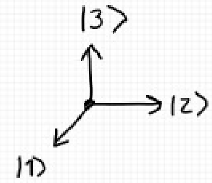
EXAMPLES: VECTOR MANIPULATIONS



(A.) $\mathbb{V}^3$ ORTHONORMAL BASIS $\{|i\rangle\}$, $i \in 1,2,3$ $\langle i|j\rangle = \delta_{ij}$

a) $\mathbb{V}^3(\mathbb{R})$

$$|v\rangle = 3|1\rangle + 2|2\rangle - 4|3\rangle$$
$$|w\rangle = -4|1\rangle + 6|2\rangle$$

- $|v\rangle + |w\rangle = -1|1\rangle + 8|2\rangle - 4|3\rangle$; $\langle v|w\rangle = -12 + 12 = 0$ ORTHOGONAL.
- $|v|^2 = 3^2 + 2^2 + 4^2 = 29$

b) $\mathbb{V}^3(\mathbb{C})$

$$|v\rangle = (2-5i)|2\rangle - 3i|3\rangle \; ; \; |z\rangle = 3i|v\rangle + 4|w\rangle$$
$$|w\rangle = i|1\rangle + (2+2i)|2\rangle$$

- $$\langle z|w\rangle = \langle 3iv + 4w|w\rangle = -3i\langle v|w\rangle + 4\langle w|w\rangle$$
$$= -3i(2+5i)(2+2i) + 4(1+4+4) = -3i(-6+14i) + 4\cdot 9$$
$$= 78 + 18i$$

(B.) $\mathbb{V}^4$ – WORKS THE SAME WAY AS $\mathbb{V}^3$, BUT LET'S CONSIDER A CONCRETE REALIZATION:

2x2 MATRICES

- POSSIBLE BASIS: $|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $|2\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $|3\rangle = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $|4\rangle = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

* CAREFUL: HERE WE EQUATE THE ABSTRACT VECTORS $\{|i\rangle\}$ WITH PARTICULAR 2x2 MATRICES. HAS NOTHING TO DO WITH VIEWING A KET $|v\rangle = \sum_i v_i|i\rangle \rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ COLUMN MATRIX $\Leftarrow$ WE CAN ALWAYS ASSOC. KET $|v\rangle$ TO COLUMN MATRIX.

HERE: REGARD 2x2 MATRICES AS
- VECTORS IN $\mathbb{V}^4$
- "PHYSICAL" OBJECTS (LIKE ARROWS ↗ IN $\mathbb{V}^3(\mathbb{R})$)

- NICER BASIS (WHICH WILL RE-APPEAR IN QUANTUM PHYSICS OF SPINS)

$$|\sigma^0\rangle \equiv \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\equiv \hat{\mathbb{1}}}, \quad |\sigma^1\rangle \equiv \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\equiv \hat{\sigma}^1}, \quad |\sigma^2\rangle \equiv \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{\equiv \hat{\sigma}^2}, \quad |\sigma^3\rangle \equiv \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\equiv \hat{\sigma}^3}$$

- ASSOCIATE SAME MATRICES TO BASIS BRAS, $\langle\sigma^1| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\langle\sigma^2| = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

ADJOINT

$$|v\rangle = a|\sigma^1\rangle + b|\sigma^2\rangle = \begin{bmatrix} 0 & a-ib \\ a+ib & 0 \end{bmatrix}$$
$$\langle v| = a^*\langle\sigma^1| + b^*\langle\sigma^2| = \begin{bmatrix} 0 & a^*-ib^* \\ a^*+ib^* & 0 \end{bmatrix}$$

