Problem 1 (2.56 P107)

Suppose $E(x,y,z)$ has

$$E_x = ax \quad E_y = 0 \quad E_z = 0$$

① What is charge density?

② Explain field point one direction but density = const

①: $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$

$$\rho = \varepsilon_0 \left(\frac{\partial}{\partial x} ax + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} 0\right)$$

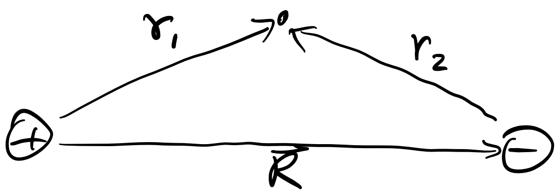
$$= \varepsilon_0 a$$

So ρ is a constant

② There is no constraint on the boundaries of the E field. Then, the endless feature of this field would lead to a lack of condition for the ODE to be solved, and thus allowing a uniform charge density, since we really do not know where this field ends.

Problem 2 (3.2 P264)

a)



$r_1 = x\hat{x} + y\hat{y}$

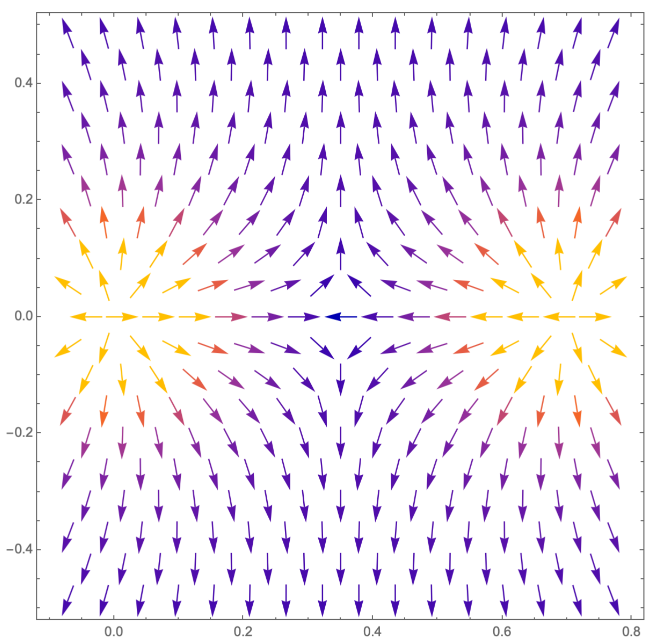
$r_2 = r_1 - \vec{R}$

$= (x-R)\hat{x} + y\hat{y}$

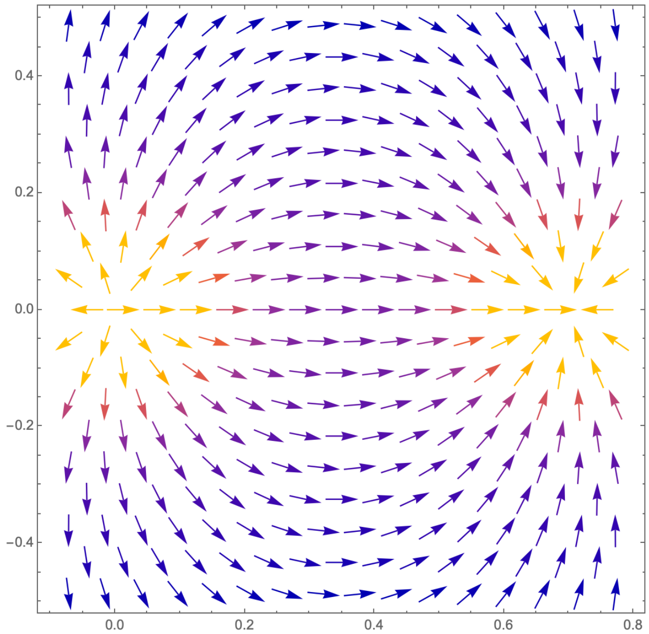
$$Ef = \frac{q}{4\pi\varepsilon_0} \quad \frac{(x-R)\hat{x} + y\hat{y}}{((x-2)^2+y^2)^{\frac{3}{2}}} + \frac{x\hat{x}+y\hat{y}}{(x^2+y^2)^{\frac{3}{2}}}$$ (same

$$Ef = \frac{q}{4\pi\varepsilon_0} \quad \frac{(x-R)\hat{x} + y\hat{y}}{((x-2)^2+y^2)^{\frac{3}{2}}} - \frac{x\hat{x}+y\hat{y}}{(x^2+y^2)^{\frac{3}{2}}}$$ (diff)

