Quantum Physics in $\mathbb{V}^2(\mathbb{C})$: Spin - $\frac{1}{2}$, or a single "Qubit"

- $\mathbb{V}^2(\mathbb{C})$: The simplest (Complex) Linear Vector Space
- Corresponds to the simplest possible Quantum System: a "Spin-$\frac{1}{2}$", or (Equiv.) a "Qubit" = Quantum Bit

① States $\quad |\psi\rangle \Rightarrow \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$, $\psi_i \in \mathbb{C} \Rightarrow$ 4 real parameters.

Quantum Mech.: • Norm $\langle\psi|\psi\rangle = 1$
• Overall phase is (usually) not impt.

$\Rightarrow$ WLoG, $|\psi\rangle \Rightarrow \frac{1}{\sqrt{1+|\psi_2|^2}}\begin{bmatrix} 1 \\ \psi_2 \end{bmatrix}$, $\psi_2 = a + ib$, $a, b \in \mathbb{R}$ only 2 "physical" real parameters

② Operators $\quad$ As discussed in HWs 1,3,4, convenient basis for a generic operator:

Pauli Matrices + Identity

$$\hat{\sigma}^1 \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}^2 \Rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}^3 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{\mathbb{I}} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

All 4 matrices are Hermitian!

(a) Generic Op: $\hat{A} \equiv a_0 \hat{\mathbb{I}} + \vec{a}\cdot\hat{\vec{\sigma}}$, $\vec{a} \equiv a_x \vec{n}_x + a_y \vec{n}_y + a_z \vec{n}_z$, $\hat{\vec{\sigma}} = \hat{\sigma}^1 \vec{n}_x + \hat{\sigma}^2 \vec{n}_y + \hat{\sigma}^3 \vec{n}_z$; $\vec{n}_a \cdot \vec{n}_b = \delta_{ab}$; $\{a_{0,x,y,z}\} \in \mathbb{C}$

(b) Hermitian Op: $\hat{H} \equiv \theta \hat{\mathbb{I}} + \vec{\phi}\cdot\hat{\vec{\sigma}}$; $\{\theta, \phi_{x,y,z}\} \in \mathbb{R}$

(c) Unitary Op: $\hat{U} = e^{-i\hat{H}} = e^{-i\theta} e^{-i\vec{\phi}\cdot\hat{\vec{\sigma}}} = e^{-i\theta}\left[\cos(\phi)\,\hat{\mathbb{I}} - i\,\vec{n}_\phi \cdot \hat{\vec{\sigma}}\,\sin(\phi)\right]$ HW #3, Problem 4

- $\theta \neq 0$: $\hat{U} \in U(2) = U(1) \otimes SU(2)$
  - overall phase: an Abelian continuous group $e^{-i\theta_1} e^{-i\theta_2} = e^{-i(\theta_1+\theta_2)}$
- $\theta = 0$: $\det \hat{U} = 1 \Rightarrow \hat{U} \in SU(2)$
  - Lie Group of "special" ($\det = 1$) unitary transf. on $\mathbb{V}^2(\mathbb{C})$
  - Group Property: $\hat{U}_3 = \hat{U}_1 \cdot \hat{U}_2$ belongs to $SU(2)$ if $\hat{U}_{1,2}$ do. $\Rightarrow \begin{bmatrix} \hat{U}_3^\dagger \hat{U}_3 = \hat{\mathbb{I}}, \\ \det \hat{U}_3 = 1 \end{bmatrix}$
  - Non-Abelian: For generic $\hat{U}_{1,2} \in SU(2)$, $[\hat{U}_1, \hat{U}_2] \neq 0 \Rightarrow$ order of operations matter

Pauli Matrices: Useful Properties

① "Anticommutation Rule" (HW 1, #4) $\qquad \hat{\sigma}^a \hat{\sigma}^b + \hat{\sigma}^b \hat{\sigma}^a = 2\,\hat{\mathbb{I}}\,\delta^{a,b}$