TRANSPOSE, COMPLEX CONJUGATE!
$\therefore$ CONSISTENT WITH NOTION OF ADJOINT W A BASIS

- INNER PRODUCT, ORTHONORMAL BASIS: MEANS ADJOINT: $\hat{A}^\dagger \equiv (\hat{A}^T)^*$

$$\langle\sigma^\mu|\sigma^\nu\rangle \equiv \tfrac{1}{2}\,\mathrm{Tr}\left[(\hat{\sigma}^\mu)^\dagger\cdot\hat{\sigma}^\nu\right] = \tfrac{1}{2}\,\mathrm{Tr}\left[\hat{\sigma}^\mu\cdot\hat{\sigma}^\nu\right] = \delta^{\mu\nu} \quad , \mu,\nu \in \{0,1,2,3\}$$

(HOMEWORK)

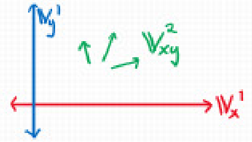
MATRIX TRACE

ORDINARY MATRIX MULTIPLICATION

$$\rightarrow |v\rangle = \sum_{\mu=0}^{3} V_\mu|\sigma^\mu\rangle, \quad |w\rangle = \sum_{\nu=0}^{3} W_\nu|\sigma^\nu\rangle \; ; \; \langle v|w\rangle = \sum_{\mu,\nu=0}^{3} V_\mu^* W_\nu \, \tfrac{1}{2}\,\mathrm{Tr}\left[\hat{\sigma}^\mu\hat{\sigma}^\nu\right]$$
$$= \sum_{\mu=0}^{3} V_\mu^* W_\mu \; ✓$$

Subspaces A subset of vectors within some LVS $\mathbb{V}$ that form a vector space themselves is called a subspace.

ex:) $\mathbb{V}^3(\mathbb{R})$ Some subspaces:



- $\mathbb{V}_x^1$: all vectors directed (or anti-directed) along x-axis
- $\mathbb{V}_y^1$: " " y-axis
- $\mathbb{V}_{xy}^2$: all vectors lying entirely in xy plane

Direct Sum of Two LVS's:

$$\mathbb{V}^{d_1+d_2} = \mathbb{V}^{d_1} \oplus \mathbb{V}^{d_2}$$

↑direct sum of ↑ and ↑

$\mathbb{V}^{d_1+d_2}$ contains: a) all $|v\rangle_1 \in \mathbb{V}^{d_1}$

b) all $|w\rangle_2 \in \mathbb{V}^{d_2}$

c) all lin. combinations $\alpha|v\rangle_1 + \beta|w\rangle_2$

ex:) $\mathbb{V}_{xy}^2(\mathbb{R})$ = all vectors in xy plane $= \mathbb{V}_x^1(\mathbb{R}) \oplus \mathbb{V}_y^1(\mathbb{R})$

3.) Linear Operators

Let $|v\rangle \in \mathbb{V}$, some LVS. Define an operator $\equiv \hat{\Omega}$, such that

$\hat{\Omega}|v\rangle = |v'\rangle$. $\hat{\Omega}$ transforms $|v\rangle$ into $|v'\rangle$; typically $|v'\rangle \neq |v\rangle$

- Restrict to $\hat{\Omega}$ such that both $|v\rangle, |v'\rangle \in \mathbb{V}$ (closure)

Operator can also act on a bra:

$\langle v|\hat{\Omega} = \langle v''|$; in general, $|v'\rangle \neq |v''\rangle$ $\Rightarrow$ "$\overleftarrow{\text{left}}$", "$\overrightarrow{\text{right}}$" actions of $\hat{\Omega}$ not generically equivalent

↑ket assoc. to $\langle v''|$

We restrict to Linear Operators:

$$\hat{\Omega}[\alpha|v_1\rangle + \beta|v_2\rangle] = \alpha\hat{\Omega}|v_1\rangle + \beta\hat{\Omega}|v_2\rangle \qquad \alpha,\beta,\lambda,\gamma \in \mathbb{C}$$

$$[\langle w_1|\lambda + \langle w_2|\gamma]\hat{\Omega} = \lambda\langle w_1|\hat{\Omega} + \gamma\langle w_2|\hat{\Omega}$$

