Two Spin-½'s: Triplet, Singlet

① Product Basis: Eigenstates of $\hat{S}^z_\sigma$ and $\hat{S}^z_K$ $\{ |\overset{\sigma K}{\uparrow\uparrow}\rangle, |\overset{\sigma K}{\downarrow\uparrow}\rangle, |\overset{\sigma K}{\uparrow\downarrow}\rangle, |\overset{\sigma K}{\downarrow\downarrow}\rangle \}$

② Total $\hat{J}^z$ Eigenbasis: $\mathbb{V}^2(\mathbb{C}) \otimes \mathbb{V}^2(\mathbb{C}) = \mathbb{V}^1(\mathbb{C}) \oplus \mathbb{V}^3(\mathbb{C})$

Ⓐ $j=0, m=0$: "singlet" state $\quad |0,0\rangle \equiv \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$; $\quad \hat{J}^z|0,0\rangle = \hat{J}^+|0,0\rangle = \hat{J}^-|0,0\rangle = 0$

∴ Transforms trivially under rotations:

$$e^{-i\frac{\hat{\vec{J}}\cdot\vec{\theta}}{\hbar}}|0,0\rangle = |0,0\rangle$$

Ⓑ $j=1$, $-j \leq m \leq j$: "Triplet" (spin-one) states : $\quad |1,1\rangle = |\uparrow\uparrow\rangle$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right]$$

$$\hat{J}^z|1,m\rangle = m\hbar|1,m\rangle \qquad |1,-1\rangle = |\downarrow\downarrow\rangle$$

$\hat{J}^\pm|1,m\rangle = \sqrt{2}\hbar|1,m\pm1\rangle$, except: $\hat{J}^+|1,1\rangle = \hat{J}^-|1,-1\rangle = 0$

∴ Triplet states transform as comp. of spin-1 under rotations

$$\Longrightarrow \boxed{\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1}$$

(Product of two spin-½'s = spin 0 ⊕ spin 1)

∴ In the singlet-triplet basis $\{|0,0\rangle, |1,1\rangle, |1,0\rangle, |1,-1\rangle\}$, the Total Ang. Mom. Op.s take the Block-Diagonal (⇒ Direct Sum) form

$$\hat{J}^z \Rightarrow \hbar\left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right], \quad \hat{J}^x \Rightarrow \frac{\hbar}{\sqrt{2}}\left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad \hat{J}^y \Rightarrow \frac{\hbar}{\sqrt{2}}\left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -i & 0 \\ 0 & i & 0 & -i \\ 0 & 0 & i & 0 \end{array}\right]$$

- This is an example of "Angular Momentum Addition:"
  - A Direct Product of state spaces corresponding to spin-j and spin-j' degrees of freedom is Equivalent to a Direct Sum of Independent Total Ang. Momentum Subspaces
  - Claim: (Not proven here)

$$\underset{\substack{\uparrow \\ \text{MAX} \\ \hat{S}^z_\sigma}}{j} \otimes \underset{\substack{\uparrow \\ \text{MAX} \\ \hat{S}^z_K}}{j'} = |j-j'| \oplus |j-j'+1| \oplus \ldots \oplus \underbrace{j+j'}_{\substack{\text{MAX} \\ \hat{J}^z}}$$

- The Block-Diag. Structure of Total Ang. Mom. Op.s will reappear in 3D [Orbital $\hat{\vec{L}}$ of a particle moving in $\mathbb{R}^3$]

Entanglement

- Suppose "Alice" and "Bob" each have a spin-$\frac{1}{2}$ qubit. States: $|AB\rangle$, $A \in \uparrow$ or $\downarrow$, $B \in \uparrow$ or $\downarrow$

  "Alice" $\Downarrow$ $\hat{\vec{S}}_A$ "Bob" $\Downarrow$ $\hat{\vec{S}}_B$

- Suppose Alice + Bob prepare a Singlet State for their combined system

$$|\psi\rangle = |0,0\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$$

- Assume $|\psi\rangle$ prepared in Houston. Then Alice takes her qubit to Beijing, ~8000 miles from Bob's qubit, which remains in Houston.

