Hermitian, Anti-Hermitian, and Unitary Operators

Definition: $\hat{\Omega}$ is <u>Hermitian</u> if $\hat{\Omega} = \hat{\Omega}^\dagger$ ("self-adjoint")

Matrix Elements (orthonormal basis for $\mathbb{V}^n(\mathbb{C})$): $\Omega_{ij} = \Omega_{ji}^*$ e.g. $\hat{H} = \hat{H}^\dagger \Rightarrow \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12}^* & h_{22} & h_{23} \\ h_{13}^* & h_{23}^* & h_{33} \end{bmatrix}$

$h_{ii} \in \mathbb{R}$, for $i \in \{1,2,3\}$

<u>Note:</u> if $\Omega_{ij} = \Omega_{ij}^* \in \mathbb{R}$ for all i,j (all matrix elem. real)

then $\Omega_{ij} = \Omega_{ji}$ or $\hat{\Omega} = \hat{\Omega}^T$ ← transpose $\Rightarrow \hat{\Omega}$ is <u>symmetric</u>

<u>Looking Forward:</u> A general change of basis that <u>preserves orthonormality</u> of the basis is called a "unitary transformation" (see below)

- Preserves Hermiticity $\Rightarrow \hat{\Omega} = \hat{\Omega}^\dagger$ is basis indept.

<u>Note:</u> $n \times n$ Hermitian Op. has n^2 indept. real parameters:

- n real diag. elem.s Ω_{ii}
- $\frac{n(n-1)}{2}$ complex "upper triangular" elements

$\therefore$ Hermitian matrix has same # param.s as a generic, real-valued $n \times n$ matrix

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Def.: $\hat{\Omega}$ is <u>Anti-Hermitian</u> if $\hat{\Omega} = -\hat{\Omega}^\dagger$

- Can make Hermitian by multiplying by i: $\hat{\Lambda} \equiv i\hat{\Omega}$; $\hat{\Lambda}^\dagger = \hat{\Lambda}$

Definition: An operator $\hat{U}$ is <u>Unitary</u> if $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger \hat{U} = \hat{\mathbb{I}}$ (identity operator)

$\therefore \hat{U}$ is invertible, and $\hat{U}^{-1} = \hat{U}^\dagger$.

<u>Theorem 5</u>: <u>Unitary Operators Preserve the Inner Products Between Vectors They Act Upon</u>

Important!

<u>Proof:</u> $|v'\rangle \equiv \hat{U}|v\rangle$; $|w'\rangle \equiv \hat{U}|w\rangle \Rightarrow \langle v'|w'\rangle = \langle v|\hat{U}^\dagger \hat{U}|w\rangle = \langle v|w\rangle$

$\therefore$ A unitary "change of basis" preserves orthonormality

$$|i'\rangle \equiv \hat{U}|i\rangle \; ; \; \langle i|j\rangle = \delta_{ij} \Rightarrow \langle i'|j'\rangle = \langle i|j\rangle = \delta_{ij}$$

$\Rightarrow$ Can view unitary operator as a "generalized" rotation

Recall: $\hat{R}_x^{-1}(\theta) = \hat{R}_x(-\theta) = \hat{R}_x^T(\theta) = \hat{R}_x^\dagger(\theta)$, because $(\hat{R}_x(\theta))_{ij}^* = (\hat{R}_x(\theta))_{ij}$, purely real

Theorem 6: Columns of an $n \times n$ Unitary Matrix are Orthonormal

Proof: Recall (Lec. 3, p. 4) that the i^{th} column of a matrix U_{pq} = image $|i'\rangle \equiv \hat{U}|i\rangle$ in original basis

But $\langle i'|j'\rangle = \delta_{ij}$ for $\hat{U}^\dagger \hat{U} = \hat{\mathbb{I}}$ ✓

$$|j'\rangle = \hat{U}|j\rangle \Rightarrow \left.\begin{bmatrix} \langle 1|\hat{U}|j\rangle \\ \langle 2|\hat{U}|j\rangle \\ \vdots \\ \langle n|\hat{U}|j\rangle \end{bmatrix}\right\} \; j^{th} \text{ column of } \langle p|\hat{U}|q\rangle = U_{pq}$$

