THEOREM 10: EIGENVALUES OF UNITARY OPERATORS HAVE MODULUS 1

i.e., $\hat{U}|\omega\rangle = \omega|\omega\rangle, \ \hat{U}^\dagger\hat{U} = \hat{\mathbb{I}} \ \Rightarrow \ |\omega| = 1. \Rightarrow \omega = e^{i\phi_\omega}$ IS A PURE PHASE $(\phi_\omega \in [0, 2\pi))$

THEOREM 11: EIGENVECTORS OF A UNITARY OPERATOR ARE MUTUALLY ORTHOGONAL (ASSUMING NO DEGENERACY)

BOTH THEOREMS 10, 11 FOLLOW SIMPLY FROM THE FACT THAT $\hat{U} \equiv e^{i\hat{H}}$, WHERE $\hat{H}^\dagger = \hat{H}$ (NOT PROVEN HERE)

$\Rightarrow$ WILL RETURN TO THIS WHEN WE STUDY FUNCTIONS OF OPERATORS

PROOF: $\hat{U}|\omega_i\rangle = \omega_i|\omega_i\rangle \xrightarrow{\text{ADJOINT}} \langle\omega_i|\omega_i^* = \langle\omega_i|\hat{U}^\dagger$

$$\therefore \ \langle\omega_j|\hat{U}^\dagger\hat{U}|\omega_i\rangle = \langle\omega_j|\omega_i\rangle = \langle\omega_j|\omega_i\rangle\,\omega_j^*\omega_i$$

a) $i = j \ \Rightarrow \ |\omega_i|^2 = 1$ IF $\langle\omega_i|\omega_i\rangle > 0$ [WLoG, $\langle\omega_i|\omega_i\rangle = 1$]

b) $i \neq j \ \Rightarrow \ \langle\omega_j|\omega_i\rangle = (\underbrace{\omega_j^*\omega_i}_{\neq 1 \text{ IF NO DEGENERACY}})\langle\omega_j|\omega_i\rangle$

$\Rightarrow \ \therefore \ \langle\omega_i|\omega_j\rangle = \delta_{ij}$ ✓ $\Rightarrow$ EIGENVECTORS OF $\hat{U}$ CAN BE TAKEN TO FORM AN ORTHONORMAL BASIS.

DIAGONALIZING HERMITIAN MATRICES

CONSIDER $\hat{\Omega}^\dagger = \hat{\Omega}$. MATRIX ELEMENTS $\Omega_{ij} \equiv \langle i|\hat{\Omega}|j\rangle$ IN SOME ORTHONORMAL BASIS $\langle i|j\rangle = \delta_{ij}$

$$\hat{\Omega}|\omega_i\rangle = \omega_i|\omega_i\rangle \quad (\omega_i = \omega_i^*)$$

WE KNOW THAT $\{|\omega_i\rangle\}$ CAN BE USED TO FORM AN ORTHONORMAL BASIS

LET $|\omega_i\rangle \equiv \hat{U}|i\rangle, \ \hat{U}^\dagger\hat{U} = \hat{\mathbb{I}}$ $\quad$ $\hat{U}$ "ROTATES" OLD BASIS VECTOR $|i\rangle$ INTO NEW VECTOR $|\omega_i\rangle$

WHAT IS $\hat{U}$? $\Downarrow$ IN THE ORIGINAL $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$ BASIS

$$\begin{bmatrix} \omega_{i,1} \\ \omega_{i,2} \\ \vdots \\ \omega_{i,i} \\ \vdots \\ \omega_{i,n} \end{bmatrix} = \begin{bmatrix} \text{---} \langle 1|\hat{U}|i\rangle \text{---} \\ \text{---} \langle 2|\hat{U}|i\rangle \text{---} \\ \vdots \\ \text{---} \langle i|\hat{U}|i\rangle \text{---} \\ \vdots \\ \text{---} \langle n|\hat{U}|i\rangle \text{---} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{\text{TH}} \text{ PLACE}$$

$\llcorner$ i^{TH} COLUMN

$$\therefore \ \hat{U} \Rightarrow \begin{bmatrix} | & | & | & & | \\ |\omega_1\rangle & |\omega_2\rangle & |\omega_3\rangle & \cdots & |\omega_n\rangle \\ | & | & | & & | \end{bmatrix}$$

FOR $|\omega_i\rangle = \hat{U}|i\rangle$, COLUMNS OF $\hat{U}$ IN $\{|i\rangle\}$ BASIS ARE EIGENVECTORS $\{|\omega_i\rangle\}$

