Direct Sum $\oplus$ vs. Direct Product $\otimes$; Particle in the Plane and 2D Orbital Ang. Momentum

① Direct Sum (Lec. $\frac{3}{7}$, p. 2)

$$\mathbb{V}^m \oplus \mathbb{V}^n = \mathbb{V}^{m+n}$$

Orthonormal Bases: $\{|i\rangle\}$, $i \in 1,2,\dots,m$; $\{|\alpha\rangle\}$, $\alpha \in 1,2,\dots,n$; $\{\{|i\rangle\}, \{|\alpha\rangle\}\}$ $m+n$ basis states

② Direct Product

$$\mathbb{V}^m \otimes \mathbb{V}^n = \mathbb{V}^{m\cdot n}$$

- Basis states: $\{ |i\rangle \otimes |\alpha\rangle \}$, $i \in 1,\dots,m$; $\alpha \in 1,\dots,n$ $\Rightarrow$ $m\cdot n$ basis states!

  Simultaneously keeps track of (1) subsystem state $|i\rangle \in \mathbb{V}^m$ and (2) sub. sys. state $|\alpha\rangle \in \mathbb{V}^n$

- Short hand notation: $|i\rangle \otimes |\alpha\rangle \equiv |i\rangle|\alpha\rangle \equiv |i,\alpha\rangle$

Ex: $\mathbb{V}^3$, basis $\{|1\rangle, |2\rangle, |3\rangle\}$; $\langle i|j\rangle = \delta_{ij}$

$\mathbb{V}^2$, basis $\{|\uparrow\rangle, |\downarrow\rangle\}$; $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$; $\langle\uparrow|\downarrow\rangle = 0$

- $\mathbb{V}^3 \oplus \mathbb{V}^2 = \mathbb{V}^5$; Basis: $\{|1\rangle, |2\rangle, |3\rangle, |\uparrow\rangle, |\downarrow\rangle\}$
- $\mathbb{V}^3 \otimes \mathbb{V}^2 = \mathbb{V}^6$; Basis: $\{|1\uparrow\rangle, |2\uparrow\rangle, |3\uparrow\rangle, |1\downarrow\rangle, |2\downarrow\rangle, |3\downarrow\rangle\}$

Operators:

$$\mathbb{R}_z^{(3)}(\theta) \Rightarrow \begin{array}{cc} \left[\begin{array}{ccc|ccc} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{c} |1\uparrow\rangle \\ |2\uparrow\rangle \\ |3\uparrow\rangle \\ |1\downarrow\rangle \\ |2\downarrow\rangle \\ |3\downarrow\rangle \end{array} \\ \begin{array}{cccccc} |1\uparrow\rangle & |2\uparrow\rangle & |3\uparrow\rangle & |1\downarrow\rangle & |2\downarrow\rangle & |3\downarrow\rangle \end{array} & \end{array}$$

$$\hat{\sigma}_2^{(2)} \Rightarrow \begin{array}{cc} \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ \hline i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} |1\uparrow\rangle \\ |2\uparrow\rangle \\ |3\uparrow\rangle \\ |1\downarrow\rangle \\ |2\downarrow\rangle \\ |3\downarrow\rangle \end{array} \\ \begin{array}{cccccc} |1\uparrow\rangle & |2\uparrow\rangle & |3\uparrow\rangle & |1\downarrow\rangle & |2\downarrow\rangle & |3\downarrow\rangle \end{array} & \end{array}$$

- Clearly, $[\hat{R}_z^{(3)}(\theta), \hat{\sigma}_2^{(2)}] = 0$, since they act on different subsystems.

- More complicated op.s that mix all states: Products of subsys. op.s

$$\mathbb{V}^m \otimes \mathbb{V}^n = \mathbb{V}^{m\cdot n}$$

$\mathbb{V}^m$: m^2 indept. Hermitian op.s $\{\hat{\Omega}_A^{(m)}\}$

$\mathbb{V}^n$: n^2 indept. Hermitian op.s $\{\hat{\Lambda}_\rho^{(n)}\}$

$\mathbb{V}^{m\cdot n}$: $m^2\cdot n^2$ indept. Hermitian op.s $\{\hat{\Omega}_A^{(m)} \otimes \hat{\Lambda}_\rho^{(n)}\}$

NOTE: IN QUANTUM PHYSICS, DIRECT SUMS USUALLY ARISE INDIRECTLY, AS DECOMPOSITIONS OF DIRECT PRODUCTS

i.e., $\mathbb{V}^m \otimes \mathbb{V}^n = \mathbb{V}^{mn} = \mathbb{V}^{P_1} \oplus \mathbb{V}^{P_2} \oplus \ldots \oplus \mathbb{V}^{P_j}$, $\quad \sum_{i=1}^{j} P_j = m \cdot n$