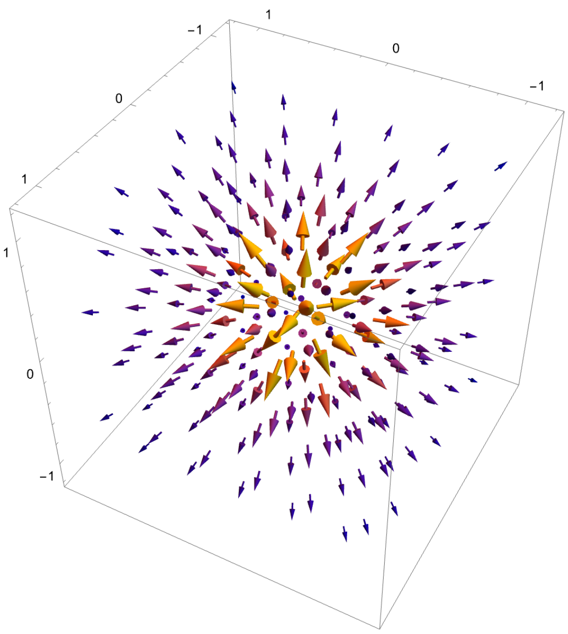
Same



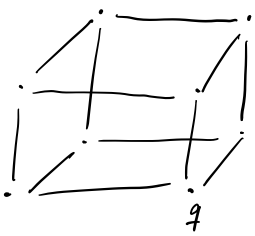
Difference



b) :

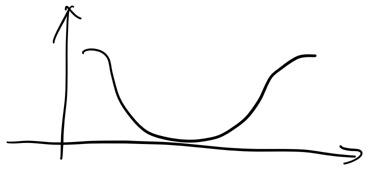


Problem 3,

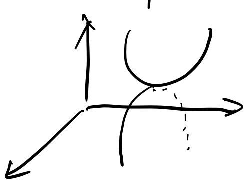


Consider such a field

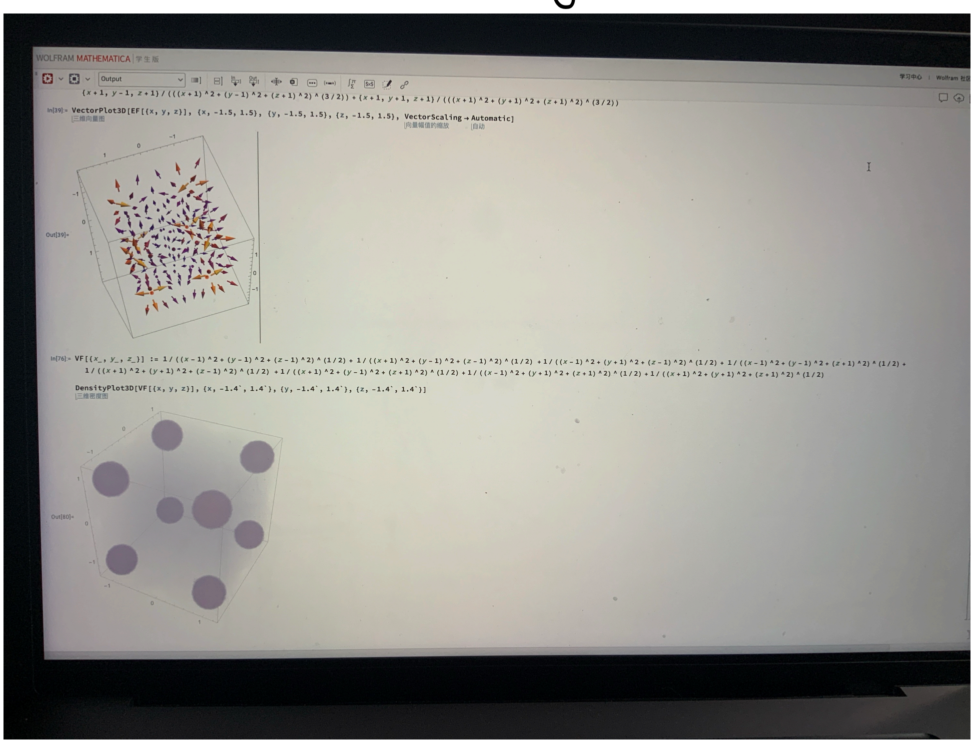
The potential needs to form a well to capture a charge. that is



However, $\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$, that is to say, potential $V(r)$ is laplacian,
A laplace equation require that no local extreme shell be fined if not on boundaries, and the center of any boundaries is average of whats in the boundary.
So we need our "potential well" to be:



So the charge slip away!