The Hard Part: Assume that "Quantum Coherence" is preserved $\Rightarrow$ No interaction with the environment

Entanglement: Assume Alice measures her qubit along the z-axis.* No matter which result she gets $\hat{S}_A^z = \pm\frac{\hbar}{2}$; Bob must get the opposite if he measures after Alice!

(*) It actually does not matter which axis Alice and Bob choose to measure their qubits (Why?), so long as they both agree.

① $t<0$: $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\underset{A}{\uparrow}\underset{B}{\downarrow}\rangle - |\underset{A}{\downarrow}\underset{B}{\uparrow}\rangle\right]$

② $t=0$: Alice Measures Qubit A.

$\hat{S}_A^z \Rightarrow \frac{\hbar}{2}$ $(\uparrow)$ $\qquad$ $\hat{S}_A^z \Rightarrow -\left(\frac{\hbar}{2}\right)$ $(\downarrow)$

$\therefore |\psi\rangle \Rightarrow |\uparrow\downarrow\rangle$ $\qquad$ $\therefore |\psi\rangle \Rightarrow |\downarrow\uparrow\rangle$

③ $t = t_B > 0$:

Bob Measures Qubit B

$\hat{S}_B^z \Rightarrow -\frac{\hbar}{2}$ $(\downarrow)$ $\qquad$ $\hat{S}_B^z \Rightarrow +\frac{\hbar}{2}$ $(\uparrow)$

"Spooky Action at a Distance" — Einstein, criticizing the Copenhagen Interpretation

- Houston – Beijing distance: 8000 miles ~ 1.3×10^7 meters (m)
- Assume Alice and Bob synchronize their clocks within 1 msec.
- Alice measures at $t = 0$. Bob measures at $t = 10$ msec, guaranteed to get opposite result from Alice.

$\Longrightarrow$ "information velocity" $\sim \dfrac{1.3 \times 10^7 \, m}{10^{-2} \, s} = 1.3 \times 10^9 \, m/s > c \simeq 3.0 \times 10^8 \, m/s$

? Does Quantum Entanglement Violate Special Relativity ?

[= Causality here: Alice and Bob's measurement events are spacelike-separated; sequence of events depends on the frame]

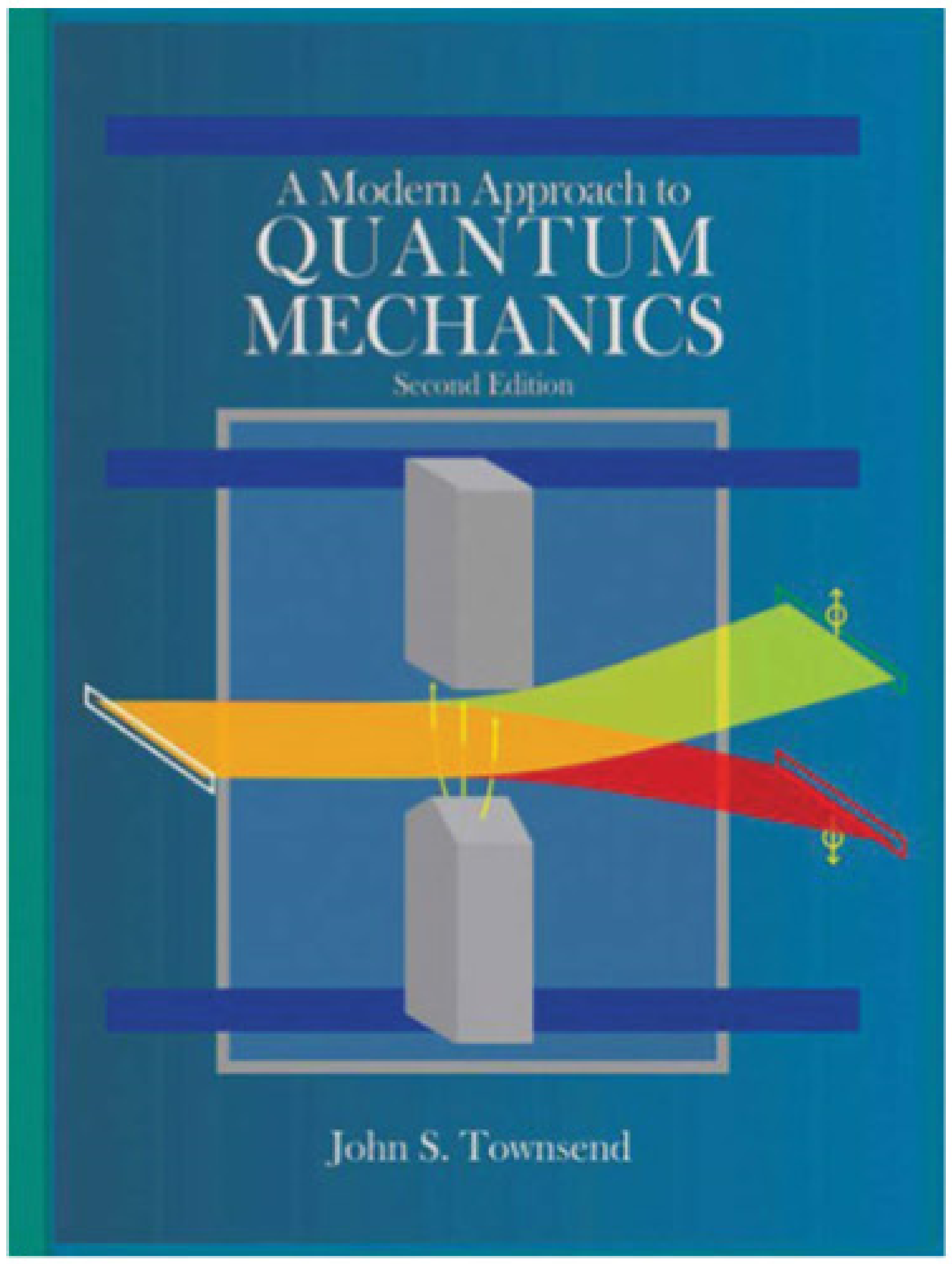
"Einstein – Podolsky – Rosen" (EPR) Paradox.

