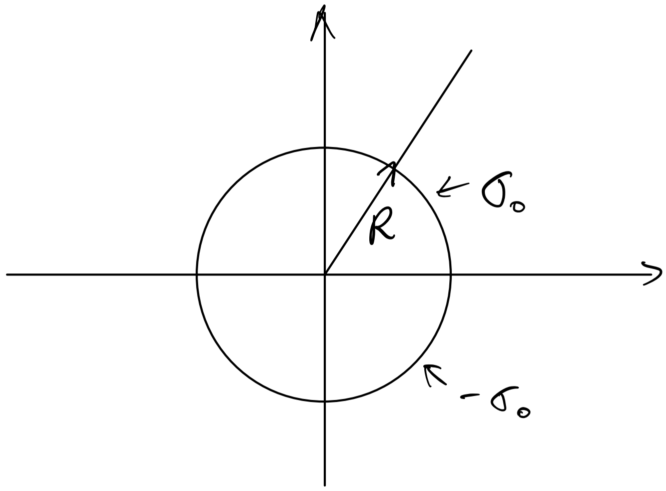
Problem 1

eq 2.36:

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{1}{\varepsilon_0}\sigma$$



$$V(s,\phi,z) = a_0 + b_0 \ln s + \sum_{k=1}^{\infty}[s^k(a_k\cos k\phi + b_k \sin k\phi) + s^{-k}(c_k\cos k\phi + d_k \sin k\phi)]$$

heres $n = s$

First consider $V(0)$:

$$V(0) = \int \vec{E}\cdot dl$$

$$= \int_0^{\pi}\int_0^{R} \frac{1}{4\pi\varepsilon_0}\frac{\sigma_0}{r^2} r\, d\theta + \int_{\pi}^{2\pi}\int_0^{R} \frac{1}{4\pi\varepsilon_0}\frac{-\sigma_0}{r^2} r\, d\theta$$

$$= 0$$

$\therefore$ $a_0 = 0$, $b_0 = 0$

Now look at the forms of V_{above} (out) V_{below} (in)

V_{out} has $\lim_{s\to\infty} V = 0$, so $s^k \to 0$, $a_k = 0, b_k = 0$

$$V_{out}(s,\phi,z) = \sum_{k=1}^{\infty}[s^{-k}(c_k \cos k\phi + d_k \sin k\phi)]$$

V_{in} has $\lim_{s\to 0} V \neq \infty$, so $s^{-k} \neq \infty$, $\ln s \neq -\infty$, $b_0 = 0, c_k = 0, d_k = 0$

$$V_{in}(s,\phi,z) = \sum_{k=1}^{\infty}[s^k(a_k \cos k\phi + b_k \sin k\phi)]$$

Now impose first BC: V is always continuous:

when $s = R$: $V_{in} = V_{out}$

$$\therefore \quad \sum_{k=1}^{\infty}[R^{-k}(c_k \cos(k\phi) + d_k \sin(k\phi))] = \sum_{k=1}^{\infty}[R^k(a_k \cos(k\phi) + B_k \sin(k\phi)] \quad BC\ 1$$

Now impose 2nd BC: $\frac{\partial V_{out}}{\partial n} - \frac{\partial V_{in}}{\partial n} = -\frac{1}{\varepsilon_0}\sigma(\phi)$ $\quad \sigma(\phi) = \begin{cases} \sigma_0, & 0 < \phi \leq \pi \\ -\sigma_0, & \pi < \phi \leq 2\pi \end{cases}$

In this case $\vec{\nabla} V \cdot \hat{n} = \frac{\partial V}{\partial s}$ by cylindrical symmetry.

Now $\frac{\partial V_{out}}{\partial s} = \frac{\partial}{\partial s}\left(a_0 + b_0 \ln s + \sum_{k=1}^{\infty}[s^{-k}(c_k \cos(k\phi) + d_k \sin(k\phi))]\right)$

$$= b_0 \frac{1}{s} - \sum_{k=1}^{\infty} k s^{-k-1}(c_k \cos(k\phi) + d_k \sin(k\phi))$$

$$\frac{\partial V_{in}}{\partial s} = \frac{\partial}{\partial s}\left(a_0 + \sum_{k=1}^{\infty} s^k(a_k \cos(k\phi) + b_k \sin(k\phi))\right)$$

