

Columbia University: CU Later Team Reference Document

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Templates

Ken's template

```
1 #include <bits/stdc++.h>
2 using namespace std;
3 #define all(v) (v).begin(), (v).end()
4 typedef long long ll;
5 typedef long double ld;
6 typedef vector<int> vi;
7 typedef vector<ll> vll;
8 typedef pair<int, int> pii;
9 typedef pair<ll, ll> pll;
10 #define pb push_back
11 #define sz(x) (int)(x).size()
12 #define fi first
13 #define se second
14 #define forn(i, n) for (int i = 0; i < int(n); i++)
15 #define endl '\n'
```

Kevin's template

```
1 // paste Ken's Template, minus last line
2 const char nl = '\n';
3 ll k, n, m, u, v, w, x, y, z;
4 string s;
5
6 bool multiTest = 1;
7 void solve(int tt){
8 }
9
10 int main(){
11     ios::sync_with_stdio(0);cin.tie(0);cout.tie(0);
12     cout<<fixed<< setprecision(14);
13
14     int t = 1;
15     if (multiTest) cin >> t;
16     forn(ii, t) solve(ii);
17 }
```

Kevin's Template Extended

- to type after the start of the contest

```
1 typedef pair<double, double> pdd;
2 const ld PI = acos(-1);
3 const ll mod7 = 1e9 + 7;
4 const ll mod9 = 998244353;
5 const ll INF = 2*1024*1024*1023;
6 #pragma GCC target("avx2,bmi,bmi2,lzcnt,popcnt")
7 #include <ext/pb_ds/assoc_container.hpp>
8 #include <ext/pb_ds/tree_policy.hpp>
9 using namespace __gnu_pbds;
10 template<class T> using ordered_set = tree<T, null_type,
11     ↪ less<T>, rb_tree_tag, tree_order_statistics_node_update>;
12 vi d4x = {1, 0, -1, 0};
13 vi d4y = {0, 1, 0, -1};
14 vi d8x = {1, 0, -1, 0, 1, 1, -1, -1};
15 vi d8y = {0, 1, 0, -1, 1, -1, 1, -1};
16 mt19937
17     ↪ rng(chrono::steady_clock::now().time_since_epoch().count());
```

Geometry

Point and vector basics

```
1 const ld EPS = 1e-9;
2
3 struct pt{
4     ld x, y;
5     pt() : x(0), y(0) {}
6     pt(ld x_, ld y_) : x(x_), y(y_) {}
7
8     pt operator+ (pt rhs) const{
9         return pt(x + rhs.x, y + rhs.y); }
```

```
10     pt operator- (pt rhs) const{
11         return pt(x - rhs.x, y - rhs.y); }
12     pt operator* (ld rhs) const{
13         return pt(x * rhs, y * rhs); }
14     pt operator/ (ld rhs) const{
15         return pt(x / rhs, y / rhs); }
16     pt ort() const{
17         return pt(-y, x); }
18     ld abs2() const{
19         return x * x + y * y; }
20     ld len() const{
21         return sqrt(abs2()); }
22     pt unit() const{
23         return pt(x, y) / len(); }
24     pt rotate(ld a) const{
25         return pt(x * cos(a) - y * sin(a), x * sin(a) + y *
26     ↪ cos(a)); }
27     friend ostream& operator<<(ostream& os, pt p){
28         return os << "(" << p.x << "," << p.y << ")";
29     }
30
31     bool operator< (pt rhs) const{
32         return make_pair(x, y) < make_pair(rhs.x, rhs.y);
33     }
34     bool operator==(pt rhs) const{
35         return abs(x - rhs.x) < EPS && abs(y - rhs.y) < EPS;
36     }
37 };
38
39 ld sq(ld a){
40     return a * a; }
41 ld dot(pt a, pt b){
42     return a.x * b.x + a.y * b.y; }
43 ld cross(pt a, pt b){
44     return a.x * b.y - a.y * b.x; }
45 ld dist(pt a, pt b){
46     return (a - b).len(); }
47 bool acw(pt a, pt b){
48     return cross(a, b) > -EPS; }
49 bool cw(pt a, pt b){
50     return cross(a, b) < EPS; }
51 int sgn(ld x){
52     return (x > EPS) - (x < EPS); } // for integer: EPS = 0
53 int half(pt p) { return p.y != 0 ? sgn(p.y) : sgn(p.x); } //
54     ↪ +1: [0, pi), -1: [pi, 2*pi)
55 bool angle_comp(pt a, pt b) { int A = half(a), B = half(b);
56     return A == B ? cross(a, b) > 0 : A > B; }
```