Answer: ① Better not, since our Standard Model of Particle Physics is built upon the fusion of Q.M. and Special Relativity!

② No – because 1st measurement (Alice or Bob) is Random. No physical quantity [Energy, Momentum, Mass, etc.] is exchanged between Alice and Bob.

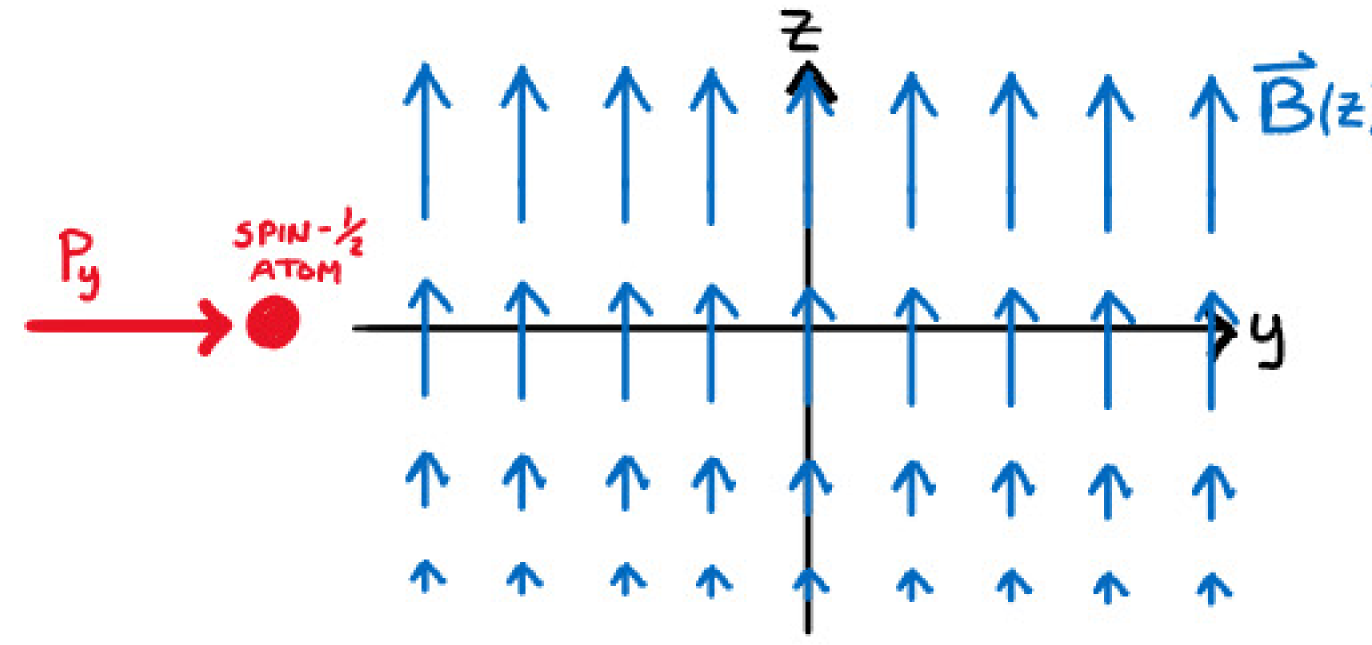
③ In order to test Quantum Theory, Alice must Communicate the results of her measurement to Bob, and this (effectively classical/macroscopic) signal cannot travel faster than light.