Problem 4 (3.3 P118)

Spherical $\nabla^2 t(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial t}{\partial r}\right) = 0$

$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} V(r)\right) = 0$

let $r \neq 0$, $r \neq \infty$

$\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} V(r) = 0$

$r^2 \frac{\partial}{\partial r} V(r) = C$

$\frac{\partial}{\partial r} V(r) = \frac{C}{r^2}$

$V(r) = -\frac{C_1}{r} + C_2$

Cylindrical:

$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial t}{\partial s}\right) = 0$

$\frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial V(s)}{\partial s}\right) = 0$

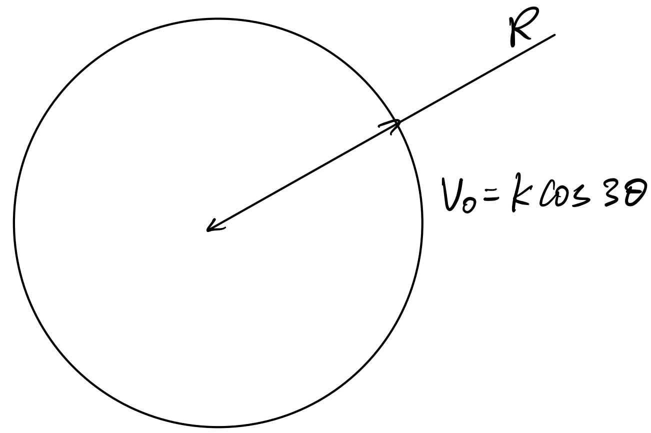
$\frac{\partial}{\partial s}\left(s \frac{\partial V(s)}{\partial s}\right) = 0$

$s \frac{\partial V(s)}{\partial s} = C$

$\frac{\partial V(s)}{\partial s} = \frac{C}{s}$

$V(r) = \ln(C_1 r) + C_2$

Problem 5 (3.21)



General SOL

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

Inside: $B_l = 0$ for $V(0,\theta) \neq \infty$

$$k\cos(3\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\int_0^{\pi} d\theta\, k\cos(3\theta)\, P_l(\cos\theta) = \int_0^{\pi} \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

$$= A_l R^l \frac{2}{2l+1}$$

$4x^3 - 3x$

$$A_l = \frac{2l+1}{R^l} \int_0^{\pi} d\theta\, k\cos(3\theta)\, P_l(\cos\theta)$$

$$= \frac{2l+1}{R^l} \int_0^{\pi} d\theta\, k\left(\cos^3\theta - 3\cos\theta(1-\cos^2\theta)\right) P_l(\cos\theta)$$

$$= \frac{2l+1}{R^l} \int_0^{\pi} d\theta\, k\left(\cos^3\theta + 3\cos^3\theta - 3\cos\theta\right) P_l(\cos\theta)$$

$$= \frac{2l+1}{R^l} \int_0^{\pi} d\theta\, k\left(4\cos^3\theta - 3\cos\theta\right) P_l(\cos\theta)$$

$$= \frac{2l+1}{R^l} \int_0^{\pi} d\theta\, k\,\frac{4}{5}\left(\frac{5}{4}4\cos^3\theta - \frac{5}{4}3\cos\theta\right) P_l(\cos\theta)$$

$$= \frac{2l+1}{R^l} \int_0^{\pi} d\theta\, \frac{4}{5}k\left(5\cos^3\theta - \frac{12}{4}\cos\theta - \frac{3}{4}\cos\theta\right) P_l(\cos\theta)$$

$$= \frac{2l+1}{R^l} \int_0^{\pi} d\theta\, \left(\frac{4}{5}k\left(5\cos^3\theta - 3\cos\theta\right) - \frac{3}{5}k\cos\theta\right) P_l(\cos\theta)$$