$$= \sum_{k=1}^{\infty} k s^{k-1}(a_k \cos(k\phi) + b_k \sin(k\phi))$$

$$\frac{\partial V_{out}}{\partial s} - \frac{\partial V_{in}}{\partial s} = \frac{b_0}{s} + \sum_{k=1}^{\infty} -ks^{-k-1}c_k \cos(k\phi) - ks^{-k-1}d_k \sin(k\phi) - ks^{k-1}a_k \cos(k\phi) - ks^{k-1}b_k \sin(k\phi) \quad 2$$

$$= \frac{b_0}{s} - \sum_{k=1}^{\infty}[\cos(k\phi)(ks^{-k-1}c_k + ks^{k-1}a_k) + \sin(k\phi)(ks^{-k-1}d_k + ks^{k-1}b_k)]$$

$-\frac{1}{\varepsilon_0}\sigma = -\frac{1}{\varepsilon_0}\sigma(\phi)$ ②

$\sin^2(k\phi)+\cos^2(k\phi)=1$

$-\sin^2(k\phi)+\cos^2(k\phi)=\cos(2k\phi)$

Use the fourier's trick: $\int_0^{2\pi} C\sin^2(k\phi) = C\int_0^{2\pi}\frac{1-\cos(2k\phi)}{2}d\phi$

$= C\pi$

$\int_0^{2\pi} C\cos^2(k\phi) = C\pi$

Apply on ②

$\therefore\ \pi(ks^{-k-1}c_k + ks^{k-1}a_k) = \frac{1}{\varepsilon_0}\left(\int_0^{\pi}\sigma_0\cos(k\phi) - \int_{\pi}^{2\pi}\sigma_0\cos(k\phi)\right)$ (cos term)

$= \frac{2\sin(k\pi)}{k} - \sin(2k\pi) = 0$

$s^{-k-1}c_k + s^{k-1}a_k = 0$

$s^{-k}c_k + s^{k}a_k = 0$

$c_k + s^{2k}a_k = 0$

$\therefore\ c_k = -s^{2k}a_k$

Now integrate both side by $\sin(k\phi)$

$\pi(d_k k s^{-k-1} + ks^{k-1}b_k) = \int_0^{\pi}\frac{1}{\varepsilon_0}\sigma_0\sin(k\phi)\,d\phi - \int_{\pi}^{2\pi}\frac{1}{\varepsilon_0}\sigma_0\sin(k\phi)\,d\phi$

$= \frac{1-2\cos(k\pi)+\cos(2k\pi)}{k}\frac{\sigma_0}{\varepsilon_0}$

$= \frac{1-2(-1)^k+1}{k}\frac{\sigma_0}{\varepsilon_0} = \frac{2-2(-1)^k}{k}\frac{\sigma_0}{\varepsilon_0}$

$d_k s^{-k-1} + b_k s^{k-1} = \frac{2\sigma_0}{\varepsilon_0\pi}\left(\frac{1-(-1)^c}{k^2}\right)$

$d_k + b_k s^{2k} = s^{k+1}\frac{2\sigma_0}{\varepsilon_0\pi}\left(\frac{1-(-1)^k}{k^2}\right)$

Now impose same method on BC ①: (V continuous cond)

$$\begin{cases} R^{-k}c_k = R^{k}a_k \\ R^{-k}d_k = R^{k}b_k \end{cases}$$

$$\therefore \begin{cases} C_k = R^{2k} a_k \\ d_k = R^{2k} b_k \\ C_k = -R^{2k} a_k \\ d_k + R^{2k} b_k = R^{k+1} \frac{2\sigma_0}{\varepsilon_0 \pi} \left(\frac{1-(-1)^k}{k^2} \right) \end{cases}$$

$$\therefore \quad C_k = 0, \quad a_k = 0$$

$$2R^{2k} b_k = R^{k+1} \frac{2\sigma_0}{\varepsilon_0} \left(\frac{1-(-1)^k}{k^2} \right)$$

$$b_k = R^{-k+1} \frac{\sigma_0}{\varepsilon_0} \left(\frac{1-(-1)^k}{k^2} \right)$$

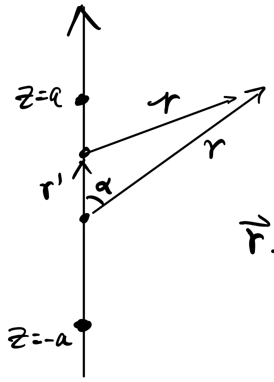
$$d_k = R^{2k} b_k$$

$$= R^{k+1} \frac{\sigma_0}{\varepsilon_0} \left(\frac{1-(-1)^k}{k^2} \right)$$

$$\therefore V_{out}(r) = \sum_{n=1}^{\infty} r^{-k} R^{k+1} \frac{\sigma_0}{\varepsilon_0} \left(\frac{1+(-1)^{k+1}}{k^2} \right) \sin(k\phi)$$

$$V_{in}(r) = \sum_{n=1}^{\infty} r^{k} R^{-k+1} \frac{\sigma_0}{\varepsilon_0} \left(\frac{1+(-1)^{k+1}}{k^2} \right) \sin(k\phi)$$

Problem 2.