Measuring Spin: Stern-Gerlach Expt.



- Consider a spin-$\frac{1}{2}$ particle moving in 2D, subject to an external $\overrightarrow{B}$-field
- $\vec{B} = B_z \vec{n}_z$ is spatially inhomogeneous: $B_z = B\frac{z}{a}$

  so that $\frac{d}{dz}\vec{B} = \frac{B}{a}\vec{n}_z$, where a is a length scale characterizing the magnetic gradient slope



$$\hat{H} = \frac{\hat{P}_y^2}{2\mu} + \frac{\hat{P}_z^2}{2\mu} - \gamma B_z(\hat{Z})\hat{S}_z$$

We use μ for the mass, to avoid confusion with spin eigenvalue m_z

NOTE:
① $[\hat{H}, \hat{P}_y] = 0$ — No $\hat{Y}$-depc.
② $[\hat{H}, \hat{S}_z] = 0$ — $\vec{B} = B_z \vec{n}_z$
③ $[\hat{P}_y, \hat{S}_z] = 0$ — Operators acting on different subspaces

$\therefore$ **Can find simultaneous eigenstates of $\hat{H}, \hat{P}_y, \hat{S}_z$:**

$$\hat{H}\,|E P_y m_z\rangle = E\,|E P_y m_z\rangle \Rightarrow \langle z y m_z'|\hat{H}|E P_y m_z\rangle = E\,\langle z y m_z'|E P_y m_z\rangle$$

$$\langle z y m_z'|E P_y m_z\rangle = \delta_{m_z', m_z}\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i P_y y}{\hbar}}\,\psi_{EP_ym_z}(z)$$

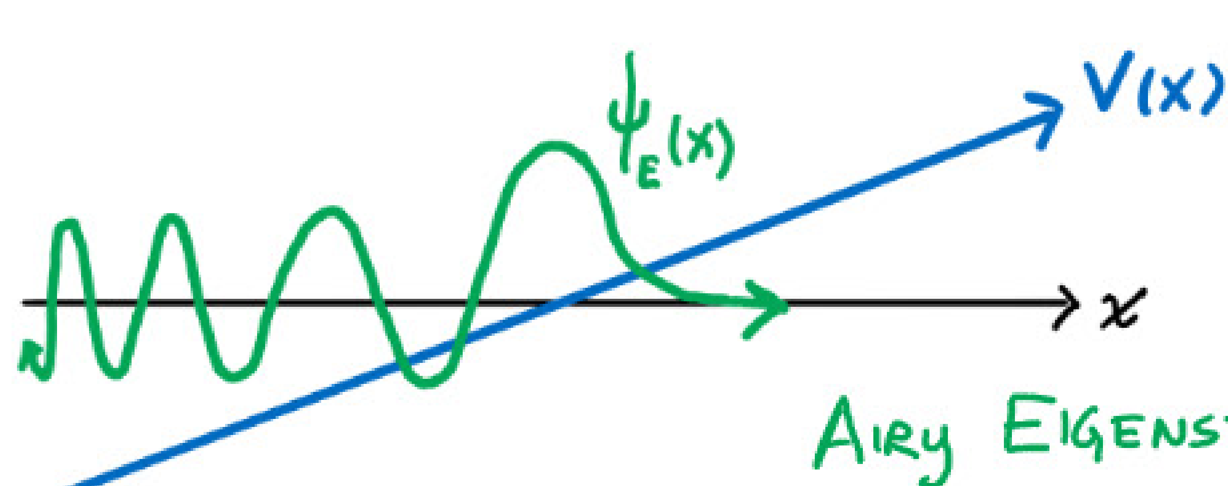
$$\Rightarrow \left[\frac{-\hbar^2}{2\mu}\frac{d^2}{dz^2} - \frac{\gamma B z}{a}(\hbar m_z)\right]\psi_{EP_ym_z}(z) = \left(E - \frac{P_y^2}{2\mu}\right)\psi_{EP_ym_z}(z) \quad ; \; m_z \in \{-\tfrac{1}{2}, +\tfrac{1}{2}\}$$

