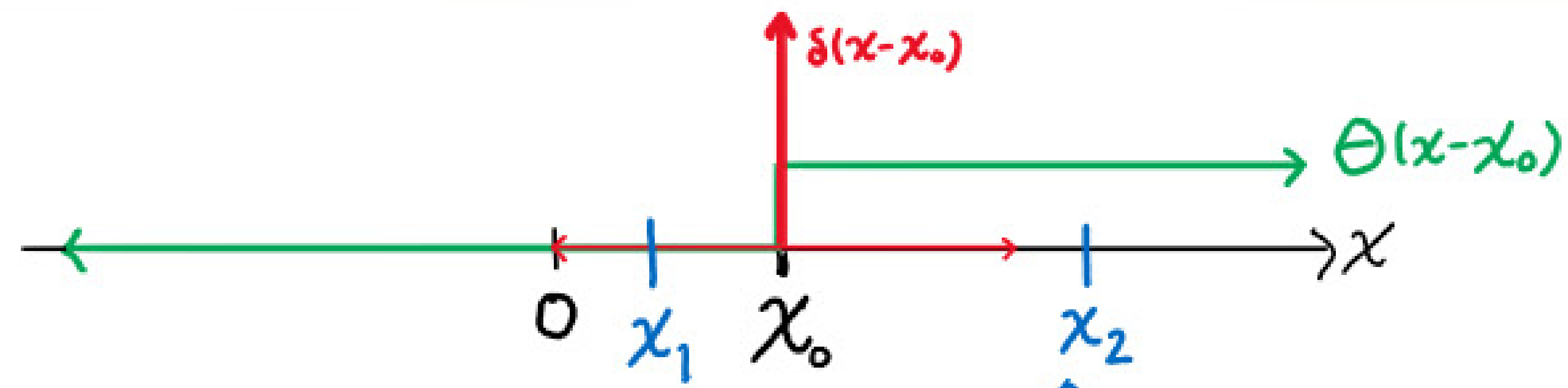
Dirac Delta Function $\delta(x)$

① $\delta(x) = \frac{d}{dx}\Theta(x)$; $\Theta(x) = \begin{cases} 1, & x \geq 1 \\ 0, & x < 0 \end{cases}$

Heaviside unit step function



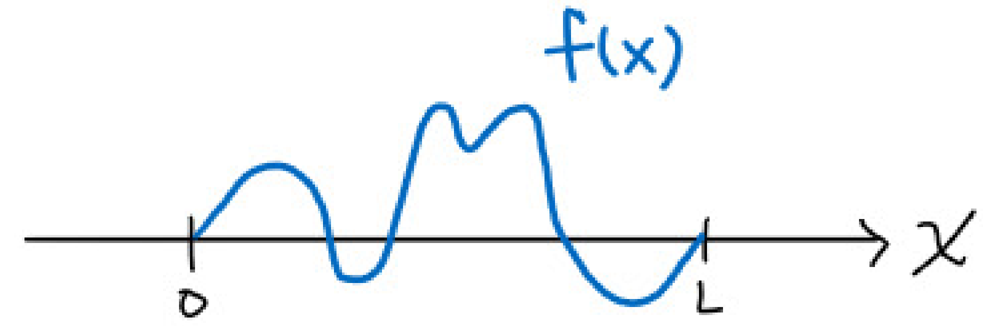
② $$\int_{x_1}^{x_2} dx\ \delta(x-x_0)\, f(x) = \begin{cases} f(x_0), & \text{if } x_1 < x_0 < x_2 \\ 0, & \text{else} \end{cases}$$

● **Warning:**

(a) $\delta(x-x_1)\,\delta(x-x_2) = 0$ for $x_1 \neq x_2$

(b) $[\delta(x-x_0)]^2$ is ~~not~~ <u>NOT</u> defined (e.g., try computing $\int dx\,[\delta_\Delta(x-x_0)]^2 f(x)$ for some model $\delta_\Delta(x)$, then take $\Delta \to 0$)