- "Clifford Algebra" — very helpful for explicit calculations
- $(\hat{\sigma}^1)^2 = (\hat{\sigma}^2)^2 = (\hat{\sigma}^3)^2 = \hat{\mathbb{I}}$

② $(\vec{n}\cdot\hat{\vec{\sigma}})^2 = \hat{\mathbb{I}}$ for any real $\vec{n}$, $\vec{n}\cdot\vec{n} = 1$ $\qquad$ (HW 3, #4)

③ $\hat{\vec{\sigma}} \times \hat{\vec{\sigma}} = 2i\,\hat{\vec{\sigma}}$, or $[\hat{\sigma}^a, \hat{\sigma}^b] = 2i\,\epsilon^{abc}\,\hat{\sigma}^c$ $\qquad$ (HW 4, #1)

Quantum Spin - 1/2 Operators

$$\hat{S}_a \equiv \frac{\hbar}{2}\hat{\sigma}^a \Rightarrow [\hat{S}_a, \hat{S}_b] = i\hbar\,\epsilon_{abc}\,\hat{S}_c$$

Same Op. (Lie) Algebra we had for Spin-1!

"SU(2)" and "SO(3)" Lie Algebras are identical.

Same Commutator Algebra as for spin 1 $\qquad \hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$

① Can Interpret $\hat{U}(\vec{\theta}) \equiv e^{-\frac{i}{\hbar}\hat{\vec{S}}\cdot\vec{\theta}} = e^{-i\frac{\hat{\vec{\sigma}}}{2}\cdot\vec{\theta}}$ as a CCW Rotation about $\vec{n}_\theta \equiv \frac{\vec{\theta}}{|\vec{\theta}|}$ by angle $\theta = |\vec{\theta}|$

$\therefore$ A Generic SU(2) Unitary Trans. can be Interpreted as an ordinary Rotation!

② Eigenbasis and Ladder Operators

As for spin 1, define $\hat{S}_\pm \equiv \hat{S}_x \pm i\hat{S}_y \Rightarrow [\hat{S}_+, \hat{S}_-] = 2\hbar\hat{S}_z$; $[\hat{S}_z, \hat{S}_\pm] = \pm\hbar\hat{S}_\pm$ $\quad$ Identical to Spin 1

Choose Eigenbasis: $\hat{S}_z|m_z\rangle = \hbar m_z|m_z\rangle$; $m_z = \pm 1/2$ $\quad$ "Spin ½"

Explicitly

$$\hat{S}_z \Rightarrow \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \hat{S}_+ \Rightarrow \hbar\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \quad \hat{S}_- \Rightarrow \hbar\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

Notation: $|m_z\rangle = \{|\tfrac{1}{2}\rangle_z, |{-\tfrac{1}{2}}\rangle_z\}$ or $\{|\uparrow\rangle_z, |\downarrow\rangle_z\}$ $\Rightarrow$

"Spin up" "Spin down"

Cannot raise spin up

Just a coefficient, not $\hat{S}_z$ e'val.

$$\hat{S}_z|\uparrow\rangle_z = \frac{\hbar}{2}|\uparrow\rangle_z \; ; \; \hat{S}_+|\uparrow\rangle_z = 0 \; ; \; \hat{S}_-|\uparrow\rangle_z = \hbar|\downarrow\rangle_z$$

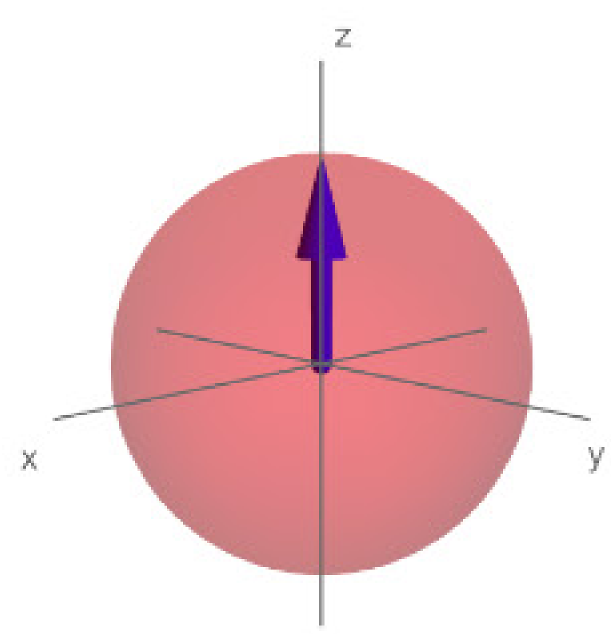
$$\hat{S}_z|\downarrow\rangle_z = -\frac{\hbar}{2}|\downarrow\rangle_z \; ; \; \hat{S}_+|\downarrow\rangle_z = \hbar|\uparrow\rangle_z \; ; \; \hat{S}_-|\downarrow\rangle_z = 0$$

