Quantum S.H.O. Part II: Operator Method

$\hat{H} = \frac{1}{2M}\hat{P}^2 + \frac{m\omega^2}{2}\hat{X}^2$; $\hat{H}|n\rangle = E_n|n\rangle$, $E_n = \hbar\omega(n+\frac{1}{2})$, $n \in \{0,1,2,\dots\}$

$\langle x|n\rangle = \psi_n(x) = c_n H_n(\frac{x}{b}) e^{-\frac{1}{2}(\frac{x}{b})^2}$, $b = \sqrt{\frac{\hbar}{m\omega}}$

Introduce Dim.less Operators:

$$\hat{\bar{X}} \equiv \frac{1}{b}\hat{X} \; ; \; \hat{\bar{P}} \equiv \frac{b}{\hbar}\hat{P} \quad ; \quad [\hat{\bar{X}}, \hat{\bar{P}}] = i\hat{\mathbb{I}}$$

$$\hat{H} = \frac{1}{2m}\left(\frac{\hbar}{b}\right)^2\hat{\bar{P}}^2 + \frac{m\omega^2}{2}b^2\hat{\bar{X}}^2 = \frac{1}{2\cancel{m}}\cancel{m}\omega\hbar\,\hat{\bar{P}}^2 + \frac{\cancel{m}\omega^{\cancel{2}}}{2}\frac{\hbar}{\cancel{m\omega}}\bar{X}^2 = \frac{\hbar\omega}{2}\left[\hat{\bar{P}}^2 + \hat{\bar{X}}^2\right]$$

Now, $E_n = \hbar\omega(n+\frac{1}{2}) \Rightarrow$ suggests $\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2}\hat{\mathbb{I}})$; Number Operator $\hat{n} = \hat{n}^\dagger$; $\hat{n}|n\rangle = n|n\rangle$, $n \in \{0,1,2,\dots\}$

$$\therefore \hat{H} = \frac{\hbar\omega}{2}\left[\hat{\bar{P}}^2 + \hat{\bar{X}}^2\right] = \hbar\omega(\hat{n} + \tfrac{1}{2}\hat{\mathbb{I}})$$

Symmetry: $\hat{H}$ is invariant under a "Rotation" $\begin{bmatrix}\hat{\bar{X}}' \\ \hat{\bar{P}}'\end{bmatrix} = \begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\hat{\bar{X}} \\ \hat{\bar{P}}\end{bmatrix}$

Consider $\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{\bar{X}} + i\hat{\bar{P}})$

$\hat{a}^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{\bar{X}} - i\hat{\bar{P}})$

Compare to SO(3) or SU(2):

$\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y$

Under the "Rotation":

- $\hat{a} \to \frac{1}{\sqrt{2}}\left(\cos\phi\,\hat{\bar{X}} - \sin\phi\,\hat{\bar{P}} + i\left[\sin\phi\,\hat{\bar{X}} + \cos\phi\,\hat{\bar{P}}\right]\right) = \frac{1}{\sqrt{2}}\left(e^{i\phi}\hat{\bar{X}} + ie^{i\phi}\hat{\bar{P}}\right)$
  $= e^{i\phi}\hat{a}$
- $\hat{a}^\dagger \to \hat{a}^\dagger e^{-i\phi}$

$\Rightarrow \hat{H} = \hbar\omega\left[c_1\hat{a}^\dagger\hat{a} + c_2\hat{a}\hat{a}^\dagger\right]$ for some real, const. $c_{1,2}$

In fact: $\hat{\bar{X}} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$; $\hat{\bar{P}} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})$

$$\Rightarrow \hat{H} = \frac{\hbar\omega}{2}\left[\hat{\bar{P}}^2 + \hat{\bar{X}}^2\right] = \frac{\hbar\omega}{2}\Big[\left(-\tfrac{1}{2}\right)\left(\cancel{\hat{a}^\dagger\hat{a}^\dagger} - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \cancel{\hat{a}^2}\right)$$
$$+ \left(\tfrac{1}{2}\right)\left(\cancel{\hat{a}^\dagger\hat{a}^\dagger} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \cancel{\hat{a}^2}\right)\Big]$$

$$= \frac{\hbar\omega}{2}\left[\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right]$$

$$\bullet\ [\hat{a},\hat{a}^\dagger] = \tfrac{1}{2}[\hat{X}+i\hat{P},\ \hat{X}-i\hat{P}] = \tfrac{1}{2}\left([i\hat{P},\hat{X}]+[\hat{X},-i\hat{P}]\right) = \hat{\mathbb{I}}$$

$$\therefore\ \hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\right) = \frac{\hbar\omega}{2}\left(2\hat{a}^\dagger\hat{a}+\hat{\mathbb{I}}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a}+\tfrac{1}{2}\hat{\mathbb{I}}\right)$$

$$\boxed{\Rightarrow \text{The Number Operator } \hat{n} \equiv \hat{a}^\dagger\hat{a}\ ;\ \hat{H} = \hbar\omega\left(\hat{n}+\tfrac{1}{2}\hat{\mathbb{I}}\right)}$$