Back to "Function Spaces" $\mathbb{V}^\infty(\mathbb{C})$



- State : $|f\rangle$
- Resolution of the Identity $\hat{\mathbb{I}} = \int_0^L dx\ |x\rangle\langle x|$
- Basis overlap: $\langle x|x'\rangle = \delta(x-x')$
- $\langle x|f\rangle = \langle x|\hat{\mathbb{I}}|f\rangle = \int_0^L dx'\ \langle x|x'\rangle\langle x'|f\rangle = \langle x|f\rangle \equiv f(x)$

● **Inner Product:** Square-normalizable functions ⊛

⊛ Excludes basis vectors!

$$\langle g|f\rangle = \int_0^L dx\ g^*_{(x)}\, f_{(x)}$$

Complex conjugate (as usual for the "bra" $\langle g|$) ⇐ Diff. from def'n on p. 5 of Lecture 1 (which assumed real functions)

Norm:

$$|f|^2 = \int_0^L dx\ f^*_{(x)}\, f_{(x)} \geq 0$$

● Space of functions normalizable to one: $\mathbb{L}^2$, an infinite-dimensional (Hilbert) vector space

● In Quantum Physics, often extend this to "Physical Hilbert Space": square-normalizable to 1 or to a Delta function ⇒ includes basis vec.s!

Linear Operators on a Hilbert Space:

① Position Operator $\hat{X}$:

Position-base kets are eigenstates $\hat{X}|x_0\rangle = x_0|x_0\rangle$

$$\therefore \langle x|\hat{X}|x'\rangle = x\,\delta(x-x') = x'\,\delta(x-x')$$

Claim: $\hat{X} = \hat{X}^\dagger$, Hermitian

Proof: $\langle f|\hat{X}|g\rangle = \langle f|Xg\rangle = \int_0^L dx \int_0^L dx' \langle f|x\rangle\langle x|\hat{X}|x'\rangle\langle x'|g\rangle = \int_0^L dx\, f^*_{(x)}\, x\, g(x)$

$$\langle g|\hat{X}^\dagger|f\rangle = \langle f|\hat{X}|g\rangle^* = \int_0^L dx\, g^*_{(x)}\, x\, f_{(x)} = \langle g|\hat{X}|f\rangle \checkmark$$

Since $\hat{X}^\dagger = \hat{X}$, an arbitrary operator function $\hat{V}(\hat{X}) \equiv \sum_{n=0}^{\infty} V_n \hat{X}^n$ is Hermitian if $V^*_{(x)} = V(x)$ (Real $\Longleftrightarrow V_n = V_n^*$)

$$\boxed{\therefore \left[\hat{V}(\hat{X})\right]^\dagger = \hat{V}(\hat{X}) \quad \text{for} \quad V(x) \in \mathbb{R}}$$

---

② Hermitian Derivative Operator $\hat{K}$

Define via matrix elements:

$$\langle x|\hat{K}|f\rangle \equiv -i\,\frac{df}{dx}$$

!

Consider $\langle f|\hat{K}|g\rangle = \int_0^L dx\,\langle f|x\rangle\langle x|\hat{K}|g\rangle = \int_0^L dx\, f^*_{(x)}\left[-i\frac{dg}{dx}\right] = \langle f|Kg\rangle$

$$\langle g|\hat{K}^\dagger|f\rangle = \langle Kg|f\rangle = \langle f|\hat{K}|g\rangle^* = \int_0^L dx\, f(x)\left[i\frac{dg^*}{dx}\right]$$

Integrate by parts:
$u = f(x)$ $\quad dv = i(g^*)'dx$
$du = f'(x)\,dx$ $\quad v = i g^*$

$$= f(x)\,i\,g^*_{(x)}\Big|_{x=0}^{x=L} - \int_0^L i\,g^*_{(x)}\,\frac{df}{dx}\,dx$$

$$= \int_0^L dx\, g^*_{(x)}\left[-i\frac{df}{dx}\right] + \underbrace{i f(x)\, g^*_{(x)}\Big|_{x=0}^{x=L}}$$