INSTEAD OF "ACTIVE" TRANSFORMATION ON BASIS, CONSIDER "PASSIVE" TRANSFORMATION OF $\hat{\Omega}$:

$$\hat{\Omega} \rightarrow \hat{\Omega}_D \equiv \hat{U}^\dagger\hat{\Omega}\hat{U} = \begin{bmatrix} \omega_1 & & & 0 \\ & \omega_2 & & \\ & & \ddots & \\ 0 & & & \omega_n \end{bmatrix}$$

DIAGONAL MATRIX OF EIGENVALUES!

ex):

$$\hat{\Omega} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \hat{\Omega}^{+}$$

; EIGENANALYSIS:

$$\det(\hat{\Omega} - \omega \hat{\mathbb{I}}) = \det \begin{bmatrix} -\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & -\omega \end{bmatrix} = \begin{matrix} -\omega^3 + 0 + 0 \\ +\omega - 0 - 0 \end{matrix} = 0$$

$$= \omega(1-\omega)(1+\omega)$$

USING STANDARD METHOD:

$$|\omega_1 = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |\omega_2 = 1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad |\omega_3 = -1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \hat{U} \Rightarrow \left[|\omega_1\rangle |\omega_2\rangle |\omega_3\rangle \right] = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\hat{U}^{+} \hat{\Omega} \hat{U} = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix} \checkmark$$

THEOREM 12: IF $\hat{\Omega}$ AND $\hat{\Lambda}$ ARE TWO HERMITIAN OPERATORS THAT COMMUTE:

IMPORTANT FOR QUANTUM MEASUREMENTS!

$$[\hat{\Omega}, \hat{\Lambda}] = 0 \quad \text{AND} \quad \hat{\Omega}^{+} = \hat{\Omega}, \; \hat{\Lambda}^{+} = \hat{\Lambda}$$

THEN: THERE EXISTS A COMMON EIGENBASIS THAT DIAGONALIZES BOTH $\hat{\Omega}$ AND $\hat{\Lambda}$.

PROOF: ASSUME $\hat{\Omega}$ IS NON-DEGENERATE; $\hat{\Omega}|\omega_i\rangle = \omega_i|\omega_i\rangle$; $\hat{\Lambda}\hat{\Omega}|\omega_i\rangle = \omega_i \hat{\Lambda}|\omega_i\rangle$

$$[\hat{\Omega}, \hat{\Lambda}] = 0 \Rightarrow \hat{\Omega}\left[\hat{\Lambda}|\omega_i\rangle\right] = \omega_i \left[\hat{\Lambda}|\omega_i\rangle\right] \Rightarrow \hat{\Lambda}|\omega_i\rangle = \text{CONST.} \times |\omega_i\rangle$$

$$= \lambda_i |\omega_i\rangle$$

$$\therefore \begin{matrix} \hat{\Omega}|\omega_i\rangle = \omega_i |\omega_i\rangle \\ \hat{\Lambda}|\omega_i\rangle = \lambda_i |\omega_i\rangle \end{matrix} \Rightarrow$$

$|\omega_i\rangle$ IS A SIMULTANEOUS EIGENVECTOR OF $\hat{\Omega}$ AND $\hat{\Lambda}$

PROOF ASSUMES NO DEGENERACY. IF A SUBSPACE $\mathbb{V}^d(\mathbb{C})$ SPANNED BY AN ORTHONORMAL (BUT DEG.) BASIS $\{|\omega,1\rangle, |\omega,2\rangle, \ldots, |\omega,d\rangle\}$, $\hat{\Omega}|\omega,i\rangle = \omega|\omega,i\rangle$; $\langle \omega,i|\omega,j\rangle = \delta_{ij}$

THEN: CAN FIND COMMON EIGENBASIS FOR $\hat{\Omega}, \hat{\Lambda}$ IN THIS SUBSPACE BY DIAGONALIZING $\hat{\Lambda}$ IN $\mathbb{V}^d$

- IN OTHER WORDS,

IF TWO HERMITIAN OPERATORS COMMUTE, THERE EXISTS A COMMON EIGENBASIS.

WITH DEGENERACY, MUST CHOOSE COMMON EIGENSTATES TO DIAGONALIZE BOTH.

$\longrightarrow$ OFTEN USEFUL IN QUANTUM, BUT IT IS ALSO ADVANTAGEOUS TO RECOGNIZE DIFF. CHOICES ARE POSSIBLE.

