Commutator of $\hat{X}$ and $\hat{K}$

Consider $\langle x | \hat{X}\hat{K} | f\rangle = \int dx' \, x \langle x | \hat{K} | x' \rangle \langle x' | f \rangle = \int dx' \, x \left[\delta(x-x')(-i)\right] \frac{df}{dx'} = -i x \frac{df}{dx}$

$$\langle x | \hat{K}\hat{X} | f\rangle = \int dx' \langle x | \hat{K} | x' \rangle x' f(x') = \int dx' \left[\delta(x-x')(-i)\right] \frac{d}{dx'}\left(x' f(x')\right)$$

$$= -i\left[f(x) + x\frac{df}{dx}\right]$$

$$\therefore \ \langle x | [\hat{X},\hat{K}] | f\rangle = (-i)\left[\cancel{x\frac{df}{dx}} - f - \cancel{x\frac{df}{dx}}\right] = i\langle x | f\rangle = i\langle x | \hat{\mathbb{I}} | f\rangle$$

$$\boxed{[\hat{X},\hat{K}] = i\,\hat{\mathbb{I}}}$$

$\Rightarrow$ $\hat{X} = \hat{X}^{+}$ and $\hat{K} = \hat{K}^{+}$ are both Hermitian Op.s with real eigenvalues

But: these operators do not commute!

Theorem 12, Lecture 5, p2: Two Hermitian operators $\hat{A}, \hat{B}$ can be simultaneously diagonalized if $[\hat{A},\hat{B}] = 0$

Here: obvious that is not true for $\hat{X}, \hat{K}$

- $[\hat{X},\hat{K}] = i\,\hat{\mathbb{I}} \neq 0$

- Consider a position eigenket $|x\rangle$

  Rep. in wavenumber basis: $\langle k | x \rangle = \frac{1}{\sqrt{2\pi}} e^{-ikx}$

  - Non-zero overlap with <u>ALL</u> $|k\rangle$ states
  - "completely localized (delocalized) in position (wavenumber) basis.

TYPICAL EIGENVALUE PROBLEM: $\hat{H}|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$

$$\hat{H} = \hat{K}^2 + \hat{V}(\hat{X}) = \hat{H}^\dagger,$$ ASSUMING ① APPROPRIATE B.C. (LEC. 7 p. 2, 3) ② $V(x) = V^*(x)$ IS A REAL FUNCTION

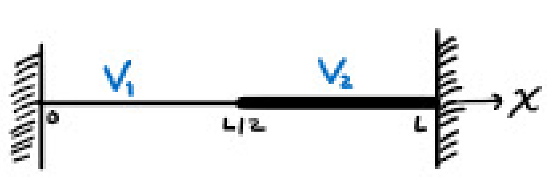
$$\langle x|\hat{H}|\varepsilon\rangle = \left[-\frac{d^2}{dx^2} + V(x)\right]\psi_\varepsilon(x) = \varepsilon\,\psi_\varepsilon(x), \quad \langle x|\varepsilon\rangle \equiv \psi_\varepsilon(x)$$

TIME-INDEPT. SCHRÖDINGER EQ.!

BUT: NO QUANTUM PHYSICS YET. THIS IS JUST A DIFFERENTIAL EIGENVALUE EQUATION

QUANTUM MECHANICS:
① PUT $\hbar$ SOMEWHERE ($\hbar \to 0$ IS "CLASSICAL LIMIT")
② INTERPRETATION OF $\psi_\varepsilon(x)$ (WAVE FUNCTION), ε
③ HOW TO PREDICT PHYSICAL MEASUREMENTS

CLASSICAL EIGENVALUE PROBLEM REDUX: INHOMOGENEOUS STRING



CONSIDER A STRING WITH TRANSVERSE DISPLACEMENT $q(x,t)$, THAT HAS AN INHOMOGENEOUS DENSITY PROFILE, S.T. THE STRING VELOCITY PARAM IS V_1, $0 \le x \le \frac{L}{2}$; V_2, $\frac{L}{2} \le x \le L$

STRING EQUATION: $-\Omega^2 q = V^2(x)\dfrac{d^2}{dx^2}q$, WHERE AGAIN WE ASSUME A RESPONSE AT FREQUENCY Ω.