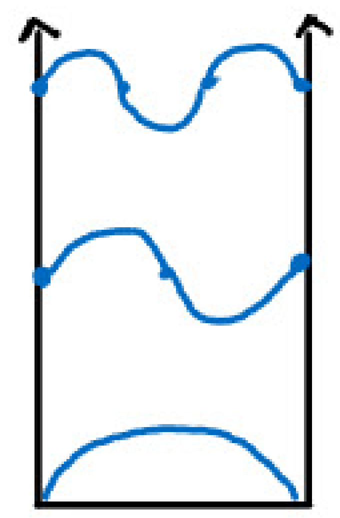
$$= \langle g|\hat{K}|f\rangle$$ If we can ignore boundary term

$\therefore \langle g|\hat{K}^{\dagger}|f\rangle = \langle g|\hat{K}|f\rangle + i\left[g^*(L)f(L) - g^*(0)f(0)\right]$

$\Rightarrow \hat{K} = \hat{K}^{\dagger}$ IF WE RESTRICT TO CERTAIN BOUNDARY CONDITIONS

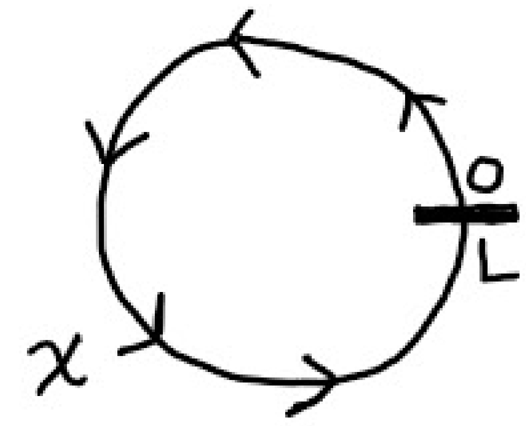
① DIRICHLET: $f(0) = f(L) = 0$ FOR ALL FUNCTIONS ON $0 \le x \le L$

- OUR CLASSICAL STRING EXAMPLE (LEC 1)
- IN QUANTUM, AN INFINITE SQUARE WELL



② PERIODIC: $f(0) = f(L)$ (BUT NOT NEC. ZERO) FOR ALL FUNCTIONS

- CORRESPONDS TO A RING OF CIRCUMFERENCE L

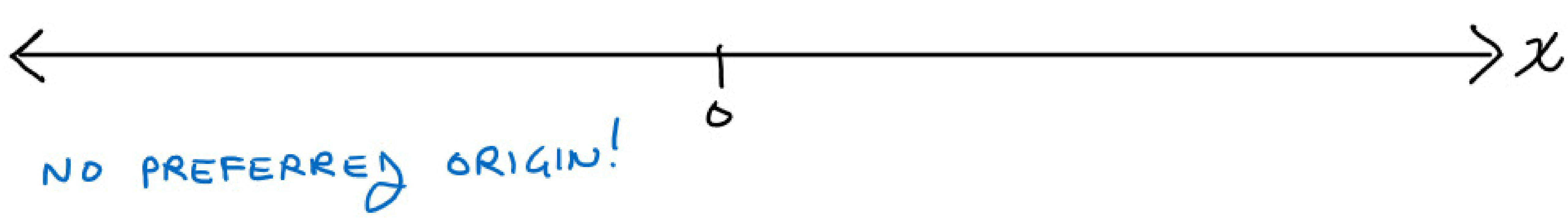


- ALL x ARE EQUIVALENT: NO NATURAL ORIGIN
- $f(x+L) = f(x)$

- SEEMS ARTIFICIAL, BUT GIVES A NATURAL WAY TO TAKE $L \to \infty$ LIMIT

$\Rightarrow$ RING WITH INFINITE CIRCUMFERENCE IS (LOCALLY) INDISTINGUISHABLE FROM THE INFINITE REAL LINE

$\mathbb{R}$:



$\therefore$ FOR BOUNDARY CONDITIONS OF TYPE ① OR ②,

$\hat{K} = \hat{K}^{\dagger}$ IS A HERMITIAN OPERATOR.

(A) Matrix Elements of $\hat{K}^p$ between Position Kets

$\langle x|\hat{K}|f\rangle \equiv -i\frac{d}{dx}f(x) \Rightarrow \langle x|\hat{K}|x'\rangle = -i\frac{d}{dx}\langle x|x'\rangle = -i\frac{d}{dx}\delta(x-x')$, Derivative of the Dirac δ-fcn.

