② LINEAR VECTOR SPACES

SET OF OBJECTS $\{|1\rangle, |2\rangle, \dots, |v\rangle, |w\rangle, \dots\}$ "VECTORS"

FORM A LINEAR VECTOR SPACE $\mathbb{V}$ IF

① $|v\rangle + |w\rangle = |z\rangle$, $|z\rangle \in \mathbb{V}$ (CLOSURE)

"BELONGS TO"

② $a(|v\rangle + |w\rangle) = a|v\rangle + b|w\rangle$, $a \in \mathbb{C}$ (SCALAR MULTIPLICATION IS DISTRIBUTIVE; a = COMPLEX SCALAR (NUMBER))

MEANS: THE "FIELD" OF COMPLEX NUMBERS

③ $(a+b)|v\rangle = a|v\rangle + b|v\rangle$, $a, b \in \mathbb{C}$

④ $a(b|v\rangle) = ab|v\rangle = ba|v\rangle$

⑤ $|v\rangle + |w\rangle = |w\rangle + |v\rangle$ (ADDITION IS COMMUTATIVE — ORDER DOESN'T MATTER)

⑥ $|v\rangle + (|w\rangle + |z\rangle) = (|v\rangle + |w\rangle) + |z\rangle$ (" " IS ASSOCIATIVE)

⑦ THERE EXISTS A <u>NULL VECTOR</u> $|0\rangle$ SUCH THAT

$$|v\rangle + |0\rangle = |v\rangle$$

⑧ FOR ANY VECTOR $|v\rangle$, THERE EXISTS UNIQUE VECTOR $|-v\rangle$ SUCH THAT

$$|v\rangle + |-v\rangle = |0\rangle \ ; \quad |-v\rangle = -|v\rangle$$ UNIQUE ADDITIVE INVERSE

⑨ $0|v\rangle = |0\rangle$ GET NULL VECTOR BY MULTIPLYING <u>ANY</u> VECTOR $|v\rangle$ BY ZERO.

<u>NOTE:</u> THE SCALAR COEFFICIENTS $a, b, c, \dots$ DEFINE THE "FIELD" OF $\mathbb{V}$.

WE CALL $|v\rangle$ A "KET", FOR REASONS Y'ALL WILL SEE SHORTLY.

EXAMPLES OF LINEAR VECTOR SPACES

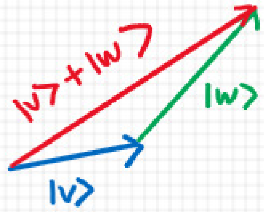
(A) REAL VECTORS IN 3D $\mathbb{V}^3(\mathbb{R})$

$\mathbb{V}^3$ ← DIM. OF VECTOR SPACE

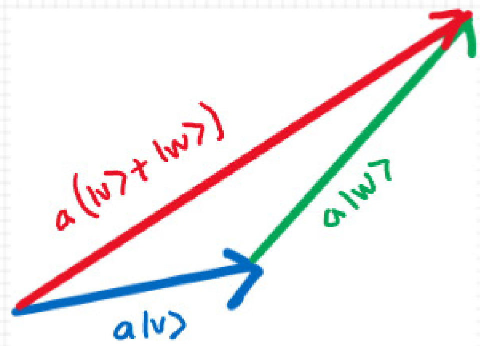
$(\mathbb{R})$ ← FIELD OF REAL NUMBERS

→ WHAT YOU ARE USED TO: "ARROWS" WITH MAGNITUDE, DIRECTION

① VECTOR ADDITION:

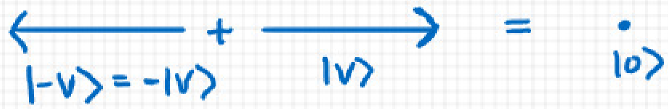


② SCALAR MULTIPLICATION



$a \in \mathbb{R}$

⑧ ADDITIVE INVERSE



* COUNTER EXAMPLE: ALL 3D VECTORS WITH POSITIVE Z-COMPONENTS. WHY NOT A LIN. VECTOR SPACE?