Raises spin down by $+\hbar$
$(-\frac{\hbar}{2} \to \frac{\hbar}{2})$

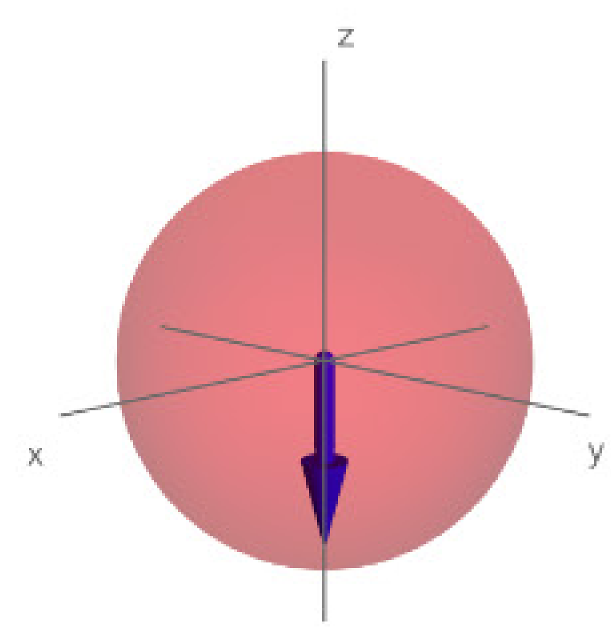
- Clearly $\;{}_z\langle\uparrow|\hat{S}_z|\uparrow\rangle_z = +\frac{\hbar}{2}$; $\;{}_z\langle\downarrow|\hat{S}_z|\downarrow\rangle_z = -\frac{\hbar}{2}$

$\Rightarrow$ Can visualize spin up, spin down states as vertical, antiparallel arrows, living on a sphere of radius $\frac{\hbar}{2}$

$|\uparrow\rangle_z \Rightarrow$



$|\downarrow\rangle_z \Rightarrow$



This Visualization is called the "Bloch Sphere"

- Note: "Spin up" $|\tfrac{1}{2}\rangle_z = |\uparrow\rangle_z$ and "spin down" $|{-\tfrac{1}{2}}\rangle_z = |\downarrow\rangle_z$ are Orthogonal

$\Rightarrow$ Same as spin-1 $\hat{S}_z$ e'states, $\langle m_z|m_z'\rangle = \delta_{m_z, m_z'}$, $m_z, m_z' \in \{-1, 0, 1\}$

- How to reconcile with Bloch Sphere picture?