- $-\gamma B \hbar m_z \equiv V_{m_z}$; $E - \frac{P_y^2}{2\mu} \equiv E_{P_y}$

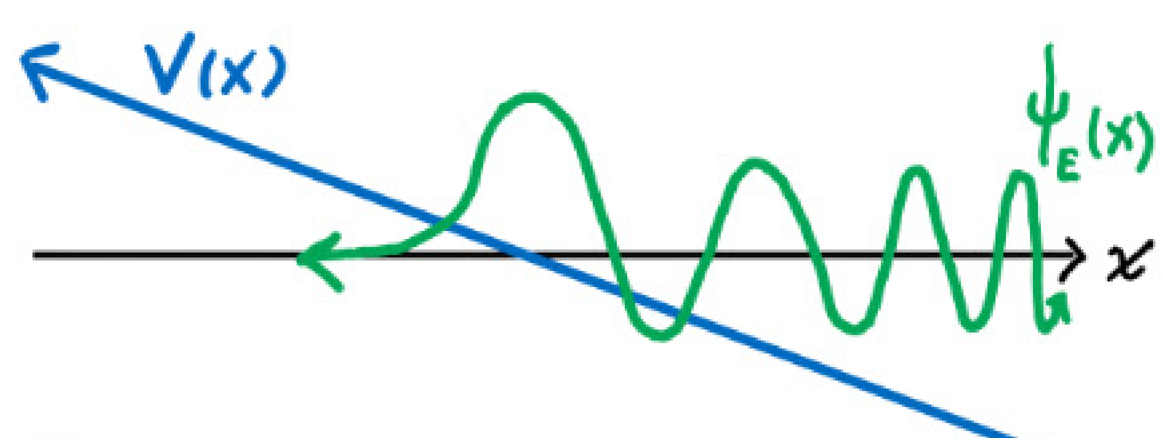
$$\boxed{\Rightarrow \left[\frac{-\hbar^2}{2\mu}\frac{d^2}{dz^2} + V(z)\right]\psi_{E_{P_y}}(z) = E_{P_y}\psi_{E_{P_y}}(z) \quad ; \; V(z) = V_{m_z}\cdot\frac{z}{a}}$$

Sign $\propto m_z$

1D spinless particle in a linear potential



Airy eigenstate, lin. incr. potential (HW#5)



Lin. decr. potential

$$\hat{H}_z = \frac{\hat{P}_z^2}{2\mu} + \hat{V}(\hat{z})$$; ALTERNATIVE TO TIME-INDEPT S.E. / EIGENSTATE SPECTRUM:

OPERATOR EQUATION OF MOTION (cf. LEC. 12, p. 4 FOR SPIN EOM)

$$\langle \hat{z} \rangle \equiv \langle \psi_{(t)} | \hat{z} | \psi_{(t)} \rangle \; ; \quad |\dot{\psi}\rangle = \frac{-i}{\hbar} \hat{H}_z |\psi\rangle$$

$$\Rightarrow \frac{d}{dt} \langle \hat{z} \rangle = \langle \dot{\psi} | \hat{z} | \psi \rangle + \langle \psi | \hat{z} | \dot{\psi} \rangle = \frac{i}{\hbar} \langle [\hat{H}_z, \hat{z}] \rangle$$

