Problem 1 Eric Cai

$$f(E) = \begin{cases} 0 & E \geq E_0 \\ \frac{f_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \left\{ \exp\left(-\frac{E-E_0}{\sigma^2}\right) - 1 \right\} & E < E_0 \end{cases}$$

$$E = \frac{v^2}{2} + \phi(r)$$

a) $d^3v = 4\pi v^2 dv$

prove: $\frac{\rho(r)}{\rho_0} = e^{x} \operatorname{erf}(\sqrt{x}) - \sqrt{\frac{4x}{\pi}}\left(1 + \frac{2x}{3}\right)$ $\quad x = \frac{E_0 - \phi(r)}{\sigma^2}$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2} du$$

$$-\frac{E-E_0}{\sigma^2} = -\frac{1}{\sigma^2}\left(\frac{v^2}{2} + \phi(r) - E_0\right)$$

$$= -\frac{1}{\sigma^2}\frac{v^2}{2} + \frac{E_0 - \phi(r)}{\sigma^2}$$

$$= -\frac{1}{2\sigma^2} v^2 + x$$

let $k = \frac{f_0}{(2\pi\sigma^2)^{\frac{3}{2}}}$

$\therefore \rho(r) = k \int (e^{-\frac{1}{2\sigma^2}v^2} e^{x} - 1)\, 4\pi v^2 dv$

get 4π:

$4\pi k = f_0\, 2^{-\frac{3}{2}} \pi^{-\frac{3}{2}} (\sigma^2)^{-\frac{3}{2}}\, 2^2 \pi^1 = f_0\, 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} \sigma^{-3}$

$= f_0 \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^3}$

$\therefore \frac{\rho(r)}{\rho_0} = \frac{1}{\sigma^3}\sqrt{\frac{2}{\pi}} \int e^{-\frac{1}{2\sigma^2}v^2} e^{x} v^2 - v^2 \, dv$

Integrate from to $E = E_0$

$\therefore \frac{v_0^2}{2} + \phi(r) = E_0$

$\frac{v^2}{2} = E_0 - \phi(r)$

$$\frac{v^2}{2\sigma^2} = \frac{E_0 - \phi(r)}{\sigma^2} = \chi$$

$$\therefore \frac{f(r)}{f_0} = \frac{1}{\sigma^3}\sqrt{\frac{2}{\pi}} \int_{v=0}^{\frac{v^2}{2\sigma^2}=\chi} e^{-\frac{v^2}{2\sigma^2}} e^{\chi} v^2 - v^2 dv$$

we want $\frac{v^2}{2\sigma^2} = u^2$

$$v^2 = u^2 2\sigma^2 \quad , \quad v = u\sqrt{2\sigma^2}$$

$$\frac{d}{dv} u^2 2\sigma^2 = \frac{d}{dv} v^2$$

$$2\sigma^2 \, 2u \frac{du}{dv} = 2v$$

$$2\sigma^2 u\,du = v\,dv = u\sqrt{2\sigma^2}\,dv$$

$$\sqrt{2\sigma^2}\,du = dv$$

$$\therefore \frac{f(r)}{f_0} = \frac{1}{\sigma^3}\sqrt{\frac{2}{\pi}} \int_{u=0}^{\sqrt{\chi}} \left(e^{-u^2} e^{\chi} u^2 2\sigma^2 - u^2 2\sigma^2\right)\sqrt{2\sigma^2}\,du$$

$$= \frac{1}{\sigma^2}\sqrt{\frac{4}{\pi}}\left(-\frac{u^3}{3} 2\sigma^2 \Big|_0^{\sqrt{\chi}} + \int_{u=0}^{\sqrt{\chi}} 2\sigma^2 e^{\chi} u^2 e^{-u^2} du\right)$$

$$= \sqrt{\frac{4\chi}{\pi}}\left(-\frac{2\chi}{3}\right) + \sqrt{\frac{4}{\pi}}\, 2e^{\chi} \int_{u=0}^{\sqrt{\chi}} u^2 e^{-u^2} du$$

$$= \sqrt{\frac{4\chi}{\pi}}\left(-\frac{2\chi}{3}\right) + \sqrt{\frac{4}{\pi}}\, 2e^{\chi} \frac{1}{4}\left(-2e^{-\chi}\sqrt{\chi} + \sqrt{\pi}\,\mathrm{erf}(\sqrt{\chi})\right)$$