Line basics

```
1 struct line{
2     ld a, b, c;
3     line() : a(0), b(0), c(0) {}
4     line(ld a_, ld b_, ld c_) : a(a_), b(b_), c(c_) {}
5     line(pt p1, pt p2){
6         a = p1.y - p2.y;
7         b = p2.x - p1.x;
8         c = -a * p1.x - b * p1.y;
9     }
10 };
11
12 ld det(ld a11, ld a12, ld a21, ld a22){
13     return a11 * a22 - a12 * a21;
14 }
15 bool parallel(line l1, line l2){
16     return abs(cross(pt(l1.a, l1.b), pt(l2.a, l2.b))) < EPS;
17 }
18 bool operator==(line l1, line l2){
19     return parallel(l1, l2) &&
20     ↪ abs(det(l1.b, l1.c, l2.b, l2.c)) < EPS &&
21     ↪ abs(det(l1.a, l1.c, l2.a, l2.c)) < EPS;
22 }
```

Line and segment intersections

```
1 // {p, 0} - unique intersection, {p, 1} - infinite, {p, 2} -
  ↳ none
2 pair<pt, int> line_inter(line l1, line l2){
3     if (parallel(l1, l2)){
4         return {pt(), 11 == 12? 1 : 2};
5     }
6     return {pt(
7         det(-l1.c, l1.b, -l2.c, l2.b) / det(l1.a, l1.b, l2.a,
  ↳ 12.b),
8         det(l1.a, -l1.c, l2.a, -l2.c) / det(l1.a, l1.b, l2.a,
  ↳ 12.b)
9     ), 0};
10 }
11
12 // Checks if p lies on ab
13 bool is_on_seg(pt p, pt a, pt b){
14     return abs(cross(p - a, p - b)) < EPS && dot(p - a, p - b) <
  ↳ EPS;
15 }
16
17 /*
18 If a unique intersection point between the line segments going
  ↳ from a to b and from c to d exists then it is returned.
19 If no intersection point exists an empty vector is returned.
20 If infinitely many exist a vector with 2 elements is returned,
  ↳ containing the endpoints of the common line segment.
21 */
22 vector<pt> segment_inter(pt a, pt b, pt c, pt d) {
23     auto oa = cross(d - c, a - c), ob = cross(d - c, b - c), oc
  ↳ = cross(b - a, c - a), od = cross(b - a, d - a);
24     if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0) return
  ↳ {(a * ob - b * oa) / (ob - oa)};
25     set<pt> s;
26     if (is_on_seg(a, c, d)) s.insert(a);
27     if (is_on_seg(b, c, d)) s.insert(b);
28     if (is_on_seg(c, a, b)) s.insert(c);
29     if (is_on_seg(d, a, b)) s.insert(d);
30     return {all(s)};
31 }
32 }
```

Distances from a point to line and segment

```
1 // Distance from p to line ab
2 ld line_dist(pt p, pt a, pt b){
3     return cross(b - a, p - a) / (b - a).len();
4 }
5
6 // Distance from p to segment ab
7 ld segment_dist(pt p, pt a, pt b){
8     if (a == b) return (p - a).len();
9     auto d = (a - b).abs2(), t = min(d, max((ld)0, dot(p - a, b
  ↳ - a)));
10     return ((p - a) * d - (b - a) * t).len() / d;
11 }
```

Polygon area and Centroid

```
1 pair<pt,ld> cenArea(const vector<pt>& v) { assert(sz(v) >= 3);
2     pt cen(0, 0); ld area = 0;
3     forn(i,sz(v)) {
4         int j = (i+1)%sz(v); ld a = cross(v[i],v[j]);
5         cen = cen + a*(v[i]+v[j]); area += a; }
6     return {cen/area/(ld)3,area/2}; // area is SIGNED
7 }
```

Convex hull

- Complexity: $O(n \log n)$.

```
1 vector<pt> convex_hull(vector<pt> pts){
2     sort(all(pts));
3     pts.erase(unique(all(pts)), pts.end());
```

```
4     vector<pt> up, down;
5     for (auto p : pts){
6         while (sz(up) > 1 && acw(up.end()[-1] - up.end()[-2], p -
  ↳ up.end()[-2])) up.pop_back();
7         while (sz(down) > 1 && cw(down.end()[-1] - down.end()[-2],
  ↳ p - down.end()[-2])) down.pop_back();
8         up.pb(p), down.pb(p);
9     }
10     for (int i = sz(up) - 2; i >= 1; i--) down.pb(up[i]);
11     return down;
12 }
```

Point location in a convex polygon

- Complexity: $O(n)$ precalculation and $O(\log n)$ query.

```
1 void prep_convex_poly(vector<pt>& pts){
2     rotate(pts.begin(), min_element(all(pts)), pts.end());
3 }
4
5 // 0 - Outside, 1 - Exclusively Inside, 2 - On the Border
6 int in_convex_poly(pt p, vector<pt>& pts){
7     int n = sz(pts);
8     if (!n) return 0;
9     if (n <= 2) return is_on_seg(p, pts[0], pts.back());
10    int l = 1, r = n - 1;
11    while (r - l > 1){
12        int mid = (l + r) / 2;
13        if (acw(pts[mid] - pts[0], p - pts[0])) l = mid;
14        else r = mid;
15    }
16    if (!in_triangle(p, pts[0], pts[l], pts[l + 1])) return 0;
17    if (is_on_seg(p, pts[l], pts[l + 1]) ||
18        is_on_seg(p, pts[0], pts.back()) ||
19        is_on_seg(p, pts[0], pts[l]))
20        return 2;
21    return 1;
22 }
```

