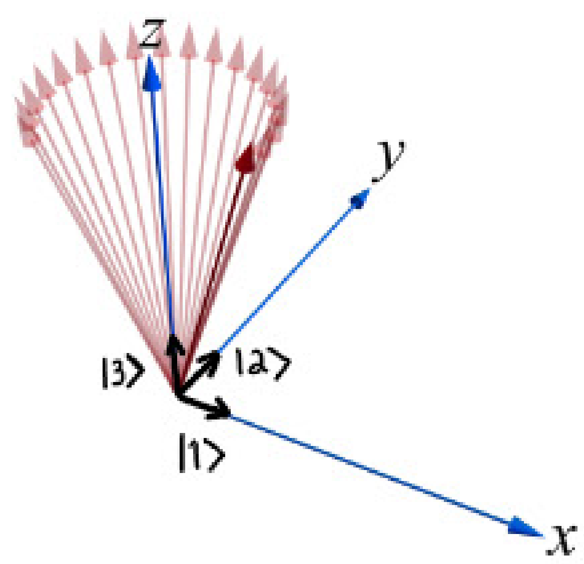
$\mathbb{V}^3(\mathbb{C})$, Generators of Rotations, and Eigenstates of Rotation



Consider $\mathbb{V}^3(\mathbb{C})$. Generic state: $|v\rangle = \sum_{i=1}^{3} v_i |i\rangle$

$\{|i\rangle\}$: orthonormal basis, can "picture" as unit vectors along x, y, z axes

Lec 3, p. 5, we introduced continuous rotation operators

e.g., $\hat{R}_z(\theta) \equiv$ CCW rotation by angle θ around z-axis

In the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis: $\hat{R}_z(\theta) \Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $|i'\rangle = \hat{R}_z(\theta)|i\rangle$

$$\begin{array}{ccc} \Uparrow & \Uparrow & \Uparrow \\ |1'\rangle & |2'\rangle & |3'\rangle \end{array}$$

Adjoint of op. Lec. 3, p. 5 = transpose, conjugate of 3x3 matrix (in a basis)

$$\hat{R}_z^{-1}(\theta) = \hat{R}_z(-\theta) = \hat{R}_z^{\dagger}(\theta)$$

$\therefore \hat{R}_z(\theta)\hat{R}_z^{\dagger}(\theta) = \hat{\mathbb{I}} \Rightarrow \hat{R}_z(\theta)$ is unitary

In fact, $\hat{R}_z(\theta)$ is a special kind of unitary operator, because $\hat{R}_z(\theta)\hat{R}_z^{T}(\theta) = \hat{\mathbb{I}}$. "Orthogonal transformation"

$T \leftarrow$ transpose, not conjugate

Lec. 6, p. 3: Can write a unitary op. $\hat{U}$ as exp. of antihermitian op. $\hat{G}$

$$\hat{U} = e^{\hat{G}} \; ; \quad \hat{G}^{\dagger} = -\hat{G}$$

Consider an Infinitesimal Rotation: $|\theta| \ll 1 \Rightarrow \cos\theta \simeq 1 - \mathcal{O}(\theta^2)$; $\sin\theta \simeq \theta - \mathcal{O}(\theta^3)$

$\therefore \hat{R}_z(\theta) = \hat{\mathbb{I}} + \theta\hat{G}_3 + \mathcal{O}(\theta^2)$; $\hat{G}_3 = -\hat{G}_3^{\dagger} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ HW: Prove $\hat{R}_z(\theta) = e^{\theta\hat{G}_3}$

Similarly, $\hat{R}_x(\theta) = e^{\theta\hat{G}_1} = \hat{\mathbb{I}} + \theta\hat{G}_1 + \cdots$

$\hat{R}_y(\theta) = e^{\theta\hat{G}_2} = \hat{\mathbb{I}} + \theta\hat{G}_2 + \cdots \quad \hat{G}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{G}_2 = \begin{bmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\{\hat{G}_i\}$: "Generators" of Rotations