(B) 2x2 MATRICES

① VECTOR ADDITION: $|v\rangle + |w\rangle \Rightarrow \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} V_{11}+W_{11} & V_{12}+W_{12} \\ V_{21}+W_{21} & V_{22}+W_{22} \end{bmatrix}$

⋮

⑦ NULL VECTOR: $|0\rangle \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

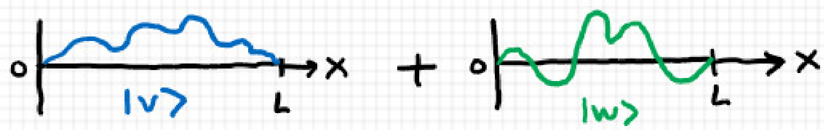
⑧ ADDITIVE INVERSE: $-|v\rangle = |-v\rangle = \begin{bmatrix} -V_{11} & -V_{12} \\ -V_{21} & -V_{22} \end{bmatrix}$

☆ SATISFIES LVS AXIOMS;

BUT: • NO OBVIOUS NOTION OF "MAGNITUDE" OR "DIRECTION"

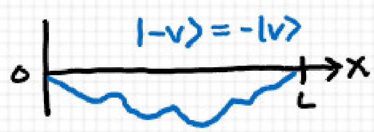
(C) FUNCTIONS $f(x)$ DEFINED OVER THE INTERVAL $0 \le x \le L$, WITH $f(0) = f(L) = 0$

① VECTOR ADDITION: $|v\rangle + |w\rangle =$



⋮

⑧ ADDITIVE INVERSE:



AGAIN: NO OBVIOUS "LENGTH" OR "DIRECTION" OF $f(x) \Rightarrow |f\rangle$

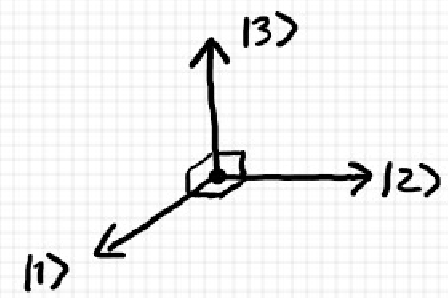
SOME USEFUL DEFINITIONS:

① LINEAR INDEPENDENCE

A SET OF VECTORS $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$ FORM A LINEARLY INDEPT. SET IFF ("IF AND ONLY IF")

$$\sum_{i=1}^{n} a_i |i\rangle = |0\rangle \text{ IMPLIES THAT } \underline{\text{ALL}} \quad a_i = 0, \; i \in \{1, 2, \ldots, n\}$$

ex: ORTHOGONAL BASIS VECTORS IN $\mathbb{V}^3(\mathbb{R})$



NO WAY OF SUMMING

$a_1|1\rangle + a_2|2\rangle + a_3|3\rangle = |0\rangle$, EXCEPT $a_i = 0$

- WOULD BE TRUE EVEN FOR NON-ORTHOG. VECTORS, AS LONG AS LIN. INDEPT.

E.G., $\left.\begin{aligned} |a\rangle &= |1\rangle + \tfrac{1}{2}|2\rangle \\ |b\rangle &= |1\rangle - \tfrac{1}{2}|2\rangle \\ |c\rangle &= |3\rangle - |1\rangle \end{aligned}\right\}$ LIN. INDEPT.

NOTE: IN $\mathbb{V}^3(\mathbb{R})$, WE HAVE A NATURAL "DOT" OR "INNER" PRODUCT: $\vec{V} \cdot \vec{W} = V_i W_i$

$= V_1 W_1 + V_2 W_2 + V_3 W_3$

EINSTEIN SUMMATION CONVENTION: SUM A DOUBLY-REPEATED INDEX OVER ALL APPROP. VALUES

- WE HAVE NOT YET DEFINED A GENERAL INNER PRODUCT FOR $\{|i\rangle\}$ KETS

② DIMENSION OF A VECTOR SPACE $\mathbb{V}^n$

A VECTOR SPACE $\mathbb{V}^n$ IS OF DIMENSION n IF IT CAN ACCOMMODATE A MAXIMUM OF n LIN. INDEPT. VECTORS

$\mathbb{V}^3(\mathbb{R})$: 3D VECTORS WITH REAL COEFFICIENTS, $n=3$

$\mathbb{V}^n(\mathbb{C})$: VECTORS WITH COMPLEX COEFFICIENTS, DIM. n ⟵ WHAT WE NEED FOR QUANTUM