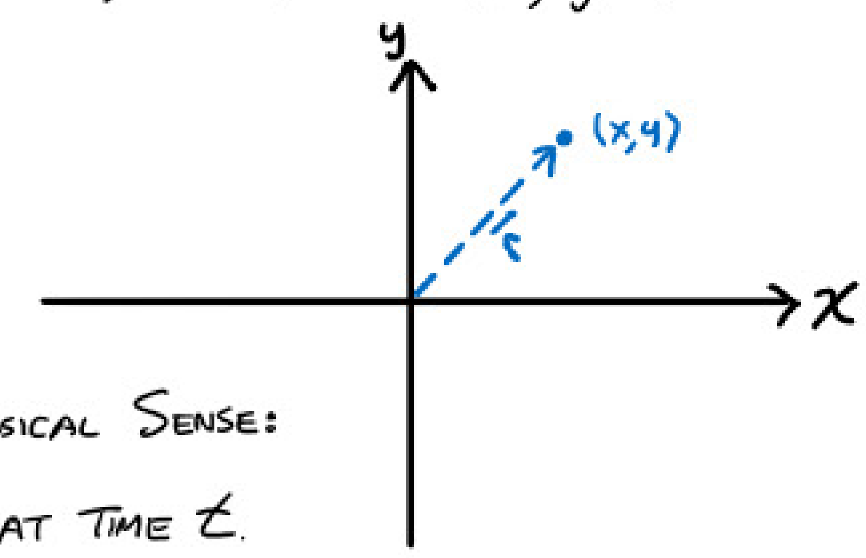
- SUCH A DECOMPOSITION IS ALWAYS POSSIBLE
- USEFUL WHEN SUBSPACES IN THE SUM TRANSFORM INDEPT.LY UNDER SYMMETRY OPERATIONS

$\longrightarrow$ WE WILL SEE AN EXAMPLE WHEN WE STUDY TWO SPIN ½'S (LEC. 16)

DIRECT PRODUCT EXAMPLE: QUANTUM PARTICLE IN THE PLANE

$\mathbb{V}_x^\infty \otimes \mathbb{V}_y^\infty \equiv \mathbb{V}_{xy}^\infty$ DIRECT PRODUCT OF FUNCTION (HILBERT) SPACES ON x-, y- REAL LINES

$\{|x\rangle\}$ $\quad$ $\{|y\rangle\}$



- POSITION BASE KETS: $|x\rangle \otimes |y\rangle \equiv |x\rangle|y\rangle \equiv |xy\rangle \equiv |\vec{r}\rangle$
- REPRESENTS A SYSTEM WITH 2 "DEGREES OF FREEDOM" IN THE CLASSICAL SENSE:
  - SIMULTANEOUSLY KEEP TRACK OF (x,y) COORDINATES LOCATING A PARTICLE AT TIME t.
- QUANTUM PHYSICS: PARTICLE CAN EXIST IN A (CONTINUOUS) SUPERPOSITION OF LOCATIONS

- GENERIC STATE: $|\psi\rangle$; $\langle \vec{r} | \psi \rangle = \psi_{(\vec{r})} = \psi_{(x,y)}$ WAVE FUNC. AMPLITUDE AT $\vec{r}$; $|\langle \vec{r}|\psi\rangle|^2 =$ PROB. DENSITY TO FIND PARTICLE AT $\vec{r}$.
- MOMENTUM BASIS: $|P_x\rangle \otimes |P_y\rangle \equiv |P_x P_y\rangle \equiv |\vec{P}\rangle$; $\langle \vec{P} | \psi \rangle = \tilde{\psi}_{(\vec{P})}$

- **BASIS OVERLAP:** $\langle \vec{r} | \vec{P} \rangle = \langle xy | P_x P_y \rangle = \frac{1}{2\pi\hbar} e^{i \frac{\vec{P}\cdot\vec{r}}{\hbar}}$