Matrix elements for expansion in original $\{|i\rangle\}$ basis (as before)

i.e.,

$$\hat{U} \Rightarrow \begin{bmatrix} \top & \top & & \top \\ |1'\rangle & |2'\rangle & \cdots & |n'\rangle \\ \bot & \bot & & \bot \end{bmatrix}$$

$\hat{U}^\dagger = \hat{U}^{-1}$ is also a unitary operator $(\hat{U}\hat{U}^\dagger = \hat{U}^\dagger \hat{U} = \hat{\mathbb{I}})$ ∴ its columns are orthonormal

- Rows are also orthonormal, since

$$\hat{U}^\dagger \Rightarrow \begin{bmatrix} \vdash\!\!-\!\!-\; \langle 1'| \;-\!\!-\!\!\dashv \\ \vdash\!\!-\!\!-\; \langle 2'| \;-\!\!-\!\!\dashv \\ \vdots \\ \vdash\!\!-\!\!-\; \langle n'| \;-\!\!-\!\!\dashv \end{bmatrix}$$

Active vs. Passive Transformations

Suppose we change basis, from $\{|i\rangle\}$ to $\{|i'\rangle\}$, with $|i'\rangle \equiv \hat{U}|i\rangle$; $\hat{U}^\dagger\hat{U} = \hat{\mathbb{I}}$

Matrix elements of any operator depend on basis:

$$\underbrace{\langle j'|\hat{\Omega}|i'\rangle} = \langle Uj|\hat{\Omega}|Ui\rangle = \underbrace{\langle j|\hat{U}^\dagger\hat{\Omega}\hat{U}|i\rangle}$$

"Active Basis Transformation":

- Deform original basis vectors $\{|i\rangle\}$ via a unitary transformation $|i'\rangle = \hat{U}|i\rangle$
- Compute matrix elements of op. $\hat{\Omega}$ in new basis

$\langle i'|\hat{\Omega}|j'\rangle$

"Passive Basis Transformation"

- Define "rotated" operator $\hat{\Omega}' \equiv \hat{U}^\dagger\hat{\Omega}\hat{U}$
- $\hat{\Omega}'$ is related to $\hat{\Omega}$ via simultaneous right (left) multiplication by $\hat{U}$ $(\hat{U}^\dagger)$

$\Rightarrow$ one can also call this a "similarity transformation" on operator $\hat{\Omega}$

- Compute matrix elements of transformed operator $\hat{\Omega}'$ in old basis: $\langle i|\hat{\Omega}'|j\rangle = \langle i|\hat{U}^\dagger\hat{\Omega}\hat{U}|j\rangle$

- **Same answer!** $\langle i'|\hat{\Omega}|j'\rangle = \langle i|\hat{\Omega}'|j\rangle$

$\Rightarrow$ Diff. interpretations: "Rotate" the basis vs. the operator

Later, in Quantum:

① Rotate basis [and any vector: $|v'\rangle \equiv \sum_i v_i \hat{U}|i\rangle = \sum_i v_i |i'\rangle$]

"Schrödinger Picture" — what you've seen in 202

② Rotate operators $\hat{\Omega}' \equiv \hat{U}^\dagger\hat{\Omega}\hat{U}$

"Heisenberg Picture" — advanced applications
- Path integrals
- QFT

5. EIGENVECTORS AND EIGENVALUES

$\hat{\Omega}$: GENERAL LINEAR OPERATOR ON $\mathbb{V}^n(\mathbb{C})$; $|v\rangle \in \mathbb{V}^n(\mathbb{C})$

$\hat{\Omega}|v\rangle = |v'\rangle$; GENERICALLY, $|v'\rangle$ IS DIFFERENT FROM $|v\rangle$ [ACTION OF OPERATOR $\hat{\Omega}$ IS NON-TRIVIAL]

SPECIAL CLASS OF VECTORS FOR A GIVEN OPERATOR $\hat{\Omega}$: EIGENVECTORS

IF: $\hat{\Omega}|v_i\rangle = \lambda_i |v_i\rangle$,
- $|v_i\rangle$ IS AN EIGENVECTOR OF $\hat{\Omega}$
- λ_i IS THE ASSOCIATED EIGENVALUE

$\mathbb{C}$ NUMBER

ex 1): $\hat{\Omega} = \hat{\mathbb{I}}$, IDENTITY OPERATOR

$\hat{\mathbb{I}}|v\rangle = |v\rangle \Rightarrow$
- ANY $|v\rangle \in \mathbb{V}^n$ IS AN EIGENVECTOR
- $\lambda_v = 1$ FOR ALL EIGENVECTORS