$$= \sqrt{\frac{4\chi}{\pi}}\left(-\frac{2\chi}{3}\right) + \sqrt{\frac{4}{\pi}}\left(-\sqrt{\chi} + 2e^{\chi}\sqrt{\pi}\,\mathrm{erf}(\sqrt{\chi})\right)$$

$$= e^{\chi}\mathrm{erf}(\sqrt{\chi}) - \sqrt{\frac{4\chi}{\pi}}\left(1 + \frac{2\chi}{3}\right)$$

$$\begin{array}{ccc} 2\pi & 2 & \cos\theta \end{array}$$

b) $$M(r_t) = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r_t} f(r)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 4\pi \int_{r=0}^{r_t} f(r) r^2 dr$$

we know $f(r_t) = 0$ such that $\chi = 0$

$\therefore \chi(r_t) \equiv 0$ is definition of r_t

c) assume $\phi(r) = -\frac{GM(r)}{r}$

$x \ll 1$, $\frac{E_0 - \phi(r_t)}{\sigma^2} \ll 1$, $\sigma^2 \propto E_k$ $\therefore$ ϕ dominance

$\therefore \frac{1}{\sigma^2}(E_0 - \frac{GM(r_t)}{r_t}) + 1 = 1$

$E_0 = \frac{GM(r_t)}{r_t}$

d) $\frac{\rho(r)}{\rho_0} = e^x \operatorname{erf}(\sqrt{x}) - \sqrt{\frac{4x}{\pi}}(1 + \frac{2x}{3})$

$\rho(r) = \rho_0 (e^x \operatorname{erf}\sqrt{x} - \sqrt{\frac{4x}{\pi}}(1 + \frac{2x}{3}))$

$\overset{\text{expand}}{\approx} \frac{8x^{\frac{5}{2}}}{15\sqrt{\pi}} \rho_0$

let $x = \frac{1}{\sigma^2}(\frac{GM(r_t)}{r_t} - \frac{GM_{r_t}}{r})$

$\rho(r) = \rho_0 \frac{8}{15\sqrt{\pi}} (\frac{1}{\sigma^2}(\frac{GM_{r_t}}{r_t} - \frac{GM_{r_t}}{r}))^{\frac{5}{2}}$

e) $M(r_t) = 4\pi \int_{r=0}^{r_t} \rho(r) r^2 dr$ $\qquad \frac{GM_{r_t}}{r_t} = E_0$, $GM_{r_t} = b$

$= 4\pi \rho_0 \frac{8}{15\sqrt{\pi}} \frac{1}{\sigma^5} \int_0^{r_t} (E_0 - \frac{b}{r})^{\frac{5}{2}} r^2 dr$

$= \frac{32\pi\rho_0}{15\sqrt{\pi}\sigma^5} \frac{5}{16} (-GM_{r_t})^{\frac{5}{2}} \pi \sqrt{r_t}$

$= \frac{2\pi\rho_0}{3\sqrt{\pi}\sigma^5} G^{\frac{5}{2}} M_{(r_t)}^{\frac{5}{2}} \pi \sqrt{r_t}$ $\qquad 1 - \frac{5}{2} = -\frac{3}{2}$

$= \frac{2\rho_0 \pi^{\frac{3}{2}}}{3\sigma^5} G^{\frac{5}{2}} M_{r_t}^{\frac{5}{2}} r_t^{\frac{1}{2}} r_t^{-\frac{5}{2}} r_t^{\frac{5}{2}}$

$= \frac{2\rho_0 \pi^{\frac{3}{2}}}{3\sigma^5} E_0^{\frac{5}{2}} r_t^3$

f) All we really know is the radial velocity σ from spectrometry

① Tully fisher → mass & distance → ρ_0

② Virial theorem & definition of E → E_0

Problem 2.

Eq 24.51 : $\rho(r) = \rho_0 \frac{1}{1+(r/a)^2}$ $\quad r >> a$

$= \rho_0 \frac{1}{\frac{a^2}{a^2}+\frac{r^2}{a^2}}$

$= \rho_0 \frac{a^2}{a^2+r^2}$ $\quad r >> a$

$= \rho_0 \frac{a^2}{r^2}$

$= \rho_0 \frac{c}{r^2}$ $\quad c \approx a^2$

now $\frac{d^2r}{dt^2} = -\frac{GM_r}{r^2}$

$M_r = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0\, r^2 \sin\theta\, dr\, d\theta\, d\phi$

$= 4\pi \int_0^{r} \rho_0 C\, dr'$

$= 4\pi \rho_0 C r$

Assume M_r never change when r reduces (like matter are in a ballon)

