Problem 1

$$\vec{r} = \hat{x}_i + \hat{x}_j + \hat{x}_k$$

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} \quad , \quad \hat{L}_i = \varepsilon_{ijk}\, \hat{x}_j \hat{p}_k$$

a) $[\hat{x}_j, \hat{L}_i]$

$= \varepsilon_{ilk}\, [\hat{x}_j, \hat{x}_l \hat{p}_k]$

$= \varepsilon_{ilk} ([\hat{x}_j, \hat{x}_l]\hat{p}_k + \hat{x}_l [\hat{x}_j, \hat{p}_k])$

$= \varepsilon_{ilk}\ (0 + \hat{x}_l\, i\hbar\, \delta_{jk})$

$= i\hbar\, \varepsilon_{ilj}\, \hat{x}_l$

b) $[\hat{p}_j, \hat{L}_i]$

$= \varepsilon_{ijk}\, [\hat{p}_j, \hat{x}_j \hat{p}_k]$

$= \varepsilon_{ijk}\ ([\hat{p}_j, \hat{x}_j]\hat{p}_k + \hat{x}_j [\hat{p}_j, \hat{p}_k])$

$= \varepsilon_{ijk}\ ([\hat{p}_j, \hat{x}_j]\hat{p}_k + 0)$

$= \varepsilon_{ijk} (-i\hbar\, \delta_{jj}\, \hat{p}_k)$

$= -i\hbar\, \varepsilon_{ijk} \hat{p}_k$

$p_k x_j - x_j p_k = -i\hbar$

$x_j p_k = p_k x_j + i\hbar$

c) $\hat{\vec{r}} \cdot (\hat{\vec{L}} \times \hat{\vec{r}})$

$= \hat{x}_i \varepsilon_{ijk} (\varepsilon_{jlm} \hat{x}_l \hat{p}_m) \hat{x}_k$

$= \varepsilon_{ijk} \varepsilon_{jlm} \hat{x}_i \hat{x}_l \hat{p}_m \hat{x}_k$

$= \varepsilon_{jki} \varepsilon_{jlm} \hat{x}_i \hat{x}_l \hat{p}_m \hat{x}_k$

$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \hat{x}_i \hat{x}_l \hat{p}_m \hat{x}_k$

$= \delta_{jl}\delta_{km} \hat{x}_i \hat{x}_l \hat{p}_m \hat{x}_k - \delta_{jm}\delta_{kl} \hat{x}_i \hat{x}_l \hat{p}_m \hat{x}_k$

$= \hat{x}_i \hat{x}_j \hat{p}_k \hat{x}_k - \hat{x}_i \hat{x}_k \hat{p}_j \hat{x}_k$

$= \hat{x}_i (\hat{x}_j \hat{p}_k - \hat{x}_k \hat{p}_j) \hat{x}_k$

$= \hat{x}_i \quad i\hbar \delta_{jk} \hat{x}_k$

$= i\hbar \hat{x}_i \hat{x}_j$

$= i\hbar \hat{r} \cdot \hat{r}$

Problem 2

$$\vec{r} = \hat{x}_1 + \hat{x}_2 + \hat{x}_3$$

$$\vec{P} = \hat{P}_1 + \hat{P}_2 + \hat{P}_3$$

$$\hat{L} = \varepsilon_{ijk}\, \hat{x}_j\, \hat{p}_k$$

a) $[\hat{L}, \vec{r}^2]$

$$= \varepsilon_{ijk}\, [\hat{x}_j \hat{P}_k ,\ \hat{x}_j \hat{x}_j]$$

$$= \varepsilon_{ijk}\, (\hat{x}_j [\hat{P}_k, \hat{x}_j] \hat{x}_j + [\hat{x}_j, \hat{x}_j] \hat{P}_k \hat{x}_j + \hat{x}_j \hat{x}_j [\hat{P}_k, \hat{x}_j] + \hat{x}_j [\hat{x}_j, \hat{x}_j] \hat{P}_k)$$