REWRITE AS $\left[-\dfrac{d^2}{dx^2} - \dfrac{\Omega^2}{V^2(x)}\right]q = 0$ TAKES THE FORM OF A HERMITIAN E'VALUE PROBLEM:

$\hat{H}|q\rangle = \varepsilon|q\rangle$, BUT WITH $\varepsilon = 0$. $\underbrace{\hat{H} = \hat{K}^2 - \frac{\Omega^2}{V^2(x)}}_{\text{WELL-DEFINED SO LONG AS } V(x) \neq 0} = \hat{H}^\dagger$

HOW TO SOLVE? ANALYZE HOMOGENEOUS SEGMENTS, THEN MATCH. SAME STRATEGY AS FINITE-DEPTH SQUARE WELL IN QUANTUM MECH.

① $0 \le x \le \frac{L}{2}$: $\quad -\dfrac{d^2}{dx^2}q_< = K_1^2 q_<$, $K_1 = \dfrac{\Omega}{V_1}$; DIRICHLET B.C.: $q_<(x=0) = 0$

$$\Rightarrow \quad q_<(x) = A\sin(K_1 x)$$

② $\frac{L}{2} \le x \le L$: $\quad -\dfrac{d^2}{dx^2}q_> = K_2^2 q_>$, $K_2 = \dfrac{\Omega}{V_2}$; $q_>(x=L) = 0$

$$\Rightarrow \quad q_>(x) = C e^{iK_2 x} + D e^{-iK_2 x}; \quad q_>(L) = 0 = C e^{iK_2 L} + D e^{-iK_2 L}$$

$$\therefore \quad q_>(x) = C\left[e^{iK_2 x} - e^{iK_2(2L - x)}\right]$$

HOW TO DETERMINE COEFFICIENTS A, C ? MATCH $q_<(x)$ AND $q_>(x)$ AT $x = \frac{L}{2}$.

DIFF. EQ: $\left[-\frac{d^2}{dx^2} - \frac{\Omega^2}{V^2(x)}\right] q = 0$ OR $q'' = -\frac{\Omega^2}{V^2(x)}$

LET'S INTEGRATE THIS DIFF. EQ. THROUGH A NARROW WINDOW CENTERED AT $x = \frac{L}{2}$, WHERE $V(x)$ IS DISCONTINUOUS

$$\int_{L/2-\varepsilon}^{L/2+\varepsilon} dx \, \frac{d^2 q}{dx^2} = \int_{L/2-\varepsilon}^{L/2+\varepsilon} dx \left[-\frac{\Omega^2}{V^2(x)}\right] q$$

IN THE LIMIT $\varepsilon \to 0^+$, RHS VANISHES (SINCE $\frac{\Omega^2}{V^2(x)} q(x)$ IS FINITE EVERYWHERE)

$$\therefore \quad \frac{dq}{dx}\left(x = \tfrac{L}{2} + \varepsilon\right) = \frac{dq}{dx}\left(x = \tfrac{L}{2} - \varepsilon\right) \quad \text{OR} \quad q_>'\left(\tfrac{L}{2}\right) = q_<'\left(\tfrac{L}{2}\right)$$

SIMILARLY, CAN INTEGRATE AGAIN TO PROVE THAT $q_>\left(\frac{L}{2}\right) = q_<\left(\frac{L}{2}\right)$

$$q_<(x) = A \sin K_1 x$$

$$q_>(x) = C\left[e^{iK_2 x} - e^{iK_2(2L - x)}\right]$$

(a) $q_<'\left(\frac{L}{2}\right) = q_>'\left(\frac{L}{2}\right)$

$$\Rightarrow A K_1 \cos\left(K_1 \tfrac{L}{2}\right) = iK_2 C\left[e^{iK_2\frac{L}{2}} + e^{iK_2\frac{3}{2}L}\right] = iK_2 C e^{iK_2\frac{L}{2}}\left[1 + e^{iK_2 L}\right] \qquad (1)$$