EX: SHOW THAT $[\hat{\Omega},\hat{\Lambda}]=0$ AND FIND COMMON EIGENBASIS

$$\hat{\Omega} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} ; \hat{\Lambda} \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \qquad \hat{\Omega}\hat{\Lambda} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} = \hat{\Lambda}\hat{\Omega} \checkmark$$

EIGENVALUES OF $\hat{\Omega}$: $\omega_1 = 2$; $\omega_2 = \omega_3 = 0$

$$|\omega_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; \hat{\Omega}|\omega_{2,3}\rangle = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0 \Rightarrow |\omega_2\rangle \equiv \underset{\text{NORMALIZATION}}{\alpha_2}\begin{bmatrix} 1 \\ a \\ -1 \end{bmatrix}, \ |\omega_3\rangle \equiv \alpha_3\begin{bmatrix} 1 \\ -\frac{2}{a} \\ -1 \end{bmatrix} \Rightarrow \langle\omega_2|\omega_3\rangle \propto 2 - \left(\tfrac{2}{a}\right)a = 0$$

WANT TO CHOOSE a S.T. $|\omega_{2,3}\rangle$ ARE EIGENVECTORS OF $\hat{\Lambda}$

① $|\omega_1\rangle$:

$$\hat{\Lambda}|\omega_1\rangle \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \Rightarrow 3|\omega_1\rangle \checkmark$$

② $|\omega_2\rangle$:

$$\hat{\Lambda}|\omega_2\rangle \Rightarrow \alpha_2\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ a \\ -1 \end{bmatrix} = \alpha_2\begin{bmatrix} 1+a \\ 2 \\ -1-a \end{bmatrix} = \alpha_2\lambda_2\begin{bmatrix} 1 \\ a \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 1+a = \lambda_2\cdot 1 \\ 2 = \lambda_2\cdot a \end{matrix} \quad \therefore \quad 1+a = \frac{2}{a} \text{ or } (a+2)(a-1) = 0$$

$a = -2$: $\lambda_2 = \frac{2}{a} = -1$

$a = 1$: $\lambda_3 = \frac{2}{a} = 2$

$$\therefore \quad |\omega_2,\lambda_2\rangle \equiv |0,-1\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$|\omega_3,\lambda_3\rangle \equiv |0,2\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

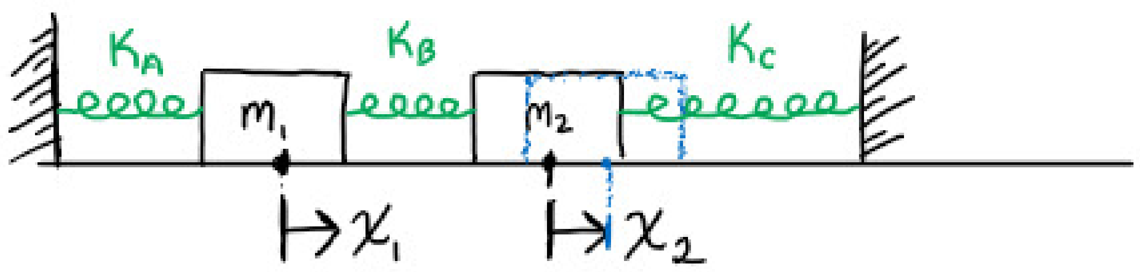
$\therefore$ $|\omega_2,\lambda_2\rangle$, $|\omega_3,\lambda_3\rangle$ ARE SIMULT. EIGENSTATES OF $\hat{\Omega},\hat{\Lambda}$

HOWEVER, $|v\rangle \equiv \alpha|\omega_2,\lambda_2\rangle + \beta|\omega_3,\lambda_3\rangle$ IS

- AN EIGENSTATE OF $\hat{\Omega}$ WITH $\omega_2 = \omega_3 = 0$
- NOT AN EIGENSTATE OF $\hat{\Lambda}$

$$\hat{\Lambda}|v\rangle = -\alpha|\omega_2,\lambda_2\rangle + 2\beta|\omega_3,\lambda_3\rangle \neq (\text{CONST.})\times|v\rangle$$

Eigenanalysis in a Simple Classical System



- Two masses can slide without friction, connected to each other and to walls with ideal springs
- Let $x_1 = 0$ ($x_2 = 0$) locate the mechanical (zero force) position of mass m_1 (mass m_2)