"EIGENVECTORS ARE DEGENERATE"

ex 2): $\hat{\Omega} = \hat{\mathbb{P}}_v \equiv |v\rangle\langle v|$; $|v|^2 = 1$

PROJECTION OPERATOR ONTO $|v\rangle$

> MATHEMATICAL ASIDE: $\hat{\Omega}|v_i\rangle = \lambda_i |v_i\rangle$
> - TECHNICALLY, $|v_i\rangle$ IS THE "RIGHT" EIGENVECTOR OF $\hat{\Omega}$
> - CAN ALSO DEFINE A "LEFT" EIGENVECTOR:
>
> $\langle \tilde{v}_i|\hat{\Omega} = \tilde{\lambda}_i \langle \tilde{v}_i|$
> - VIA ADJOINT OP, CAN CONVERT $\langle \tilde{v}_i| \to |\tilde{v}_i\rangle$
> - FOR A GENERIC OPERATOR $\hat{\Omega}$, i.e., cannot find $|\tilde{v}_i\rangle = |v_j\rangle$ FOR ANY i,j
>
> $\{|v_i\rangle\} \neq \{|\tilde{v}_i\rangle\}$ SET OF RIGHT EIGENVECTORS NOT EQUAL TO (ADJOINT OF) THE SET OF LEFT EIGENVECTORS
>
> ✪ IN QUANTUM, MOSTLY INTERESTED IN EIGENVECTORS OF HERMITIAN $\hat{\Omega} = \hat{\Omega}^\dagger$; THEN CAN TAKE $\{|v_i\rangle\} = \{|\tilde{v}_i\rangle\}$ (SEE BELOW)

3 CASES:

① $|w\rangle = \alpha|v\rangle$: $\hat{\mathbb{P}}_v|w\rangle = |w\rangle$ • EIGENVECTOR WITH EIGENVALUE 1 $\Leftarrow$ TRIVIAL CASE. LATER WE WILL ALWAYS NORMALIZE EIGENVECTORS, WHICH JUST REPLACES $|w\rangle \to e^{i\phi}|v\rangle$ POSSIBLE PURE PHASE, $\phi \in \mathbb{R}$

② $|w\rangle = |v_\perp\rangle$, SUCH THAT $\langle v|v_\perp\rangle = 0$

$\hat{\mathbb{P}}_v|w\rangle = 0$ • EIGENVECTOR WITH EIGENVALUE 0 ($|w\rangle$ IS "ANNIHILATED" BY $\hat{\mathbb{P}}_v$)

③ $|w\rangle = \alpha|v\rangle + \beta|v_\perp\rangle$; $\hat{\mathbb{P}}_v|w\rangle = \alpha|v\rangle \neq (\#)\cdot|w\rangle$ NOT AN EIGENVECTOR

CHARACTERISTIC EQUATION: HOW TO SOLVE AN EIGENVALUE PROBLEM

$(\hat{\Omega} - \lambda\hat{\mathbb{I}})|v\rangle = |0\rangle$ MUST BE TRUE FOR $|v\rangle$ EIGENVECTOR WITH EIGENVALUE λ.

WHAT IF $(\hat{\Omega} - \lambda\hat{\mathbb{I}})^{-1}$ EXISTS? $\Rightarrow |v\rangle = (\hat{\Omega} - \lambda\hat{\mathbb{I}})^{-1}|0\rangle = |0\rangle \longrightarrow$ TRIVIAL SOLUTION!

$\therefore$ LOOK FOR SOLUTION SUCH THAT $(\hat{\Omega} - \lambda\hat{\mathbb{I}})^{-1}$ DOES NOT EXIST.

IN AN ORTHONORMAL BASIS $\{|i\rangle\}$ FOR $\mathbb{V}^n(\mathbb{C})$, $\langle i|j\rangle = \delta_{ij}$ $i,j \in \{1,2,\dots,n\}$

$\Rightarrow$ MATRIX $\langle i|(\hat{\Omega} - \lambda\hat{\mathbb{I}})|j\rangle = (\hat{\Omega} - \lambda\hat{\mathbb{I}})_{ij}$ IS NOT INVERTIBLE.