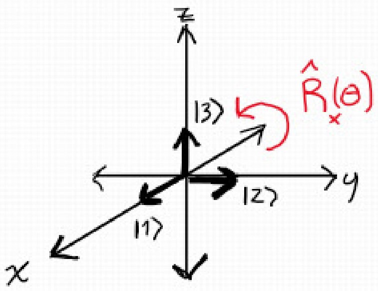
Examples: (A) Identity Op. $\hat{\mathbb{I}}$: $\hat{\mathbb{I}}|v\rangle = |v\rangle$, $\langle v|\hat{\mathbb{I}} = \langle v|$

Claim: always exists, acts same way to left, right ("Hermitian" — discussed later)

(B) Rotation Operators on $\mathbb{V}^3(\mathbb{R})$

orthonormal basis $|1\rangle, |2\rangle, |3\rangle$

Define $\hat{R}_x(\theta) \equiv$ rotates any vector $|v\rangle$ by θ, CCW, around x-axis



- e.g., $\pi/2$ (90°) rotation : $\hat{R}_x(\frac{\pi}{2})|1\rangle = |1\rangle$, $\hat{R}_x(\frac{\pi}{2})|2\rangle = |3\rangle$, $\hat{R}_x|3\rangle = -|2\rangle$

Linearity: $\hat{R}_x(\frac{\pi}{2})(|2\rangle + |3\rangle) = (|3\rangle - |2\rangle) = \hat{R}_x(\frac{\pi}{2})|2\rangle + \hat{R}_x(\frac{\pi}{2})|3\rangle$

- Action of $\hat{\Omega}$ on basis $\{|i\rangle\}$ defines action on any vector:

$$\hat{\Omega}|i\rangle = |i'\rangle \Rightarrow \hat{\Omega}|v\rangle = \sum_{i=1}^{n} v_i\hat{\Omega}|i\rangle = \sum_{i=1}^{n} v_i|i'\rangle \equiv |v'\rangle$$

Product of Two Operators: $\hat{\Lambda}\cdot\hat{\Omega}|v\rangle = \hat{\Lambda}(\hat{\Omega}|v\rangle) = \hat{\Lambda}\underbrace{|\Omega v\rangle}_{\equiv |v'\rangle} \equiv |v''\rangle$

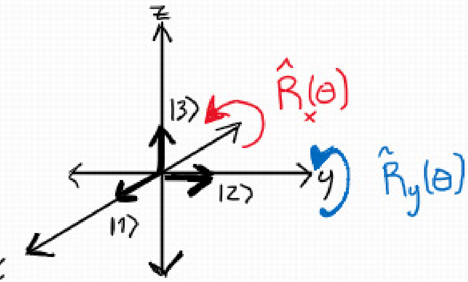
• Key Def: "Commutator" of $\hat{A}$ and $\hat{B}$:

$$[\hat{A},\hat{B}] \equiv \hat{A}\cdot\hat{B} - \hat{B}\cdot\hat{A} \neq 0 \text{ in general} \longrightarrow \text{ORDER OF OPERATORS MATTERS!}$$

ex): Rotations in $\mathbb{V}^3(\mathbb{R})$ again;

$\hat{R}_x(\frac{\pi}{2})$: $\frac{\pi}{2}$ CCW rot around x-axis

$\hat{R}_y(\frac{\pi}{2})$: " " y-axis



• $\hat{R}_x(\frac{\pi}{2})\hat{R}_y(\frac{\pi}{2})|2\rangle = \hat{R}_x(\frac{\pi}{2})|2\rangle = |3\rangle$

• $\hat{R}_y(\frac{\pi}{2})\hat{R}_x(\frac{\pi}{2})|2\rangle = \hat{R}_y(\frac{\pi}{2})|3\rangle = |1\rangle$

$\therefore [\hat{R}_x(\frac{\pi}{2}), \hat{R}_y(\frac{\pi}{2})]|2\rangle = |3\rangle - |1\rangle \neq 0 \Rightarrow$ Rotations of "Ordinary" 3D Vectors About Different Rotation Axes

{ie, rotations performed in diff. planes}

Do NOT Commute!