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\alpha) \rho(r') d\tau$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$r^{(n+1)} = (x^2+y^2+z^2)^{\frac{n+1}{2}}$$

$$\rho(r') = \delta x' \delta y' \lambda(z')$$

leading term, $n=0$

$$V'(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int_{z'=-a}^{a} P_n(\cos\alpha) \lambda(z') dz'$$

now $r = \sqrt{x^2+y^2+z^2}$ $\quad \cos\alpha = \frac{z}{r}$

$$\therefore V'(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \left(\int_0^a P_0\left(\frac{z}{r}\right) \lambda(z') dz' + \int_{-a}^{0} P_0\left(-\frac{z}{r}\right) \lambda(z') dz' \right)$$

However $P_0(x) = 1$

$$V'(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int_{z=-a}^{a} \lambda(z) dz$$

a) $\lambda = k\cos\left(\frac{\pi z}{2a}\right)$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int_{z=-a}^{a} k\cos\left(\frac{\pi z}{2a}\right) dz$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \frac{4ak}{\pi}$$

$$= \frac{ak}{\pi^2 \varepsilon_0 r}$$

b) $V(r) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int_{-a}^{a} k \sin\left(\frac{\pi z}{a}\right) dz$

$= \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \times 0$

$= 0$

c) $V(r) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int_{-a}^{a} k \cos\left(\frac{\pi z}{a}\right) dz$

$= 0$

Problem 3 (3.51 P159-P160)

a)

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

q at center:

$$E(r) = \frac{1}{\frac{4}{3}\pi R^3}\frac{1}{4\pi\varepsilon_0}\int \frac{q}{r^2}\hat{r}\,d\tau$$

ρ at sphere as $\rho = -q/(\frac{4}{3}\pi R^3)$

$$E_s = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\frac{4}{3}\pi R^3}\frac{-q}{r^2}\hat{r}\,d\tau'$$

Now we say outwards is positive, sign change required:

$$E_s = \frac{1}{\frac{4}{3}\pi R^3}\frac{1}{4\pi\varepsilon_0}\int \frac{q}{r^2}\hat{r}\,d\tau'$$

b) Since $E_s(r) = \frac{\rho r}{3\varepsilon_0}\hat{r}$

$$\vec{P} = \int \vec{r}\rho(\vec{r})\,d\tau' \qquad \vec{r} = \vec{r}' - \vec{r}$$

$$= \rho\int_0^{2\pi}\int_0^{\pi}\int_0^{R}(\hat{r}' - \hat{r})\,r'^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= -\rho\,\frac{4}{3}\pi R^3\,\hat{r}$$

$$\rho r\hat{r} = -\vec{P}\,\frac{3}{4\pi R^3}$$

$$E_s(r) = \frac{1}{3\varepsilon_0}\,-\vec{P}\,\frac{3}{4\pi R^3}$$

$$= -\frac{\vec{P}}{4\pi R^3\varepsilon_0}$$

c) Now we want to add up all charges within the sphere.

$$\vec{E}_{tot} = \sum_i \overrightarrow{E_{AVE}(r_i)} = \sum_i \overrightarrow{E_s(r_i)}$$

$$= \sum_i -\frac{1}{4\pi R^3 \varepsilon_0} \vec{P_i}$$

$$= -\frac{1}{4\pi R^3 \varepsilon_0} \sum_i \vec{P_i}$$

$$= -\frac{\vec{P}_{tot}}{4\pi R^3 \varepsilon_0}$$

d) look at 1 point charge

$$E_{average} = \frac{1}{4\pi\varepsilon_0} \frac{1}{\frac{4}{3}\pi R^3} \int \frac{q}{r^2} \hat{r}\, d\tau'$$

which exactly is of the form of $\rho = -\frac{q}{\frac{4}{3}\pi R^3}$ on sphere at point r

calculated in part a. so $E_{ave} = E_s$

Now we want to see what is E_s at center

$$\oint E_s \cdot d\hat{A} = \frac{Q}{\varepsilon_0}$$

$$E_s = \frac{1}{4\pi\varepsilon_0} \frac{-q}{r^2} = E_{central\ charge}$$

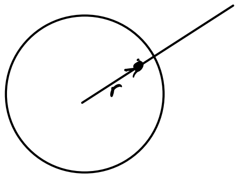
By principle of superposition, just pile up as much

as you want.

