Rotation Generators and Eigenstates, Continued...

$\hat{J}_z |m\rangle = m|m\rangle$, $m \in \{1,0,-1\}$; $|m=\pm 1\rangle = \frac{1}{\sqrt{2}}(|\vec{n}_x\rangle \pm i|\vec{n}_y\rangle)$, $|m=0\rangle = |\vec{n}_z\rangle$

$\hat{R}_z(\theta)|m\rangle = e^{-i\theta \hat{J}_z}|m\rangle = e^{-im\theta}|m\rangle$ $\Rightarrow$ States $|m=\pm 1\rangle$ acquire phases under Rotation

Action of other Generators: Not diagonalized by $\{|m\rangle\}$ basis

$[\hat{J}_a, \hat{J}_b] = i\epsilon_{abc}\hat{J}_c$; $\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y$

- $[\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm$; $[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z$

Let's consider the action of $\hat{J}_\pm$ on a $\hat{J}_z$ eigenstate:

$\hat{J}_\pm |m\rangle \equiv |m,\pm\rangle$; $\hat{J}_z\hat{J}_\pm|m\rangle = [\hat{J}_\pm\hat{J}_z \pm \hat{J}_\pm]|m\rangle = (m\pm 1)\hat{J}_\pm|m\rangle$

$\therefore$ $|m,\pm\rangle$ is an eigenstate of $\hat{J}_z$ with eigenvalue $(m\pm 1)$

$\Rightarrow$ $\hat{J}_+$ raises the E'value of $\hat{J}_z$ by 1 "Raising operator"

$\hat{J}_-$ lowers the E'value of $\hat{J}_z$ by 1 "Lowering operator"

Can also see this explicitly:

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \underset{\substack{\{|m\rangle\}\\ \text{basis}}}{\Longrightarrow} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} ; \quad |m=1\rangle \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |m=0\rangle \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |m=-1\rangle \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $\hat{J}_+|m=1\rangle = 0$ $|m=1\rangle$ is annihilated by the raising op.: cannot be raised further

"Highest weight" (biggest m) state

$\hat{J}_+|m=0\rangle = \sqrt{2}|m=1\rangle$

$\hat{J}_+|m=-1\rangle = \sqrt{2}|m=0\rangle$

$\hat{J}_- = \hat{J}_x - i\hat{J}_y = \hat{J}_+^\dagger$

Summary: Eigenstates of a Rotation Generator; Action of other (Non-commuting) Generators

$\hat{J}_z|m\rangle = m|m\rangle$; $\hat{J}_+|m\rangle = \sqrt{2}|m+1\rangle$

$m \in \{1,0,-1\}$ $\hat{J}_-|m\rangle = \sqrt{2}|m-1\rangle$

Except: $\hat{J}_+|1\rangle = 0$

$\hat{J}_-|-1\rangle = 0$

Quantum Physics: Rotation Generators, Angular Momentum, and SPIN

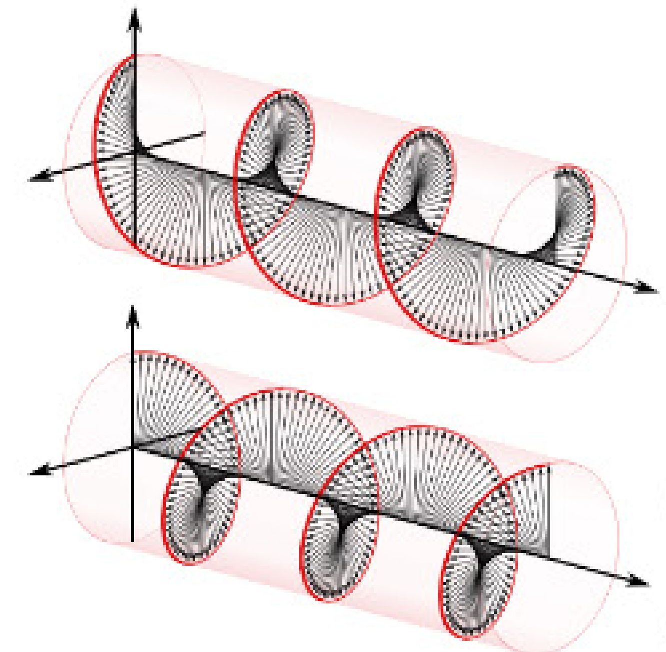
Under z-axis Rotation: $e^{-i\theta \hat{J}_z} |m\rangle = e^{-im\theta} |m\rangle$

$|m=\pm 1\rangle = \frac{|\vec{n}_x\rangle \pm i |\vec{n}_y\rangle}{\sqrt{2}}$

- Acquire a phase under a z-rotation
- "Phasor" notation for Circularly Polarized Light

$m=1 \iff$ CW (RH) Circ. Pol.