(b) $q_<\left(\frac{L}{2}\right) = q_>\left(\frac{L}{2}\right)$

$$\Rightarrow A \sin\left(K_1 \tfrac{L}{2}\right) = C e^{iK_2\frac{L}{2}}\left[1 - e^{iK_2 L}\right] \qquad (2)$$

CONSIDER THE RATIO OF $\frac{\text{EQ. (1)}}{\text{EQ. (2)}}$:

$$\frac{\cancel{A} K_1 \cos\left(K_1 \frac{L}{2}\right)}{\cancel{A} \sin\left(K_1 \frac{L}{2}\right)} = \frac{iK_2 \cancel{C} \cancel{e^{iK_2\frac{L}{2}}}\left[1 + e^{iK_2 L}\right]}{\cancel{C} \cancel{e^{iK_2\frac{L}{2}}}\left[1 - e^{iK_2 L}\right]}$$

$$K_1 \cot\left(K_1 \tfrac{L}{2}\right) = iK_2 \frac{\left(e^{iK_2\frac{L}{2}} + e^{-iK_2\frac{L}{2}}\right)}{\left(e^{-iK_2\frac{L}{2}} - e^{iK_2\frac{L}{2}}\right)} = \cancel{i} K_2 \frac{\cancel{2} \cos\left(K_2 \frac{L}{2}\right)}{-\cancel{2}\cancel{i} \sin\left(K_2 \frac{L}{2}\right)}$$

$$K_{1,2} = \frac{\Omega}{V_{1,2}}$$

$$\boxed{\therefore \quad \frac{1}{V_1} \cot\left(\Omega \frac{L}{2V_1}\right) = -\frac{1}{V_2} \cot\left(\Omega \frac{L}{2V_2}\right)}$$

CONSTRAINS (DETERMINES) THE ALLOWED MODE FREQ. Ω!

SANITY CHECK: UNIFORM CASE, $V_1 = V_2 \equiv V$

$$\Rightarrow \cot\left(\Omega \frac{L}{2V}\right) = -\cot\left(\Omega \frac{L}{2V}\right) = 0 \,!$$

$$\therefore \quad \Omega = \frac{\cancel{2}V}{L} \frac{\pi}{\cancel{2}}(2n+1), \quad n \in \mathbb{Z}$$

$$\Rightarrow K_n = \frac{\Omega}{V} = (2n+1)\frac{\pi}{L} \; ; \quad q_n(x) = A \sin(K_n x) \Leftarrow ?$$

RECALL SOL'N FROM LEC. 1: $\phi_m(x) = \sin\left(\frac{m\pi x}{L}\right)$

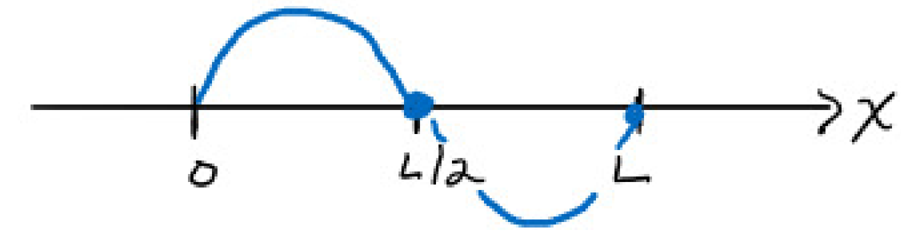
ONLY ODD-M MODES ?!

UNIFORM CASE:

LEC $\frac{1}{III}$: $\phi_m(x) = \sin(K_m x)$; $K_m = \frac{m\pi}{L}$, $m \in \{1, 2, 3 \ldots\}$

HERE: $K_n = \frac{\Omega_n}{V} = (2n+1)\frac{\pi}{L}$ $\Rightarrow$ ONLY ODD M!

? WHY ARE WE MISSING EVEN M? CONSIDER $\phi_2(x)$:



$\Rightarrow \phi_m\left(\frac{L}{2}\right) = 0$ FOR $m \in 2n = \{2, 4, 6 \ldots\}$

THIS IS A CONSEQUENCE OF PARITY OR REFLECTION SYMMETRY:

- UNIFORM STRING SATISFIES $\left(-\frac{d^2}{dx^2} - K^2\right) q = 0$.