Point location in a simple polygon

- Complexity: $O(n)$.

```
1 // 0 - Outside, 1 - Exclusively Inside, 2 - On the Border
2 int in_simple_poly(pt p, vector<pt>& pts){
3     int n = sz(pts);
4     bool res = 0;
5     for (int i = 0; i < n; i++){
6         auto a = pts[i], b = pts[(i + 1) % n];
7         if (is_on_seg(p, a, b)) return 2;
8         if (((a.y > p.y) - (b.y > p.y)) * cross(b - p, a - p) >
  ↳ EPS){
9             res ^= 1;
10        }
11    }
12    return res;
13 }
```

Minkowski Sum

- For two convex polygons P and Q , returns the set of points $(p + q)$, where $p \in P, q \in Q$.
- This set is also a convex polygon.
- Complexity: $O(n)$.

```
1 void minkowski_rotate(vector<pt>& P){
2     int pos = 0;
3     for (int i = 1; i < sz(P); i++){
4         if (abs(P[i].y - P[pos].y) <= EPS){
5             if (P[i].x < P[pos].x) pos = i;
6         }
7         else if (P[i].y < P[pos].y) pos = i;
8     }
9     rotate(P.begin(), P.begin() + pos, P.end());
10 }
```

```

11 // P and Q are strictly convex, points given in
12   ↪ counterclockwise order.
13 vector<pt> minkowski_sum(vector<pt> P, vector<pt> Q){
14     minkowski_rotate(P);
15     minkowski_rotate(Q);
16     P.pb(P[0]);
17     Q.pb(Q[0]);
18     vector<pt> ans;
19     int i = 0, j = 0;
20     while (i < sz(P) - 1 || j < sz(Q) - 1){
21         ans.pb(P[i] + Q[j]);
22         ld curmul;
23         if (i == sz(P) - 1) curmul = -1;
24         else if (j == sz(Q) - 1) curmul = +1;
25         else curmul = cross(P[i + 1] - P[i], Q[j + 1] - Q[j]);
26         if (abs(curmul) < EPS || curmul > 0) i++;
27         if (abs(curmul) < EPS || curmul < 0) j++;
28     }
29     return ans;
}

```

Half-plane intersection

- Given N half-plane conditions in the form of a ray, computes the vertices of their intersection polygon.
- Complexity: $O(N \log N)$.
- A ray is defined by a point p and direction vector dp . The half-plane is to the **left** of the direction vector.

```

1 // Extra functions needed: point operations, dot, cross
2 const ld EPS = 1e-9;
3
4 int sgn(ld a){
5     return (a > EPS) - (a < -EPS);
6 }
7 int half(pt p){
8     return p.y != 0 ? sgn(p.y) : -sgn(p.x);
9 }
10 bool angle_comp(pt a, pt b){
11     int A = half(a), B = half(b);
12     return A == B ? cross(a, b) > 0 : A < B;
13 }
14 struct ray{
15     pt p, dp; // origin, direction
16     ray(pt p_, pt dp_){
17         p = p_, dp = dp_;
18     }
19     pt isect(ray l){
20         return p + dp * (cross(l.dp, l.p - p) / cross(l.dp, dp));
21     }
22     bool operator<(ray l){
23         return angle_comp(dp, l.dp);
24     }
25 };
26 vector<pt> half_plane_isect(vector<ray> rays, ld DX = 1e9, ld
27   ↪ DY = 1e9){
28     // constrain the area to [0, DX] x [0, DY]
29     rays.pb({pt(0, 0), pt(1, 0)});
30     rays.pb({pt(DX, 0), pt(0, 1)});
31     rays.pb({pt(DX, DY), pt(-1, 0)});
32     rays.pb({pt(0, DY), pt(0, -1)});
33     sort(all(rays));
34     {
35         vector<ray> nrays;
36         for (auto t : rays){
37             if (nrays.empty() || cross(nrays.back().dp, t.dp) >
38   ↪ EPS){
39                 nrays.pb(t);
40                 continue;
41             }
42             if (cross(t.dp, t.p - nrays.back().p) > 0) nrays.back()
43   ↪ = t;
44         }
45         swap(rays, nrays);
46     }
47     auto bad = [&] (ray a, ray b, ray c){

```

```

45     pt p1 = a.isect(b), p2 = b.isect(c);
46     if (dot(p2 - p1, b.dp) <= EPS){
47         if (cross(a.dp, c.dp) <= 0) return 2;
48         return 1;
49     }
50     return 0;
51 };
52 #define reduce(t) \
53     while (sz(poly) > 1){ \
54         int b = bad(poly[sz(poly) - 2], poly.back(), t); \
55         if (b == 2) return {}; \
56         if (b == 1) poly.pop_back(); \
57         else break; \
58     }
59 deque<ray> poly;
60 for (auto t : rays){
61     reduce(t);
62     poly.pb(t);
63 }
64 for (; poly.pop_front()){
65     reduce(poly[0]);
66     if (!bad(poly.back(), poly[0], poly[1])) break;
67 }
68 assert(sz(poly) >= 3); // expect nonzero area
69 vector<pt> poly_points;
70 for (int i = 0; i < sz(poly); i++){
71     poly_points.pb(poly[i].isect(poly[(i + 1) % sz(poly)]));
72 }
73 return poly_points;
74 }