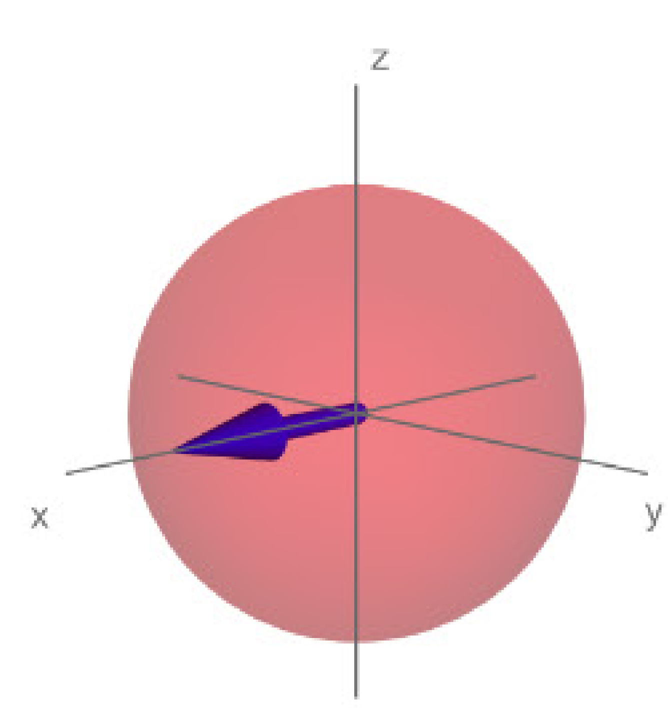
Remember that $|m_z = \pm\tfrac{1}{2}\rangle$ has angular momentum $\pm\frac{1}{2}\hbar$ $\Rightarrow$ Should imagine "orthogonal" CCW, CW "spinning around" z.

- Can use Bloch Sphere to depict other eigenstates

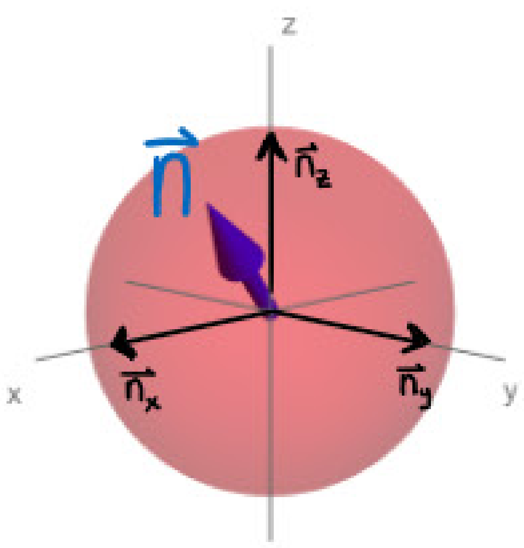
$\hat{S}_x|m_x\rangle = m_x\hbar|m_x\rangle$; $|\uparrow\rangle_x \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$, $|\downarrow\rangle_x \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$

In the $\{|\uparrow\rangle_z, |\downarrow\rangle_z\}$ basis

$|\uparrow\rangle_x \Rightarrow$



In fact, **any** quantum state $|\psi\rangle$ in $\mathbb{V}^2(\mathbb{C})$ can be visualized as pointing somewhere on the Bloch sphere



$$\Rightarrow |\vec{n}\rangle$$

Another (Bloch Sph.) Notation: label state by where it "points"

- $|\uparrow\rangle_z = |\vec{n}_z\rangle$
- $|\downarrow\rangle_z = |-\vec{n}_z\rangle$
- $|\uparrow\rangle_x = |\vec{n}_x\rangle$

etc.

- How to see? Consider state $|\psi\rangle \equiv \alpha|\uparrow\rangle_z + \beta|\downarrow\rangle_z$

Claim: (HW!)

$$\langle\psi|\hat{\sigma}^1|\psi\rangle = 2\,\mathrm{Re}(\alpha^*\beta) \equiv (\vec{n})_x$$
$$\langle\psi|\hat{\sigma}^2|\psi\rangle = 2\,\mathrm{Im}(\alpha^*\beta) \equiv (\vec{n})_y$$
$$\langle\psi|\hat{\sigma}^3|\psi\rangle = |\alpha|^2 - |\beta|^2 \equiv (\vec{n})_z$$

Can show: (HW!)

$$[\vec{n}\cdot\hat{\vec{\sigma}}]|\psi\rangle = \underbrace{(|\alpha|^2+|\beta|^2)}_{=1 \text{ (normalization!)}}|\psi\rangle$$

$$\therefore \quad \vec{n}\cdot\hat{\vec{S}}\,|\psi\rangle = \left(+\tfrac{\hbar}{2}\right)|\psi\rangle$$

