = \int dI = \int_0^{2\pi}\frac{I}{2\pi}\,d\phi$




- Lorentz Force:

$$d\vec{F} = \left(\frac{2\pi R}{c}\right)d\vec{I}\times\vec{B}_{\parallel} = \left(\frac{2\pi R}{c}\right)dI\;\hat{\phi}\times B_{\parallel}\hat{y} = -\hat{z}\left(\frac{\cancel{2\pi} R}{c}\right)\frac{I}{\cancel{2\pi}}\,d\phi\,B_{\parallel}\sin\phi$$

- Torque:

$$d\vec{N} = R\hat{r}\times d\vec{F} = -\left(\frac{R^2 I}{c}\right)B_{\parallel}\sin\phi\,d\phi\,(\hat{r}\times\hat{z}) = \left(\frac{R^2 I}{c}\right)B_{\parallel}\sin\phi\,\hat{\phi}\,d\phi$$

$$\therefore \vec{N} = \int d\vec{N} = \left(\frac{R^2 I}{c}\right)B_{\parallel}\int_0^{2\pi}\sin\phi\left[-\sin\phi\,\hat{x} + \cancel{\cos\phi\,\hat{y}}^{\,0}\right]d\phi = -\left(\frac{\pi R^2 I}{c}\right)B_{\parallel}\,\hat{x}$$

Classical Mag. Moment (Lec. 11, p. 5)

$$\vec{\mu} = \frac{IA}{c}\,\hat{z} \implies \vec{N} = -\mu B_{\parallel}\,\hat{x}\;;\quad \text{But } \vec{B} = B_{\parallel}\hat{y} + B_\perp\hat{z}$$

$$\implies \vec{\mu}\times\vec{B} = -\mu B_{\parallel}\,\hat{x}$$

$$\therefore \vec{N} = \vec{\mu}\times\vec{B}$$

# Classical E.O.M. for Angular Momentum:

$\vec{L} = \vec{r}\times\vec{p}\;;\quad \frac{d\vec{L}}{dt} = \cancel{\dot{\vec{r}}\times\vec{p}}^{\,0} + \vec{r}\times\dot{\vec{p}} = \vec{r}\times\vec{F} = \vec{N}$

$$\implies \frac{d\vec{L}}{dt} = \vec{N} = \vec{\mu}\times\vec{B}\;;\quad \text{But } \vec{\mu} = \gamma\vec{L} \quad \text{Lec. 11, p. 5}$$

$$\therefore \frac{d\vec{\mu}}{dt} = \gamma\,\vec{\mu}\times\vec{B}$$

- Same form as Quantum Op. Expectation Eqn.:

$$\frac{d}{dt}\langle\hat{\vec{\mu}}\rangle = \gamma\,\langle\hat{\vec{\mu}}\rangle\times\vec{B}(t)$$