```

Circles

- Finds minimum enclosing circle of vector of points in expected $O(N)$

```

1 // necessary point functions
2 ld sq(ld a) { return a*a; }
3 pt operator+(const pt& l, const pt& r) {
4     return pt(l.x+r.x, l.y+r.y); }
5 pt operator*(const pt& l, const ld& r) {
6     return pt(l.x*r, l.y*r); }
7 pt operator*(const ld& l, const pt& r) { return r*l; }
8 ld abs2(const pt& p) { return sq(p.x)+sq(p.y); }
9 ld abs(const pt& p) { return sqrt(abs2(p)); }
10 pt conj(const pt& p) { return pt(p.x, -p.y); }
11 pt operator-(const pt& l, const pt& r) {
12     return pt(l.x-r.x, l.y-r.y); }
13 pt operator*(const pt& l, const pt& r) {
14     return pt(l.x*r.x-l.y*r.y, l.y*r.x+l.x*r.y); }
15 pt operator/(const pt& l, const ld& r) {
16     return pt(l.x/r, l.y/r); }
17 pt operator/(const pt& l, const pt& r) {
18     return l*conj(r)/abs2(r); }
19
20 // circle code
21 using circ = pair<pt, ld>;
22
23 circ ccCenter(pt a, pt b, pt c) {
24     b = b-a; c = c-a;
25     pt res = b*c*(conj(c)-conj(b))/(b*conj(c)-conj(b)*c);
26     return {a+res, abs(res)};
27 }
28
29 circ mec(vector<pt> ps) {
30     // expected O(N)
31     shuffle(all(ps), rng);
32     pt o = ps[0]; ld r = 0, EPS = 1+1e-8;
33     forn(i, sz(ps)) if (abs(o-ps[i]) > r*EPS) {
34         o = ps[i], r = 0; // point is on MEC
35         forn(j, i) if (abs(o-ps[j]) > r*EPS) {
36             o = (ps[i]+ps[j])/2, r = abs(o-ps[i]);
37             forn(k, j) if (abs(o-ps[k]) > r*EPS)
38                 tie(o, r) = ccCenter(ps[i], ps[j], ps[k]);
39         }
40     }

```

```

41     return {o,r};
42 }

```

- Circle tangents, external and internal

```

1 pt unit(const pt& p) { return p * (1/abs(p)); }
2
3 pt tangent(pt p, circ c, int t = 0) {
4     c.se = abs(c.se); // abs needed because internal calls y.s <
    ↪ 0
5     if (c.se == 0) return c.fi;
6     ld d = abs(p-c.fi);
7     pt a = pow(c.se/d,2)*(p-c.fi)+c.fi;
8     pt b = sqrt(d*d-c.se*c.se)/d*c.se*unit(p-c.fi)*pt(0,1);
9     return t == 0 ? a+b : a-b;
10 }
11 vector<pair<pt,pt>> external(circ a, circ b) {
12     vector<pair<pt,pt>> v;
13     if (a.se == b.se) {
14         pt tmp = unit(a.fi-b.fi)*a.se*pt(0, 1);
15         v.emplace_back(a.fi+tmp,b.fi+tmp);
16         v.emplace_back(a.fi-tmp,b.fi-tmp);
17     } else {
18         pt p = (b.se*a.fi-a.se*b.fi)/(b.se-a.se);
19         forn(i,2) v.emplace_back(tangent(p,a,i),tangent(p,b,i));
20     }
21     return v;
22 }
23 vector<pair<pt,pt>> internal(circ a, circ b) {
24     return external({a.fi,-a.se},b); }

```

Strings

```

1 vi prefix_function(string s){
2     int n = sz(s);
3     vi pi(n);
4     for (int i = 1; i < n; i++){
5         int k = pi[i - 1];
6         while (k > 0 && s[i] != s[k]){
7             k = pi[k - 1];
8         }
9         pi[i] = k + (s[i] == s[k]);
10    }
11    return pi;
12 }
13 // Returns the positions of the first character
14 vi kmp(string s, string k){
15     string st = k + "#" + s;
16     vi res;
17     auto pi = prefix_function(st);
18     forn(i, sz(st)){
19         if (pi[i] == sz(k)){
20             res.pb(i - 2 * sz(k));
21         }
22     }
23     return res;
24 }
25 vi z_function(string s){
26     int n = sz(s);
27     vi z(n);
28     int l = 0, r = 0;
29     for (int i = 1; i < n; i++){
30         if (r >= i) z[i] = min(z[i - l], r - i + 1);
31         while (i + z[i] < n && s[z[i]] == s[i + z[i]]){
32             z[i]++;
33         }
34         if (i + z[i] - 1 > r){
35             l = i, r = i + z[i] - 1;
36         }
37     }
38     return z;
39 }