USING $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$: $[\hat{H}_z, \hat{z}] = \frac{1}{2\mu}[\hat{P}_z^2, \hat{z}] = \frac{-i\hbar}{\mu} \hat{P}_z$

$$\therefore \frac{d\langle \hat{z} \rangle}{dt} = \frac{1}{\mu} \langle \hat{P}_z \rangle$$ SAME AS CLASSICAL EOM!

$$\frac{d}{dt} \langle \hat{P}_z \rangle = \frac{i}{\hbar} \langle [\hat{H}_z, \hat{P}_z] \rangle$$

CLAIM: $[\hat{A}, \hat{f}(\hat{B})] = \hat{f}'(\hat{B}) [\hat{A}, \hat{B}]$, $f'(b) \equiv \frac{df}{db}(b)$, IF $[[\hat{A}, \hat{B}], \hat{B}] = 0$.

$$\Rightarrow [\hat{H}_z, \hat{P}_z] = [\hat{V}(\hat{z}), \hat{P}_z] = \frac{d\hat{V}}{d\hat{z}}(\hat{z}) \, i\hbar$$

$$\boxed{\therefore \frac{d\langle \hat{P}_z \rangle}{dt} = - \left\langle \frac{d\hat{V}(\hat{z})}{d\hat{z}} \right\rangle} \neq - \left. \frac{dV}{dz} \right|_{z = \langle \hat{z} \rangle}$$

$\Rightarrow$ FOR GENERAL $V(z)$, $\frac{d\langle \hat{P}_z \rangle}{dt}$ IS NOT DETERMINED BY CLASSICAL E.O.M. WITH $z \rightarrow \langle \hat{z} \rangle$

- **EXCEPTION:** $V(z) = V_0 + V_1 \left(\frac{z}{a} \right) + V_2 \left(\frac{z}{a} \right)^2$ SIMPLE HARMONIC OSCILLATOR IN A UNIFORM FORCE FIELD

IN THIS CASE, $$\left\langle \frac{dV}{dz} \right\rangle = \left\langle \frac{V_1}{a} \hat{\mathbb{I}} + \frac{2V_2}{a^2} \hat{z} \right\rangle = \left. \frac{dV}{dz} \right|_{z = \langle \hat{z} \rangle}$$

- STERN-GERLACH:

$$\frac{d}{dt} \langle \hat{z} \rangle = \frac{1}{\mu} \langle \hat{P}_z \rangle \; ; \quad \frac{d}{dt} \langle \hat{P}_z \rangle = - \left. \frac{dV}{dz} \right|_{z = \langle z \rangle} = - \frac{V_{m_z}}{a}$$

$$\Rightarrow \langle z \rangle (t) = - \frac{V_{m_z}}{2\mu a} t^2$$ UNIFORM ACCELERATION OF AVG. Z POSITION DUE TO CONSTANT FORCE FIELD $F = -V_{m_z}/a$