This is consistent with what we said on p.1 of this lecture: Physical quantum states in $\mathbb{V}^2(\mathbb{C})$ char. by two real params — e.g., azimuthal θ, polar angle ϕ for

$$\vec{n} = \cos\phi\sin\theta\,\vec{n}_x + \sin\phi\sin\theta\,\vec{n}_y + \cos\theta\,\vec{n}_z$$

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③ Bloch Sphere and Rotations

Consider a rotation around the y-axis of $|\uparrow\rangle_z = |\vec{n}_z\rangle$

$$\hat{U}(\theta\vec{n}_y) = e^{-i\frac{\hat{S}_y\theta}{\hbar}} = e^{-i\frac{\theta}{2}\hat{\sigma}^2} \underset{\text{HW 3, \#4}}{=} \hat{\mathbb{I}}\cos\left(\tfrac{\theta}{2}\right) - i\,\hat{\sigma}^2\sin\left(\tfrac{\theta}{2}\right) \Rightarrow \begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{bmatrix}$$

$$\therefore \hat{U}(\theta\vec{n}_y)|\uparrow\rangle_z \Rightarrow \begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \cos\left(\tfrac{\theta}{2}\right)|\uparrow\rangle_z + \sin\left(\tfrac{\theta}{2}\right)|\downarrow\rangle_z \equiv |\psi(\theta)\rangle$$

(a) $\theta = \frac{\pi}{2}$: should rotate $|\uparrow\rangle_z \Rightarrow |\uparrow\rangle_x$

$$|\psi(\tfrac{\pi}{2})\rangle = \tfrac{1}{\sqrt{2}}[|\uparrow\rangle_z + |\downarrow\rangle_z] = |\uparrow\rangle_x \;\checkmark$$

(b) $\theta = \pi$: should rotate $|\uparrow\rangle_z \Rightarrow |\downarrow\rangle_z$

$$|\psi(\pi)\rangle = |\downarrow\rangle_z \;\checkmark$$

(c) $\theta = 2\pi$: $|\uparrow\rangle_z \Rightarrow |\uparrow\rangle_z$

$$|\psi(2\pi)\rangle = \cos(\pi)|\uparrow\rangle_z = -|\uparrow\rangle_z$$

?! $\therefore$ A spin-½ rotated through 2π goes to minus itself!

- 4π rotation needed to send $|\uparrow\rangle_z \Rightarrow +1|\uparrow\rangle_z$

$\therefore$ Although SU(2), SO(3) Lie groups are "locally" the same (same Lie algebra) the groups differ globally

- Since in quantum physics, we mainly label states using Lie alg. of Hermitian (observable) operators, one can instead simply distinguish integer $j = \{1, 2, 3, \dots\}$ vs. half-integer $j = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$ angular momentum

$$\hat{S}_z|m_z\rangle = \hbar m_z|m_z\rangle\;;\; m_z \in \overbrace{\{-j, -j-1, \dots, j-1, j\}}^{(2j+1)\text{ states}}$$

- We will study general (spin and orbital) j-angular momentum towards the end of the semester $\Rightarrow$ needed for the hydrogen atom.

NOTE: BLOCH SPHERE DOESN'T GENERALLY WORK FOR SPIN 1

e.g. $\hat{S}_z |m_z\rangle = \hbar m_z |m_z\rangle$, $m_z \in \{-1, 0, 1\}$ $\Rightarrow$ CONSIDER $|m_z = 0\rangle \equiv |0\rangle_z$ STATE: ${}_z\langle 0 | \hat{S}_x | 0\rangle_z = {}_z\langle 0 | \hat{S}_y | 0\rangle_z = {}_z\langle 0 | \hat{S}_z | 0\rangle_z = 0$

- ALTHOUGH $|m_z = 0\rangle \equiv |0\rangle_z$ STATE IS NOT THE NULL VECTOR $[{}_z\langle 0|0\rangle_z = 1 \neq 0]$, IT HAS VANISHING EXP. VALUE IN $\hat{S}_{x,y,z}$
- CAN CREATE STATE WITH NONZERO $\langle \hat{S}_z \rangle = m_z \hbar$ BY RAISING OR LOWERING: $\hat{S}_\pm |0\rangle_z = \sqrt{2}\,\hbar\, |\pm 1\rangle_z$

$\Rightarrow$ RESERVE BLOCH SPHERE NOTATION $|\vec{n}\rangle$, s.t. $\hat{\vec{S}} \cdot \vec{n}\, |\vec{n}\rangle = \left(+\frac{\hbar}{2}\right) |\vec{n}\rangle$ FOR SPIN-½ !