THEOREM 1: ANY $|v\rangle$ IN $\mathbb{V}^n$ CAN BE EXPANDED IN n LINEARLY INDEPT. VECTORS $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$

(1) $$|v\rangle = \sum_{i=1}^{n} V_i |i\rangle$$

- V_i : "COMPONENT" OF $|v\rangle$ IN "$|i\rangle$" DIRECTION
- $|i\rangle$: A "BASIS" VECTOR (FOR EXPANDING $|v\rangle$)

PROOF (YAWN): IF THERE EXISTS $|v\rangle$ THAT CANNOT BE WRITTEN AS IN (1), ABOVE, THEN $\{|1\rangle, |2\rangle, \ldots, |n\rangle, |v\rangle\}$ FORM A LIN. INDEPT. BASIS FOR $(n+1)$-D SPACE ⟶ NOT $\mathbb{V}^n$ (CONTRADICTION)

VECTOR ADDITION: $|v\rangle = \sum_{i=1}^{n} v_i |i\rangle$, $|w\rangle = \sum_{j=1}^{n} w_j |j\rangle$; $|v\rangle + |w\rangle = \sum_{i=1}^{n} (v_i + w_i)|i\rangle$

SCALAR MULTIPLICATION: $a|v\rangle = \sum_{i=1}^{n} (a v_i)|i\rangle$

INNER PRODUCT

$\mathbb{V}^3(\mathbb{R})$: $\vec{A}\cdot\vec{B} = |A||B|\cos\theta$ DOT PRODUCT

$\sqrt{\vec{A}\cdot\vec{A}} = \sqrt{|A|^2} = |A|$ "NORM" OF $\vec{A}$

TRY TO GENERALIZE THIS IN A NATURAL WAY:

DEFINE THE INNER PRODUCT OF $|v\rangle, |w\rangle \in \mathbb{V}^n(\mathbb{C})$

$\hookrightarrow \equiv \langle v|w\rangle$

COMPLEX CONJUGATE: $z = x + iy$; $z^* = x - iy$ x, y REAL

PROPERTIES (THAT WE ARE ASSERTING):

① $\langle v|w\rangle = \langle w|v\rangle^*$ "SKEW SYMMETRY"

FOR REAL VECTORS IN $\mathbb{V}^n(\mathbb{R})$, REDUCES TO $\langle v|w\rangle = \langle w|v\rangle$

② $\langle v|v\rangle \equiv |v|^2 \geq 0$; $|v|^2 = 0$ ONLY FOR $|v\rangle = |0\rangle \equiv 0$

NULL VECTOR

SLIGHT ABUSE OF NOTATION

DEF: NORM OF $|v\rangle$: $|v| \equiv \sqrt{\langle v|v\rangle}$

③ $\langle v|\left(a|w\rangle + b|z\rangle\right) = a\langle v|w\rangle + b\langle v|z\rangle$, $a, b \in \mathbb{C}$ "LINEARITY"

④ $$\left(\langle aW| + \langle bZ|\right)|v\rangle = \langle aW + bZ|v\rangle$$
$$= \langle v|aW + bZ\rangle^* \quad \text{(USING ①, ABOVE)}$$
$$= a^*\langle v|w\rangle^* + b^*\langle v|z\rangle^*$$
$$= a^*\langle w|v\rangle + b^*\langle z|v\rangle \quad \text{"ANTI-LINEARITY"}$$

COEFF.'S FOR VECTOR ON LHS ARE COMPLEX CONJUGATED!