$$= \varepsilon_{ijk}\, (\hat{x}_j [\hat{P}_k, \hat{x}_j] \hat{x}_j + 0 + x_j x_j [\hat{P}_k, \hat{x}_j] + 0)$$

$$= \varepsilon_{ijk}\, (\hat{x}_j (-i\hbar \delta_{jk}) \hat{x}_j + i\hbar \delta_{jk}\ \hat{x}_j \hat{x}_j)$$

$$= \varepsilon_{ijk}\, (-i\hbar \delta_{jk} - i\hbar \delta_{jk})\, \hat{x}_j \hat{x}_j$$

$$= 0$$

b) $\hat{\vec{L}} \cdot \hat{\vec{r}} = \varepsilon_{ijk}\, \hat{x}_j \hat{P}_k\ \hat{x}_l\ \delta_{il}$

$$= \varepsilon_{ijk}\, (i\hbar \delta_{jk} + P_k x_j)\, \hat{x}_i$$

$$= (\varepsilon_{ijk}\, \hat{P}_k \hat{x}_j)\, \hat{x}_i$$

$$= 0$$

$$\hat{\vec{r}} \cdot \hat{L} = \hat{x}_l\ \varepsilon_{ijk}\ \hat{x}_j \hat{P}_k\ \delta_{il}$$

$$= 0$$

$\therefore \hat{L}\cdot\hat{r} = \hat{r}\cdot\hat{L} = 0$

c) $\vec{W} \equiv \frac{1}{2m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{e^2}{r}\vec{r}$

Prove $\vec{L}\cdot\vec{W} = 0$

$\vec{W} = \frac{1}{2m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{e^2}{r}\vec{r}$

$\vec{L}\cdot\vec{W} = \frac{1}{2m}(L.\vec{p}\times\vec{L} - L.\vec{L}\times\vec{p}) - 0$

$= \frac{1}{2m}(\vec{x}\times\vec{p}\cdot\vec{p}\times\vec{L} - \vec{x}\times\vec{p}\cdot\vec{L}\times\vec{p})$

$= \frac{1}{2m}((\vec{x}\cdot\vec{p})(\vec{p}\cdot\vec{L}) - \underset{\times}{(\vec{x}\cdot\vec{L})}(\vec{p}\cdot\vec{p}) - \underset{\times}{(\vec{x}\cdot\vec{L})}(\vec{p}\cdot\vec{p}) + (\vec{x}\cdot\vec{p})(\vec{p}\cdot\vec{L}))$

$= \frac{1}{m}(\vec{x}\cdot\vec{p})(\vec{p}\cdot\vec{L})$

$= \frac{1}{m}\varepsilon_{ijk}(\hat{x}_a\hat{p}_a \;\; \hat{p}_i\hat{x}_j\hat{p}_k)$ $\qquad x_jp_k - p_kx_j = i\hbar\delta_{jk}$

$= \frac{1}{m}\vec{x}\cdot\vec{p}\;\varepsilon_{ijk}(\hat{p}_i(i\hbar\delta_{jk} + p_kx_j)$

$= \frac{1}{m}\vec{x}\cdot\vec{p}\;\varepsilon_{ijk}\;p_ip_kx_j$

$= \frac{1}{m}\vec{x}\cdot\vec{p}\;0$

$= 0$

Problem 3

a) $j_1 = \frac{5}{2}, \quad j_2 = \frac{3}{2}$

$Max(j_t) = \frac{8}{2} = 4$

$Min(j_t) = \frac{2}{2} = 1$

$\frac{5}{2} \times \frac{3}{2}$

m_1, m_2 \ J, M	4, +4
$+\frac{5}{2}, +\frac{3}{2}$	1

m_1, m_2 \ J, M	4, +3	3, +3
$+\frac{5}{2}$, $+\frac{1}{2}$		
$+\frac{3}{2}$, $+\frac{3}{2}$		