EXAMPLES OF PARTICLES WITH SPIN-½

- ALL ELEMENTARY FERMIONS IN PARTICLE PHYSICS: electrons, muons, taus, neutrinos, quarks
- COMPOSITE PARTICLES: PROTONS, NEUTRONS, ^6Li ATOMS (USEFUL IN ULTRACOLD ATOM EXPT.s), ...

OTHER ASPECTS OF SPIN-½ WORK SIMILAR TO SPIN-1

4. HAMILTONIAN

$$\hat{H} = -\hat{\vec{\mu}} \cdot \vec{B}, \quad \hat{\vec{\mu}} = \gamma \hat{\vec{S}} \; ; \quad \gamma = \underbrace{g}_{\text{g-FACTOR}} \times \left(\frac{q}{2mc}\right)$$

ALT. CONVENTION: $\hat{\vec{\mu}} = g \times \mu_q \times \left(\frac{\hat{\vec{S}}}{\hbar}\right)$; $\mu_q = \frac{\hbar \gamma}{g} = \frac{q\hbar}{2mc}$ BOHR MAGNETON

- ELECTRONS: $\mu_e = -\mu_B = -\frac{e\hbar}{2m_e c}$; $g_e \sim 2 +$ SMALL CORRECTIONS (QUANTUM E+M)

electron charge is $-e<0$

* μ_B DEFINED AS A POSITIVE QUANTITY

FROM THE NON-RELATIVISTIC LIMIT OF THE DIRAC EQN. IN 3 SPATIAL DIMENSIONS.

$$\mu_B = 9.274 \cdot 10^{-21} \text{ erg/Gauss}$$
$$= 5.788 \cdot 10^{-5} \text{ eV/Tesla}$$

5. TIME-EVOLUTION IN A TIME-DEPT. FIELD

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle = -\gamma \vec{B}(t) \cdot \hat{\vec{S}}\, |\psi(t)\rangle$$

AGAIN,

$$\frac{d}{dt} \langle \hat{\vec{\mu}} \rangle(t) = \gamma \langle \vec{\mu}(t) \rangle \times \vec{B}(t)$$

; NOW, HOWEVER, $\langle \hat{\vec{\mu}} \rangle \cdot \langle \hat{\vec{\mu}} \rangle = \left(\frac{\gamma\hbar}{2}\right)^2$ FOR ANY INITIAL STATE *

- CONSEQUENCE OF BLOCH SPHERE PICTURE

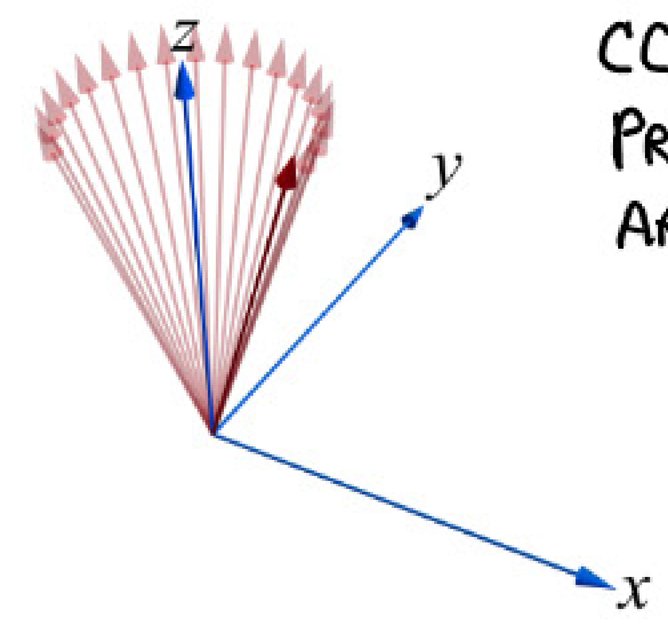
(*) $\langle \hat{\vec{\mu}} \rangle \cdot \langle \hat{\vec{\mu}} \rangle$ IS CONSERVED FOR ANY SPIN, AS A CONSEQ. OF THE E.O.M. ABOVE (PROVE IT!)