Springs have spring constants $K_{A,B,C}$

Zero rest length

Given that $x_1 = 0$ and $x_2 = 0$ locate equilibrium, Newton's 2nd Law yields

① $m_1 \ddot{x}_1 = -K_A x_1 + K_B(x_2 - x_1)$

② $m_2 \ddot{x}_2 = -K_B(x_2 - x_1) - K_C x_2$

Simplification: Choose $m_1 = m_2 \equiv m$

[Otherwise, generalized eigenanalysis → 301]

Assume: solutions with well-defined frequency exist, as in string problem in Lecture 1

$\Rightarrow$ can write $x_1(t) = A_1 e^{-i\omega t}$, $x_2(t) = A_2 e^{-i\omega t}$

- "Ansatz" inspired by string example
- More formal: ①,② constitute a system of coupled linear equations that do not depend explicitly on time
- ↳ implies that solution can be obtained via Fourier (or Laplace) transform: Time domain ⇓ Freq. domain
- We will study Fourier transforms in position space, as a <u>basis change in function space</u> (later...)

Using ansatz, rewrite ①,② as a matrix equation:

$$-m\omega^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -K_A - K_B & K_B \\ K_B & -K_B - K_C \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

or $\hat{K}|A\rangle = -m\omega^2 |A\rangle$

$\hat{K}^\dagger = \hat{K}$

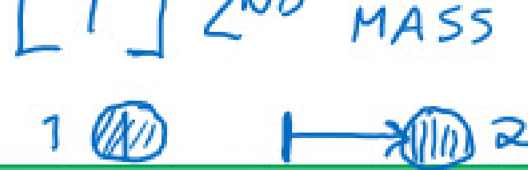
∴ This is a Hermitian operator eigenvalue problem

Simple version: $K_A = K_B = K_C \equiv K$ $\Longrightarrow$ $\hat{K} \Rightarrow K \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

- Physical interpretation of basis vectors

$|1\rangle \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ First mass displaced, 2nd mass at $x_2 = 0$

$|2\rangle \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ First mass at $x_1 = 0$, 2nd mass displaced



$$\hat{K}|A\rangle = -m\omega^2|A\rangle \,, \quad \hat{K} \Rightarrow k\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = k\left(-2\hat{\mathbb{I}} + \hat{\sigma}^1\right)$$

2x2 MATRIX NOTATION OF LEC. 3, p.1

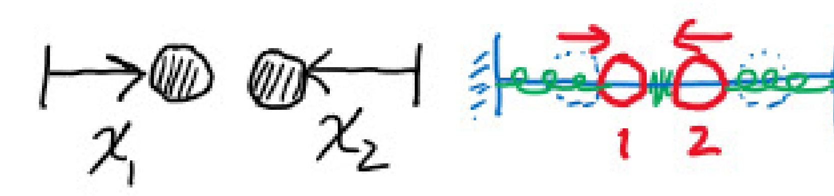
$$\hat{\mathbb{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{\sigma}^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

NOTE: $[\hat{\mathbb{I}}, \hat{\sigma}^1] = 0$

$\Longrightarrow$ EIGENVECTORS OF $\hat{K}$ ARE EIGENVECTORS OF $\hat{\sigma}^1$

$\hat{\sigma}^1 |s\rangle = (+1)|s\rangle$ $\qquad |s\rangle \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ "SYMMETRIC" MODE: BOTH MASSES DISPLACE IN SAME DIRECTION:

$\hat{\sigma}^1 |A\rangle = (-1)|A\rangle$ $\qquad |A\rangle \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ "ANTISYMMETRIC" MODE: MASSES DISPLACE IN OPPOSITE DIRECTIONS:



EIGENFREQUENCIES:

① $|s\rangle$: $-m\omega_s^2|s\rangle = \hat{K}|s\rangle = k(-2\hat{\mathbb{I}} + \hat{\sigma}^1)|s\rangle = -k|s\rangle \Rightarrow \omega_s = \sqrt{\frac{k}{m}}$

② $|A\rangle$: $-m\omega_A^2|A\rangle = \hat{K}|A\rangle = k(-2\hat{\mathbb{I}} + \hat{\sigma}^1)|A\rangle = -3k|A\rangle \Rightarrow \omega_A = \sqrt{\frac{3k}{m}} > \omega_s$

GENERAL SOLUTION: $|x(t)\rangle = \alpha e^{-i\omega_s t}|s\rangle + \beta e^{-i\omega_A t}|A\rangle$, $\alpha, \beta \in \mathbb{C}$

