Summary: Hermitian Operators and Eigenspectra on the Infinite Real Line

① **Position Operator** $\hat{X}$ $\qquad$ $\hat{X}|x\rangle = x|x\rangle$

    a) Eigenvectors $|x\rangle$

- Overlap: $\langle x|x'\rangle = \delta(x-x')$

- Wavenumber Basis: $\langle k|x\rangle = \frac{1}{\sqrt{2\pi}} e^{-ikx}$

    b) Matrix Elements

- $|x\rangle$ Basis: $\langle x|\hat{X}|x'\rangle = x\,\delta(x-x') = x'\,\delta(x-x')$

- $|k\rangle$ Basis: $\langle k|\hat{X}|k'\rangle = i\frac{d}{dk}\delta(k-k') = \delta(k-k')\,i\frac{d}{dk'}$

    c) Effect on Generic state $|f\rangle$

- $|x\rangle$ Basis: $\langle x|\hat{X}|f\rangle = \int dx'\,\langle x|\hat{X}|x'\rangle\langle x'|f\rangle = x\,f(x)$

- $|k\rangle$ Basis: $\langle k|\hat{X}|f\rangle = \int dk'\,\langle k|\hat{X}|k'\rangle\underbrace{\langle k'|f\rangle}_{\equiv\ \tilde{f}(k')} = i\frac{d}{dk}\tilde{f}(k)$

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② **Wavenumber / Hermitian Derivative** $\hat{K}$ $\qquad$ $\hat{K}|k\rangle = k|k\rangle$

    a) Eigenvectors $|k\rangle$

- Overlap $\langle k|k'\rangle = \delta(k-k')$

- Position Basis: $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{ikx}$

    b) Matrix Elements

- $|x\rangle$ Basis: $\langle x|\hat{K}|x'\rangle = -i\frac{d}{dx}\delta(x-x') = \delta(x-x')\left[-i\frac{d}{dx'}\right]$

- $|k\rangle$ Basis: $\langle k|\hat{K}|k'\rangle = k\,\delta(k-k') = k'\,\delta(k-k')$

    c) Effect on a Generic State $|f\rangle$

- $|x\rangle$ Basis: $\langle x|\hat{K}|f\rangle = \int dx'\,\langle x|\hat{K}|x'\rangle\langle x'|f\rangle = -i\frac{df(x)}{dx}$

- $|k\rangle$ Basis: $\langle k|\hat{K}|f\rangle = \int dk'\,\langle k|\hat{K}|k'\rangle\underbrace{\langle k'|f\rangle}_{\equiv\ \tilde{f}(k')} = k\,\tilde{f}(k)$