- ASSUME THAT $q(x)$ SOLVES THIS EQUATION
- DEFINE $q_R(x) \equiv q(L-x)$, REFLECTION OF $q(x)$ ABOUT MIDPOINT $x = \frac{L}{2}$

✓ IF $q(0) = q(L) = 0 \Rightarrow q_R(0) = q_R(L) = 0$ BC STILL SATISFIED

✓ $\left(-\frac{d^2}{dx^2} - K^2\right) q_R(x) = 0$ (WHY?)

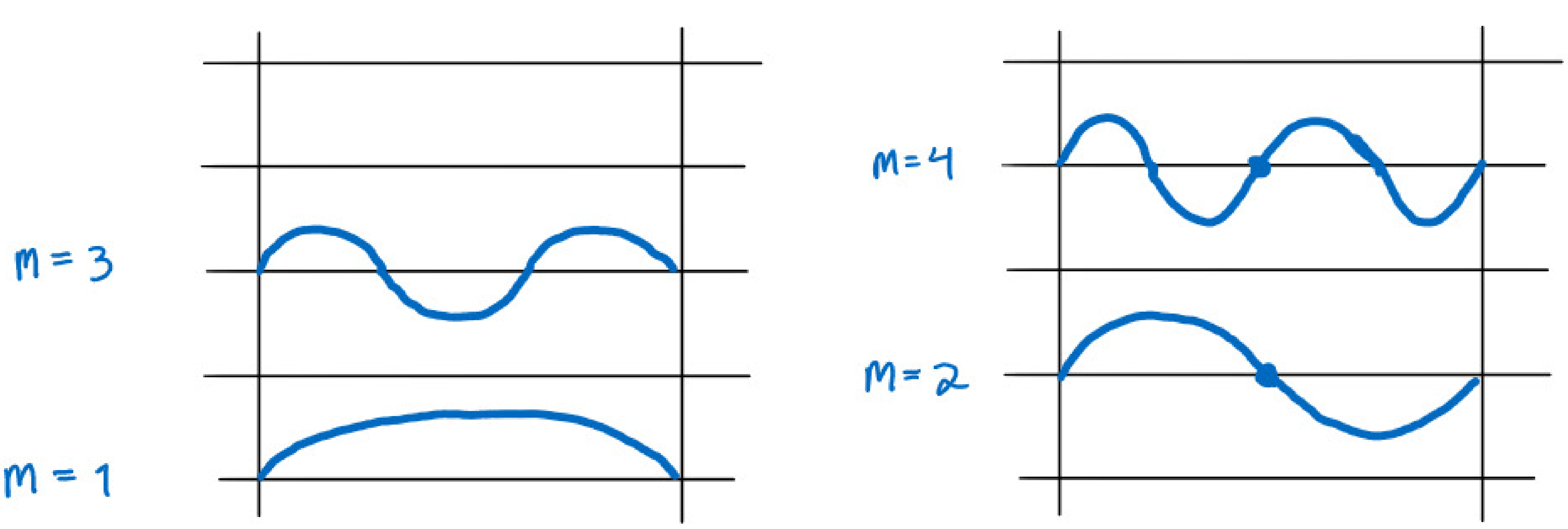
$\therefore$ BOTH $q(x)$ AND $q_R(x)$ ARE SOLUTIONS, DUE TO THE REFLECTION SYMMETRY OF THE PROBLEM

CONSEQUENCE: CAN FIND EIGENSTATES WITH DEFINITE PARITY.

- EVEN PARITY: $q_E(x) \equiv \frac{q(x) + q_R(x)}{2} = q_E(L-x)$.
- ODD PARITY: $q_O(x) \equiv \frac{q(x) - q_R(x)}{2} = -q_E(L-x) \Rightarrow q_O\left(\frac{L}{2}\right) = 0$ ✓

FOR UNIFORM STRING, EIGENMODES ALREADY HAVE DEFINITE PARITY, ARE NON-DEGENERATE

EVEN: $\phi_{2n+1}(x) = \sin\left(\frac{(2n+1)\pi x}{L}\right)$ ODD: $\phi_{2n}(x) = \sin\left(\frac{2n\pi x}{L}\right)$



$\Rightarrow$ PARITY IS ALSO AN IMPORTANT SYMMETRY IN QUANTUM MECH!