$$= \frac{2l+1}{R^l} \int_0^\pi d\theta \left(\frac{4}{5} k \, 2P_3(\cos\theta) - \frac{3}{5} k \, P_1(\cos\theta) \right) P_l(\cos\theta)$$

$$A_1 = -\frac{2+1}{R2} \int_0^\pi d\theta \, \frac{8}{5} k \, P_1^2(\cos\theta)$$

$$= -\frac{3}{5} \frac{k}{R}$$

$$A_3 = \frac{6+1}{2} \frac{8}{R^3 5} k \frac{2}{6+1}$$

$$= \frac{8k}{5R^3}$$

$$\therefore V(r,\theta) = -\frac{3}{5}\frac{k}{R} r \cos\theta + \frac{8}{5}\frac{k}{R^3} r^3 (5\cos^3\theta - 3\cos\theta)/2$$

$$= -\frac{3}{5}\frac{k}{R} r \cos\theta + \frac{4}{5}\frac{k}{R^3} r^3 (5\cos^3\theta - 3\cos\theta)$$

Now look at outside, then A vanishes.

$$V(r,\theta) = \Sigma \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

with BC

$$V(R,\theta) = k\cos(3\theta) = \Sigma \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

that is $\int k\cos(3\theta) P_l(\cos\theta) = \frac{B_l}{R^{l+1}} \frac{2}{2l+1}$ $\frac{4}{5}\left(\frac{5}{4}(4\cos^3\theta) - \frac{5}{4} 3\cos\theta\right)$

$\frac{4}{5}\left(5\cos^3\theta - \frac{12}{4}\cos\theta - \frac{3}{4}\cos\theta\right)$

$$B_l = \frac{2l+1}{2} R^{l+1} \int k (4\cos^3\theta - 3\cos\theta) P_l(\cos\theta) \, d\theta$$

$$= \frac{2l+1}{2} R^{l+1} \int k \left(\frac{8}{5} P_3(\cos\theta) - \frac{3}{5} P_1(\cos\theta) \right) P_l(\cos\theta) \, d\theta$$

$$B_1 = \frac{2+1}{2} R^{1+1} \left(-\frac{3}{5}k\right) \frac{2}{2+1}$$

$$= -\frac{3}{5}kR^2$$

$$B_3 = \frac{6+1}{2} R^{3+1} \; \frac{8}{5}k \; \frac{2}{6+1}$$

$$= \frac{8}{5}kR^4$$

$$V(r,\theta) = -\frac{3}{5}kR^2 \frac{1}{r^2}\cos\theta + \frac{8}{5}kR^4 \frac{1}{r^3}(5\cos^3\theta - 3\cos\theta)/2$$

$$= -\frac{3}{5}kR^2 \frac{1}{r^2}\cos\theta + \frac{4}{5}kR^4 \frac{1}{r^4}(5\cos^3\theta - 3\cos\theta)$$

By discontinuity BC,

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\Big|_R = -\frac{1}{\varepsilon_0}\sigma$$

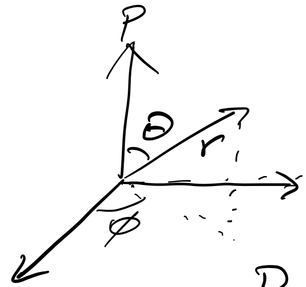
$$\therefore \; -\frac{3}{5}kR^2(-2)r^{-3}\cos\theta + \frac{4}{5}kR^4(-4)r^{-5}(5\cos^3 - 3\cos)$$
$$+\frac{3}{5}\frac{k}{R}\cos\theta - \frac{4}{5}\frac{k}{R^3}\, 3r^2(5\cos^3 - 3\cos) \qquad \Big|_R = -\frac{1}{\varepsilon_0}\sigma$$

$$\frac{3+6}{5}\frac{k}{R}\cos\theta - \frac{12+16}{5}\frac{k}{R}(5\cos^3\theta - 3\cos\theta) = -\frac{1}{\varepsilon_0}\sigma$$