In fact, can generate a CCW rotation about an arbitrary axis:

$$\hat{R}(\vec{\theta}) \equiv e^{\sum_{i=1}^{3}\theta_i\hat{G}_i} = e^{\vec{\theta}\cdot\vec{\hat{G}}}, \text{ where } \vec{\hat{G}} \equiv \hat{G}_1\vec{n}_1 + \hat{G}_2\vec{n}_2 + \hat{G}_3\vec{n}_3$$

"Vector of Operators"

NOTATION: $\hat{R}(\vec{\theta}) = e^{\vec{\theta}\cdot\hat{\vec{G}}}$; $\vec{\theta} = \theta_1 \vec{n}_1 + \theta_2 \vec{n}_2 + \theta_3 \vec{n}_3$ ORDINARY REAL 3-VECTOR ; $\hat{\vec{G}} = \hat{G}_1 \vec{n}_1 + \hat{G}_2 \vec{n}_2 + \hat{G}_3 \vec{n}_3$ VECTOR OF OPERATORS ; HERE, $\{\vec{n}_i\}$ DENOTE AN ORTHONORMAL SET OF UNIT VECTORS ALONG THE x-, y-, z- AXES: $\vec{n}_i \cdot \vec{n}_j = \delta_{ij}$, $i,j \in \{1,2,3\}$

- WE DO NOT USE "KET" NOTATION FOR THE BASIS VECTORS $\{\vec{n}_i\}$
- WHY? BECAUSE THESE APPEAR ONLY IN THE PARAMETERIZATION (DEF'N) OF THE OPERATORS $\hat{R}(\vec{\theta})$ OR $\hat{\vec{G}}\cdot\vec{\theta}$
- OPERATORS ACT ON KETS $|v\rangle \in \mathbb{V}^d$, HERE $d=3$

IN OTHER WORDS,

① A GENERIC ROTATION OP. $\hat{R}(\vec{\theta}) \equiv e^{\theta_1 \hat{G}_1 + \theta_2 \hat{G}_2 + \theta_3 \hat{G}_3}$

$= e^{\vec{\theta}\cdot\hat{\vec{G}}}$

JUST SHORT-HAND NOTATION USING $\{\vec{n}_i\}$ BASIS VECTORS

② $\hat{R}(\theta)$ ACTS ON $|v\rangle \in \mathbb{V}^3$, i.e. HAS

MATRIX ELEMENTS $\langle i | \hat{R}(\theta) | j \rangle = \langle i | \left[\hat{\mathbb{I}} + \vec{\theta}\cdot\hat{\vec{G}} + \ldots \right] | j \rangle$

$= \delta_{ij} + \theta_k (\hat{G}_k)_{ij} + \ldots$

SUM OVER REPEATED DUMMY INDEX k

"GENERATORS" OF ROTATIONS

$\hat{G}_1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$; $\hat{G}_2 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$; $\hat{G}_3 \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

LEVI-CIVITA SYMBOL: ϵ_{ijk}, $i,j,k \in \{1,2,3\}$ "FULLY ANTISYMMETRIC TENSOR IN 3D"

PROPERTIES: ① $\epsilon_{123} \equiv +1$

② ϵ_{ijk} IS ANTISYMMETRIC UNDER ANY ODD PERMUTATION OF ITS 3 INDICES

$\Rightarrow \epsilon_{ijk} = -\epsilon_{jik} = +\epsilon_{jki}$

$= -\epsilon_{ikj} = +\epsilon_{kij}$

$= -\epsilon_{kji} =$

ONE PERM. FROM ϵ_{ijk}, = 1 SWAP OF ANY 2 INDICES

TWO PERM. FROM ϵ_{ijk}

= 2 PAIRWISE INDEX SWAPS, OR

= A CYCLIC PERMUTATION $(ijk) \to (jki) \to (kij)$

(SPECIAL TO 3 DIMENSIONS)

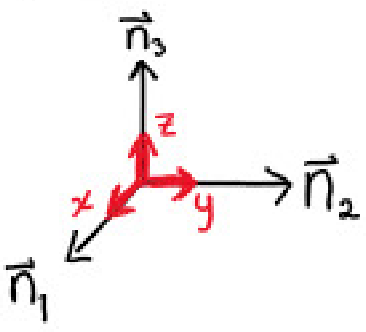
Levi-Civita Symbol... Continued

$\therefore\ \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$; ALL OTHER ELEMENTS VANISH! $\Rightarrow$ ALL OTHER ELEMENTS HAVE A "REPEATED INDEX"

$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$

i.e. $\underbrace{\epsilon_{112}}_{\text{SWAP!}} = -\epsilon_{112} = 0$!