• The fact that rotations do not in general commute will be of fundamental importance when we study quantum spins, and later, angular momentum in general in quantum physics.

Technically, rotations form a "Non-Abelian" Lie (~Infinite) Group. In quantum, most impt. will be the Lie Algebra (Finite) which generates this group, "SO(3)"

Operator Identities

① $[\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C}$

② $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

Define: Inverse of $\hat{\Omega}$: $\equiv \hat{\Omega}^{-1}$

• NOT Guaranteed to Exist
→ will view ops in terms of square matrices (in a basis); NOT all sq. matrices invertible!

If $\hat{\Omega}^{-1}$ exists, then

$$\hat{\Omega}\hat{\Omega}^{-1} = \hat{\Omega}^{-1}\hat{\Omega} = \hat{\mathbb{I}} \quad \text{Identity operator}$$

Left, right inverses are identical

Theorem 4: $\hat{\Omega}^{-1}$ exists if $\hat{\Omega}|v\rangle = |0\rangle$ implies $|v\rangle = 0$

Assume $\hat{A}^{-1}$ and $\hat{B}^{-1}$ exist.

Then: • $(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$ Proof: $(\hat{A}\hat{B})(\hat{A}\hat{B})^{-1} = \hat{A}\hat{B}\hat{B}^{-1}\hat{A}^{-1} = \hat{\mathbb{I}}$ ✓

(4.) Linear Operators as Square Matrices

Let $\{|i\rangle\}$ be an orthonormal basis for $\mathbb{V}^n(\mathbb{C})$ $\quad \langle i|j\rangle = \delta_{ij}$

$\hat{\Omega}|i\rangle = |i'\rangle$ transforms each $|i\rangle \to |i'\rangle$

NOTE: In general, $\{|i'\rangle\}$ are not mutually orthogonal, nor even lin. indept.
ex): $\hat{\Omega}|i\rangle = |1\rangle$ for all i

We can project each $|i'\rangle$ onto the original basis:

$$\boxed{\langle j|i'\rangle = \langle j|\hat{\Omega}|i\rangle \equiv \Omega_{ji} \quad \text{MATRIX ELEMENTS OF } \hat{\Omega}!}$$

- in $\mathbb{V}^n(\mathbb{C})$, $i \in 1,2,\ldots,n \Rightarrow \Omega_{ji}$ is a $\mathbb{C}$-valued element of an $n\times n$ matrix.

Action on a generic vector:

$$\hat{\Omega}|v\rangle = \sum_{i=1}^{n} v_i\,\hat{\Omega}|i\rangle = |v'\rangle\ ;\ \underset{\overset{\parallel}{v_j'}}{\langle j|v'\rangle} = \sum_{i=1}^{n} \underset{\overset{\parallel}{\langle i|v\rangle}}{v_i}\,\langle j|\hat{\Omega}|i\rangle = \sum_{i=1}^{n} v_i\,\Omega_{ji}$$

v_j' : expansion coeff. for transformed vector in original basis

$$\therefore\ |v'\rangle = \sum_{j=1}^{n} |j\rangle\underbrace{\langle j|v'\rangle}$$

$$= \sum_{j,i=1}^{n} |j\rangle\langle j|\hat{\Omega}|i\rangle\langle i|v\rangle = \sum_{i,j=1}^{n} |j\rangle\, v_i\,\Omega_{ji}$$

$$|v'\rangle \Rightarrow \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ \\ v_n' \end{bmatrix} = \begin{bmatrix} \langle 1|\hat{\Omega}|1\rangle & \langle 1|\hat{\Omega}|2\rangle & \cdots & \langle 1|\hat{\Omega}|n\rangle \\ \langle 2|\hat{\Omega}|1\rangle & \langle 2|\hat{\Omega}|2\rangle & \cdots & \\ \vdots & & & \vdots \\ \langle n|\hat{\Omega}|1\rangle & & \cdots & \langle n|\hat{\Omega}|n\rangle \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- Working in original basis $\{|i\rangle\}$, action of $\hat{\Omega}$ on $|v\rangle$ is just ord. matrix multiplication.