Check: $\langle x|\hat{K}|f\rangle = \int_0^L dy\, \langle x|\hat{K}|y\rangle\langle y|f\rangle = \int_0^L dy\left[-i\frac{d}{dx}\delta(x-y)\right]f(y)$

① But: $\frac{d}{dx}\delta(x-y) = -\frac{d}{dy}\delta(x-y)$

Proof: $z \equiv x-y$

Holding y const, $\frac{d}{dz} = \frac{d}{dx}$

$\frac{d}{dx}\delta(z) = \frac{d}{dz}\delta(z)$

Hold x const., $\frac{d}{dz} = -\frac{d}{dy}$

δ-fcn is even

$\therefore \frac{d}{dx}\delta(x-y) = -\frac{d}{dy}\delta(x-y) = -\frac{d}{dy}\delta(y-x)$

Using ①: $= \int_0^L dy\left[i\frac{d}{dy}\delta(x-y)\right]f(y)$

Integrate by parts, No boundary term (by assumption)

$$= \int_0^L dy\, \delta(y-x)\left[-i\frac{d}{dy}f(y)\right] = -i\frac{df}{dx} \checkmark$$

$$\therefore \quad \langle x|\hat{K}|x'\rangle = -i\frac{d}{dx}\delta(x-x') = \delta(x-x')\left[-i\frac{d}{dx}\right]$$

Derivative acting to right (on another function)

Similarly

$$\langle x|\hat{K}^p|f\rangle = (-i)^p\frac{d^p}{dx^p}f(x), \quad p\in\{1,2,3,\dots\}$$

$$\Rightarrow \langle x|\hat{K}^p|x'\rangle = (-i)^p\frac{d^p}{dx^p}\delta(x-x') = (-i)^p\delta(x-x')\frac{d^p}{dx^p}$$

p-fold deriv. acting to the right

(B) Eigenmodes of $\hat{K} = \hat{K}^\dagger$

① Dirichlet BC.

Consider the operator $\hat{K}^2$: $\hat{K}^2|\phi_n\rangle = \underbrace{\mathcal{E}_n}_{\text{eigenvalue}}|\phi_n\rangle$, over the interval $0\le x\le L$, $\langle x|f\rangle = f(x)$ s.t. $f(L) = f(0) = 0$ for all allowed states

$$\langle x|\hat{K}^2|\phi_n\rangle = \mathcal{E}_n\langle x|\phi_n\rangle = -\frac{d^2}{dx^2}\phi_n(x) = \mathcal{E}_n\phi_n(x), \quad \phi_n(0) = \phi_n(L) = 0$$

- Same string problem from Lec. 1 !
- Same as infinite square-well problem in Quantum (but interp. of states is different)

$\phi_n(x) = \sqrt{\frac{2}{L}}\sin(K_n x)$

$\langle\phi_m|\phi_n\rangle = \frac{2}{L}\int_0^L dx\,\sin(K_m x)\sin(K_n x) = \delta_{m,n}$

$K_n = \frac{n\pi}{L}$; $\mathcal{E}_n = K_n^2$

$\Rightarrow$ Countably infinite orthonormal basis ($n\in\{1,2,3,\dots\}$)

$=$ Eigenspectrum of $\hat{K}^2$ on $0\le x\le L$ with Dirichlet ("hard wall") BC

② Periodic B.C.

$-\frac{L}{2} \le x \le \frac{L}{2}$; $\quad f(\frac{L}{2}) = f(-\frac{L}{2})$, $\quad f(x+L) = f(x)$

$$\hat{k}|k\rangle = k|k\rangle \Rightarrow \langle x|\hat{k}|k\rangle = k \underbrace{\langle x|k\rangle}_{\equiv \psi_k(x)} \quad \text{or} \quad -i\frac{d}{dx}\psi_k(x) = k\,\psi_k(x)$$

$$\therefore \boxed{\psi_k(x) = \frac{1}{\sqrt{L}} e^{ikx}}; \qquad \frac{\psi_k(L)}{\psi_k(0)} = 1 = e^{ikL} \Rightarrow k_n = \frac{2\pi n}{L}, \quad n \in \mathbb{Z} \text{ (any integer, positive, zero, or negative)}$$