SO WHAT WENT WRONG?

FROM p. 3

(a) $q'_<\left(\frac{L}{2}\right) = q'_>\left(\frac{L}{2}\right) \Rightarrow$ EQ. (1)

(b) $q_<\left(\frac{L}{2}\right) = q_>\left(\frac{L}{2}\right) \Rightarrow$ EQ. (2)

$$\frac{\text{EQ. (1)}}{\text{EQ. (2)}} : \frac{1}{V_1}\cot\left(\Omega\frac{L}{2V_1}\right) = -\frac{1}{V_2}\cot\left(\Omega\frac{L}{2V_2}\right) \quad (*)$$

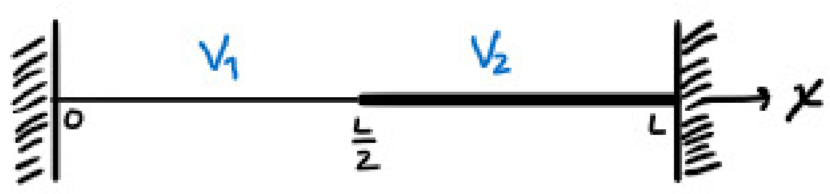
BUT: IN UNIFORM CASE $V_1 = V_2 = V$, ODD-PARITY MODES VANISH AT $q_>\left(\frac{L}{2}\right) = q_<\left(\frac{L}{2}\right) = 0$

$\Rightarrow$ EQ.(2) $\to 0$; $\frac{\text{EQ.(1)}}{\text{EQ.(2)}} \to \infty$!

$\therefore$ ODD PARITY MODES: $\tan\left(\Omega\frac{L}{2V}\right) = 0$

$\Rightarrow$ POLES OF $\cot\left(\Omega\frac{L}{2V}\right)$!

INHOMOGENEOUS STRING: NO PARITY SYMMETRY!



$$\left[-\frac{d^2}{dx^2} - \frac{\Omega^2}{V^2(x)}\right] q(x) = 0 \; ;$$

IF $q(x)$ IS A SOLUTION, $q_R(x) \equiv q(L-x)$ IS NOT A SOLUTION, BECAUSE $V(x) \neq V(L-x)$