$\hat{A}$: $n\times n$ MATRIX; $\hat{A}^{-1} = \frac{1}{\det\hat{A}}[\cdots]$ $\Rightarrow$ $\hat{A}^{-1}$ DOES NOT EXIST IF $\det\hat{A} = 0$

$$\boxed{\therefore\ \det(\hat{\Omega} - \lambda\hat{\mathbb{I}}) = 0 \qquad \text{"CHARACTERISTIC EQN."}}$$

DET. COMPUTED IN ANY ORTHONORMAL BASIS

SOLVING E'VALUE PROBLEM:

⓪ CHOOSE ORTHONORMAL BASIS $\langle i|j\rangle = \delta_{ij}$ TO WORK IN

① COMPUTE $\det(\hat{\Omega} - \lambda \hat{\mathbb{I}})$ IN THIS BASIS → GIVES A "CHARACTERISTIC POLYNOMIAL" IN λ : $P_n(\lambda)$ — TYPICALLY n^{th} ORDER IN $\mathbb{V}^n(\mathbb{C})$

② SET $\det(\hat{\Omega} - \lambda \hat{\mathbb{I}}) = P_n(\lambda) = 0$.

FIND n (NOT NECESSARILY) DISTINCT ROOTS $\{\lambda_i\}$, $P_n(\lambda_i) = 0$.

$\{\lambda_i\}$ ARE EIGENVALUES.

③ FOR EACH λ_i, SOLVE THE MATRIX EQN.

$$(\hat{\Omega} - \lambda_i \hat{\mathbb{I}})|V_i\rangle = |0\rangle \underset{\substack{\text{IN BASIS} \\ \langle p|q\rangle = \delta_{pq}}}{\Longrightarrow} \underbrace{(\Omega_{pq} - \lambda_i \delta_{pq})(V_i)_q}_{\text{SUM } q = 1 \dots n \text{ (MATRIX MULTIPLICATION)}} = 0$$

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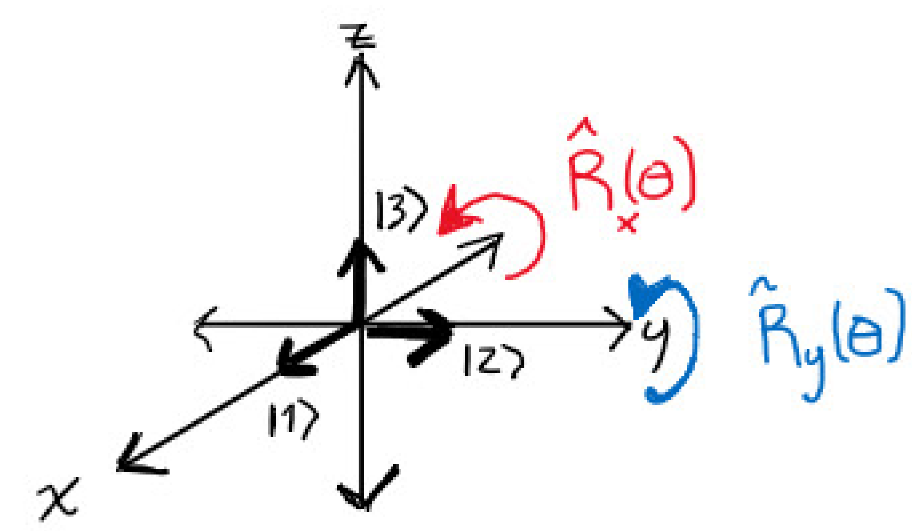
ex): $\hat{R}_x(\theta = \frac{\pi}{2}) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

SEE LEC. 3, p. 3 AND 5



$$\det(\hat{R}_x(\tfrac{\pi}{2}) - \lambda \hat{\mathbb{I}}) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda)^2 + 0 + 0 - 0 + 1(+1)(1-\lambda) - 0$$

$$= \lambda^2(1-\lambda) + (1-\lambda) = (\lambda^2+1)(1-\lambda) = P_3(\lambda)$$

EIGENVALUES:

$\lambda_1 = 1$

$\lambda_2 = i$

$\lambda_3 = -i$

EIGENVECTORS:

① $\lambda_1 = 1$: OBVIOUSLY $|V_1\rangle \equiv |\lambda_1\rangle = |1\rangle$ NOT AFFECTED BY X-ROTATION

② $\lambda_2 = i$: $(\hat{R}_x(\frac{\pi}{2}) - i\hat{\mathbb{I}})|\lambda_2\rangle = |0\rangle$

$$\begin{bmatrix} 1-i & 0 & 0 \\ 0 & -i & -1 \\ & 1 & -i \end{bmatrix} \begin{bmatrix} V_{2,1} \\ V_{2,2} \\ V_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} V_{2,1} &= 0 \\ -i V_{2,2} - V_{2,3} &= 0 \\ V_{2,2} - i V_{2,3} &= 0 \end{aligned}$$

SAME EQN.