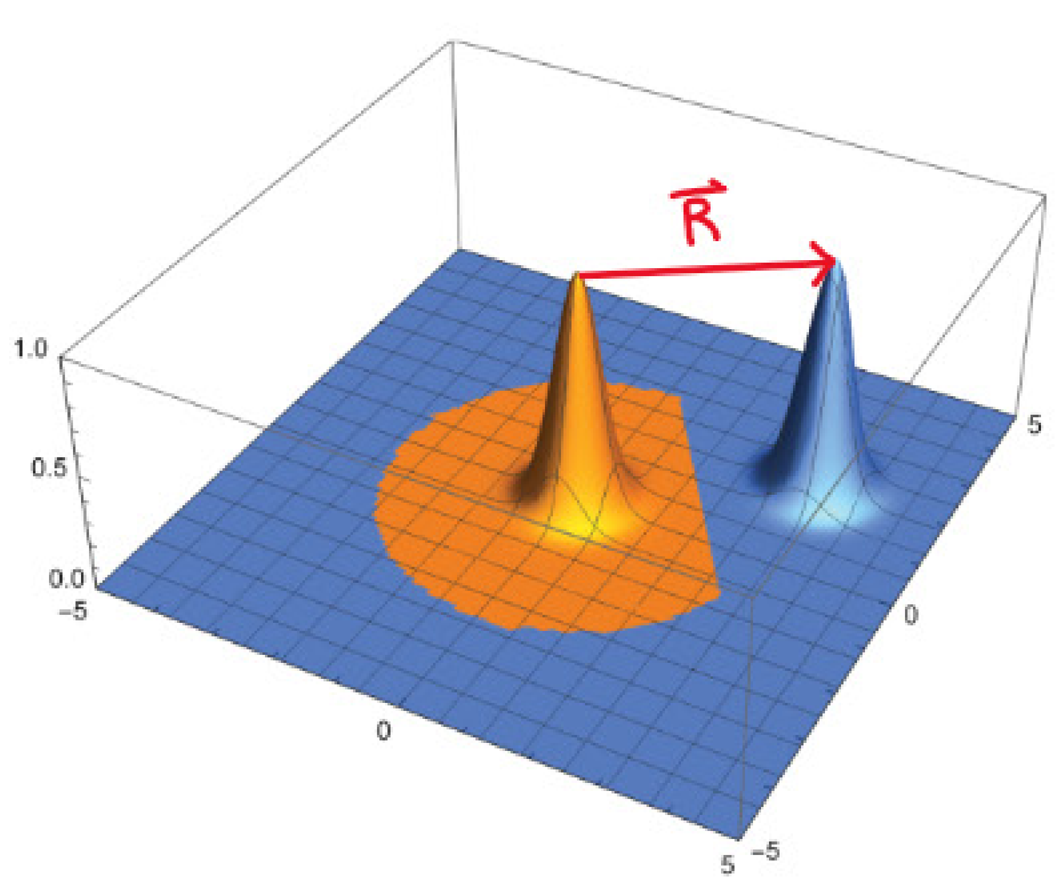
- **OPERATORS:** $\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y$; $[\hat{X}, \hat{Y}] = [\hat{X}, \hat{P}_y] = [\hat{P}_x, \hat{Y}] = [\hat{P}_x, \hat{P}_y] = 0$

$$[\hat{X}_a, \hat{P}_b] = i\hbar\, \delta_{a,b}\, \hat{\mathbb{I}} \quad ; \; a,b \in \{x,y\}$$

MOMENTUM OPERATORS: GENERATORS OF TRANSLATIONS IN THE PLANE

$$\langle \vec{r} | \, e^{-i \frac{\vec{R}\cdot\hat{\vec{P}}}{\hbar}} | \psi \rangle = e^{-i R_a \left(-i \frac{\partial}{\partial x^a}\right)} \psi_{(\vec{r})} = e^{-\vec{R}\cdot\vec{\nabla}} \psi_{(\vec{r})}$$

$$= \psi_{(\vec{r} - \vec{R})}$$



RIGID TRANSLATION OF A FUNCTION

$\psi_{(\vec{r})} \Longrightarrow \psi_{(\vec{r} - \vec{R})}$ (ACTIVE TRANSFORMATION)

"Orbital" Angular Momentum in 2D : Expect to Generate Rotations of Functions!

Angular Momentum in Classical Mech for Particle Confined to the Plane : $L_z = X P_y - Y P_x$ or $L_3 = \epsilon_{3ab} X_a P_b$; $a,b \in 1,2$

Operator: Hermitian Observable: $\hat{L}_z = \epsilon_{3ab} \hat{X}_a \hat{P}_b = \underbrace{\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x}$

Order in products does NOT matter, since $[\hat{X},\hat{P}_y] = [\hat{Y},\hat{P}_x] = 0$

Useful Commutator Identity:

$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$ (check it!)

- $[\hat{L}_z, \hat{X}_a] = \epsilon_{3bc}[\hat{X}_b \hat{P}_c, \hat{X}_a] = \epsilon_{3bc} \hat{X}_b [\hat{P}_c, \hat{X}_a] + \epsilon_{3bc} \underbrace{[\hat{X}_b, \hat{X}_a]}_{0} \hat{P}_c$

$$= -i\hbar \epsilon_{3ba} \hat{X}_b = i\hbar \epsilon_{3ab} \hat{X}_b$$

similarly,

- $[\hat{L}_z, \hat{P}_a] = i\hbar \epsilon_{3ab} \hat{P}_b$

Infinitesimal Rotation: $e^{i\theta \frac{\hat{L}_z}{\hbar}} \hat{X}_a e^{-i\theta \frac{\hat{L}_z}{\hbar}} \simeq \left[\hat{\mathbb{I}} + \frac{i\theta}{\hbar}\hat{L}_z + \dots\right] \hat{X}_a \left[\hat{\mathbb{I}} - \frac{i\theta}{\hbar}\hat{L}_z + \dots\right]$