Problem 4 (4.10 P175)

$$\vec{P}(\vec{r}) = k\vec{r} = kr\hat{r}$$

a) $\sigma_b \equiv \vec{P}\cdot\hat{n}$



$$= kr\hat{r}\cdot\hat{n}\big|_{r=R}$$

$$= kR$$

$$\rho_b \equiv -\nabla\cdot\vec{P}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr)$$

$$= -\frac{1}{r^2} 3kr^2$$

$$= -3k$$

b)

$$E_{in} = \frac{1}{4\pi r^2}\left(-3k \times \frac{4}{3}\pi r^3\right)\hat{r}\,\frac{1}{\varepsilon_0} \qquad \text{(gauss's law)}$$

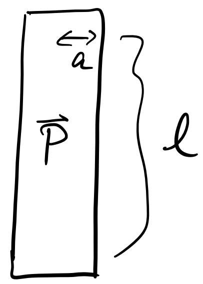
$$= -\frac{kr}{\varepsilon_0}\hat{r}$$

$$E_{out} = \frac{1}{4\pi r^2}\frac{\hat{r}}{\varepsilon_0}\left(-3k \times \frac{4}{3}\pi R^3 + kR \times 4\pi R^2\right)$$

$$= 0$$

$$\therefore \begin{cases} E(r) = -\frac{kr}{\varepsilon_0}\hat{r} & r<R \\ E(r) = 0 & r \geq R \end{cases}$$

Problem 5 (4.11 P175)



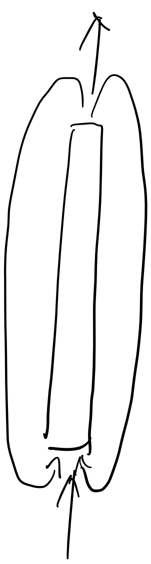
$\sigma_b = \vec{P} \cdot \hat{n}$

$= \pm P$

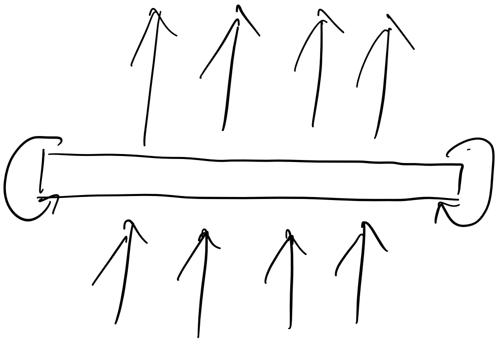
side : 0
up : +
down : −

$\rho_b = -\nabla \cdot P$

$= 0$ (uniform ... sad)



$L \gg a$



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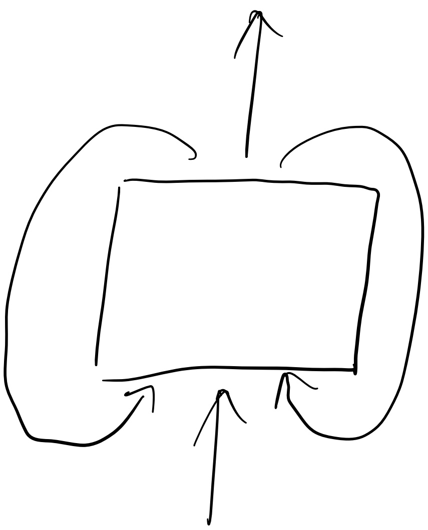

$L \approx a$

Problem 6 (4.14 P178)

$$\sigma_b \equiv P\cdot\vec{n}$$

$$\rho_b \equiv -\nabla\cdot P$$

$$Q_{tot} = \int d\tau' \rho_b + \oint dA\, \sigma_b$$

$$= \int d\tau' -\nabla\cdot P + \oint dA\, P\cdot\vec{n}$$

$$= -\int d\tau' \nabla\cdot P + \int d\tau' \nabla\cdot P$$

$$= 0$$