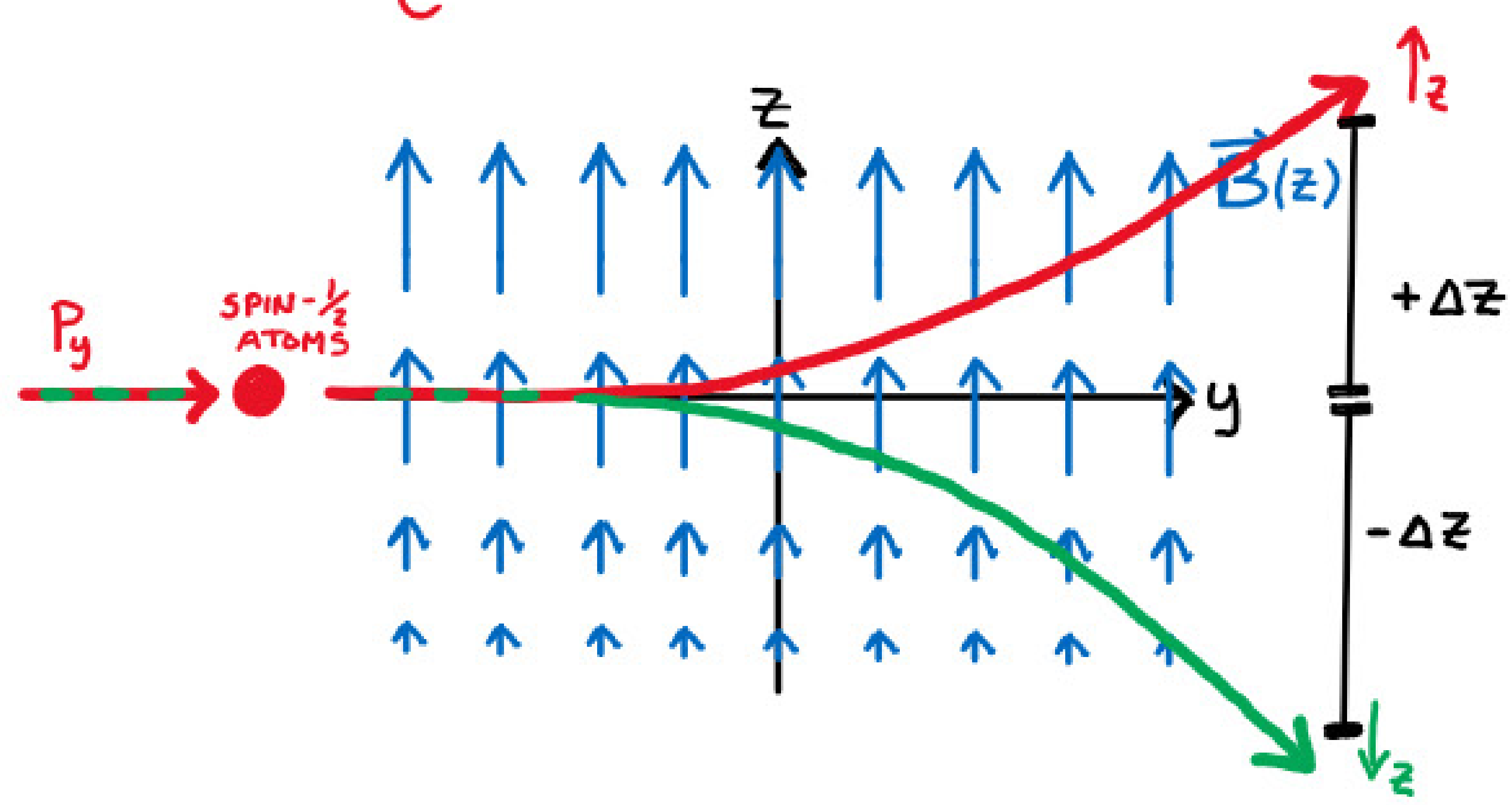
FOR FIXED (CONSERVED) P_y, TRANSIT TIME THROUGH S-G APPARATUS OF LENGTH L_y IS $t = \frac{L_y \mu}{P_y}$.

$$\therefore \langle z \rangle = - \frac{V_{m_z}}{2\mu a}\left(\frac{L_y \mu}{P_y}\right)^2 = + \frac{\left(+\gamma B \hbar \frac{1}{2}\, \mathrm{sgn}(m_z)\right)}{2\cancel{\mu}\, a} \frac{L_y^2 \mu^{\cancel{2}}}{P_y^2}$$

OR

$$\langle Z \rangle = \frac{L_y^2}{8a}\, \mathrm{sgn}(m_z) \left[\frac{\hbar \omega_L}{P_y^2/2\mu} \right], \quad \omega_L \equiv \gamma B$$

- FOR FIXED GEOMETRY L_y, FIELD GRADIENT (B/a) AND KINETIC ENERGY $P_y^2/2\mu$, Z-DISPLACEMENT IS QUANTIZED



- CAN USE AN ABSORBER TO BLOCK (e.g.) $\downarrow_z$ ATOMS EMERGING FROM APPARATUS $\Longrightarrow$ PROJ. MEAS. FOR ATOMS THAT "GET THROUGH" $(\uparrow_z)$