WHICH EIGENVECTOR $|V_2\rangle$

WHICH COMPONENT IN ORIGINAL BASIS

$\therefore V_{2,3} = -i V_{2,2}$

CAN FIX ALL 3 COMPONENTS BY NORMALIZING $\langle V_2|V_2\rangle = 1$

$\therefore |V_{2,2}|^2 + |V_{2,3}|^2 = 2|V_{2,2}|^2 = 1 \Rightarrow |V_{2,2}| = \frac{1}{\sqrt{2}}$

$\therefore |V_2\rangle \Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$

- NORMALIZED EIGENVECTOR WITH EIGENVALUE $\lambda_2 = i$
- EVEN WHEN NORMALIZED, DEFINITION NOT COMPLETELY UNIQUE: OVERALL PHASE OF $|V_2\rangle$ CAN BE FREELY CHOSEN $|V_2\rangle \to e^{i\phi}|V_2\rangle$

ALWAYS CHECK ANSWER

$\hat{\Omega}|V_i\rangle = \lambda_i |V_i\rangle$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = i \times \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} \quad ✓$$

NOTE: EIGENVALUES AND EIGENVECTORS OF AN ORDINARY ROTATION MATRIX INVOLVE COMPLEX NUMBERS! ⇒ QUANTUM SPIN (LATER)

③ THIRD EIGENVALUE $\lambda_3 = -i$

CLAIM: (CHECK!)

$|V_3\rangle \Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}$

- EIGENVECTORS OF $\hat{R}_x(\frac{\pi}{2})$, A UNITARY OP., ARE ORTHOGONAL:

$$\langle V_i | V_j \rangle = \delta_{ij} \qquad \begin{aligned} & i,j \in \{1,2,3\} \\ & \lambda_i \in \{1, i, -i\} \end{aligned}$$

Theorem 7: Eigenvalues of a Hermitian Operator are Real

$$\hat{\Omega}|\omega_i\rangle = \omega_i |\omega_i\rangle \; ; \; \hat{\Omega} = \hat{\Omega}^\dagger \text{ Hermitian}$$

① $\langle\omega_j|\hat{\Omega}|\omega_i\rangle = \omega_i \langle\omega_j|\omega_i\rangle$

② $\langle\omega_i|\hat{\Omega}|\omega_j\rangle = \omega_j \langle\omega_i|\omega_j\rangle \xrightarrow[\text{Adjoint}]{} \langle\omega_j|\hat{\Omega}^\dagger|\omega_i\rangle = \omega_j^* \langle\omega_j|\omega_i\rangle$

① - ②: $(\omega_i - \omega_j^*)\langle\omega_j|\omega_i\rangle = 0$

$i = j \Rightarrow \langle\omega_i|\omega_i\rangle > 0$ [can take $\langle\omega_i|\omega_i\rangle = 1$] $\therefore \; \omega_i = \omega_i^* \in \mathbb{R}$ purely real

Theorem 8: Non-deg. Eigenvectors of a Hermitian Operator are Orthogonal

- Assume no degenerate eigenvalues $\omega_i = \omega_i^*$

$$\Rightarrow (\omega_i - \omega_j)\langle\omega_j|\omega_i\rangle = 0 \Rightarrow \langle\omega_j|\omega_i\rangle = 0 \text{ for } i \neq j \; ✓$$

Theorem 9: In the basis of its eigenvectors, a Hermitian Operator is diagonal, with diagonal elements given by its eigenvalues

Start with $|\omega_1\rangle$: $\hat{\Omega}|\omega_1\rangle = \omega_1|\omega_1\rangle$; $\langle\omega_1|\omega_1\rangle = 1$

Choose $|\omega_1\rangle$ as a basis vector: $|\omega_1\rangle \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$