Useful in Math / Physics? Cross Products!



LET $\{\vec{n}_i\}$ AGAIN DENOTE UNIT VECTORS ALONG x,y,z DIRECTIONS.

$$\Rightarrow \text{RH RULE:}\quad \left.\begin{aligned} \vec{n}_1\times\vec{n}_2 &= \vec{n}_3 = -\vec{n}_2\times\vec{n}_1 \\ \vec{n}_2\times\vec{n}_3 &= \vec{n}_1 = -\vec{n}_3\times\vec{n}_2 \\ \vec{n}_3\times\vec{n}_1 &= \vec{n}_2 = -\vec{n}_1\times\vec{n}_3 \end{aligned}\right\}$$

CAN WRITE SUCCINCTLY:

$$\vec{n}_i \times \vec{n}_j = \epsilon_{ijK}\,\vec{n}_K$$

EINSTEIN SUM ON "DUMMY" REPEATED INDEX K

MORE GENERAL: LET $\vec{V} = \sum_{i=1}^{3} V_i\,\vec{n}_i$

$$\vec{V}\times\vec{W} = \sum_{i=1}^{3}\sum_{j=1}^{3} V_i W_j\,(\vec{n}_i\times\vec{n}_j) = \sum_{i,j,K=1}^{3} V_i W_j\,\epsilon_{ijK}\,\vec{n}_K$$

$$\boxed{\therefore\ (\vec{V}\times\vec{W})_K = \vec{n}_K\cdot(\vec{V}\times\vec{W}) = \epsilon_{ijK}\,V_i\,W_j}$$

EINSTEIN SUM ON DUMMY INDICES $i,j = 1,2,3$

Now: Back To Rotation Generators

$$\hat{G}_1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} ;\quad \hat{G}_2 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} ;\quad \hat{G}_3 \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ \langle i|\hat{G}_1|K\rangle = (\hat{G}_1)_{iK} = \epsilon_{i1K}$ $\qquad (\hat{G}_2)_{iK} = \epsilon_{i2K}$ $\qquad (\hat{G}_3)_{iK} = \epsilon_{i3K}$

CHECK: $\epsilon_{213} = -1$ ✓, $\epsilon_{312} = +1$ ✓ $\qquad$ CHECK: $\epsilon_{123} = +1$ ✓, $\epsilon_{321} = -1$ ✓ $\qquad$ CHECK: $\epsilon_{132} = -1$ ✓, $\epsilon_{231} = +1$ ✓

$$\boxed{\therefore\ (\hat{G}_j)_{iK} = \epsilon_{ijK}}$$

INFINITESIMAL ROTATION: $\hat{R}(\vec{\theta}) = \hat{\mathbb{I}} + \theta_j(\hat{G}_j) + \ldots$

- ACTION ON A BASIS VECTOR: $(\hat{\mathbb{I}} + \theta_j\,\hat{G}_j + \ldots)|K\rangle \equiv |K'\rangle$
- COMPONENTS IN ORIGINAL BASIS: $\langle i|K'\rangle = \delta_{iK} + \theta_j(\hat{G}_j)_{iK}$

$$= \delta_{iK} + \theta_j\,\epsilon_{ijK}$$

$$\Rightarrow |K'\rangle = \sum_i |i\rangle\langle i|K'\rangle = |K\rangle + \sum_{i=1}^{3}\sum_{j=1}^{3} |i\rangle\,\epsilon_{ijK}\,\theta_j + \ldots \qquad (1)$$

ALTERNATE NOTATION FOR BASIS KETS:

$|1\rangle \Rightarrow |\vec{n}_x\rangle$

$|2\rangle \Rightarrow |\vec{n}_y\rangle$

$|3\rangle \Rightarrow |\vec{n}_z\rangle$

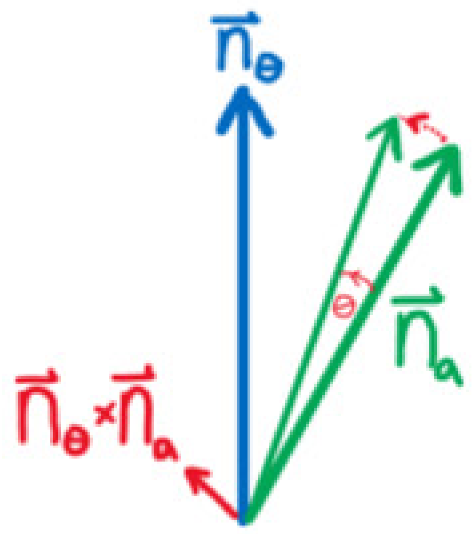
- EQ. (1): $|\vec{n}_a'\rangle = |\vec{n}_a\rangle + \epsilon_{bcd}\,|\vec{n}_b\rangle\,(\vec{\theta})_c\,\overbrace{(\vec{n}_a)_d}^{=\,\delta_{a,d}\,=\,\vec{n}_d\cdot\vec{n}_a} + \ldots$ $\Leftarrow$ EINSTEIN SUM ON b,c,d

$\qquad = |\vec{n}_a\rangle + |\vec{n}_b\rangle\,(\vec{\theta}\times\vec{n}_a)_b$ $\Leftarrow$ EINSTEIN SUM ON b

$$\boxed{\therefore\ |\vec{n}_a'\rangle = |\vec{n}_a\rangle + |\vec{\theta}\times\vec{n}_a\rangle + \mathcal{O}(\vec{\theta})^2}$$

INFINITESIMAL ROTATION: $|\vec{n}_a'\rangle = (\hat{\mathbb{I}} + \vec{\theta}\cdot\hat{\vec{G}} + \ldots)|\vec{n}_a\rangle = |\vec{n}_a\rangle + |\vec{\theta}\times\vec{n}_a\rangle + \ldots$

THIS IS WHAT WE EXPECT FOR THE ACTIVE CCW ROTATION OF A VECTOR $\vec{n}_a$ AROUND AXIS $\vec{n}_\theta$, BY ANGLE θ $(\vec{\theta} = \theta\vec{n}_\theta;\ |\vec{n}_\theta| = 1)$



- 1^{ST} ORDER CHANGE IN VECTOR $\vec{n}_a$ IS $\perp$ TO THAT VECTOR

$\Rightarrow$ PRESERVES (CONVENTIONAL $\mathbb{V}^3(\mathbb{R})$) NORM:

$$\vec{n}_a'\cdot\vec{n}_a' = (\vec{n}_a + \vec{\theta}\times\vec{n}_a + \ldots)\cdot(\vec{n}_a + \vec{\theta}\times\vec{n}_a + \ldots)$$
$$= 1 + \cancelto{0}{2(\vec{\theta}\times\vec{n}_a)\cdot\vec{n}_a} + \mathcal{O}(\theta^2) \checkmark$$

IN THE USUAL ORTHONORMAL BASIS

$|1\rangle \Rightarrow |\vec{n}_x\rangle$ UNIT VEC. ALONG X-AXIS

$|2\rangle \Rightarrow |\vec{n}_y\rangle$ " " y-AXIS

$|3\rangle \Rightarrow |\vec{n}_z\rangle$ " " Z-AXIS

ALL GENERATORS $\hat{G}_1, \hat{G}_2, \hat{G}_3$ ARE OFF-DIAGONAL

- ALREADY KNEW THIS, FROM EXPLICIT MATRIX FORM IN $\{|i\rangle\}$ BASIS

$\Rightarrow$ • BUT NOW WE KNOW WHY: $\hat{G}_j\,\theta_j\,|\vec{n}_a\rangle = |\vec{\theta}\times\vec{n}_a\rangle$,

$\Rightarrow$ INF. ROT. INVOLVES THE CROSS PRODUCT, CHANGE IS $\perp$ TO ORIGINAL VECTOR.