DEF.: ORTHOGONALITY

VECTORS $|v\rangle, |w\rangle$ ARE ORTHOGONAL IF

$$\langle v|w\rangle = 0.$$

EXPANSION IN AN ORTHONORMAL BASIS

LET $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$ BE A LINEARLY INDEPT. BASIS FOR $\mathbb{V}^n(\mathbb{C})$

A NICE CHOICE (WHICH IN QUANTUM WE WILL ALMOST ALWAYS MAKE):

ORTHONORMAL BASIS:
$$\left.\begin{array}{l} \langle i|j\rangle = 0, \ i \neq j \\ \langle i|i\rangle = 1, \ \text{FOR ALL } i \end{array}\right\} \quad \langle i|j\rangle = \delta_{ij}$$

KRONECKER DELTA
= i,j MATRIX ELEMENT OF IDENTITY $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$
= 1 FOR $i = j$, 0 ELSE

THEN, $|v\rangle = \sum_{i=1}^{n} v_i |i\rangle$

$|w\rangle = \sum_{j=1}^{n} w_j |j\rangle$;

- $\langle v|w\rangle = \sum_{i,j=1}^{n} v_i^* w_j \langle i|j\rangle = \sum_{i=1}^{n} v_i^* w_i$
- $\langle v|v\rangle = |v|^2 = \sum_{i=1}^{n} |v_i|^2 \geq 0$ ✓ SQUARED NORM OF $|v\rangle$

ASIDE: HOW TO NORMALIZE A VECTOR

$$|v\rangle_N \equiv \frac{|v\rangle}{|v|}$$

$$_N\langle v|v\rangle_N = \frac{\langle v|v\rangle}{\langle v|v\rangle} = 1 ✓$$

EXPANSION IN ORTHONORMAL BASIS, CONTINUED:

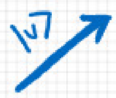
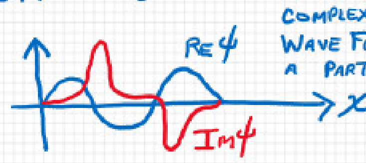
$$|v\rangle = \sum_i V_i |i\rangle \; ; \; \langle j|v\rangle = \sum_i V_i \langle j|i\rangle = V_j$$

$$\therefore |v\rangle = \sum_i |i\rangle V_i = \underbrace{\sum_i |i\rangle\langle i|}_{\text{MUST BE AN IDENTITY OPERATOR } \hat{\mathbb{I}}} v\rangle$$

MUST BE AN IDENTITY OPERATOR $\hat{\mathbb{I}}$: $\hat{\mathbb{I}}|v\rangle = |v\rangle$ (LATER)

DUAL SPACES, DIRAC BRA $\langle v|$, KET $|w\rangle$ NOTATION

"PHYSICAL" IDEA OF A VECTOR: $|v\rangle$ REPRESENTS A "PHYSICAL" OR GEOMETRIC OBJECT

ex: $|v\rangle$  ARROW WITH MAGNITUDE, DIRECTION ; $|\psi\rangle$: 

COMPLEX-VALUED WAVE FUNCTION FOR A PARTICLE

$\therefore$ A VECTOR $|v\rangle$ IS INDEPT. OF BASIS CHOICE.

IN PRACTICE, ALWAYS NEED A BASIS TO MANIPULATE MATHEMATICALLY

$$|v\rangle = \sum_{i=1}^{n} \underset{\text{ASSOC. COEFF.}}{V_i} \underset{\text{SOME BASIS}}{|i\rangle} = \sum_{j=1}^{n} \underset{\text{COEFF.S}}{V_j'} \underset{\text{SOME OTHER BASIS}}{|j'\rangle}$$

● WE CAN ASSOCIATE A KET $|v\rangle$ TO A COLUMN MATRIX OF COEFFICIENTS IN PARTICULAR BASIS:

$$|v\rangle = \sum_{i=1}^{n} V_i |i\rangle \Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix} = V_1 \underset{|1\rangle}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + V_2 \underset{|2\rangle}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + V_3 \underset{|3\rangle}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + \dots + V_n \underset{|n\rangle}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}$$

ARROW MEANS: IN A PARTICULAR BASIS

INNER PRODUCT OF $|v\rangle, |w\rangle$:

$$\langle v|w\rangle = \sum_{i=1}^{n} V_i^* W_i = \begin{bmatrix} V_1^* & V_2^* & V_3^* & \cdots & V_n^* \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_n \end{bmatrix}$$

ORDINARY MATRIX MULTIPLICATION: $(1 \times n) \cdot (n \times 1)$ = SCALAR (1×1)

$\therefore$ ASSOCIATE THE "BRA" $\langle v|$ TO ROW VECTOR

$$\langle v| \Rightarrow \begin{bmatrix} V_1^* & V_2^* & V_3^* & \cdots & V_n^* \end{bmatrix}$$

IN A PARTICULAR BASIS.