$$= \hat{X}_a + \frac{i\theta}{\hbar}[\hat{L}_z, \hat{X}_a] + O(\theta)^2$$

$(\vec{n}_z \times \hat{\vec{X}})_a = \epsilon_{a3b} \hat{X}_b$

$$= \hat{X}_a + \frac{i\theta}{\cancel{\hbar}}\left(i\cancel{\hbar}\epsilon_{3ab}\hat{X}_b\right) = \hat{X}_a + \theta \epsilon_{3ba} \hat{X}_b$$

$$= \hat{X}_a + \theta (\vec{n}_z \times \hat{\vec{X}})_a + O(\theta^2)$$

✓ CCW Rotation of $\hat{\vec{X}}$ around $\vec{n}_z$ by angle θ (Lec. 10, p. 4 ; Lec. 10A)

- $[\hat{L}_z, \hat{P}_a \hat{P}_a] = 0$, $[\hat{L}_z, \hat{X}_a \hat{X}_a] = 0$ Einstein sum $a \in 1,2$

Typical Hamiltonian: $\hat{H} = \frac{\hat{\vec{P}}^2}{2\mu} + \hat{V}(\hat{\vec{X}})$; IF: $V(\vec{x}) = V(|\vec{x}|) = V(\sqrt{\vec{x}\cdot\vec{x}})$

(Central Potential)

Use "μ" for mass to avoid confusion with $\frac{L_z}{\hbar}$ e'value m.

Then $[\hat{H}, \hat{L}_z] = 0$

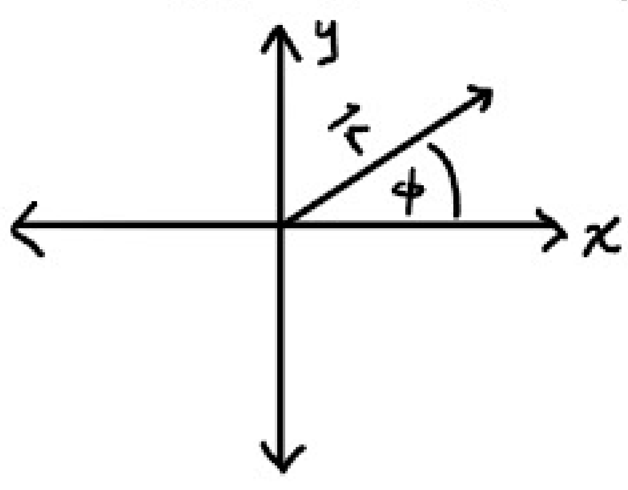
$\Rightarrow$ For a Rotationally Invariant $\hat{H}$,

$\hat{H}$ and $\hat{L}_z$ can be simultaneously diagonalized

Position Space: $\langle \vec{r} | \hat{L}_z | m \rangle = m\hbar \langle \vec{r} | m \rangle \Rightarrow -i\hbar(x\partial_y - y\partial_x)\psi_m = m\hbar \psi_m$

$\langle \vec{r} | m \rangle \equiv \psi_m(\vec{r})$

Since $\hat{L}_z$ generates rotations in the plane, makes sense to switch to Polar Coordinates



$x = r\cos\phi$
$y = r\sin\phi$; $\frac{\partial}{\partial\phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y}$

$$= -r\sin\phi\, \partial_x + r\cos\phi\, \partial_y = x\,\partial_y - y\,\partial_x$$

$$\therefore \quad \hat{L}_z \Rightarrow -i\hbar \frac{\partial}{\partial\phi} \quad ; \quad -i\cancel{\hbar}\frac{\partial}{\partial\phi}\psi_m(r,\phi) = m\cancel{\hbar}\,\psi_m(r,\phi) \Rightarrow \psi_m(r,\phi) = \psi_m(r)\, e^{im\phi}$$

Periodic B.C.: $\psi_m(r,\phi+2\pi) = \psi_m(r,\phi)$

$$\boxed{\therefore \; m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\} \quad \text{Any Integer!}}$$

- No surprise; same PBC on ring
- Orbital Angular Momentum is Quantized in units of $\hbar$!