Also conceptually helpful: Let $|v\rangle = |i\rangle$ (basis vector; $\langle i|j\rangle = \delta_{ij}$)

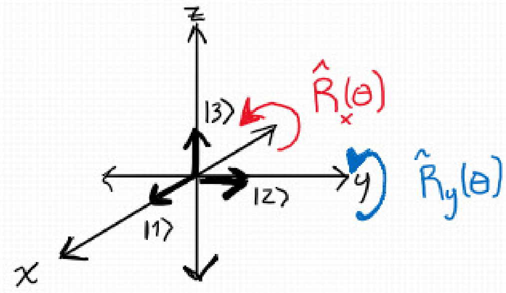
$$|v\rangle = \sum_j v_j|j\rangle = |i\rangle \Rightarrow v_j = \delta_{ji}$$

$$\therefore\ |v'\rangle = |i'\rangle \Rightarrow \begin{array}{c} \\ \begin{bmatrix} \text{———} & \langle 1|\hat{\Omega}|i\rangle & \text{——} \\ & \langle 2|\hat{\Omega}|i\rangle & \\ & \vdots & \\ & \langle i|\hat{\Omega}|i\rangle & \\ & \vdots & \\ & \langle n-1|\hat{\Omega}|i\rangle & \\ \text{———} & \langle n|\hat{\Omega}|i\rangle & \text{——} \end{bmatrix} \\ \uparrow i^{\text{th}}\text{ column} \end{array} \begin{array}{cc} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} \\ \\ \\ \\ \\ \leftarrow i^{\text{th}}\text{ place} \\ \\ \\ \\ \end{array} \end{array} = \begin{array}{c} \begin{bmatrix} \langle 1|\hat{\Omega}|i\rangle \\ \langle 2|\hat{\Omega}|i\rangle \\ \vdots \\ \\ \langle n|\hat{\Omega}|i\rangle \end{bmatrix} \\ \Uparrow \\ |i'\rangle \end{array}$$

$\therefore$ i^{th} column of Ω_{pq} = image of $|i'\rangle = \hat{\Omega}|i\rangle$ in original basis!

Examples: Operators as Square Matrices

① Rotations in $\mathbb{V}^3(\mathbb{R})$



$$\hat{R}_x(\theta)\,|1\rangle = |1\rangle \equiv |1'\rangle$$

$$\hat{R}_x(\theta)\,|2\rangle = \cos\theta\,|2\rangle + \sin\theta\,|3\rangle = |2'\rangle$$

$$\hat{R}_x(\theta)\,|3\rangle = \cos\theta\,|3\rangle - \sin\theta\,|2\rangle \equiv |3'\rangle$$

$$\therefore\ \hat{R}_x(\theta) \Rightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \\ \begin{array}{ccc} \Uparrow & \Uparrow & \Uparrow \\ |1'\rangle & |2'\rangle & |3'\rangle \end{array} \end{array}$$

NOTE: $\hat{R}_x^{-1}(\theta) = \hat{R}_x(-\theta) = \hat{R}_x^{T}(\theta)$ — TRANSPOSE: REALLY MEANS TRANSPOSE OF 3x3 MATRIX IN THIS BASIS

- INVERSE OF A ROTATION EXISTS

- PRODUCT OF ANY TWO ROTATIONS IS ITSELF A ROTATION

(1) SAME AXIS: $\hat{R}_x(\theta_1)\,\hat{R}_x(\theta_2) = \hat{R}_x(\theta_2)\,\hat{R}_x(\theta_1) = \hat{R}_x(\theta_1+\theta_2)$ INTUITIVELY OBVIOUS; CHECK IN HW

→ ROTATIONS ABOUT A FIXED AXIS (= IN A FIXED 2D PLANE) COMMUTE

FORM AN "ABELIAN GROUP" SO(2)

↳ ORDER OF OPERATOR MULTIPLICATION DOESN'T MATTER

(2) DIFFERENT AXES: CLAIM: $\hat{R}_x(\theta_1)\cdot\hat{R}_y(\theta_2) = \underbrace{\hat{R}_{\hat{n}}(\theta_3)}_{} \neq \hat{R}_y(\theta_2)\cdot\hat{R}_x(\theta_1)$ ROTATIONS ABOUT DIFF. AXES DO NOT IN GENERAL COMMUTE!