Note: $\hat{n}^\dagger = (\hat{a}^\dagger\hat{a})^\dagger$
$= (\hat{a})^\dagger(\hat{a}^\dagger)^\dagger$
$= \hat{a}^\dagger\hat{a} = \hat{n}$ ✓

What is the Relationship Between $\hat{n}, \hat{a}, \hat{a}^\dagger$?

Recall $SO(3)$ or $SU(2)$:

$\hat{J}_z|m_z\rangle = m_z|m_z\rangle$; $\quad [\hat{J}_z,\hat{J}_\pm] = \pm\hat{J}_\pm \quad \Rightarrow \quad \hat{J}_\pm|m_z\rangle \propto |m_z \pm 1\rangle$

$\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y \quad [\hat{J}_+,\hat{J}_-] = 2\hat{J}_z \quad$ "Ladder Operators"

using: $[\hat{A}\hat{B},\hat{C}] = [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}]$

$\bullet\ [\hat{n},\hat{a}] = [\hat{a}^\dagger\hat{a},\hat{a}] = [\hat{a}^\dagger,\hat{a}]\hat{a} + \hat{a}^\dagger\cancel{[\hat{a},\hat{a}]} = -[\hat{a},\hat{a}^\dagger]\hat{a} = -\hat{a}$

$\bullet\ [\hat{n},\hat{a}^\dagger] = [\hat{a}^\dagger\hat{a},\hat{a}^\dagger] = +\hat{a}^\dagger$

Suggests (imprecise) Analogy:
$\hat{n} \leftrightarrow \hat{J}_z$
$\hat{a} \leftrightarrow \hat{J}_-$, $\hat{a}^\dagger \leftrightarrow \hat{J}_+$

Do $\hat{a}$ and $\hat{a}^\dagger$ act as Ladder Operators on Number States?

Let $\hat{n}|n\rangle = n|n\rangle$ Number Operator Eigenspectrum

Consider the states $\hat{a}|n\rangle$, $\hat{a}^\dagger|n\rangle$:

$\bullet\ \hat{n}\hat{a}|n\rangle = \left[\hat{a}\hat{n} - \hat{a}\right]|n\rangle = (n-1)\hat{a}|n\rangle$

$\therefore\ \hat{a}|n\rangle \propto |n-1\rangle \Rightarrow \hat{a}$ Lowers # of "Quanta" (n) by one: "Annihilation Operator"

$\bullet\ \hat{n}\hat{a}^\dagger|n\rangle = \left[\hat{a}^\dagger\hat{n} + \hat{a}^\dagger\right]|n\rangle = (n+1)\hat{a}^\dagger|n\rangle$

$\therefore\ \hat{a}^\dagger|n\rangle \propto |n+1\rangle \Rightarrow \hat{a}^\dagger$ Raises # of "Quanta" (n) by one: "Creation Operator"

$\hat{a}^\dagger|n\rangle \propto |n+1\rangle \Rightarrow \hat{a}^\dagger|n\rangle \equiv \alpha_n |n+1\rangle$, $\alpha_n \in \mathbb{C}$ SOME CONSTANT.

$$\therefore \langle n|\hat{a}\hat{a}^\dagger|n\rangle = |\alpha_n|^2 \langle n+1|n+1\rangle = |\alpha_n|^2$$

$$= \langle n|\hat{a}^\dagger\hat{a} + \hat{\mathbb{I}}|n\rangle = (n+1)\langle n|n\rangle = (n+1) \Rightarrow \text{CAN TAKE } \alpha_n = \sqrt{n+1}$$

SIMILARLY, LET $\hat{a}|n\rangle \equiv \beta_n |n-1\rangle$

$$\therefore \langle n|\hat{a}^\dagger\hat{a}|n\rangle = n = |\beta_n|^2 \langle n-1|n-1\rangle = |\beta_n|^2 \Rightarrow \text{CAN TAKE } \beta_n = \sqrt{n}$$