- WE CAN CASCADE MULTIPLE S-G APPARATUSES [WITH DIFF. FIELD AND ABSORBER ORIENTATIONS] IN ORDER TO AFFECT SEQUENTIAL PROJECTIVE MEASUREMENTS ON THE BEAM (ENSEMBLE) OF INCOMING SPIN-½ ATOMS

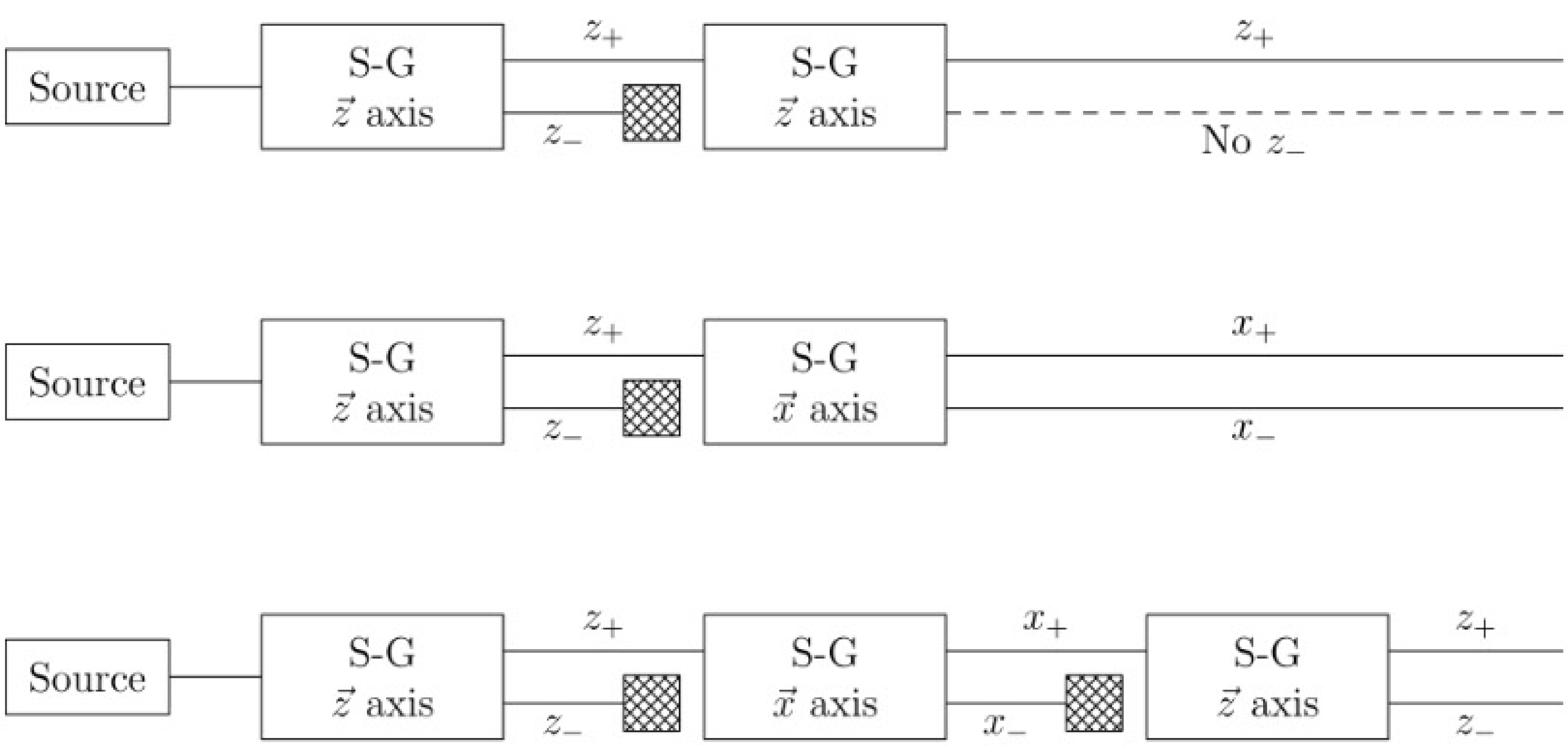


IMAGE CREDIT: WIKIPEDIA