Problem 1

a) $j_1 = 5$, $j_2 = 4$, $j_3 = 2$

First do 1,2 then 3

$j_1 + j_2 = 9$

$|j_1 - j_2| = 1$

$j_1 j_2$	$j_1 j_2 + j_3$	$\lvert j_1 j_2 - j_3 \rvert$	j_{tot}
9	11	7	7, 8, 9, 10, 11
8	10	6	6 ~ 10
7	9	5	5 ~ 9
6	8	4	4 ~ 8
5	7	3	3 ~ 7
4	6	2	2 ~ 6
3	5	1	1, 2, 3, 4, 5
2	4	0	0, 1, 2, 3, 4
1	3	1	1, 2, 3

allowed: 0 ~ 11 , 12 j_{tot} s

Then do $j_1 j_3$, j_2

$j_1 + j_3 = 7$

$j_1 - j_3 = 3$

$j_1 j_3$	$j_1 j_3 + j_2$	$\lvert j_1 j_3 - j_2 \rvert$	j_{tot}
7	11	3	3 4 5 6 7 8 9 10 11
6	10	2	2 ~ 10
5	9	1	1 ~ 9
4	8	0	0 ~ 8
3	7	1	1 ~ 7

j_{tot} : 0 ~ 11 , 12 Values

Finally do $j_2 j_3$, j_1

$j_2 + j_3 = 6$

$|j_2 - j_3| = 2$

$j_2 j_3$	$j_2 j_3 + j_1$	$\lvert j_2 j_3 - j_1 \rvert$	j_{tot}
6	11	1	1~11
5	10	0	0~10
4	9	1	1~9
3	8	2	2~8
2	7	3	3~7

allow 0~11 12 values

∴ it doesn't matter how to combine, always 12 values

b) $Tot = N(m_1) \times N(m_2) \times N(m_3)$

$= (2j_1+1)(2j_2+1)(2j_3+1)$

$= (2\times5+1)(2\times4+1)(2\times2+1)$

$= 11 \times 9 \times 5$

$= 11 \times 45$

$= 495$

c) total states($|j, m, 5, 4, 2\rangle$)

$$= \sum_{j=0}^{11} (2j+1)$$

$$= 144 < 495$$

$\therefore$ Not Enough! There are too many ways to get j_{tot} from 3 spins!

How many copies? (Using computer program go through all combs)

j_{tot}	copies
1	3
2	4
3	5
4	5
5	5
6	5
7	5
8	4
9	3
10	2
11	1
0	1

total state of each j is just $(2j+1) \times$ "copies"

do $j_{tot} = \sum_{i=0}^{11} (2j_i+1) \times$ "Copies_i"

$$= 495$$

Problem 2.

(Followed the note a bit)

$|\uparrow\uparrow\Uparrow\rangle = |2,2\rangle$ is initial state.

$\frac{1}{2} \oplus \frac{1}{2} \oplus 1$

$$= \left\{ \begin{array}{lll} 0 & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) & \\ 1 & |\uparrow\uparrow\rangle & \Uparrow \\ & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) & 0 \\ & |\downarrow\downarrow\rangle & \Downarrow \end{array} \right\} \oplus 1$$

$\therefore$

$$0 \oplus 1 = \left\{ \begin{array}{ll} 1 & \left\{ \begin{array}{l} |1,1\rangle = (0)\Uparrow = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\Uparrow\rangle - |\downarrow\uparrow\Uparrow\rangle) \\ |1,0\rangle = (0)0 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow 0\rangle - |\downarrow\uparrow 0\rangle) \\ |1,-1\rangle = (0)\Downarrow = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\Downarrow\rangle - |\downarrow\uparrow\Downarrow\rangle) \end{array} \right. \\ 0 & |0,0\rangle = (0)0 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow 0\rangle - |\downarrow\uparrow 0\rangle) \end{array} \right.$$

$\Leftarrow$ (This HW is pledged so there is no way for me to know if this is right, pls leave note if seen)

$$1 \oplus 1 = \left\{ \begin{array}{lll} 2 & |\uparrow\uparrow\rangle & = \Uparrow \\ 1 & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) & = 0 \\ 0 & |\downarrow\downarrow\rangle & = \Downarrow \end{array} \right.$$

2: $|2,2\rangle = |\uparrow\uparrow\Uparrow\rangle$

$|2,1\rangle = \sqrt{\frac{1}{2}}|\uparrow\uparrow 0\rangle + \sqrt{\frac{1}{2}} \times \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)|\Uparrow\rangle$

$= \sqrt{\frac{1}{2}}|\uparrow\uparrow 0\rangle + \frac{1}{2}|\uparrow\downarrow\Uparrow\rangle + \frac{1}{2}|\downarrow\uparrow\Uparrow\rangle$

$|2,0\rangle = \sqrt{\frac{1}{6}}|\Uparrow\Downarrow\rangle + \sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{6}}|\Downarrow\Uparrow\rangle$