WHAT ARE EIGENSTATES OF ROTATIONS?

DEFINE HERMITIAN GENERATORS OF ROTATION: $\hat{J}_a \equiv i\,\hat{G}_a\ ;\ \hat{J}_a^\dagger = -i\,\hat{G}_a^\dagger = \hat{J}_a$

EXPLICITLY,

$$\hat{J}_x \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}\ ;\quad \hat{J}_y \Rightarrow \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}\ ;\quad \hat{J}_z \Rightarrow \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A FINITE Z-ROTATION IS

$$\hat{R}_z(\theta) = e^{\theta\,\hat{G}_z} = e^{-i\theta\,\hat{J}_z}$$

$\Rightarrow$ ORTHONORMAL EIGENSTATES OF $\hat{J}_z$ ARE ALSO EIGENSTATES OF $\hat{R}_z(\theta)$

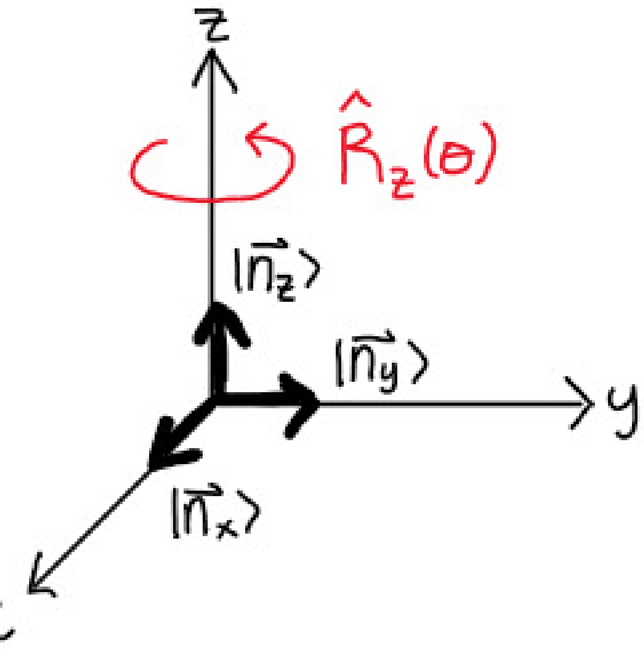
Eigenstates of $\hat{J}_z$, Gen. of Rotations about z-Axis

$\{|\vec{n}_x\rangle, |\vec{n}_y\rangle, |\vec{n}_z\rangle\}$ basis:

$$\hat{J}_z \Rightarrow \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; ; \quad \hat{J}_z |m\rangle = m|m\rangle$$

Usual Business: $\det(\hat{J}_z - m\hat{\mathbb{I}}) = 0 = P_3(m) = -m(m^2-1)$

$\therefore \; m = \{-1, 0, 1\}$



Eigenstates:

① $|m=0\rangle = |\vec{n}_z\rangle \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Makes sense:
Rotation of $|\vec{n}_z\rangle$ along z-axis = No rotation

$$\hat{R}_z(\theta)\,|\vec{n}_z\rangle = e^{-i\theta\hat{J}_z}|m=0\rangle = e^{-i\theta\cdot 0}|m=0\rangle = |\vec{n}_z\rangle \; \checkmark$$

② What about nonzero $m = \pm 1$?

$$\hat{R}_z(\theta)\,|m\rangle = e^{-i\theta\hat{J}_z}|m\rangle = e^{-im\theta}|m\rangle \Rightarrow \left.\begin{matrix} |m=+1\rangle \text{ acquires phase } e^{-i\theta} \\ |m=-1\rangle \text{ acquires phase } e^{i\theta} \end{matrix}\right\} \text{ under a CCW z-axis rotation.}$$

Claim: (Check!)

$$|m=\pm 1\rangle = \frac{1}{\sqrt{2}}\left(|\vec{n}_x\rangle \pm i|\vec{n}_y\rangle\right) \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix}$$

; Check: $\hat{R}_z(\theta)|m\rangle \Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos\theta \mp i\sin\theta \\ \sin\theta \pm i\cos\theta \\ 0 \end{bmatrix} = \frac{e^{\mp i\theta}}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix}$

∴ Eigenvectors of the purely real rotation Op. $\hat{R}_z(\theta)$ [or, equiv., $\hat{G}_z$] necessarily involve complex #'s!