In this new basis of $\hat{\Omega}$'s eigenvectors

Then

$$\hat{\Omega} \Rightarrow \begin{bmatrix} \omega_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & & & & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}$$

- 1st column: $\omega_1|\omega_1\rangle$ in the eigenbasis of $\hat{\Omega}$ (Lec. 3, page 4)
- 1st row: follows from $\hat{\Omega}^\dagger = \hat{\Omega}$

Repeat with $|\omega_2\rangle$

$$\hat{\Omega} \Rightarrow \begin{bmatrix} \omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{bmatrix}$$

Continue ...

$$\hat{\Omega} \Rightarrow \begin{bmatrix} \omega_1 & & & & 0 \\ & \omega_2 & & & \\ & & \omega_3 & & \\ & & & \ddots & \\ 0 & & & & \omega_n \end{bmatrix}$$

In the basis $\{|\omega_1\rangle, |\omega_2\rangle, \ldots, |\omega_n\rangle\}$

Hermitian Operators with Degenerate Eigenvalues: Choosing Orthonormal Basis

Let $\hat{\Omega}^\dagger = \hat{\Omega}$. Suppose $\hat{\Omega}|\omega_1\rangle = \omega|\omega_1\rangle$, $\hat{\Omega}|\omega_2\rangle = \omega|\omega_2\rangle$

$$\therefore \hat{\Omega}\left[\alpha|\omega_1\rangle + \beta|\omega_2\rangle\right] = \omega\left[\alpha|\omega_1\rangle + \beta|\omega_2\rangle\right]$$

$\Rightarrow$ Any vector in the 2D subspace spanned by $|\omega_1\rangle$, $|\omega_2\rangle$ is an eigenvector.

$\Rightarrow$ No "natural" basis for eigenvectors in a degenerate subspace.

I.e., if you ask Mathematica to diagonalize a Hermitian operator, it will typically return lin. indept., but not orthogonal eigenvectors in the degen. subspace.

Degeneracy Example

$$\hat{\Omega} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \hat{\Omega}^\dagger = \hat{\Omega}^* = \hat{\Omega}^T$$

Special case: $\hat{\Omega}$ not only Hermitian, but also real

- $\hat{\Omega} = \hat{\Omega}^\dagger$ is true in any orth. basis
- $\hat{\Omega} = \hat{\Omega}^T$ or $\hat{\Omega} = \Omega^*$ is not

i.e. $\hat{\Omega}' = \hat{U}^\dagger\hat{\Omega}\hat{U}$, $\hat{U}^\dagger\hat{U} = \hat{\mathbb{I}}$

$\Rightarrow$ ① $\hat{\Omega}'^\dagger = \hat{\Omega}'$

② But $\hat{\Omega}' \neq \hat{\Omega}'^*$ for gen. $\hat{U}$

$$(\hat{\Omega} - \omega\hat{\mathbb{I}})|v\rangle = 0$$

$$\det(\hat{\Omega} - \omega\hat{\mathbb{I}}) = (1-\omega)^2(2-\omega) + 0 + 0$$
$$- (2-\omega) - 0 - 0$$
$$= (2-\omega)(\cancel{1} - 2\omega + \omega^2 - \cancel{1})$$
$$= (2-\omega)^2(-\omega)$$

$\therefore$ $\omega_1 = 0$; $\omega_2 = \omega_3 = 2$ Plug ω_1 into $(\hat{\Omega} - \omega_1\hat{\mathbb{I}})|\omega_1\rangle = 0$

$$\Rightarrow |\omega_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Plug ω_2 into $(\hat{\Omega} - \omega_2\hat{\mathbb{I}})|\omega_2\rangle = 0$:

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = 0$$

$\Rightarrow$ $V_1 = V_3$; V_2 is arbitrary!

Choose $|\omega_2\rangle \equiv |\omega_2 = 2, 1\rangle \Rightarrow \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $|\omega_3\rangle \equiv |2,2\rangle \equiv \frac{1}{\sqrt{2+|a|^2}}\begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}$

E'value

Which vector

- Any $a \neq 1$ gives $|2,2\rangle$ that is orthog. to $|\omega_1 = 1\rangle$, lin. indept. from $|2,1\rangle$
- Choose a s.t. all basis states are orthonormal

$$\langle 2,1|2,2\rangle = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2+|a|^2}}(2+a) \Rightarrow \boxed{a = -2}$$