	4, +2	3, +2	2, +2

	4, +1	3, +1	2, +1	1, +1

	4, −1	3, −1	2, −1	1, −1

	4, −2	3, −2	2, −2

	4, −3	3, −3

m_1, m_2 \ J, M	4, −4
$-\frac{5}{2}, -\frac{3}{2}$	1

$\hat{J}_- |j, m\rangle = \sqrt{j(j+1) - m(m-1)}\, \hbar\, |j, m-1\rangle$

b) $\frac{5}{2} \times \frac{3}{2}$

m_1, m_2 \ J, M	4, +4	4, +3	3, +3
$+\frac{5}{2}$, $+\frac{3}{2}$	1		
$+\frac{5}{2}$, $+\frac{1}{2}$			
$+\frac{3}{2}$, $+\frac{3}{2}$			

$\hat{J}_- |4, 4\rangle = (\hat{J}_{1-} + \hat{J}_{2-}) |\frac{5}{2}, \frac{5}{2}\rangle |\frac{3}{2}, \frac{3}{2}\rangle$

$LHS = \hbar\sqrt{20 - 12}\, |4, 3\rangle = \hbar\sqrt{8}\, |4, 3\rangle = 2\hbar\sqrt{2}\, |4, 3\rangle$

$RHS = \hbar\left(\sqrt{\frac{5}{2}\times\frac{7}{2} - \frac{5}{2}\times\frac{3}{2}}\, |\frac{5}{2}, \frac{3}{2}\rangle |\frac{3}{2}, \frac{3}{2}\rangle + \sqrt{\frac{3}{2}\times\frac{5}{2} - \frac{3}{2}\times\frac{1}{2}}\, |\frac{5}{2}, \frac{5}{2}\rangle |\frac{3}{2}, \frac{1}{2}\rangle\right)$

$= \hbar\left(\sqrt{5}\, |\frac{5}{2}, \frac{3}{2}\rangle\rangle + \sqrt{3}\, |\frac{3}{2}, \frac{1}{2}\rangle\right)$

$\therefore |4, 3\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{5}\, |\frac{5}{2}, \frac{3}{2}\rangle\rangle + \sqrt{3}\, |\frac{3}{2}, \frac{1}{2}\rangle\right)$

$= \sqrt{\frac{5}{8}}\, |\frac{5}{2}, \frac{3}{2}\rangle\rangle + \sqrt{\frac{3}{8}}\, |\frac{3}{2}, \frac{1}{2}\rangle$

$$\begin{array}{cc|c} \boxed{1} & & \begin{matrix}4\\+3\end{matrix} \\ \hline \frac{5}{2} & \frac{1}{2} & \frac{3}{8} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{8} \\ \hline \end{array}$$

now

$$\langle 4,3 | 3,3\rangle = \alpha \left|\tfrac{5}{2},\tfrac{5}{2}\right\rangle\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle + \beta\left|\tfrac{5}{2},\tfrac{3}{2}\right\rangle\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle = 0$$

$$\alpha^2+\beta^2=1$$

$$\alpha=\sqrt{\frac{5}{8}}\ , \quad \beta=-\frac{3}{8}$$

$$\begin{array}{cc|cc} \boxed{1} & & \begin{matrix}4\\+3\end{matrix} & \begin{matrix}3\\+3\end{matrix} \\ \hline \frac{5}{2} & \frac{1}{2} & \frac{3}{8} & \frac{5}{8} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{8} & -\frac{3}{8} \end{array}$$

c) $\quad |4,-4\rangle_t = \left|\tfrac{5}{2},-\tfrac{5}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle$

$$J^+|4,-4\rangle_t = J_1^+J_2^+\left|\tfrac{5}{2},-\tfrac{5}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle \qquad J^+|j,m\rangle=\hbar\sqrt{j(j+1)-m(m+1)}\ |j,m+1\rangle$$