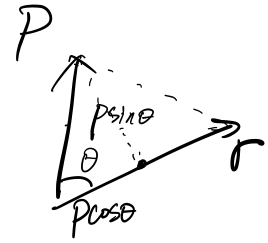
$$\sigma = -\varepsilon_0 \frac{k}{R}\left(\frac{9}{5}\cos\theta - \frac{28}{5}(5\cos^3\theta - 3\cos\theta)\right)$$

$$\therefore \begin{cases} V_{in} = -\frac{3}{5}\frac{k}{R} r\cos\theta + \frac{4}{5}\frac{k}{R^3} r^3 (5\cos^3\theta - 3\cos\theta) \\ V_{out} = -\frac{3}{5}k\frac{R^2}{r^2}\cos\theta + \frac{4}{5}k\frac{R^4}{r^4}(5\cos^3\theta - 3\cos\theta) \\ \sigma = -\varepsilon_0\frac{k}{R}\left(\frac{9}{5}\cos\theta - \frac{28}{5}(5\cos^3\theta - 3\cos\theta)\right) \end{cases}$$

Problem 6:



$$E_{dip}(r,\theta) = \frac{P}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(2P\cos\theta\,\hat{r} + P\sin\theta\,\hat{\theta}\right)$$

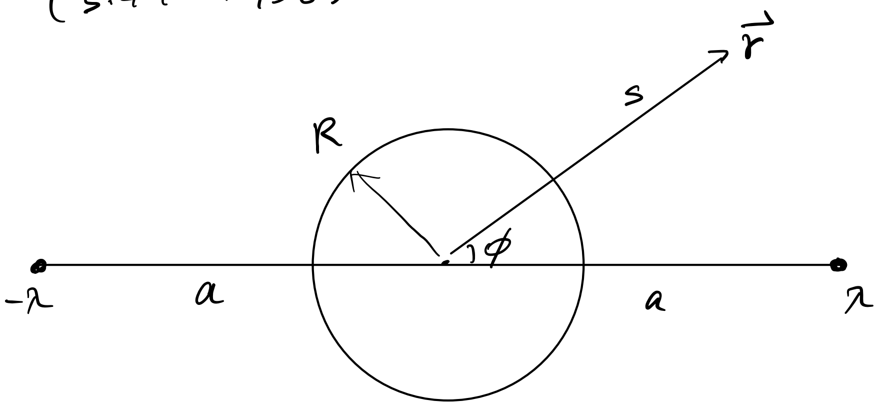
$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(2P\cos\theta\,\hat{r} + P\sin\theta\,\hat{\theta} + P\cos\theta\,\hat{r} - P\cos\theta\,\hat{r}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3P\cos\theta\,\hat{r} + P\sin\theta\,\hat{\theta} - P\cos\theta\,\hat{r}\right)$$

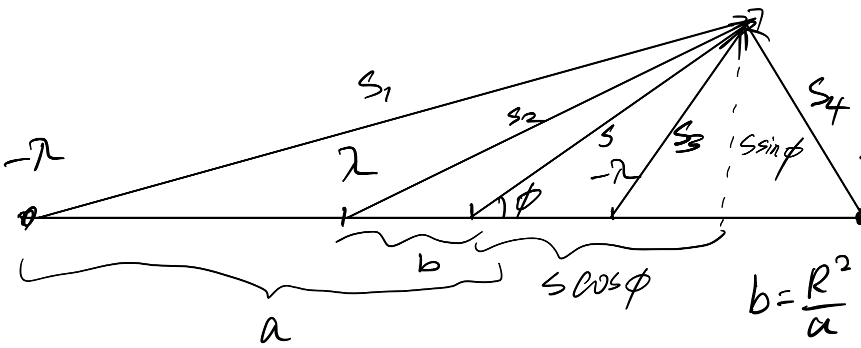
$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3(\vec{P}\cdot\hat{r})\hat{r} + \hat{P}\times\hat{r} - (\hat{P}\cdot\hat{r})\hat{r}\right) \qquad \varepsilon_{ijk} P_i r_j\,\hat{\theta} - \delta_{Pi} r_i\,\hat{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3(\hat{P}\cdot\hat{r})\hat{r} - \hat{P}\right)$$