BUT: VALUE DEPENDS ON INITIAL STATE FOR SPIN-j, $j \geq 1$.

Static $\vec{B}$-Field: Larmor Precession Again

$$\langle \hat{\mu}_z \rangle(t) = \langle \hat{\mu}_z \rangle(0)$$
$$\langle \hat{\mu}_x \rangle(t) = \langle \hat{\mu}_x \rangle(0) \cos(\omega_L t)$$
$$\langle \hat{\mu}_y \rangle(t) = -\langle \hat{\mu}_x \rangle(0) \sin(\omega_L t)$$

$$\omega_L = \gamma B$$



CCW Precession Around Z

- For electrons, charge $q = -e < 0$

$$\Rightarrow \gamma_e = \frac{g_e}{\hbar}\mu_e = \frac{-g_e \cdot e}{2 m_e c} \; ; \; \omega_L = \gamma B \text{ is Negative for } \vec{B} = B\vec{n}_z, \; B > 0.$$

$$|\gamma_e| = 1.761 \cdot 10^{11} \frac{\text{radians}}{\text{Tesla} \times \text{sec.}}$$

$$\frac{|\gamma_e|}{2\pi} = 28{,}030 \frac{\text{MHz}}{\text{Tesla}}$$

What About More Complicated $\vec{B}(t)$?

- Since $\langle \hat{\vec{\mu}} \rangle^2 = \left(\frac{\gamma\hbar}{2}\right)^2$, irresp. of initial state, we can consider $\langle \hat{\vec{\mu}} \rangle$ to be a classical vector of fixed length (Bloch Sphere Again)

E.O.M.: $\frac{d\vec{\mu}}{dt} = \gamma \vec{\mu} \times \vec{B}(t)$, or $i\hbar \frac{d}{dt}|\psi(t)\rangle = -\hat{\vec{\mu}} \cdot \vec{B}(t)|\psi(t)\rangle$

- Surprisingly, No Analytic Solution for General $\vec{B}(t)$!
- Motion on Bloch Sphere can be complex or Chaotic, depending on $\vec{B}(t)$
- One way to see complexity: write $\vec{\mu} = |\vec{\mu}| \left[\cos\phi \sin\theta\, \vec{n}_x + \sin\phi \sin\theta\, \vec{n}_y + \cos\theta\, \vec{n}_z\right]$

  Plug into EOM: Get Two Coupled, Horribly Nonlinear ODEs for

$$\dot{\phi} = (\text{blah})$$
$$\dot{\theta} = (\widetilde{\text{blah}})$$

Impt. Exception: <u>Paramagnetic Resonance</u>

Time-Dept. $\vec{B}$-Fields, Rotating Frame

$\hat{H} = -\gamma \hat{\vec{S}} \cdot \vec{B}(t)$; $\quad i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$

- WLoG, can write $\vec{B}(t) = B(t)\left(\hat{R}(t)\vec{n}\right)$, where
  - $B(t)$ is the time-dept. amplitude $|\vec{B}(t)|$
  - $\hat{R}(t)\vec{n} \equiv \vec{n}'(t)$ is the instantaneous direction $[\vec{n}'\cdot\vec{n}' = 1]$, expressed as a ($t$-dept.) 3x3 (ordinary) rotation matrix $\hat{R}(t)$, acting on a static unit vector $\vec{n}$

  e.g., for $\vec{B}$ confined to the xy plane,
  $$\begin{bmatrix} B^x(t) \\ B^y(t) \\ B^z(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\phi(t)) & -\sin(\phi(t)) & 0 \\ \sin(\phi(t)) & \cos(\phi(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\hat{R}(\phi(t)\vec{n}_z)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} B(t)$$