$= \sqrt{\frac{1}{6}}|\uparrow\uparrow\Downarrow\rangle + \sqrt{\frac{1}{3}}|\uparrow\downarrow 0\rangle + \sqrt{\frac{1}{3}}|\downarrow\uparrow 0\rangle + \sqrt{\frac{1}{6}}|\downarrow\downarrow\Uparrow\rangle$

$$|2,-1\rangle = \sqrt{\tfrac{1}{2}}\,|0\Downarrow\rangle + \sqrt{\tfrac{1}{2}}\,|\Downarrow 0\rangle$$

$$= \tfrac{1}{2}\,|\uparrow\downarrow\Downarrow\rangle + \tfrac{1}{2}\,|\downarrow\uparrow\Downarrow\rangle + \sqrt{\tfrac{1}{2}}\,|\downarrow\downarrow 0\rangle$$

$$|2,-2\rangle = |\Downarrow\Downarrow\rangle$$

$$= |\downarrow\downarrow\Downarrow\rangle$$

1 :

$$|1,+1\rangle = \sqrt{\tfrac{1}{2}}\,|\Uparrow 0\rangle - \sqrt{\tfrac{1}{2}}\,|0\Uparrow\rangle$$

$$= \sqrt{\tfrac{1}{2}}\,|\uparrow\uparrow 0\rangle - \tfrac{1}{2}\,|\downarrow\uparrow\Uparrow\rangle - \tfrac{1}{2}\,|\uparrow\downarrow\Uparrow\rangle$$

$$|1,0\rangle = \sqrt{\tfrac{1}{2}}\,|\Uparrow\Downarrow\rangle - \sqrt{\tfrac{1}{2}}\,|\Downarrow\Uparrow\rangle$$

$$= \sqrt{\tfrac{1}{2}}\,|\uparrow\uparrow\Downarrow\rangle - \sqrt{\tfrac{1}{2}}\,|\downarrow\downarrow\Uparrow\rangle$$

$$|1,-1\rangle = \sqrt{\tfrac{1}{2}}\,|0\Downarrow\rangle - \sqrt{\tfrac{1}{2}}\,|\Downarrow 0\rangle$$

$$= \tfrac{1}{2}\,|\uparrow\downarrow\Downarrow\rangle + \tfrac{1}{2}\,|\downarrow\uparrow\Uparrow\rangle - \sqrt{\tfrac{1}{2}}\,|\downarrow\downarrow 0\rangle$$

Problem 3

Choose $\psi(x) \propto x e^{-\beta x}$

$$V(x) = \begin{cases} ax & x>0 \\ \infty & x<0 \end{cases}$$

Original $\hat{H}$ of form $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m w^2}{2}\hat{x}^2$

has $\hat{H}|\psi\rangle = E|\psi\rangle$

the new hamoltonian take the form

$$\hat{H}' = \frac{\hat{p}^2}{2m} + \frac{1}{2}m w^2 \hat{x}^2 + V(x)$$

$$\therefore \quad \hat{H}'|\psi\rangle = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}m w^2 x^2 \psi + V(x)\psi$$

$$\langle\psi|\hat{H}|\psi\rangle = \int_0^\infty \psi^* \hat{H}\psi \, dx$$

$$= \int_0^\infty -\frac{\hbar^2}{2m}\psi^*\frac{\partial^2}{\partial x^2}\psi + \psi^*\frac{1}{2}m w^2 x^2\psi + \psi^* a x \psi \, dx$$

$$= \frac{3am\beta + \hbar\beta^4 + 3m^2w^2}{8m\beta^5}$$

This equation does not have a minimum for positive values of β.

The minimum should follow $\beta \geq 0$ or $Im(\beta) \neq 0$.

$$\frac{d}{d\beta}\frac{3am\beta + \hbar\beta^4 + 3m^2w^2}{8m\beta^5} = -\frac{1}{8m\beta^6}\left(12am\beta + \hbar\beta^4 + 15m^2w^2\right)$$

One possible way by observation is such $12am\beta + \hbar\beta^4 + 15m^2w^2 = 0$

$$\therefore \hbar\beta^4 = -12am\beta - 15m^2w^2$$

$$\therefore E' = \frac{-9am\beta - 12m^2w^2}{8m\beta^5}$$

$\frac{d}{d\beta}E' = 0$, $\beta = \frac{\sqrt[6]{\frac{5}{6}}\, w^{\frac{1}{3}}}{a^{\frac{1}{6}}}$, plug back for E

if take $\psi = xe^{-\gamma x^2}$, then

$$E' = \langle \psi | H' | \psi \rangle$$

$$= \frac{16 am\sqrt{\beta} + 3\sqrt{2\pi}\,(4\hbar^2\beta^2 + m^2\omega^2)}{128 m \beta^{\frac{5}{2}}}$$