- States $|m=\pm 1\rangle$ would not make sense for position vector of a classical particle

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We can trade the basis $\{|\vec{n}_x\rangle, |\vec{n}_y\rangle, |\vec{n}_z\rangle\}$ for $\{|m\rangle\}$ $(m \in -1, 0, 1)$ via unitary XFM:

Lec. $\frac{5}{m}$, p.1: To diagonalize $\hat{J}_z$, use $\hat{U} \equiv \begin{bmatrix} | & | & | \\ |m=1\rangle & |m=0\rangle & |m=-1\rangle \\ | & | & | \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$

Overall minus sign – choice so that $\hat{J}_x, \hat{J}_y$ take "standard form" (next page)

$$\hat{U}^{\dagger}\hat{J}_z\hat{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Leftarrow \hat{J}_z$$

Now expressed in the $\{|1\rangle, |0\rangle, |-1\rangle\}$ eigenbasis.

? How to think about $|m=\pm 1\rangle$ eigenstates.

$$\hat{R}_z(\theta)\,|\pm 1\rangle = e^{\mp i\theta}|\pm 1\rangle \Rightarrow$$

Suggests states with nonzero m are "spinning" around the z-axis

WHAT ABOUT THE OTHER GENERATORS $\hat{J}_x, \hat{J}_y$? CAN THEY BE SIMULTANEOUSLY DIAGONALIZED?

NO. TWO WAYS TO SEE:

① CONVERT TO $\{|m\rangle\}$ BASIS $[\hat{J}_z|m\rangle = m|m\rangle]$

$$\hat{J}_x \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{J}_y \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

NOT DIAGONAL

② COMMUTATOR (LIE) ALGEBRA

USING EXPLICIT FORM OF $\{\hat{J}_{x,y,z}\}$ AS 3x3 MATRICES (IN EITHER $\{|\vec{n}_{x,y,z}\rangle\}$ OR $\{|m\rangle\}$ BASIS), CAN SHOW:

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z \quad ; \quad [\hat{J}_y, \hat{J}_z] = i\hat{J}_x \quad ; \quad [\hat{J}_z, \hat{J}_x] = i\hat{J}_y$$

⇒ ROTATIONS ABOUT DIFF. AXES <u>DO NOT COMMUTE</u> (LEC. 3, P. 3)

∴ NEITHER DO THE GENERATORS.

- NICE ABBREVIATED NOTATION:

$$\boxed{[\hat{J}_a, \hat{J}_b] = i\epsilon_{abc}\hat{J}_c}$$

"LIE ALGEBRA FOR SO(3)" – SEE DISCUSSION, HW #2

⇒ SO $\hat{J}_x$ AND $\hat{J}_y$ ACT NONTRIVIALLY ON $\hat{J}_z$ E'STATES

RECALL: $|m=\pm 1\rangle = \frac{1}{\sqrt{2}}\left(|\vec{n}_x\rangle \pm i|\vec{n}_y\rangle\right)$; THESE COMBOS ACQUIRE SIMPLE PHASES $e^{\mp i\theta}$ UNDER A Z-ROTATION.

<u>GUESS:</u> CONSIDER ACTION OF $\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y$ ON $\{|m\rangle\}$

LIE ALGEBRA:

- $[\hat{J}_+, \hat{J}_-] = [\hat{J}_x + i\hat{J}_y, \hat{J}_x - i\hat{J}_y] = -i[\hat{J}_x, \hat{J}_y] + i[\hat{J}_y, \hat{J}_x]$
$= 2\cdot\hat{J}_z$

- $[\hat{J}_z, \hat{J}_\pm] = [\hat{J}_z, \hat{J}_x \pm i\hat{J}_y] = i\hat{J}_y \pm i(-i\hat{J}_x) = \pm\hat{J}_\pm$