MEANS A ROTATION BY SOME ANGLE θ_3 AROUND SOME AXIS $\hat{n}$ (UNIT VECTOR IN $\mathbb{V}^3(\mathbb{R})$, CONVENTIONAL NOTATION)

- WE WILL NOT PROVE THIS HERE, BUT IMPLIES SET OF ALL ROTATIONS ACTING ON $\mathbb{V}^3(\mathbb{R})$ FORM A "NON-ABELIAN GROUP" SO(3)

OP. MULTIPLICATION ORDER MATTERS

→ SPIN, ANGULAR MOMENTA (LATER...)

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② Identity Operator $\hat{\mathbb{I}}$: $\langle i|\hat{\mathbb{I}}|j\rangle = \langle i|j\rangle = \delta_{ij}$ (ORTHONORMAL BASIS FOR $\mathbb{V}^n$)

$$\text{ex:}\quad \hat{R}_x(\theta=0) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ Projection Operators: WILL BE CRUCIAL FOR UNDERSTANDING MEASUREMENTS IN QUANTUM

EXPANSION OF $|v\rangle$ IN ORTHONORMAL BASIS: $|v\rangle = \sum_{i=1}^{n} |i\rangle\langle i|v\rangle = \underbrace{\left(\sum_{i=1}^{n} |i\rangle\langle i|\right)}_{\text{MUST BE THE IDENTITY OP.!}} |v\rangle$

$$\boxed{\therefore\ \hat{\mathbb{I}} = \sum_{i=1}^{n} |i\rangle\langle i|}$$ "RESOLUTION OF THE IDENTITY OPERATOR IN ORTHONORMAL BASIS $\langle i|j\rangle = \delta_{ij}$ FOR $\mathbb{V}^n$"

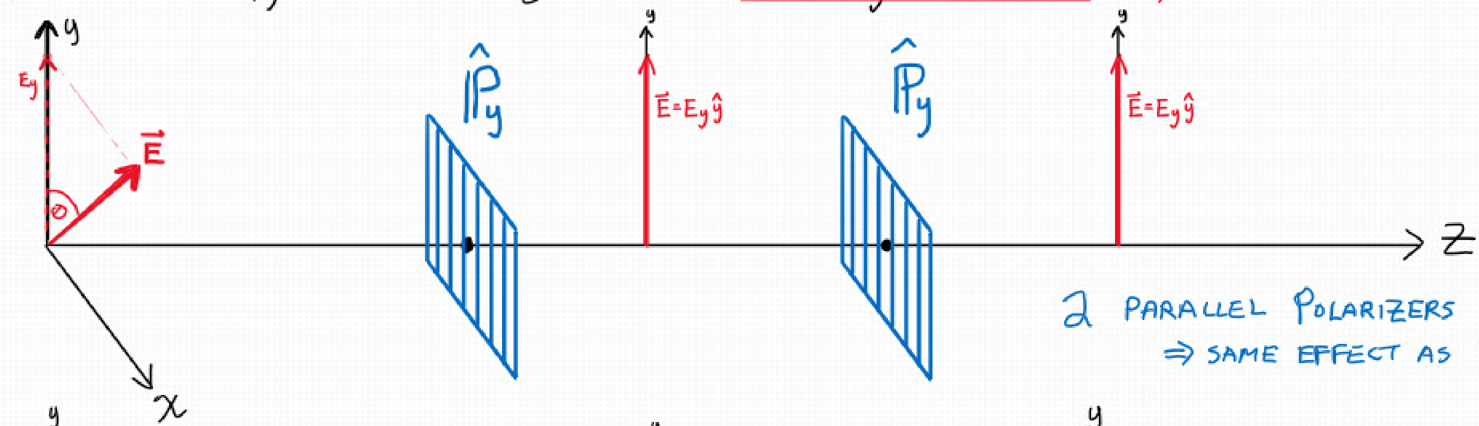
PROJECTION OPERATOR: $\hat{\mathbb{P}}_i \equiv |i\rangle\langle i|$; $\hat{\mathbb{P}}_i^2 = |i\rangle\langle i|i\rangle\langle i| = \hat{\mathbb{P}}_i \Rightarrow$