Rotationally Inv. Time-Indept. Schrödinger Eq in 2D

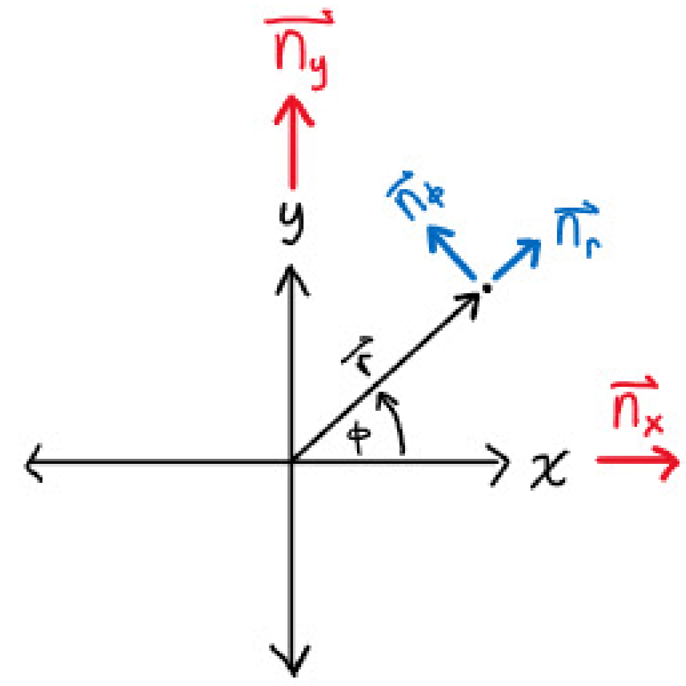
$$\left[\frac{\hat{\vec{P}}^2}{2\mu} + \hat{V}(|\hat{\vec{x}}|)\right]|E,m\rangle = E\,|E,m\rangle \quad ; \quad \hat{L}_z\,|E,m\rangle = m\hbar\,|E,m\rangle$$

$\Downarrow$ Position Basis

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\psi_{E,m}(r)\,e^{im\phi} = E\,\psi_{E,m}(r)\,e^{im\phi}$$

Aside: Gradient, Laplacian in Planar Polar Coordinates

$$\vec{\nabla} = \vec{n}_x\,\partial_x + \vec{n}_y\,\partial_y = \vec{n}_r\,\partial_r + \vec{n}_\phi\,\frac{1}{r}\partial_\phi$$



- $\vec{n}_r = \cos\phi\,\vec{n}_x + \sin\phi\,\vec{n}_y = -\partial_\phi\,\vec{n}_\phi$
- $\vec{n}_\phi = -\sin\phi\,\vec{n}_x + \cos\phi\,\vec{n}_y = \partial_\phi\,\vec{n}_r$

} Basis Vectors in Non-Cartesian (Curvilinear) Coordinates depend on coordinates!

$$\vec{\nabla}\cdot\vec{\nabla} = \left(\vec{n}_r\,\partial_r + \vec{n}_\phi\,\tfrac{1}{r}\partial_\phi\right)\cdot\left(\vec{n}_r\,\partial_r + \vec{n}_\phi\,\tfrac{1}{r}\partial_\phi\right) = \partial_r^2 + \frac{1}{r^2}\partial_\phi^2 + \frac{1}{r}\vec{n}_\phi\cdot\overbrace{(\partial_\phi\vec{n}_r)}^{=\vec{n}_\phi}\partial_r$$

$$= \partial_r^2 + \frac{1}{r^2}\partial_\phi^2 + \frac{1}{r}\partial_r$$ From the Deriv. of a Basis Vector!

Returning to the Rot. Invt. 2D Schrödinger Eqn.

$$\boxed{\left[-\frac{\hbar^2}{2\mu}\left(\partial_r^2 - \frac{m^2}{r^2} + \frac{1}{r}\partial_r\right) + V(r)\right]\psi_{E,m}(r) = E\,\psi_{E,m}(r)}$$

- Angular Part of Wave Function $e^{im\phi}$ Decouples
- However, Energies E can depend on $\hat{L}_z$ E'value m

$V=0$ (Free Particle with Ang. Momentum $m\hbar$):

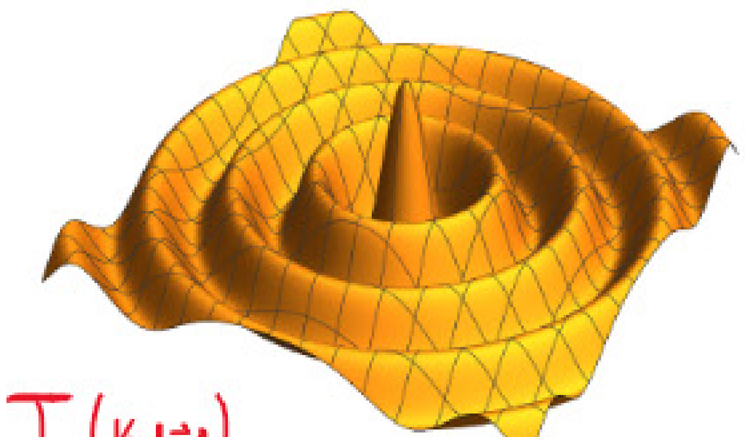
$$k^2 \equiv \frac{2\mu E}{\hbar^2} = \frac{1}{\text{Length}^2} \quad ; \quad y \equiv kr \quad \text{(dim.less)}$$

$$\Rightarrow \left(\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + \left[1 - \frac{m^2}{y^2}\right]\right)\psi_m(y) = 0 \qquad \text{Bessel's Equation}$$