$\Rightarrow$ WE THINK OF A KET $|v\rangle$ AS A COLUMN OF $\{V_i\}$, BRA $\langle v|$ AS A ROW OF $\{V_i^*\}$

$\rightarrow$ INNER PRODUCT $\underset{\text{"BRA""KET"}}{\langle v | w \rangle} = [V_1^* \cdots V_n^*] \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix}$ ORDINARY MATRIX PRODUCT OF THESE

DEFINE: **ADJOINT OPERATION**: CONVERTS A KET $|v\rangle$ INTO A BRA $\langle v|$

ABSTRACTLY: $|v\rangle \to \langle v|$

THINKING IN TERMS OF COLUMNS, ROWS: TRANSPOSE AND COMPLEX CONJUGATE

$$a|v\rangle \Rightarrow \begin{bmatrix} a\,v_1 \\ a\,v_2 \\ a\,v_3 \\ \vdots \\ a\,v_n \end{bmatrix} \to \begin{bmatrix} a^*v_1^* & a^*v_2^* & a^*v_3^* & \cdots & a^*v_n^* \end{bmatrix} \Leftarrow a^*\langle v|$$

MORE GENERAL

$$a|v\rangle + b|w\rangle + c|z\rangle \xrightarrow{\text{ADJOINT}} \langle v|a^* + \langle w|b^* + \langle z|c^*$$

NOTE: $|v\rangle = \sum_{i=1}^{n} \underbrace{|i\rangle}_{\text{VECTOR}} \underbrace{\langle i|v\rangle}_{\text{"OVERLAP" — COMPLEX \#}}$, $\langle i|v\rangle = v_i$ ORTHONORMAL BASIS $\langle i|j\rangle = \delta_{ij}$

$$\downarrow$$

$$\langle v| = \sum_{i=1}^{n} \langle i|v\rangle^* \langle i| = \sum_{i=1}^{n} \langle v|i\rangle\langle i| = \sum_{i=1}^{n} v_i^* \langle i|$$

"DUAL SPACES":

KETS $\{|i\rangle\}$ AND BRAS $\{\langle i|\}$ REALLY BELONG TO TWO DIFF. VECTOR SPACES

- ADJOINT CONVERTS $|v\rangle \leftrightarrow \langle v|$
- INNER PRODUCT ONLY DEF. BETWEEN BRA $\langle v|$, KET $|w\rangle$

SCHWARZ, TRIANGLE INEQUALITIES

THEOREM 2: SCHWARZ INEQUALITY $|\langle v|w\rangle| \le |v|\cdot|w|$; $|v| = \sqrt{\langle v|v\rangle}$

ABSOLUTE VALUE OF A COMPLEX #

THEOREM 3: TRIANGLE INEQUALITY $|v+w| \le |v| + |w|$

PROOF OF 2: $|z\rangle \equiv |v\rangle - \underbrace{\frac{\langle w|v\rangle}{|w|^2}|w\rangle}_{\text{PROJECTION OF } |v\rangle \text{ ONTO } |w\rangle}$

$$\langle z|z\rangle = \left\langle v - \frac{\langle w|v\rangle}{|w|^2} w \,\middle|\, v - \frac{\langle w|v\rangle}{|w|^2} w \right\rangle = |v|^2 - \frac{|\langle v|w\rangle|^2}{|w|^2} - \cancel{\frac{|\langle w|v\rangle|^2}{|w|^2}} + \cancel{\frac{|\langle w|v\rangle|^2}{|w|^2}}$$

$$\ge 0$$

$$\therefore |v|^2|w|^2 \ge |\langle v|w\rangle|^2$$