For Periodic B.C., eigenstates of the Hermitian Deriv. Op. $\hat{k}$ are Plane Waves with Quantized k_n

Wavenumber

$[k_n] = \frac{1}{[L]}$

① Orthonormality:

$$\langle k_n|k_m\rangle = \int_0^L dx\, \langle k_n|x\rangle\langle x|k_m\rangle = \frac{1}{L}\int_0^L dx\, e^{ix\frac{2\pi}{L}(m-n)} = \delta_{m,n}$$

$\frac{1}{\sqrt{L}} e^{-ik_n x}$, complex conjugate of $\langle x|k_n\rangle$

② Resolution of the Identity:

$$\hat{\mathbb{I}} = \sum_{n=-\infty}^{\infty} |k_n\rangle\langle k_n|$$

$$\therefore \langle x|x'\rangle = \delta(x-x') = \langle x|\hat{\mathbb{I}}|x'\rangle = \sum_{n=-\infty}^{\infty} \langle x|k_n\rangle\langle k_n|x'\rangle = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{ik_n(x-x')}$$

Note: $G(x) \equiv \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{L}x} = G(x+L)$

$$\therefore \boxed{\sum_{n=-\infty}^{\infty} \langle x|k_n\rangle\langle k_n|x'\rangle = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{ik_n(x-x')} = \sum_{m=-\infty}^{\infty} \delta(x-x'-mL)}$$

Resolution of Identity using $\hat{k}$ eigenstates on the interval $|x| \le \frac{L}{2}$

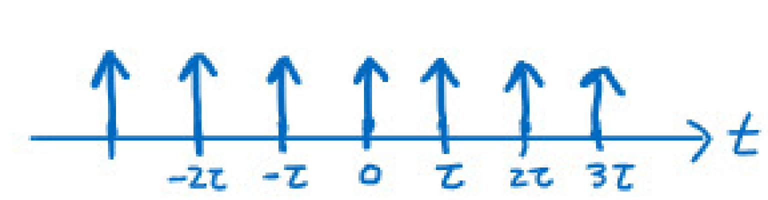
- "Poisson Resummation Formula"

- Mathematical Aside: can define Fourier Transform of function $f(t)$ on entire real line $-\infty \le t \le \infty$:

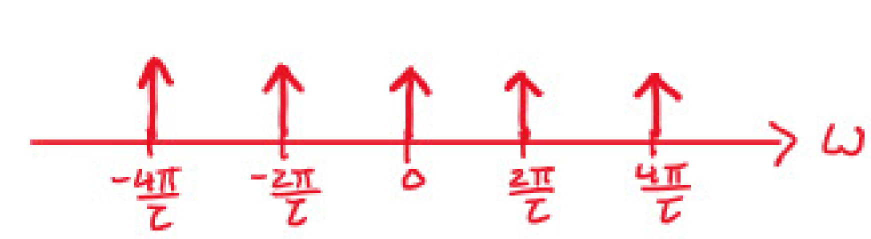
$$\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} dt\, e^{i\omega t} f(t)$$

- We will discuss Fourier Transforms next, in context of functions on $\mathbb{R}$

- Consider $f(t) \equiv \sum_{m=-\infty}^{\infty} \delta(t - m\tau)$ "Picket Fence"



- $$\tilde{f}(\omega) = \sum_{m=-\infty}^{\infty} e^{i\omega m\tau} = \sum_{n=-\infty}^{\infty} \left(\frac{2\pi}{\tau}\right)\delta\left(\omega - \left[\frac{2\pi}{\tau}\right]n\right)$$

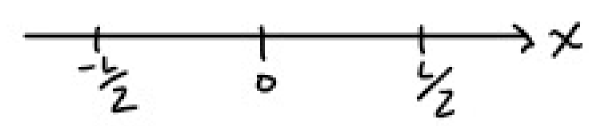


using Poisson Resummation Formula

∴ "F.T. of Picket Fence is a Picket Fence"