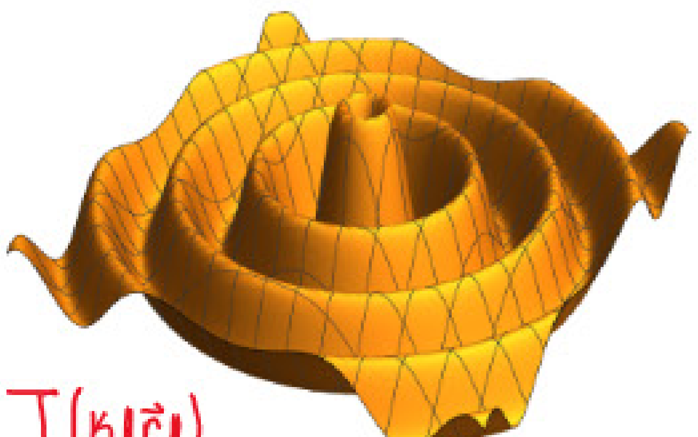
$$\therefore \quad \psi_{E,m}(r) \propto e^{im\phi}\,J_m(kr) \quad ; \quad k = \sqrt{\frac{2\mu E}{\hbar^2}}$$

$\Uparrow$ Bessel Function of the First Kind

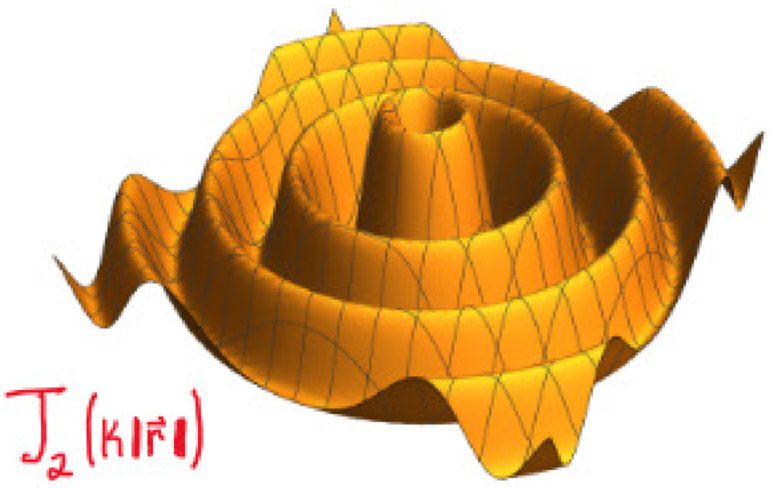
Circular Wave with orbital Ang. m



$J_0(k|\vec{r}|)$



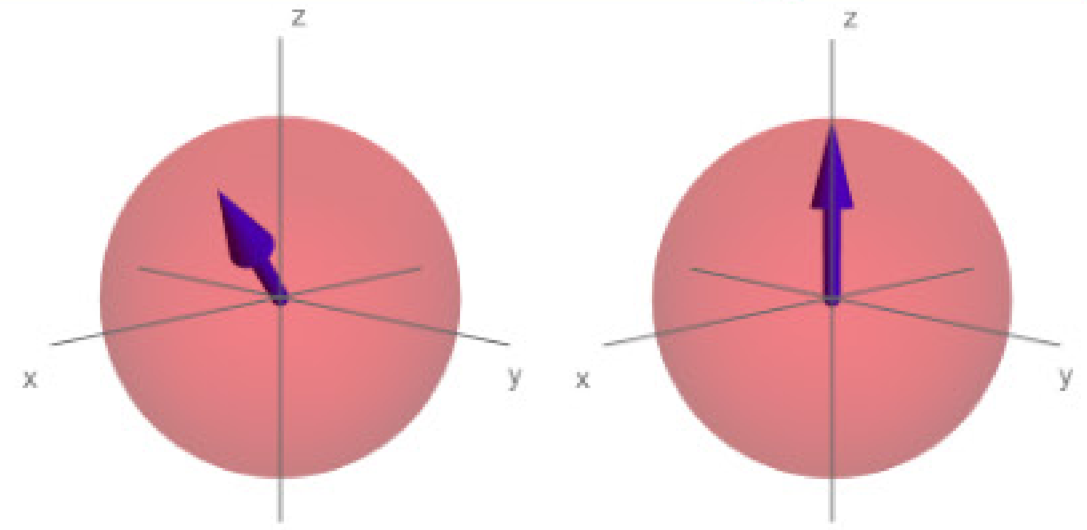
$J_1(k|\vec{r}|)$



$J_2(k|\vec{r}|)$

Another Quantum Direct Product: Two Spin-½'s (or Two "Qubits")