$m=-1 \iff$ CCW (LH) Circ. Pol.



$\Rightarrow$ Suggests states $|m=\pm 1\rangle$ are "spinning"

Rotation in Classical Mechanics $\Rightarrow$ Angular Momentum

Ang. Momentum $\vec{L} = \vec{r}\times\vec{p} = (\text{length})\times(\text{mass})\left(\frac{\text{length}}{\text{time}}\right) = \cancel{\text{length}} \times (\text{energy}) \times \left(\frac{\text{time}}{\cancel{\text{length}}}\right) = \text{Energy}\times\text{Time}$ = Units of $\hbar$!

Postulate: Rotation Generators correspond to Components of Angular Momentum in Q.M.

$$\hat{S}_a \equiv \hbar \hat{J}_a \;;\; [\hat{S}_a, \hat{S}_b] = i\hbar \epsilon_{abc} \hat{S}_c \;;\; \hat{S}_z |m\rangle = (m\hbar)|m\rangle$$

- Angular momentum is Quantized! Here $S_z = m\hbar$, $m \in \{-1,0,1\}$
- $\hat{S}_a$ act on a finite (3)-dim. LVS. $\Rightarrow$ Cannot describe "orbital" Ang. Momentum of a particle in Position Space [∞-dim. Hilbert space]

  Later, will study $\hat{L}_a = \epsilon_{abc} \hat{X}_b \hat{P}_c \Rightarrow \epsilon_{abc} x_b \left(-i\hbar \frac{\partial}{\partial x_c}\right)$

  Orbital Ang. Momentum

- Diff. Components of Angular Momentum Do Not Commute!

  $\Rightarrow$ Cannot find simultaneous eigenstate of $\hat{S}_z$ and $\hat{S}_x$ or $\hat{S}_y$

- For Eigenstates of $\hat{S}_z |m\rangle = \hbar m |m\rangle$, $-j \le m \le j$

  $\hat{S}_\pm \equiv \hat{S}_x \pm i \hat{S}_y$ act as "Ladder Operators", Raising, Lowering Ang. Mom. by $\hbar$

$$[\hat{S}_z, \hat{S}_\pm] = \pm(\hbar)\hat{S}_\pm \;;\; [\hat{S}_+, \hat{S}_-] = 2\hbar \hat{S}_z$$

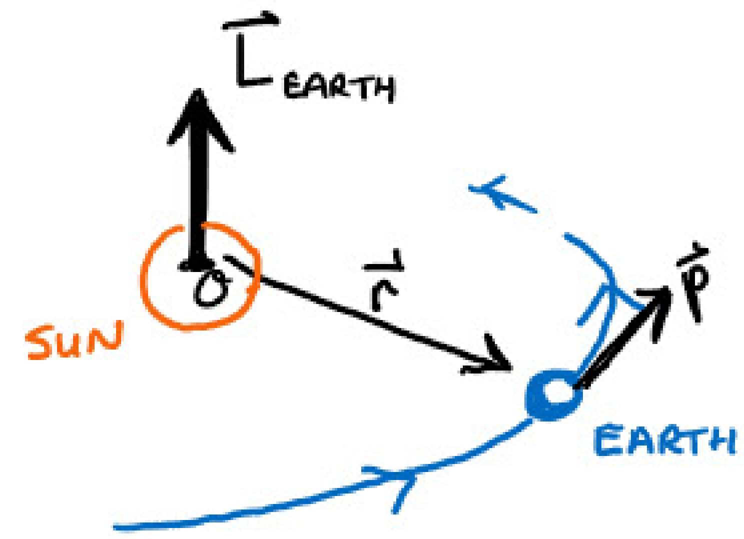
$$\hat{S}_+ |m\rangle = \sqrt{2}\hbar |m+1\rangle \;;\; \hat{S}_+ |j\rangle = 0$$

$$\hat{S}_- |m\rangle = \underbrace{\sqrt{2}\hbar}_{\text{Just a Coeff.}} |m-1\rangle \;;\; \hat{S}_- |-j\rangle = 0$$

z-comp. Ang. Mom. $= \hbar(m-1)$

Here $j=1$

In Classical Mechanics, can measure all 3 components of Ang. Mom. $\{L_x, L_y, L_z\}$ simultaneously



$$; \quad [\hat{S}_a, \hat{S}_b] = \underbrace{i\hbar}\, \hat{S}_c$$

$\Uparrow$

Not true in Quantum!