SUMMARY: CREATION, ANNIHILATION, AND NUMBER OPERATORS

1. $\hat{n}|n\rangle = n|n\rangle$

2. $\hat{n} = \hat{a}^\dagger\hat{a}$; $[\hat{n},\hat{a}] = -\hat{a}$, $[\hat{n},\hat{a}^\dagger] = +\hat{a}^\dagger$; $[\hat{a},\hat{a}^\dagger] = \hat{\mathbb{I}}$

   SIMILAR, BUT NOT IDENTICAL TO SO(3) or SU(2): $\hat{J}_z|m_z\rangle = m_z|m_z\rangle$

   $[\hat{J}_z, \hat{J}_-] = -\hat{J}_-$, $[\hat{J}_z, \hat{J}_+] = +\hat{J}_+$; $[\hat{J}_-, \hat{J}_+] = -2\hat{J}_z$

3. $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

   $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

   MNEMONIC FOR COEFFICIENT: THE LARGER OF THE n's LABELING THE TWO KETS ON EITHER SIDE OF THE EQN. GOES UNDER THE SQUARE ROOT.

NOTE: $\langle n|\hat{a}^\dagger\hat{a}|n\rangle = n\langle n-1|n-1\rangle \Rightarrow \langle 0|\hat{a}^\dagger\hat{a}|0\rangle = \langle 0|\hat{n}|0\rangle = 0$

SHO GROUND STATE WITH $n=0$, NOT THE NULL VECTOR.

$$\therefore n \in \{0, 1, 2, 3, \ldots\} \checkmark$$

CREATING NUMBER EIGENSTATES OUT OF THE "VACUUM"

$|n=0\rangle$: GROUND STATE OF QUANTUM S.H.O. $\equiv$ "VACUUM"

$\rightarrow$ STATE WITH $n=0$ QUANTA OF EXCITATION.

- $\hat{a}^\dagger|0\rangle = |1\rangle$
- $\hat{a}^\dagger|1\rangle = \sqrt{2}\,|2\rangle \Rightarrow |2\rangle = \frac{1}{\sqrt{2}}\hat{a}^\dagger|1\rangle = \frac{1}{\sqrt{2}}(\hat{a}^\dagger)^2|0\rangle$
- $\hat{a}^\dagger|2\rangle = \sqrt{3}\,|3\rangle \Rightarrow |3\rangle = \frac{1}{\sqrt{3}}\hat{a}^\dagger|2\rangle = \frac{1}{\sqrt{3!}}(\hat{a}^\dagger)^3|0\rangle$

$\therefore\ |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$. ACTING n TIMES WITH $\hat{a}^\dagger$ ON $|0\rangle$ GIVES $\sqrt{n!}\cdot|n\rangle$.

BACK TO THE POSITION BASIS:

- SHO GROUND STATE: ANNIHILATED BY $\hat{a}$ $\qquad \hat{a}|0\rangle = 0.$

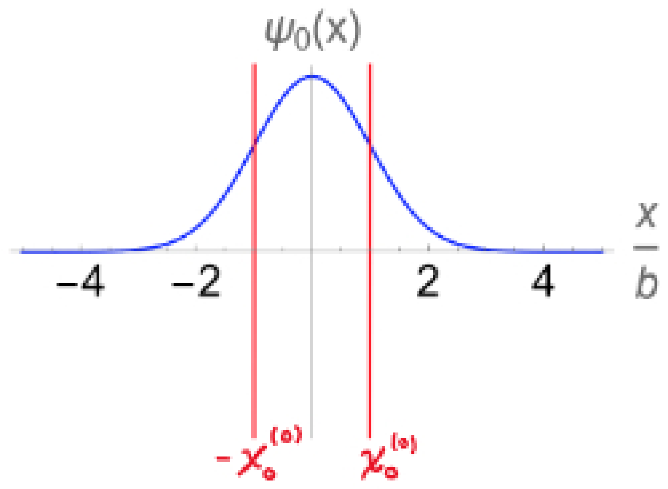
$$\therefore\ \langle x|\hat{a}|0\rangle = 0 = \langle x|\frac{1}{\sqrt{2}}\left[\hat{X} + i\hat{P}\right]|0\rangle = \frac{1}{\sqrt{2}}\langle x|\left[\frac{1}{b}\hat{X} + \frac{ib}{\hbar}\hat{P}\right]|0\rangle$$

$$\therefore\ \left(\frac{x}{b} + \frac{\cancel{i}b}{\cancel{\hbar}}(-\cancel{i}\cancel{\hbar})\frac{d}{dx}\right)\psi_0(x) = b\left(\frac{x}{b^2} + \frac{d}{dx}\right)\psi_0(x) = 0.$$

$$\Rightarrow\ \langle x|0\rangle = \psi_0(x) = \underbrace{\frac{1}{\pi^{1/4}b^{1/2}}}\, e^{-\frac{x^2}{2b^2}}$$