Consider $\mathbb{V}^2(\mathbb{C}) \otimes \mathbb{V}^2(\mathbb{C})$:



- **States**: $|\sigma\rangle \otimes |\kappa\rangle$; explicitly

  $\uparrow$ 1ST SPIN-½ ; $\uparrow$ 2ND SPIN-½

  $\sigma\ \kappa$
  $|\uparrow\uparrow\rangle$
  $|\downarrow\uparrow\rangle$
  $|\uparrow\downarrow\rangle$
  $|\downarrow\downarrow\rangle$

- **Operators:** (HW 3, #1)

$$\hat{\sigma}^1 \Rightarrow \begin{array}{c} \\ \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} \sigma\,\kappa \\ |\uparrow\uparrow\rangle \\ |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\downarrow\rangle \end{array} ; \quad \hat{\kappa}^1 \Rightarrow \begin{array}{c} \\ \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \sigma\,\kappa \\ |\uparrow\uparrow\rangle \\ |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\downarrow\rangle \end{array}$$

$[\hat{\sigma}^a, \hat{\kappa}^b] = 0$ for all $a, b \in 1,2,3$.

Basis for Generic Hermitian Op.s:

$$\{\hat{\mathbb{1}}, \{\hat{\sigma}^a\}, \{\hat{\kappa}^b\}, \{\hat{\sigma}^a \otimes \hat{\kappa}^b\}\} \qquad a,b \in 1,2,3$$

$$1 + 3 + 3 + 9 = 16 = 4^2 \text{ TOTAL} \checkmark$$

e.g., Two spin-½ particles (electrons, ^{6}Li atoms) trapped in the orbital ground state of two spatially separated quantum wells.

$\Longrightarrow$ Diff. spin-½'s are "distinguishable," labeled by location.

[The description of identical particles (e.g. electrons) is more complicated if there is no spatial separation. Then, the fundamental indistinguishability of identical quantum particles imposes constraints on allowed multiparticle wavefunctions. The Pauli Exclusion Principle is a consequence of these constraints, called "Particle Statistics" (Fermions vs. Bosons)]

Angular Momentum

$$\hat{S}^a_\sigma \equiv \frac{\hbar}{2}\hat{\sigma}^a \quad ; \quad \hat{S}^a_\kappa \equiv \frac{\hbar}{2}\hat{\kappa}^a$$

(a: WHICH GENERATOR; σ: WHICH SPIN D.O.F.)

- The product basis states $\{|\sigma\kappa\rangle\}$ ($\sigma \in \uparrow,\downarrow$) ($\kappa \in \uparrow,\downarrow$) are **simultaneous eigenstates** of $\hat{S}^z_\sigma$ and $\hat{S}^z_\kappa$.

- **Alternative**: Introduce **Total Angular Momentum Operators**: $\hat{J}^a \equiv \hat{S}^a_\sigma + \hat{S}^a_\kappa$

  Lie Brackets (Commutation Relations): $[\hat{J}^a, \hat{J}^b] = [\hat{S}^a_\sigma + \hat{S}^a_\kappa, \hat{S}^b_\sigma + \hat{S}^b_\kappa]$

$$= i\hbar\,\epsilon_{abc}(\hat{S}^c_\sigma + \hat{S}^c_\kappa) = i\hbar\,\epsilon_{abc}\,\hat{J}^c \qquad \text{SAME } SO(3) = SU(2) \text{ LIE ALGEBRA}$$

- Introduce total raising, lowering op.s

$$\hat{J}^\pm \equiv \hat{J}^x \pm i\hat{J}^y \ ; \ [\hat{J}^z, \hat{J}^\pm] = \pm\hbar\hat{J}^\pm, \ [\hat{J}^+, \hat{J}^-] = 2\hbar\hat{J}^z \qquad \text{(AS USUAL)}$$

What state has highest total $\hat{J}^z$? $\quad \hat{J}^z|\uparrow\uparrow\rangle = (\frac{\hbar}{2}+\frac{\hbar}{2})|\uparrow\uparrow\rangle = \hbar|\uparrow\uparrow\rangle$

Clearly $\hat{J}^+|\uparrow\uparrow\rangle = 0$. What happens when we lower with $\hat{J}^-$?

Lec. 13, p.2: $\hat{S}^-|\uparrow\rangle = \hbar|\downarrow\rangle \Rightarrow$

$$① \ \hat{J}^-|\uparrow\uparrow\rangle = (\hat{S}^-_\sigma + \hat{S}^-_\kappa)|\uparrow\uparrow\rangle = \hbar(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \sqrt{2}\hbar\underbrace{\left[\frac{|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle}{\sqrt{2}}\right]}_{\text{NORMALIZED TO 1}}$$

$$② \ \hat{J}^-\frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \frac{\hbar}{\sqrt{2}}(|\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle) = \sqrt{2}\hbar\,|\downarrow\downarrow\rangle$$

$$③ \ \hat{J}^-|\downarrow\downarrow\rangle = 0\,!$$

∴ STATES $\{ \underset{|1,1\rangle}{\underset{\parallel}{|\uparrow\uparrow\rangle}}, \underset{|1,0\rangle}{\underset{\parallel}{\tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)}}, \underset{|1,-1\rangle}{\underset{\parallel}{|\downarrow\downarrow\rangle}} \}$ FORM A SPIN-ONE REP. OF $SO(3)$!