$$\begin{array}{cc} & i^{\text{TH}}\ \text{COLUMN} \\ \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ i^{\text{TH}}\ \text{ROW} \\ \\ \text{ALL OTHER ELEM. VANISH} \end{array} \end{array}$$

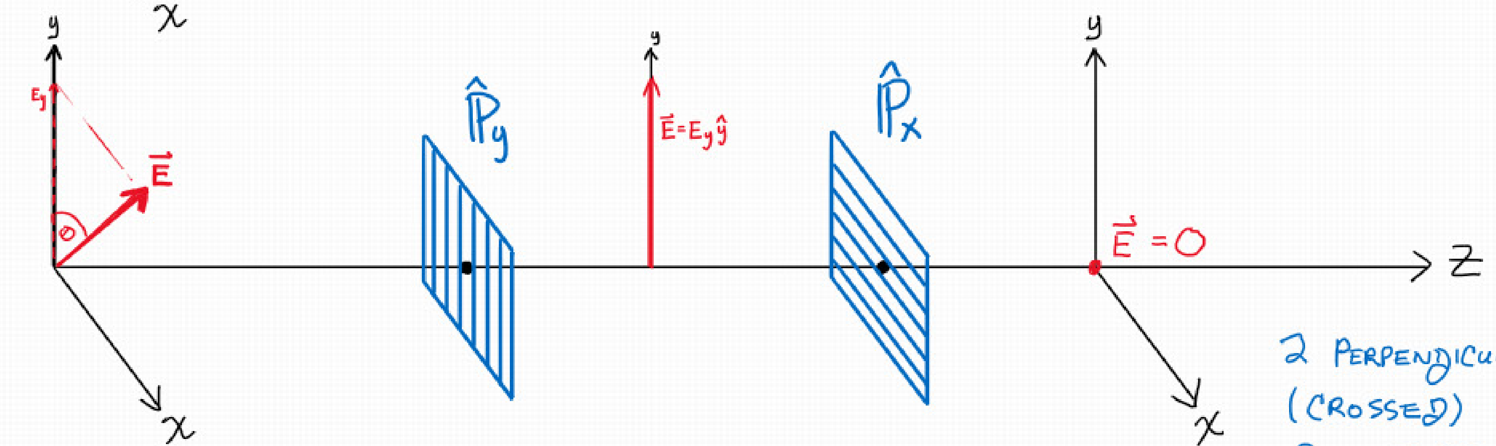
$$\langle j|\hat{\mathbb{P}}_i|k\rangle = \delta_{ik}\,\delta_{ji} \quad (\text{NO EINSTEIN SUM OVER } i)$$

$$\hat{\mathbb{P}}_i \hat{\mathbb{P}}_j = |i\rangle\langle i|j\rangle\langle j| = \delta_{ij}\, \hat{\mathbb{P}}_i$$ ORTHOGONAL PROJECTORS "ANNIHILATE"

PICTURE: $\mathbb{V}^3(\mathbb{R})$, ELECTRIC FIELD THAT IS LINEARLY POLARIZED i.e., ELECTRIC COMPONENT OF TRANSVERSE E+M WAVE



2 PARALLEL POLARIZERS
$\Rightarrow$ SAME EFFECT AS 1



2 PERPENDICULAR (CROSSED) POLARIZERS
$\Rightarrow$ PROJECTS TO ZERO.

THINKING IN TERMS OF A BASIS, $|i\rangle \Rightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$ $\leftarrow i^{TH}$ PLACE