```

Manacher's algorithm

```

1 /*
2  Finds longest palindromes centered at each index
3  even[i] = d --> [i - d, i + d - 1] is a max-palindrome
4  odd[i] = d --> [i - d, i + d] is a max-palindrome
5  */
6 pair<vi, vi> manacher(string s) {
7     vector<char> t{'^', '#'};
8     for (char c : s) t.push_back(c), t.push_back('#');
9     t.push_back('$');
10    int n = t.size(), r = 0, c = 0;
11    vi p(n, 0);
12    for (int i = 1; i < n - 1; i++) {
13        if (i < r + c) p[i] = min(p[2 * c - i], r + c - i);
14        while (t[i + p[i] + 1] == t[i - p[i] - 1]) p[i]++;
15        if (i + p[i] > r + c) r = p[i], c = i;
16    }
17    vi even(sz(s)), odd(sz(s));
18    forn(i, sz(s)){
19        even[i] = p[2 * i + 1] / 2, odd[i] = p[2 * i + 2] / 2;
20    }
21    return {even, odd};
22 }

```

Aho-Corasick Trie

- Given a set of strings, constructs a trie with suffix links.
- For a particular node, *link* points to the longest proper suffix of this node that's contained in the trie.
- *nxt* encodes suffix links in a compressed format:
 - If vertex *v* has a child by letter *x*, then *trie[v].nxt[x]* points to that child.
 - If vertex *v* doesn't have such child, then *trie[v].nxt[x]* points to the suffix link of that child if we would actually have it.
- **Facts:** suffix link graph can be seen as a tree; terminal link tree has height $O(\sqrt{N})$, where *N* is the sum of strings' lengths.
- **Usage:** add all strings, then call *add_links()*.

```

1 const int S = 26;
2
3 // Function converting char to int.
4 int ctoi(char c){
5     return c - 'a';
6 }
7
8 // To add terminal links, use DFS
9 struct Node{
10     vi nxt;
11     int link;
12     bool terminal;
13
14     Node() {
15         nxt.assign(S, -1), link = 0, terminal = 0;
16     }
17 };
18
19 vector<Node> trie(1);
20
21 // add_string returns the terminal vertex.
22 int add_string(string& s){
23     int v = 0;
24     for (auto c : s){
25         int cur = ctoi(c);
26         if (trie[v].nxt[cur] == -1){
27             trie[v].nxt[cur] = sz(trie);
28             trie.emplace_back();
29         }
30         v = trie[v].nxt[cur];
31     }
32     trie[v].terminal = 1;
33     return v;

```

```

34 }
35
36 void add_links(){
37     queue<int> q;
38     q.push(0);
39     while (!q.empty()){
40         auto v = q.front();
41         int u = trie[v].link;
42         q.pop();
43         forn(i, S){
44             int& ch = trie[v].nxt[i];
45             if (ch == -1){
46                 ch = v? trie[u].nxt[i] : 0;
47             }
48             else{
49                 trie[ch].link = v? trie[u].nxt[i] : 0;
50                 q.push(ch);
51             }
52         }
53     }
54 }
55
56 bool is_terminal(int v){
57     return trie[v].terminal;
58 }
59
60 int get_link(int v){
61     return trie[v].link;
62 }
63
64 int go(int v, char c){
65     return trie[v].nxt[toi(c)];
66 }

```

Suffix Automaton

- Given a string S , constructs a DAG that is an automaton of all suffixes of S .
- The automaton has $\leq 2n$ nodes and $\leq 3n$ edges.
- Properties (let all paths start at node 0):
 - Every path represents a unique substring of S .
 - A path ends at a terminal node iff it represents a suffix of S .
 - All paths ending at a fixed node v have the same set of right endpoints of their occurrences in S .
 - Let $endpos(v)$ represent this set. Then, $link(v) := u$ such that $endpos(v) \subset endpos(u)$ and $|endpos(u)|$ is smallest possible. $link(0) := -1$. Links form a tree.
 - Let $len(v)$ be the longest path ending at v . All paths ending at v have distinct lengths: every length from interval $[len(link(v)) + 1, len(v)]$.
- One of the main applications is dealing with **distinct** substrings. Such problems can be solved with DFS and DP.
- Complexity: $O(|S| \cdot \log |\Sigma|)$. Perhaps replace map with vector if $|\Sigma|$ is small.

```

1  const int MAXLEN = 1e5 + 20;
2
3  struct suffix_automaton{
4      struct state {
5          int len, link;
6          bool terminal = 0, used = 0;
7          map<char, int> next;
8      };
9
10     state st[MAXLEN * 2];
11     int sz = 0, last;
12
13     suffix_automaton(){