HERE $|j,m\rangle$ DESCRIBES A SPIN-j STATE WITH $\hat{J}^z$ EIGENVALUE m ; $-j \le m \le j$

THIS SET OF THREE STATES THAT TRANSFORM UNDER ROTATIONS AS SPIN-1 STATES IS CALLED THE "TRIPLET"

$\hat{J}_z |1,m\rangle = m\hbar |1,m\rangle$; $\hat{J}_\pm |1,m\rangle = \sqrt{2}\hbar |1,m\pm 1\rangle$ EXCEPT $\hat{J}_+|1,1\rangle = \hat{J}_-|1,-1\rangle = 0$

✓ SAME EXACT RULES WE FOUND FOR SPIN ONE, LEC. 11, p. 6.

NOTE: • STATES $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ ARE SIMULTANEOUS EIGENSTATES OF

$$\hat{S}^z_\sigma, \ \hat{S}^z_\kappa, \text{ AND } \hat{J}^z = \hat{S}^z_\sigma + \hat{S}^z_\kappa$$

⇒ THESE PRODUCT BASIS STATES ARE ALSO TOTAL $\hat{J}^z$ EIGENSTATES.

$$\overset{j,\,m_z}{|1,1\rangle} = \overset{\sigma\ \kappa}{|\uparrow\uparrow\rangle} \ ; \quad \overset{j,\,m_z}{|1,-1\rangle} = \overset{\sigma\ \kappa}{|\downarrow\downarrow\rangle}$$

• STATE $\overset{j,\,m_z}{|1,0\rangle} = \frac{1}{\sqrt{2}}\left(\overset{\sigma\ \kappa}{|\uparrow\downarrow\rangle} + \overset{\sigma\ \kappa}{|\downarrow\uparrow\rangle}\right)$ IS AN EIGENSTATE OF $\hat{J}^z$ **ONLY!**

$$\Delta J^z = \left[\langle 1,0|(\hat{J}^z)^2|1,0\rangle - \left(\langle 1,0|\hat{J}^z|1,0\rangle\right)^2\right]^{1/2} = 0.$$

$$\Delta S^z_\sigma = \left[\langle 1,0|\underset{(\frac{\hbar}{2})^2 \hat{\mathbb{1}}}{\underset{\parallel}{(\hat{S}^z_\sigma)^2}}|1,0\rangle - \left(\underbrace{\langle 1,0|\hat{S}^z_\sigma|1,0\rangle}_{0}\right)^2\right]^{1/2} = \frac{\hbar}{2} = \Delta S^z_\kappa\ !$$

∴ S^z_σ AND S^z_κ ARE UNCERTAIN.

WE HAVE IDENTIFIED THREE $\hat{J}^z$ EIGENSTATES THAT FORM AN EFFECTIVE SPIN-ONE OBJECT. TO COMPLETE THE BASIS, NEED A FOURTH STATE.

"SINGLET" PAIR STATE

$$|s\rangle \equiv \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

① $\hat{J}^z |s\rangle = 0$ ⇒ ANOTHER $\hat{J}^z$ EIGENSTATE WITH $m_z = 0$.

② $\langle 1, m_z | s\rangle = 0$ FOR ALL $m_z \in \{-1,0,1\}$

∴ ORTHOGONAL TO SPIN-ONE TRIPLET STATES

ACTION OF $\hat{J}^\pm$?

$$\hat{J}^+|s\rangle = (\hat{S}^+_\sigma + \hat{S}^+_\kappa)\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = \frac{\cancelto{0}{\hat{S}^+_\sigma|\uparrow\downarrow\rangle} + \hat{S}^+_\kappa|\uparrow\downarrow\rangle - \hat{S}^+_\sigma|\downarrow\uparrow\rangle - \cancelto{0}{\hat{S}^+_\kappa|\downarrow\uparrow\rangle}}{\sqrt{2}}$$

$$= \frac{\hbar}{\sqrt{2}}\left(|\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle\right) = 0\,!$$

SIMILARLY

$\hat{J}^-|s\rangle = 0$ ∴ $|s\rangle \equiv \overset{j\ m_z}{|0,0\rangle}$ IS A SCALAR (SPIN $j=0$) STATE ⇒ DOES NOT CHANGE UNDER TOTAL ROTATIONS!