- Define an SU(2) "rotation" $\hat{U}(t)$, such that

$$\hat{U}^\dagger_{(t)}\, \hat{\vec{S}}\, \hat{U}(t) \equiv R_{ab}(t)\, \hat{S}^b\, \vec{n}_a \quad \Leftarrow \text{ i.e., } \hat{U}(t) = e^{-i\frac{\vec{\Theta}(t)\cdot\hat{\vec{S}}}{\hbar}}$$

Implements the same rotation on the components of $\hat{\vec{S}}$ as $(\hat{R}\vec{n})$ in $\vec{B}(t)$.

- Define the "rotating" frame state

$|\psi_R(t)\rangle \equiv \hat{U}^\dagger_{(t)}|\psi(t)\rangle$; plug into S.E.: $\quad i\hbar\frac{d}{dt}\left(\hat{U}(t)|\psi_R(t)\rangle\right) = -\gamma B(t)\, R_{ab}(t)\,(\vec{n})_b\, \hat{S}_a\, \hat{U}(t)|\psi_R(t)\rangle$

Multiply both sides by $\hat{U}^\dagger_{(t)}$:

$$i\hbar\frac{d}{dt}|\psi_R(t)\rangle + \hat{U}^\dagger_{(t)}\left[i\hbar\frac{d}{dt}\hat{U}(t)\right]|\psi_R(t)\rangle = -\gamma B(t)\, R_{ab}(t)(\vec{n})_b\, \hat{U}^\dagger \hat{S}_a \hat{U}|\psi_R\rangle = -\gamma B(t)\, \underbrace{R_{ab}(\vec{n})_b R_{ac}(\hat{\vec{S}})_c}_{\substack{= n^T \hat{R}^T \hat{R} \hat{S} = n^T\hat{S} \\ = \hat{\vec{S}}\cdot\vec{n} \ \text{ time indept.!}}}|\psi_R\rangle$$

$\therefore$
$$i\hbar\frac{d}{dt}|\psi_R(t)\rangle = \hat{H}_R(t)|\psi_R(t)\rangle$$

$$\hat{H}_R(t) = -\gamma\left[B(t)\vec{n}\cdot\hat{\vec{S}} + \underbrace{\frac{i\hbar}{\gamma}\hat{U}^\dagger_{(t)}\frac{d}{dt}\hat{U}(t)}_{\substack{\text{Claim: can be written as} \\ \delta\vec{B}(t)\cdot\hat{\vec{S}}}}\right]$$

$$\delta\vec{B}\cdot\hat{\vec{S}} = \frac{i\hbar}{\gamma}\hat{U}^\dagger\frac{d}{dt}\hat{U}$$

$$\Rightarrow (\delta\vec{B}\cdot\hat{\vec{S}})^\dagger = -\frac{i\hbar}{\gamma}\left(\frac{d}{dt}\hat{U}^\dagger\right)\hat{U} \overset{\text{Why?}}{=} \frac{i\hbar}{\gamma}\hat{U}^\dagger\frac{d}{dt}\hat{U} = (\delta\vec{B}\cdot\hat{\vec{S}})$$

Hermitian! $\therefore$ can be written as $\delta\vec{B}(t)\cdot\hat{\vec{S}}$ [No identity piece for $\hat{U} \in SU(2)$]

$\therefore$ In rotating frame, effective $\vec{B}_R = \underbrace{B(t)\vec{n}}_{\substack{\text{Simple:} \\ \text{fixed orientation,} \\ \text{var. amplitude}}} + \delta\vec{B}(t)$

$\Uparrow$ Not simple; in general amp. + direction vary in time!

$\Rightarrow$ Usually doesn't help.

But, for simple enough $\vec{B}(t)$, "boosting" to rotating frame can solve dynamics... (next time)