```

```

14         st[0].len = 0;
15         st[0].link = -1;
16         sz++;
17         last = 0;
18     };
19
20     void extend(char c) {
21         int cur = sz++;
22         st[cur].len = st[last].len + 1;
23         int p = last;
24         while (p != -1 && !st[p].next.count(c)) {
25             st[p].next[c] = cur;
26             p = st[p].link;
27         }
28         if (p == -1) {
29             st[cur].link = 0;
30         } else {
31             int q = st[p].next[c];
32             if (st[p].len + 1 == st[q].len) {
33                 st[cur].link = q;
34             } else {
35                 int clone = sz++;
36                 st[clone].len = st[p].len + 1;
37                 st[clone].next = st[q].next;
38                 st[clone].link = st[q].link;
39                 while (p != -1 && st[p].next[c] == q) {
40                     st[p].next[c] = clone;
41                     p = st[p].link;
42                 }
43                 st[q].link = st[cur].link = clone;
44             }
45         }
46         last = cur;
47     }
48
49     void mark_terminal(){
50         int cur = last;
51         while (cur) st[cur].terminal = 1, cur = st[cur].link;
52     }
53 };
54 /*
55 Usage:
56 suffix_automaton sa;
57 for (int i = 0; i < sz(str); i++) sa.extend(str[i]);
58 sa.mark_terminal();
59 */

```

Flows

$O(N^2M)$, on unit networks $O(N^{1/2}M)$

```

1  struct FlowEdge {
2      int from, to;
3      ll cap, flow = 0;
4      FlowEdge(int u, int v, ll cap) : from(u), to(v), cap(cap) {}
5  };
6
7  struct Dinic {
8      const ll flow_inf = 1e18;
9      vector<FlowEdge> edges;
10     vector<vi> adj;
11     int n, m = 0;
12     int s, t;
13     vi level, ptr;
14     vector<bool> used;
15     queue<int> q;
16     Dinic(int n, int s, int t) : n(n), s(s), t(t) {
17         adj.resize(n);
18         level.resize(n);
19         ptr.resize(n);
20     }
21     void add_edge(int u, int v, ll cap) {
22         edges.emplace_back(u, v, cap);
23         edges.emplace_back(v, u, 0);
24         adj[u].push_back(m);
25         adj[v].push_back(m + 1);

```

```

25     m += 2;
26 }
27 bool bfs() {
28     while (!q.empty()) {
29         int v = q.front();
30         q.pop();
31         for (int id : adj[v]) {
32             if (edges[id].cap - edges[id].flow < 1)
33                 continue;
34             if (level[edges[id].to] != -1)
35                 continue;
36             level[edges[id].to] = level[v] + 1;
37             q.push(edges[id].to);
38         }
39     }
40     return level[t] != -1;
41 }
42 ll dfs(int v, ll pushed) {
43     if (pushed == 0)
44         return 0;
45     if (v == t)
46         return pushed;
47     for (int& cid = ptr[v]; cid < (int)adj[v].size(); cid++) {
48         int id = adj[v][cid];
49         int u = edges[id].to;
50         if (level[v] + 1 != level[u] || edges[id].cap -
↪ edges[id].flow < 1)
51             continue;
52         ll tr = dfs(u, min(pushed, edges[id].cap -
↪ edges[id].flow));
53         if (tr == 0)
54             continue;
55         edges[id].flow += tr;
56         edges[id ^ 1].flow -= tr;
57         return tr;
58     }
59     return 0;
60 }
61 ll flow() {
62     ll f = 0;
63     while (true) {
64         fill(level.begin(), level.end(), -1);
65         level[s] = 0;
66         q.push(s);
67         if (!bfs())
68             break;
69         fill(ptr.begin(), ptr.end(), 0);
70         while (ll pushed = dfs(s, flow_inf)) {
71             f += pushed;
72         }
73     }
74     return f;
75 }
76
77 void cut_dfs(int v){
78     used[v] = 1;
79     for (auto i : adj[v]){
80         if (edges[i].flow < edges[i].cap && !used[edges[i].to]){
81             cut_dfs(edges[i].to);
82         }
83     }
84 }
85
86 // Assumes that max flow is already calculated
87 // true -> vertex is in S, false -> vertex is in T
88 vector<bool> min_cut(){
89     used = vector<bool>(n);
90     cut_dfs(s);
91     return used;
92 }
93 };
94 // To recover flow through original edges: iterate over even
↪ indices in edges.