$\therefore \frac{d^2r}{dt^2} = -\frac{G4\pi\rho_0 C}{r}$

$v\frac{dv}{dt} = -\frac{G4\pi\rho_0 C}{r}\frac{dr}{dt}$

$v\frac{dv}{dt}dt = -G4\pi\rho_0 C \frac{1}{r} dr$

$\frac{1}{2}\frac{dv^2}{dt}dt = -G4\pi\rho_0 C \frac{1}{r} dr$

$\int_0^t \frac{1}{2}\frac{d}{dt}v^2 dt = -G4\pi\rho_0 C \int_{r_0}^{r} \frac{1}{r} dr$

$\frac{v^2}{2} = G4\pi\rho_0 C \ln\left(\frac{r_0}{r}\right)$

$v^2 = G8\pi\rho_0 C \ln\left(\frac{r_0}{r}\right)$

$$\therefore \left(\frac{dr}{dt}\right)^2 = G\, 8\pi \rho_0 C \ln\left(\frac{r_0}{r}\right)$$

$$\frac{dr}{dt} = \sqrt{8G\pi\rho_0 C}\,\sqrt{\ln\left(\frac{r_0}{r}\right)}$$

$$\frac{1}{\sqrt{\ln\left(\frac{r_0}{r}\right)}}\,dr = \sqrt{8G\pi\rho_0 C}\,dt$$

$$u = \frac{r}{r_0}, \qquad \frac{du}{dr} = \frac{1}{r_0}, \qquad dr = r_0\,du$$

$$\therefore \frac{1}{\sqrt{\ln\left(\frac{r_0}{r}\right)}}\,dr = \frac{1}{\sqrt{\ln\left(\frac{1}{u}\right)}}\,r_0\,du$$

$$u = 1:\ r = r_0\ ; \qquad u = 0\ \ r = 0$$

$$\therefore \int_0^1 \frac{1}{\sqrt{\ln\left(\frac{1}{u}\right)}}\,r_0\,du = \int_0^{t_{ff}} \sqrt{8G\pi\rho_0 C}\,dt$$

$$r_0\sqrt{\pi} = \sqrt{8G\pi\rho_0 C}\ t_{ff}$$

$$t_{ff} = \frac{r_0}{\sqrt{8G\rho_0 C}} \qquad C \approx a^2$$

b) $r_i = r_0 = 10a = 300\,kpc$

$M_r = 4\pi\rho_0 C r = 3\times10^{11} M_\odot$

$\rho_0 C = \frac{M_r}{4\pi r}$

$$\therefore t_{ff} = \frac{300\,kpc}{\sqrt{8G\,\frac{3\times10^{11}M_\odot}{4\pi\,300\,kpc}}} = 1.77\times10^{17}\,s$$

$$= 5.614\times10^{9}\,yr$$

Hubble is 14.4×10^9, so galaxy freefall is less than age of universe, about same order of magnitude though.

Problem 3 27.12

eq : 27.20: $L_x = \frac{4}{3}\pi R^3 L_{vol}$

a) $n_e = \left[\frac{3L_x}{4\pi R^3 T^{\frac{1}{2}} (1.42\times10^{-40})} \right]^{\frac{1}{2}}$

$L_x = 1.5\times10^{36}$

$R = 1.5\ Mpc = 1.5\times10^{6}\ pc$

$T = 70\ MK = 7\times10^{7}\ K$

$\therefore n_e = 54.9\ m^{-3}$

$M_{gas} = \frac{4}{3}\pi R^3 n_e m_H$, $m_H = 1.67\times10^{-27}$

$= 3.84\times10^{43}\ kg$

$= 1.9\times10^{13} M_\odot$

b) $L_v = 1.2\times10^{12} L_\odot$,

$\frac{M}{L} \approx \frac{3M_\odot}{L_\odot}$

$M \approx 3.6\times10^{12} M_\odot$, about 19% mass.

c) $L_{vol} = \frac{3L_x}{4\pi R^3} = 3.5824\times10^{-33}\ W/m^3$

$\frac{E}{m^3} = 2\times n_e \left(\frac{3}{2}kT\right) = 1.59\times10^{-13}\ J/m^3$

Cosmological Neutro Charge

$$t = \frac{E}{m^3}/L_{vol} = \frac{1.59e{-13}}{3.5824e{-33}} = 4.438\times10^{19}\,s$$

$$= 1.4\times10^{12}\,yr, \quad 100 \text{ time } t_H$$