GAUSSIAN WAVEFUNCTION NORMALIZATION — e.g. LEC. 17, p.4

GAUSSIAN!



WHAT ABOUT EXCITED STATES?

$$\langle x|n\rangle = \langle x|\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle = \frac{1}{2^{n/2}\sqrt{n!}}\langle x|\left[\frac{1}{b}\hat{X} - \frac{ib}{\hbar}\hat{P}\right]^n|0\rangle$$

$$= \frac{1}{2^{n/2}\sqrt{n!}}\left(\frac{x}{b} - b\frac{d}{dx}\right)^n\psi_0(x)$$

$$= \frac{1}{2^{n/2}\sqrt{n!}\,\pi^{1/4}b^{1/2}}\left(y - \frac{d}{dy}\right)^n e^{-\frac{y^2}{2}}, \quad y \equiv \frac{x}{b}$$

$$= (\text{CONST.}) \times (n^{\text{TH}} \text{ ORDER POLYNOMIAL IN } y) \times e^{-\frac{y^2}{2}}$$

$$= c_n H_n\left(\frac{x}{b}\right) e^{-\frac{1}{2}\left(\frac{x}{b}\right)^2} \ !$$

$\Rightarrow$ ALLOWS US TO GET THE NORMALIZATION CONSTANT c_n!

HERMITE POLYNOMIALS: $\quad H_n(y) \equiv e^{\frac{y^2}{2}}\left(y - \frac{d}{dy}\right)^n e^{-\frac{y^2}{2}}$

$$\therefore \quad C_n = \frac{1}{2^{n/2}\sqrt{n!}\,\pi^{1/4}\,b^{1/2}} = \frac{1}{(2^n \cdot n! \cdot \pi^{1/2})^{1/2}} \left(\frac{m\omega}{\hbar}\right)^{1/4}$$

IN THE NUMBER BASIS $\{|n\rangle\}$ $(n \in \{0,1,2,\dots\})$, CAN VIEW $\hat{n}, \hat{a}, \hat{a}^\dagger$ AS (INFINITE) MATRICES

$$\hat{n} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ \vdots & & & & & & & \ddots \end{bmatrix}, \quad \hat{a} \Rightarrow \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \cdots \\ \vdots & & & & & \ddots \end{bmatrix}$$

$$\hat{a}^\dagger \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \cdots \\ \vdots & & & & \ddots \end{bmatrix}$$

ADVANTAGES OF THE CREATION / ANNIHILATION OPERATOR FORMALISM

① SUCCINT FORMULA FOR HERMITE POLYNOMIALS: $H_n(y) = e^{\frac{y^2}{2}}\left(y - \frac{d}{dy}\right)^n e^{-\frac{y^2}{2}}$

② EASILY DETERMINES NORMALIZATION CONSTANT C_n (ABOVE)

③ CAN BE USED TO COMPUTE EXPECTATION VALUES OR MATRIX ELEMENTS OF OBSERVABLES WITHOUT PERFORMING POSITION- (OR MOMENTUM-) SPACE INTEGRATIONS

e.g. $\quad \langle n|\hat{X}|n\rangle = \frac{b}{\sqrt{2}}\langle n|\hat{a}+\hat{a}^\dagger|n\rangle = 0$

$$\langle n_1|\hat{P}|n_2\rangle = \frac{\hbar}{b}\frac{i}{\sqrt{2}}\langle n_1|\hat{a}^\dagger - \hat{a}|n_2\rangle = \left(\frac{\hbar i}{b\sqrt{2}}\right)\langle n_1|\left[\sqrt{n_2+1}\,|n_2+1\rangle - \sqrt{n_2}\,|n_2-1\rangle\right]$$

$$= \left(\frac{\hbar i}{b\sqrt{2}}\right)\left[\delta_{n_1,n_2+1}\cdot\sqrt{n_1} - \delta_{n_1,n_2-1}\sqrt{n_2}\right]$$