```

MCMF – maximize flow, then minimize its cost. $O(mn + Fm \log n)$.

```

1  #include <bits/stdc++.h> /// include-line, keep-include
2
3  const ll INF = LLONG_MAX / 4;
4
5  struct MCMF {
6      struct edge {
7          int from, to, rev;
8          ll cap, cost, flow;
9      };
10     int N;
11     vector<vector<edge>> ed;
12     vi seen;
13     vll dist, pi;
14     vector<edge*> par;
15
16     MCMF(int N) : N(N), ed(N), seen(N), dist(N), pi(N), par(N)
↪ {}
17
18     void add_edge(int from, int to, ll cap, ll cost) {
19         if (from == to) return;
20         ed[from].push_back(edge{ from, to, sz(ed[to]), cap, cost, 0 });
21         ed[to].push_back(edge{ to, from, sz(ed[from])-1, 0, -cost, 0
↪ });
22     }
23
24     void path(int s) {
25         fill(all(seen), 0);
26         fill(all(dist), INF);
27         dist[s] = 0; ll di;
28
29         __gnu_pbds::priority_queue<pair<ll, int>> q;
30         vector<decltype(q)::point_iterator> its(N);
31         q.push({ 0, s });
32
33         while (!q.empty()) {
34             s = q.top().second; q.pop();
35             seen[s] = 1; di = dist[s] + pi[s];
36             for (edge& e : ed[s]) if (!seen[e.to]) {
37                 ll val = di - pi[e.to] + e.cost;
38                 if (e.cap - e.flow > 0 && val < dist[e.to]) {
39                     dist[e.to] = val;
40                     par[e.to] = &e;
41                     if (its[e.to] == q.end())
42                         its[e.to] = q.push({ -dist[e.to], e.to });
43                     else
44                         q.modify(its[e.to], { -dist[e.to], e.to });
45                 }
46             }
47         }
48         forn(i, N) pi[i] = min(pi[i] + dist[i], INF);
49     }
50
51     pair<ll, ll> max_flow(int s, int t) {
52         ll totflow = 0, totcost = 0;
53         while (path(s), seen[t]) {
54             ll fl = INF;
55             for (edge* x = par[t]; x; x = par[x->from])
56                 fl = min(fl, x->cap - x->flow);
57
58             totflow += fl;
59             for (edge* x = par[t]; x; x = par[x->from]) {
60                 x->flow += fl;
61                 ed[x->to][x->rev].flow -= fl;
62             }
63         }
64         forn(i, N) for(edge& e : ed[i]) totcost += e.cost *
↪ e.flow;
65         return {totflow, totcost/2};
66     }
67
68     // If some costs can be negative, call this before maxflow:
69     void setpi(int s) { // (otherwise, leave this out)
70         fill(all(pi), INF); pi[s] = 0;
71         int it = N, ch = 1; ll v;

```



```

72 while (ch-- && it--)
73     forn(i, N) if (pi[i] != INF)
74         for (edge& e : ed[i]) if (e.cap)
75             if ((v = pi[i] + e.cost) < pi[e.to])
76                 pi[e.to] = v, ch = 1;
77     assert(it >= 0); // negative cost cycle
78 }
79 };
80 // Usage: MCMF g(n); g.add_edge(u,v,c,w); g.max_flow(s,t).
81 // To recover flow through original edges: iterate over even
    ↪ indices in edges.

```

Graphs

Kuhn's algorithm for bipartite matching

```

1  /*
2  The graph is split into 2 halves of n1 and n2 vertices.
3  Complexity: O(n1 * m). Usually runs much faster. MUCH
    ↪ FASTER!!!
4  */
5  const int N = 305;
6
7  vi g[N]; // Stores edges from left half to right.
8  bool used[N]; // Stores if vertex from left half is used.
9  int mt[N]; // For every vertex in right half, stores to which
    ↪ vertex in left half it's matched (-1 if not matched).
10
11 bool try_dfs(int v){
12     if (used[v]) return false;
13     used[v] = 1;
14     for (auto u : g[v]){
15         if (mt[u] == -1 || try_dfs(mt[u])){
16             mt[u] = v;
17             return true;
18         }
19     }
20     return false;
21 }
22
23 int main(){
24     // .....
25     for (int i = 1; i <= n2; i++) mt[i] = -1;
26     for (int i = 1; i <= n1; i++) used[i] = 0;
27     for (int i = 1; i <= n1; i++){
28         if (try_dfs(i)){
29             for (int j = 1; j <= n1; j++) used[j] = 0;
30         }
31     }
32     vector<pair<int, int>> ans;
33     for (int i = 1; i <= n2; i++){
34         if (mt[i] != -1) ans.pb({mt[i], i});
35     }
36 }
37
38 // Finding maximal independent set: size = # of nodes - # of
    ↪ edges in matching.
39 // To construct: launch Kuhn-like DFS from unmatched nodes in
    ↪ the left half.
40 // Independent set = visited nodes in left half + unvisited in
    ↪ right half.
41 // Finding minimal vertex cover: complement of maximal
    ↪ independent set.