$$\hbar\sqrt{8}\ |4,-3\rangle_t = \hbar\left(\sqrt{5}\left|\tfrac{5}{2},-\tfrac{3}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle+\sqrt{3}\left|\tfrac{5}{2},-\tfrac{5}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle\right)$$

$$|4,-3\rangle_t = \sqrt{\frac{5}{8}}\left|\tfrac{5}{2},-\tfrac{3}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle+\sqrt{\frac{3}{8}}\left|\tfrac{5}{2},-\tfrac{5}{2}\right\rangle\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle$$

$$\begin{array}{cc|c|c} & & \begin{matrix}4\\-3\end{matrix} & \\ \hline -\frac{3}{2} & -\frac{3}{2} & \frac{5}{8} & 4 \\ -\frac{5}{2} & -\frac{1}{2} & \frac{3}{8} & -4 \\ \hline & & -\frac{5}{2}\ -\frac{3}{2} & \boxed{1} \end{array}$$

change notation to $|m_1,m_2\rangle$

$$\langle 3,-3|4,-3\rangle = \alpha\sqrt{\tfrac{5}{8}}\left|-\tfrac{3}{2},-\tfrac{3}{2}\right\rangle+\beta\sqrt{\tfrac{3}{8}}\left|-\tfrac{5}{2},-\tfrac{1}{2}\right\rangle = 0$$

$$\alpha^2+\beta^2=1$$

$$\alpha=-\sqrt{\frac{3}{8}}\ , \quad \beta=\sqrt{\frac{5}{8}}$$

$$\begin{array}{cc|cc|c} & & \begin{matrix}4\\-3\end{matrix} & & \\ \hline -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{8} & \frac{5}{8} & 4 \\ -\frac{5}{2} & -\frac{1}{2} & \frac{5}{8} & \frac{3}{8} & -4 \\ \hline & & -\frac{5}{2}\ -\frac{3}{2} & & \boxed{1} \end{array}$$

Problem 4

$$\hat{H} = g\hat{S}_1 \hat{S}_2$$

a) $$\hat{S}^2 = \hat{S}_1^2 + S_1S_2 + S_2S_1 + S_2^2$$

$$\therefore \hat{S}_1\hat{S}_2 = (\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)\frac{1}{2} = \frac{1}{g}\hat{H}$$ As $[S_1, S_2] = 0$

$$\hat{H} = \frac{g}{2}(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)$$

b) $$\hat{H}|j,m\rangle_t = \frac{g}{2}(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)|j_1,m_1\rangle|j_2,m_2\rangle$$

$$= \frac{g}{2}(\hbar^2(j_t)(j_t+1)|j,m\rangle_t - \hbar^2 j_1(j_1+1)|j,m\rangle_t - \hbar^2 j_2(j_2+1)|j_2,m\rangle_t)$$

$$= \frac{\hbar^2 g}{2}((j_t)(j_t+1) - j_1(j_1+1) - j_2(j_2+1))|j,m\rangle_t$$

$$= (\text{number}) \cdot |j,m\rangle_t$$

$\therefore$ are energy eigenstates as they equates constant times the vector

Now $j_t \sim [1,4]$ as $\frac{5}{2}+\frac{3}{2}=4$, $|\frac{5}{2}-\frac{3}{2}|=1$

		E	deg
$j_t=1$,	$m_t=-1,0,1,$	$-\frac{21}{4}\hbar^2 g$	3
$j_t=2$,	$m_t=-2,-1,0,1,2$	$-\frac{13}{4}\hbar^2 g$	5
$j_t=3$,	$m_t=-3,-2,-1,0,1,2,3$	$-\frac{1}{4}\hbar^2 g$	7
$j_t=4$,	$m_t=-4,-3,-2,-1,0,1,2,3,4$	$\frac{15}{4}\hbar^2 g$	9