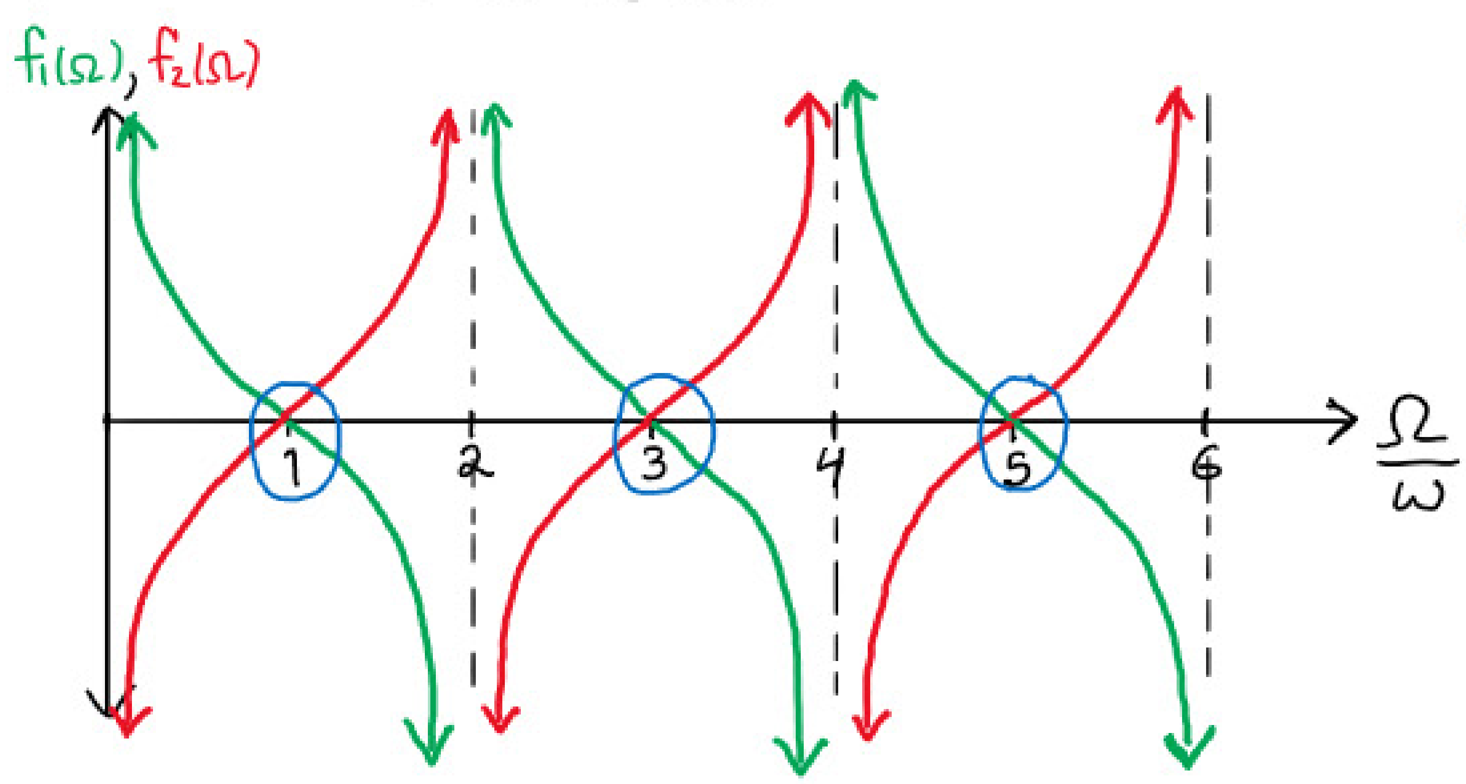
$\therefore$ DO NOT EXPECT $q\left(\frac{L}{2}\right) = 0$ FOR GENERIC EIGENMODES.

EQ. (*), ABOVE: DEFINE $\omega_i \equiv \frac{\pi V_i}{L} \Rightarrow \frac{1}{\omega_1}\cot\left[\frac{\pi}{2}\frac{\Omega}{\omega_1}\right] = -\frac{1}{\omega_2}\cot\left[\frac{\pi}{2}\frac{\Omega}{\omega_2}\right]$

$\cot\left[\frac{\pi}{2}x\right]$: (a) ZEROES AT $x = (2n+1)$
(b) POLES AT $x = 2n$

; DEFINE $f_1(\Omega) \equiv \frac{1}{\omega_1}\cot\left[\frac{\pi}{2}\frac{\Omega}{\omega_1}\right]$; $f_2(\Omega) \equiv -\frac{1}{\omega_2}\cot\left[\frac{\pi}{2}\frac{\Omega}{\omega_2}\right]$