[PHASOR NOTATION]

EXPLICITLY:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \underbrace{\mathrm{Re}\left[\alpha e^{-i\omega_s t}\right]}_{=|\alpha|\cos(\omega_s t + \phi_\alpha)} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \underbrace{\mathrm{Re}\left[\beta e^{-i\omega_A t}\right]}_{=|\beta|\cos(\omega_A t + \phi_\beta)}$$

- TAKING THE REAL PART OF THIS SOLUTION IS SPECIFIC TO THIS CLASSICAL PHYSICS PROBLEM. WE WILL <u>NOT</u> DO THIS WHEN WE SOLVE SCHRÖDINGER'S EQN. IN QUANTUM
- REAL PHYSICAL OBSERVABLES OBTAINED A DIFFERENT WAY IN QUANTUM (LATER...)

THEOREM 13: A UNITARY CHANGE OF BASIS PRESERVES

IMPORTANT !!

① HERMITICITY, AND
② UNITARITY
OF LINEAR OPERATORS

} IN OTHER WORDS, THESE ARE BASIS-INDEPT. ATTRIBUTES!

WE HAVE ALREADY SEEN HOW TO DIAGONALIZE A HERMITIAN OPERATOR $\hat{\Omega} = \hat{\Omega}^\dagger$ BY CONSTRUCTING UNITARY OP. $\hat{U}$, SUCH THAT

IF $\hat{\Omega}|\omega_i\rangle = \omega_i|\omega_i\rangle$, $\quad \underbrace{\hat{U}|i\rangle}_{\text{OLD BASIS VECTOR}} = \underbrace{|\omega_i\rangle}_{\text{EIGENBASIS OF } \hat{\Omega}}$ (PAGE 1 OF THIS LEC. 5)

THEN $\hat{\Omega}_D \equiv \hat{U}^\dagger \hat{\Omega} \hat{U} \Rightarrow \begin{bmatrix} \omega_1 & & & 0 \\ & \omega_2 & & \\ & & \ddots & \\ 0 & & & \omega_n \end{bmatrix}$

NOW, SUPPOSE WE "PASSIVELY" TRANSFORM ALL OPERATORS IN THIS FASHION: $\quad \hat{\Lambda} \to \hat{\Lambda}' \equiv \hat{U}^\dagger \hat{\Lambda} \hat{U}$

- THIS CORRESPONDS TO CHANGING FROM $\langle i|\hat{\Lambda}|j\rangle$ MATRIX ELEMENTS TO $\langle \omega_i|\hat{\Lambda}|\omega_j\rangle$ ELEMENTS

1. $\hat{\Lambda} = \hat{\Lambda}^\dagger$. <u>CLAIM</u>: $\hat{\Lambda}'^\dagger = \hat{\Lambda}'$

LEC. 3, P7: $(\hat{A}\hat{B}\hat{C}\cdots\hat{N})^\dagger = \hat{N}^\dagger\hat{M}^\dagger\cdots\hat{C}^\dagger\hat{B}^\dagger\hat{A}^\dagger$

PROOF: $\hat{\Lambda}'^\dagger = (\hat{U}^\dagger\hat{\Lambda}\hat{U})^\dagger = \hat{U}^\dagger\hat{\Lambda}^\dagger\hat{U} = \hat{\Lambda}'$ ✓

2. $\hat{\Lambda}^\dagger\hat{\Lambda} = \hat{\mathbb{I}}$. <u>CLAIM</u>: $\hat{\Lambda}'^\dagger\hat{\Lambda}' = \hat{\mathbb{I}}$

PROOF: $\hat{\Lambda}'^\dagger\hat{\Lambda}' = (\hat{U}^\dagger\hat{\Lambda}\hat{U})^\dagger(\hat{U}^\dagger\hat{\Lambda}\hat{U}) = (\hat{U}^\dagger\hat{\Lambda}^\dagger\hat{U})(\hat{U}^\dagger\hat{\Lambda}\hat{U})$

$= \hat{U}^\dagger\hat{\Lambda}^\dagger\underbrace{\hat{U}\hat{U}^\dagger}_{=\hat{\mathbb{I}}}\hat{\Lambda}\hat{U} = \hat{U}^\dagger\hat{\Lambda}^\dagger\hat{\Lambda}\hat{U} = \hat{\mathbb{I}}$ ✓