In a classical mech. problem with Rotational Symmetry [i.e., particle in central force field],

$\vec{L}$ is conserved (both $\hat{L}$ and $|\vec{L}|$)

Quantum version of $\vec{L}^2$: $\quad \hat{\vec{S}}^2 = (\hat{S}^x)^2 + (\hat{S}^y)^2 + (\hat{S}^z)^2 = \hbar^2 j(j+1)\, \hat{\mathbb{I}}$

where $j = \max(m)$

Conserved! Commutes with everything!

For our 3x3 generators

$$\hat{S}_x \Rightarrow \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{S}_y \Rightarrow \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \hat{S}_z \Rightarrow \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$j = 1$. We call this a <u>Spin 1 Quantum System</u>

<u>What is "Spin" Angular Momentum?</u>

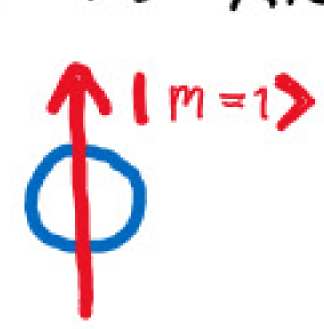
- An intrinsic property of quantum particles
- In relativistic limit, spin and orbital ang. mom. are coupled.
- For non-relativistic (hence, nonzero mass) particles, either elementary (electrons, muons, quarks) or composite (protons, neutrons, atoms)

  Spin is an "internal" degree of freedom, <u>decoupled</u> (or weakly coupled) to orbital d.o.f.

[In fact, the spin of elem. particles (electrons, quarks, photons, etc.) originates from the Representation Theory of the Lorentz Group = Group of Boosts, Rotations. In a deep sense, this is the <u>definition</u> of an elementary particle: a representation of the Lorentz Group! → Q.F.T. (Phys 622)]

– Crucial Exception: spin does interact with MAGNETIC FIELDS, as we will see...

- If you "freeze" orbital motion, e.g. trap an atom [Randy Hulet @ Rice!], can treat spin as an indept., localized degree of freedom



Trapped spin-1 atom with spin state $|m_z = 1\rangle$

Which axis

• SPIN-1 PARTICLES — EXAMPLES: ($j = 1$; $m \in \{-1, 0, 1\}$)

- ^{87}Rb ATOM (USED IN SPINOR BOSE-EINSTEIN COND. EXPERIMENTS)
- $^{3}O_2$ OXYGEN MOLECULE
- PHOTONS [HOWEVER: COMPLICATION DUE TO ZERO MASS, SPIN-ORBIT COUPLING: ONLY $m = \pm 1$ STATES ALLOWED, CW AND CCW CIRCULAR POL. LIGHT. $\Rightarrow m = 0$ NOT A POSSIBLE PHOTON STATE]

• MOST "FAMILIAR" MASSIVE PARTICLES — electrons, protons, neutrons, quarks — HAVE SPIN $j = \frac{1}{2}$. FORMALLY SIMPLER, BUT PHYSICALLY WEIRDER (WE WILL STUDY $j = \frac{1}{2}$ NEXT)

ENERGY EIGENSTATES OF A QUANTUM SPIN: $H|E\rangle = E|E\rangle$

WHAT IS THE HAMILTONIAN (ENERGY) OPERATOR?