(3) Eigenmodes of $\hat{K}$ on Entire Real Line $-\infty \le x \le \infty$

- View as $L \to \infty$ limit of Periodic Case



$f(\tfrac{L}{2}) = f(-\tfrac{L}{2})$

(1) Resolution of the Identity

$$\langle x|\hat{\mathbb{I}}|x'\rangle = \delta(x-x') = \lim_{L\to\infty} \frac{1}{L}\sum_{m=-\infty}^{\infty} e^{iK_m(x-x')} \quad ; \quad K_m = \left(\frac{2\pi}{L}\right)m. \quad \text{As } L\to\infty,\ \Delta K = \left(\frac{2\pi}{L}\right) \longrightarrow 0$$

$$= \lim_{L\to\infty}\left(\frac{1}{\Delta K L}\right)\sum_{m=-\infty}^{\infty}\Delta K\, e^{iK_m(x-x')} \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ik(x-x')}$$

$$\boxed{\therefore\quad \delta(x-x') = \int_{-\infty}^{\infty}\frac{dK}{2\pi}\, e^{iK(x-x')} = \int_{-\infty}^{\infty} dK\; \langle x|K\rangle\langle K|x'\rangle}$$

$\Rightarrow$ (a) $\hat{K}|k\rangle = k|k\rangle$ has eigenfunctions $\langle x|k\rangle = \dfrac{1}{\sqrt{2\pi}}\, e^{ikx}$, $k \in \mathbb{R}$ Not Quantized!

(b) Resolution of Identity: $\hat{\mathbb{I}} = \displaystyle\int_{-\infty}^{\infty} dk\; |k\rangle\langle k|$

(c) Basis State Overlap: $\langle k|k'\rangle = \langle k|\hat{\mathbb{I}}|k'\rangle = \displaystyle\int_{-\infty}^{\infty} dx\,\langle k|x\rangle\langle x|k'\rangle = \int_{-\infty}^{\infty}\frac{dx}{2\pi}\, e^{ix(k-k')} = \delta(k-k')$

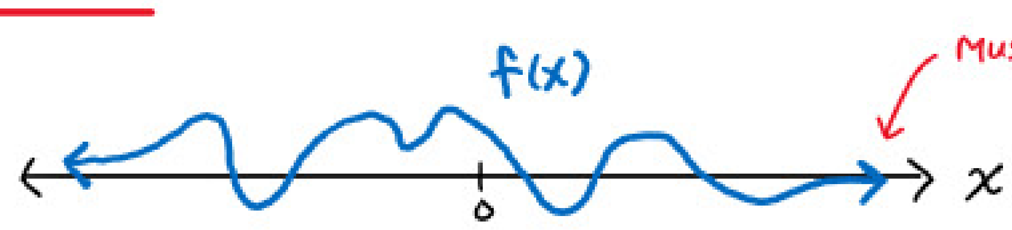
Fourier Transforms: Change of Basis

Let $|f\rangle$ represent a square-normalizable function over the entire real line:

$$\langle f|f\rangle = \langle f|\hat{\mathbb{I}}|f\rangle = \int_{-\infty}^{\infty} dx\;\langle f|x\rangle\langle x|f\rangle = \int_{-\infty}^{\infty} dx\;|f(x)|^2 = \text{Finite}$$

(1) Position Space Representation

$$\langle x|f\rangle = f(x)$$



Must decay "fast enough" as $|x| \to \infty$ s.t. $|f|^2 = $ Finite

(2) K-space ("Momentum Space in Quantum") Representation

$$\langle k|f\rangle = \tilde{f}(k) = \langle k|\hat{\mathbb{I}}|f\rangle = \int_{-\infty}^{\infty} dx\;\langle k|x\rangle\langle x|f\rangle = \int_{-\infty}^{\infty} dx\;\frac{e^{-ikx}}{\sqrt{2\pi}}\, f(x)$$

Fourier Transform! Position $\Rightarrow$ "Momentum"

Can also define **Inverse Fourier Transform**: "Momentum" $\Rightarrow$ Position

$$\langle x|f\rangle = f(x) = \langle x|\hat{\mathbb{I}}|f\rangle = \int_{-\infty}^{\infty} dk\;\langle x|k\rangle\langle k|f\rangle = \int_{-\infty}^{\infty} dk\;\frac{e^{ikx}}{\sqrt{2\pi}}\,\tilde{f}(k)$$

$\therefore$ Fourier Transforms = Basis Change Formulae on the Real Line!