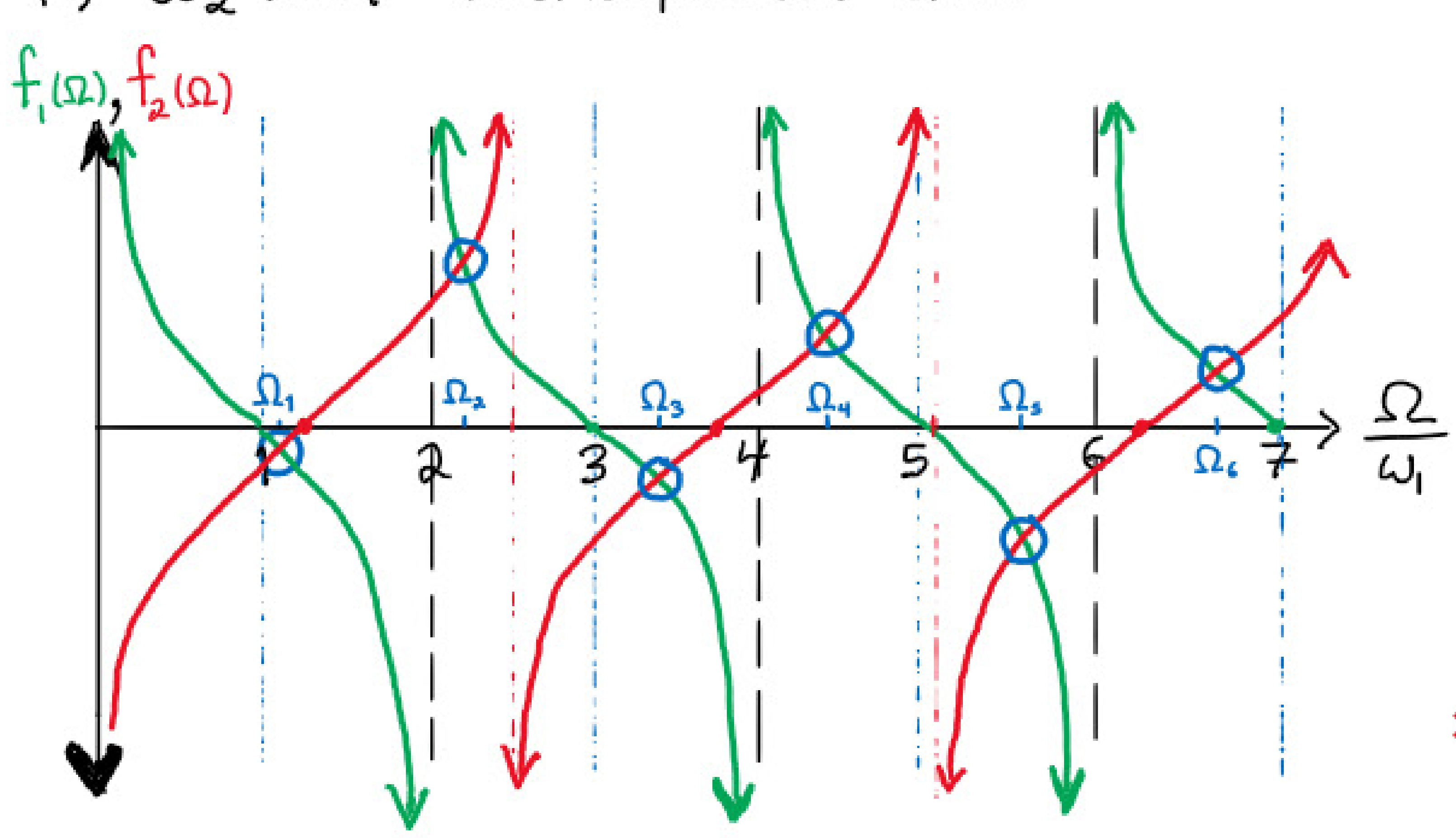
(1) UNIFORM CASE: $\omega_1 = \omega_2 \equiv \omega$



AS EXPECTED,

- $f_1(\Omega) = f_2(\Omega) = 0$ LOCATES EVEN-PARITY SOLUTIONS $\Omega = (2n+1)\frac{\pi V}{L}$
- POLES OF $f_1(\Omega) = -f_2(\Omega)$ LOCATE ODD-PARITY SOLUTIONS $\Omega = (2n)\frac{\pi V}{L}$

(2) $\omega_2 > \omega_1$ INHOMOGENEOUS CASE



$\therefore$ IN THIS CASE, ALL "EIGENFREQUENCIES" ARE OBTAINED: $\{\Omega_n\}$, $n \in \{1,2,3,\ldots\}$

= INTERSECTIONS OF $f_1(\Omega) = f_2(\Omega)$

$\Rightarrow$ SOLUTIONS LOCATED GRAPHICALLY

- CONCEPTUALLY USEFUL WAY TO IDENTIFY SOLUTIONS
- CAN USE SOFTWARE TO NUMERICALLY GET $\{\Omega_n\}$