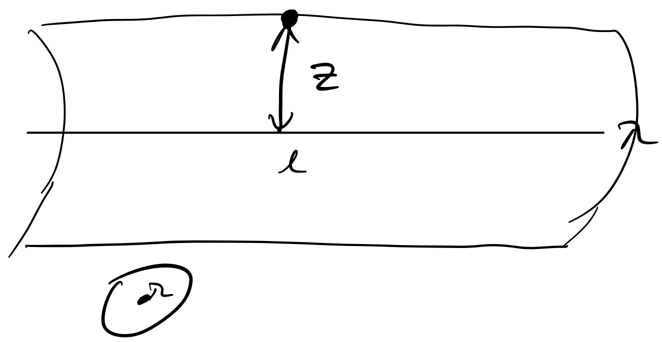
Problem 7 (3.44 P158)



from example 3.2:



examine one wire



$$V = -\int_0^r E \cdot dl$$

$$\oint E\, da = \frac{Q_{enc}}{\varepsilon_0}$$

$$\therefore E \cdot 2\pi z l = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi z \varepsilon_0}$$

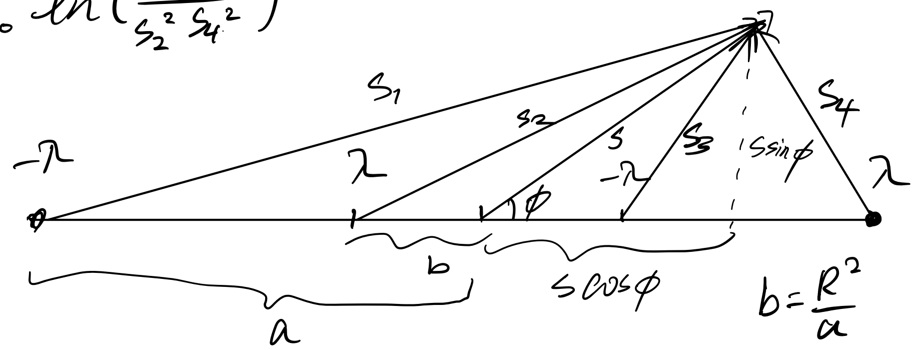
$$\vec{E} = \frac{\lambda}{2\pi z \varepsilon_0} \hat{z}$$

$$\therefore V = -\int \frac{\lambda}{2\pi z' \varepsilon_0} dz' \Big|_z = \frac{\lambda}{2\pi\varepsilon_0} \ln(z') \Big|_{z'=z} = -\frac{\lambda}{2\pi\varepsilon_0} \ln(z) \quad \text{as } V(\infty) = 0$$

$$\therefore \quad V(s,\phi) = -\frac{\lambda}{2\pi\varepsilon_0}\left(-\ln(s_1) + \ln(s_2) - \ln(s_3) + \ln(s_4)\right)$$

$$= -\frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{s_2 s_4}{s_1 s_3}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{s_1^2 s_3^2}{s_2^2 s_4^2}\right)$$



$$s_1^2 = (a + s\cos\phi)^2 + s^2\sin^2\phi$$

$$= a^2 + 2sa\cos\phi + s^2$$

$$s_2^2 = b^2 + 2sb\cos\phi + s^2 = \frac{R^4}{a^2} + 2s\frac{R^2}{a}\cos\phi + s^2 = \frac{a^2}{R^2}\left(R^2 + 2sa\cos\phi + \left(\frac{sa}{R}\right)^2\right)$$

$$s_3^2 = b^2 - 2sb\cos\phi + s^2 = \frac{R^4}{a^2} - 2s\frac{R^2}{a}\cos\phi + s^2 = \frac{a^2}{R^2}\left(R^2 - 2sa\cos\phi + \left(\frac{sa}{R}\right)^2\right)$$

$$s_4^2 = a^2 - 2sa\cos\phi + s^2$$

$$\therefore \quad V(s,\phi) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{(s^2 + a^2 + 2sa\cos\phi)\left[(sa/R)^2 + R^2 - 2sa\cos\phi\right]}{(s^2 + a^2 - 2sa\cos\phi)\left[(sa/R)^2 + R^2 + 2sa\cos\phi\right]}\right)$$

Now we need to force BC:

$V(0,\phi) = 0$: $\quad V = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{(s^2)\left(\left(\frac{sa}{R}\right)^2\right)}{(s^2)\left(\left(\frac{sa}{R}\right)^2\right)}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln(1) = 0$

$V(\infty,\phi) = 0 \quad V = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{a^2 R^2}{a^2 R^2}\right) = 0$

$\therefore$ BC, fulfilled, only solution.