CONSIDER A CLASSICAL ANALOG FOR A "SPINNING" PARTICLE: PARTICLE CIRCULATING IN A WIRE LOOP

TOP VIEW:

SUPPOSE PARTICLE HAS CHARGE q, CIRCULATES WITH PERIOD $T = \frac{2\pi R}{v}$; R = RADIUS, v = VELOCITY

CURRENT: $I = \frac{q}{T} = \frac{qv}{2\pi R}$; IF MASS IS m, $L^z = Rmv$

$$\therefore \quad I = \frac{q}{2\pi R}\left(\frac{L^z}{Rm}\right)$$

A BRIEF DIGRESSION: ELECTROMAGNETISM IN "GAUSSIAN" UNITS

STUDENTS USED TO SI ARE OFTEN TERRIFIED BY GAUSSIAN UNITS

QUANTITY	SI UNIT	GAUSSIAN UNIT
CHARGE q	COULOMBS	"STAT COULOMBS" ?? WTF!
VOLTAGE Φ	VOLTS	"STAT VOLTS" ??
	etc.	

KEY: YOU NEVER NEED TO USE THESE UNFAMILIAR UNITS.

JUST FORMULATE IN TERMS OF ENERGIES, MEASURE IN JOULES [OR BETTER FOR QUANTUM, electron-volts eV]

KEY E+M FORMULAE IN GAUSSIAN FOR THIS CLASS:

① LORENTZ FORCE ON CHARGE q: $\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$ c = SPEED OF LIGHT $\sim 3 \times 10^8$ m/s; $\vec{v}$ = PARTICLE VELOCITY

$\vec{E}, \vec{B}$ HAVE SAME UNITS: FORCE / CHARGE

② $\vec{E} = -\vec{\nabla}\Phi$, Φ = ELECTRIC POTENTIAL

③ ELECTRIC POTENTIAL ENERGY: $V(\vec{r}) = \frac{q_1 q_2}{|\vec{r}|} = q_1 \Phi(\vec{r})$ COULOMB POT. ENERGY, CHARGES q_1, q_2 SEPARATED BY DISTANCE $|\vec{r}|$

• UNITS? – MEASURE q_1, q_2 IN TERMS OF $e \equiv 1.602 \times 10^{-19}$ COULOMBS [= MINUS CHARGE OF ELECTRON]

$[V]$ = ENERGY; $\Rightarrow$ $[e^2]$ = ENERGY × LENGTH

• NATURAL UNIT OF ENERGY × LENGTH? $\hbar c$!

$\therefore$ $\frac{e^2}{\hbar c}$ IS DIMENSIONLESS, FUNDAMENTAL CONST.: "FINE STRUCTURE CONSTANT" $\frac{e^2}{\hbar c} \simeq \frac{1}{137}$

$\Longrightarrow$ CAN ALWAYS WRITE $V(\vec{r}) = \underbrace{\left[\frac{q_1 \cdot q_2}{e^2}\right]}_{\text{DIM.LESS}}\underbrace{\left[\frac{e^2}{\hbar c}\right]}_{\text{FUND. DIMLESS CONST.}}\left[\frac{\hbar c}{|\vec{r}|}\right]$; NOW USE SI UNITS FOR $\hbar c$

• $\hbar c = 3.16 \times 10^{-26}$ J·m

$= 1.97 \times 10^{-7}$ eV·m

$= 1975$ eV·Å,

$1\,Å = 10^{-10}$ m "ANGSTROM", NAT. LENGTH SCALE FOR QUANTUM ("SIZE" OF HYDROGEN ATOM)

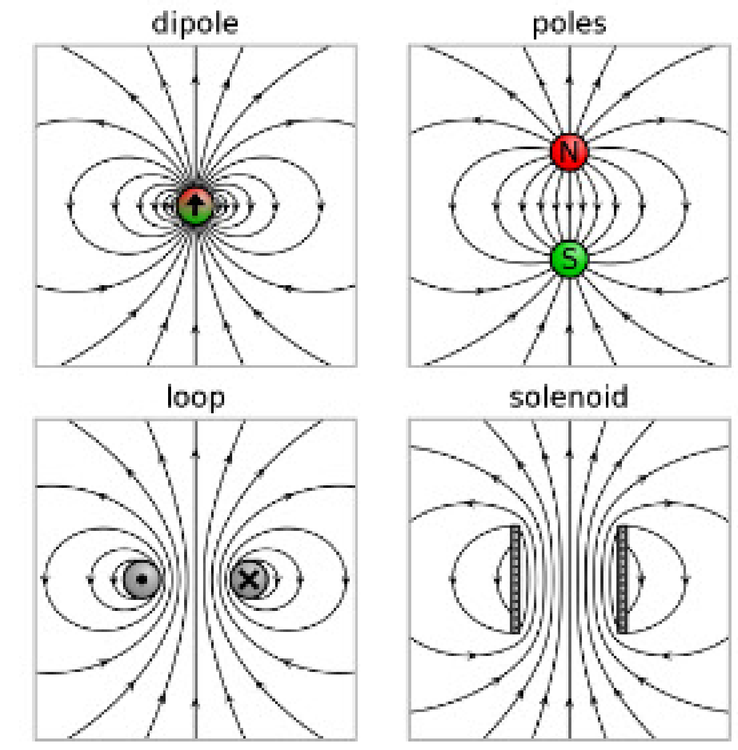
Little More E+M, in Gaussian Units

Magnetic Field due to a Dipole Magnetic Moment:

$$\vec{B}(\vec{r}) = \frac{3\hat{r}(\vec{\mu}\cdot\hat{r}) - \vec{\mu}}{r^3}$$

Mag. Dipole Fields

Image Credit: Wikipedia

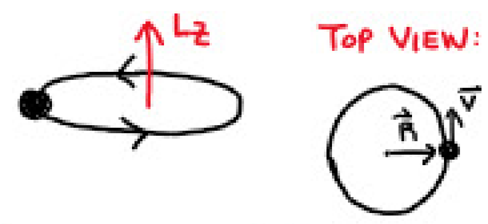


$$[B] = \frac{\text{Force}}{\text{Charge}} = \frac{\text{Energy}}{\text{Length} \times \text{Charge}}$$

Gaussian: $\frac{e^2}{r} = \frac{\text{Charge}^2}{\text{Length}} = \text{Energy}!$

$\vec{\mu}$: Magnetic Dipole Moment ; $[\mu] = \frac{\text{Energy} \times \text{Length}^2}{\text{Charge}} = \text{Charge} \times \text{Length}$

Claim : For current loop with area A



$$\mu = \frac{IA}{c}$$

Derivation: E+M class or Textbook

Prev. Page: $I = \frac{qL^2}{2\pi R^2 m} \Rightarrow \mu = \frac{\cancel{\pi R^2}}{c}\frac{qL^2}{2\cancel{\pi R^2} m}$

$\therefore$ For Classical Current Loop: $\vec{\mu} = \gamma \vec{L}$

$\vec{\mu}$ = Magnetic Moment

$\vec{L}$ = Angular Momentum

$\gamma = \frac{q}{2mc}$ "Gyromagnetic Ratio"

Quantum Spin:

$$\hat{\vec{\mu}} \equiv \gamma \hat{\vec{S}} = \gamma\hbar\left(\frac{\hat{\vec{S}}}{\hbar}\right) ; \quad \gamma = \frac{gq}{2mc}$$

$\therefore$ Quantum spins are "intrinsic" Magnetic Moments!

$\Rightarrow$ Generate Dipole Mag. Fields

$\Rightarrow$ Primarily Responsible for Magnetism in Magnetic Insulating Materials

g : "g-factor"

- Dimensionless Const.
- Value derives from High Energy Physics (QFT), + interaction corrections...

Note: $\gamma\hbar$ = natural unit for intrinsic magnetic moment

- Convention: $\gamma\hbar \equiv g \times \mu_q$, $\mu_q = \frac{q\hbar}{2mc}$ "Bohr Magneton" (dimless g-factor)

Units: $[\mu]$ = Charge $\times$ Length

- Special Case: "Electron Bohr Magneton" $\mu_B = \frac{e\hbar}{2m_e c}$, m_e = electron mass

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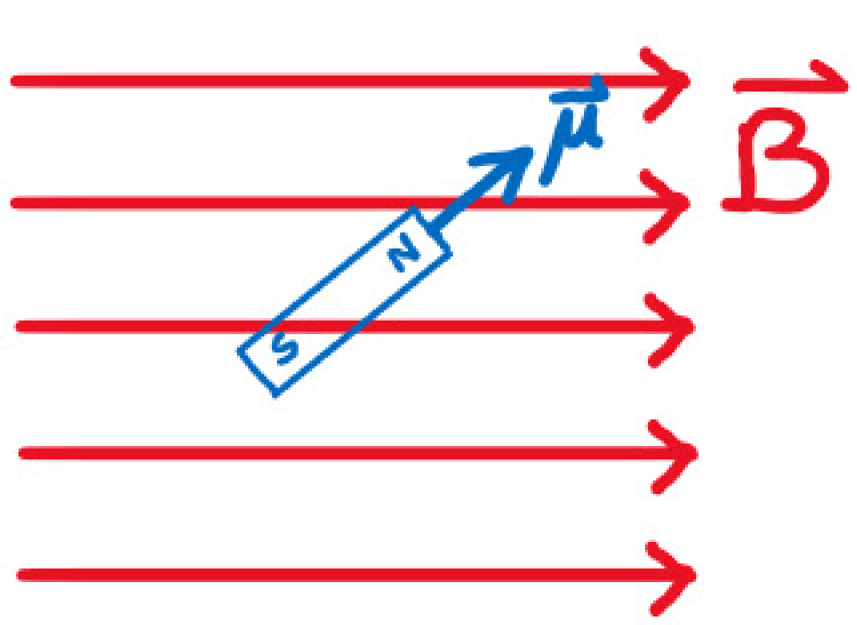
Energy of a Magnetic Moment in an External Magnetic Field:



$$E = -\vec{\mu}\cdot\vec{B}$$

$$[B] = \frac{\text{Energy}}{\text{Length} \times \text{Charge}}$$

$\therefore [\mu][B] = \text{Energy}!$

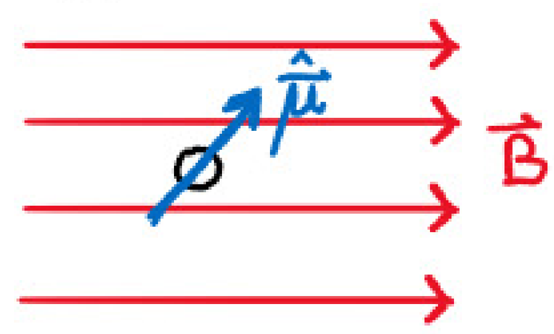
Summary: Quantum Spin-1 Magnetic Moment

- States: $\hat{S}_z |m\rangle = \hbar m |m\rangle$, $-j \le m \le j$, $j=1$ SPIN ONE.

- Ladder operators: $[\hat{S}_z, \hat{S}_\pm] = \pm\hbar \hat{S}_\pm$, $[\hat{S}_+, \hat{S}_-] = 2\hbar \hat{S}_z$

$$\hat{S}_\pm |m\rangle = \sqrt{2}\,\hbar |m \pm 1\rangle \; ; \quad \hat{S}_+ |1\rangle = \hat{S}_- |-1\rangle = 0$$

- Hamiltonian: spin in external magnetic field $\vec{B}$



Classical gyromag. ratio

$$\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B} \; ; \; \hat{\vec{\mu}} = \gamma \hat{\vec{S}} \; ; \; \gamma = \underset{\text{ratio}}{\text{spin gyromagnetic}} = g \times \left(\frac{q}{2mc}\right)$$

"g factor" (dim.less)

Alternate convention:

$$\hat{\vec{\mu}} = \underset{\text{dimless.}}{g} \times \mu_q \times \underset{\text{dim less operator}}{\left(\frac{\hat{\vec{S}}}{\hbar}\right)} \; ; \; \mu_q \equiv \frac{\hbar\gamma}{g} = \frac{q\hbar}{2mc}$$

"Bohr magneton" (units of mag. moment) = charge × length

- Time-Indept. (Static) $\vec{B}$: Energy Eigenstates

Without Loss of Generality (WLoG), choose $\vec{B} = B\,\vec{n}_z$ (or equivalently, rotate into basis s.t. this is true)

$$\therefore \; \hat{H} = -\gamma B \hat{S}_z \; ; \; \hat{H}|m\rangle = E_m |m\rangle, \; E_m = -\gamma B \hbar m = -g(\mu_q) B\, m$$

$\hat{S}_z |m\rangle = m|m\rangle$

- State with $m=1$ ($m=-1$), corresponding to magnetic moment $\hat{\vec{\mu}}$ parallel (antiparallel) to $\vec{B}$ has the lowest (highest) energy.

  - Recall: $|m = \pm 1\rangle = \frac{1}{\sqrt{2}} \left(|\vec{n}_x\rangle \pm i |\vec{n}_y\rangle \right)$

- State $|m=0\rangle = |\vec{n}_z\rangle$ has energy $E_0 = 0$

Although the state is the z-directed basis vector, no magnetic moment along z (no spin projection: $\hat{S}_z |m=0\rangle = 0$.)