PROJECTOR IS AN "OUTER PRODUCT":

$$\hat{\mathbb{P}}_i = |i\rangle\langle i| \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 & 0 & \cdots \end{bmatrix} = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

$(n\times 1)\cdot(1\times n)$ MATRIX MULTIPLICATION $\quad n\times n$ PROJECTOR MATRIX

IDENTITY IS A SUM OVER ALL PROJECTORS

$$\hat{\mathbb{I}} = \sum_{i=1}^{n} \hat{\mathbb{P}}_i = \sum_{i=1}^{n} |i\rangle\langle i|$$

IN $\mathbb{V}^n$

PRODUCTS OF OPERATORS

$\mathbb{V}^n(\mathbb{C})$

$$(\hat{A}\hat{B})_{ij} = \langle i|\hat{A}\hat{B}|j\rangle = \langle i|\hat{A}\hat{\mathbb{I}}\hat{B}|j\rangle = \sum_{k=1}^{n} \langle i|\hat{A}|k\rangle\langle k|\hat{B}|j\rangle$$

$$= \sum_{k=1}^{n} A_{ik} B_{kj}$$ ORDINARY MATRIX MULTIPLICATION OF $\hat{A}, \hat{B}$, EXPRESSED IN ORTHONORMAL BASIS

ADJOINT OF AN OPERATOR

$$a|v\rangle \equiv |av\rangle \xrightarrow[\text{ADJOINT}]{} \langle av| = a^*\langle v| \quad , \; a\in\mathbb{C}$$

CONSIDER $\hat{\Omega}|v\rangle \equiv |\Omega v\rangle \equiv |v'\rangle$; $\xrightarrow[\text{ADJOINT}]{}$ $\langle v'| = \langle \Omega v| \equiv \underbrace{\langle v|\hat{\Omega}^\dagger}$ } THE ADJOINT OF LIN. OPERATOR $\hat{\Omega}$

MATRIX ELEMENTS:

$$(\hat{\Omega}^\dagger)_{ij} = \langle i|\hat{\Omega}^\dagger|j\rangle = \langle \Omega i|j\rangle \underset{\uparrow}{=} \langle j|\Omega i\rangle^* = (\langle j|\hat{\Omega}|i\rangle)^* = \Omega^*_{ji}$$ TRANSPOSE, CONJUGATE!

SKEW SYM. OF OUR INNER PRODUCT

$\langle i|\hat{\Omega}|j\rangle = \Omega_{ij}$

ADJOINT OF $\hat{\Omega}$: $\hat{\Omega}^\dagger$, ALSO CALLED THE "HERMITIAN CONJUGATE" (FOR REASONS THAT WILL BECOME APPARENT)

- ADJOINT OF A PRODUCT OF OPERATORS: $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$

PROOF: $\langle ABv| = \langle v|(AB)^\dagger = \langle A(Bv)| = \langle Bv|\hat{A}^\dagger = \langle v|\hat{B}^\dagger\hat{A}^\dagger$

- SIMILAR TO RULE FOR INVERSE OF A PRODUCT: $(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$
(ASSUMING $\hat{A}^{-1}, \hat{B}^{-1}$ EXIST; $\hat{A}^\dagger$ ALWAYS EXISTS)

EXAMPLE: ADJOINT OF AN EQUATION WITH KETS, OP.S, SCALARS:

$$|v\rangle \equiv \alpha_1 |v_1\rangle + \alpha_2 |v_2\rangle\langle v_3|v_4\rangle + \alpha_3 \hat{A}\hat{B}|v_5\rangle$$

$\downarrow$ ADJOINT

$$\langle v| = \langle v_1|\alpha_1^* + \alpha_2^* \langle v_4|v_3\rangle\langle v_2| + \alpha_3^* \langle v_5|\hat{B}^\dagger\hat{A}^\dagger$$