```

Hungarian algorithm for Assignment Problem

- Given a 1-indexed $(n \times m)$ matrix A , select a number in each row such that each column has at most 1 number selected, and the sum of the selected numbers is minimized.

```

1  int INF = 1e9; // constant greater than any number in the
    ↪ matrix

```

```

2  vi u(n+1), v(m+1), p(m+1), way(m+1);
3  for (int i=1; i<=n; ++i) {
4      p[0] = i;
5      int j0 = 0;
6      vi minv (m+1, INF);
7      vector<bool> used (m+1, false);
8      do {
9          used[j0] = true;
10         int i0 = p[j0], delta = INF, j1;
11         for (int j=1; j<=m; ++j)
12             if (!used[j]) {
13                 int cur = A[i0][j]-u[i0]-v[j];
14                 if (cur < minv[j])
15                     minv[j] = cur, way[j] = j0;
16                 if (minv[j] < delta)
17                     delta = minv[j], j1 = j;
18             }
19         for (int j=0; j<=m; ++j)
20             if (used[j])
21                 u[p[j]] += delta, v[j] -= delta;
22         else
23             minv[j] -= delta;
24         j0 = j1;
25     } while (p[j0] != 0);
26     do {
27         int j1 = way[j0];
28         p[j0] = p[j1];
29         j0 = j1;
30     } while (j0);
31 }
32 vi ans (n+1); // ans[i] stores the column selected for row i
33 for (int j=1; j<=m; ++j)
34     ans[p[j]] = j;
35 int cost = -v[0]; // the total cost of the matching

```

Dijkstra's Algorithm

```

1  priority_queue<pair<ll, ll>, vector<pair<ll, ll>>,
    ↪ greater<pair<ll, ll>>> q;
2  dist[start] = 0;
3  q.push({0, start});
4  while (!q.empty()){
5      auto [d, v] = q.top();
6      q.pop();
7      if (d != dist[v]) continue;
8      for (auto [u, w] : g[v]){
9          if (dist[u] > dist[v] + w){
10             dist[u] = dist[v] + w;
11             q.push({dist[u], u});
12         }
13     }
14 }

```

Bellman-Ford Algorithm

- Finds single-source shortest paths with negative edge weights.
- Returns the vector of distances to 0-indexed vertices, or empty vector if a negative cycle is reachable from source.

```

1  const ll bf_inf = 1e18;
2
3  struct edge {
4      ll a, b, w;
5  };
6
7  vector<ll> bellman_ford(int n, vector<edge> edges, int src)
8  {
9      vector<ll> d(n, bf_inf);
10     d[src] = 0;
11     vector<ll> p(n, -1);
12     int x;
13     forn(i, n) {
14         x = -1;
15         for (edge e : edges)

```

```

16         if (d[e.a] < bf_inf)
17             if (d[e.b] > d[e.a] + e.w) {
18                 d[e.b] = max(-bf_inf, d[e.a] + e.w);
19                 p[e.b] = e.a;
20                 x = e.b;
21             }
22     }
23
24     if (x != -1){
25         // negative cycle reachable from src
26         return {};
27     }
28     return d;
29 }

```

Eulerian Cycle DFS

```

1 void dfs(int v){
2     while (!g[v].empty()){
3         int u = g[v].back();
4         g[v].pop_back();
5         dfs(u);
6         ans.pb(v);
7     }
8 }

```

SCC and 2-SAT

```

1 void scc(vector<vi>& g, int* idx) {
2     int n = g.size(), ct = 0;
3     int out[n];
4     vi ginv[n];
5     memset(out, -1, sizeof out);
6     memset(idx, -1, n * sizeof(int));
7     function<void(int)> dfs = [&](int cur) {
8         out[cur] = INT_MAX;
9         for(int v : g[cur]) {
10             ginv[v].push_back(cur);
11             if(out[v] == -1) dfs(v);
12         }
13         ct++; out[cur] = ct;
14     };
15     vi order;
16     for(int i = 0; i < n; i++) {
17         order.push_back(i);
18         if(out[i] == -1) dfs(i);
19     }
20     sort(order.begin(), order.end(), [&](int& u, int& v) {
21         return out[u] > out[v];
22     });
23     ct = 0;
24     stack<int> s;
25     auto dfs2 = [&](int start) {
26         s.push(start);
27         while(!s.empty()) {
28             int cur = s.top();
29             s.pop();
30             idx[cur] = ct;
31             for(int v : ginv[cur])
32                 if(idx[v] == -1) s.push(v);
33         }
34     };
35     for(int v : order) {
36         if(idx[v] == -1) {
37             dfs2(v);
38             ct++;
39         }
40     }
41 }
42
43 // 0 => impossible, 1 => possible
44 pair<int,vi> sat2(int n, vector<pii>& clauses) {
45     vi ans(n);
46     vector<vi> g(2*n + 1);
47     for(auto [x, y] : clauses) {
48         x = x < 0 ? -x + n : x;

```

```

49         y = y < 0 ? -y + n : y;
50         int nx = x <= n ? x + n : x - n;
51         int ny = y <= n ? y + n : y - n;
52         g[nx].push_back(y);
53         g[ny].push_back(x);
54     }
55     int idx[2*n + 1];
56     scc(g, idx);
57     for(int i = 1; i <= n; i++) {
58         if(idx[i] == idx[i + n]) return {0, {}};
59         ans[i - 1] = idx[i + n] < idx[i];
60     }
61     return {1, ans};
62 }

```

Finding Bridges

```

1 /*
2  Bridges.
3  Results are stored in a map "is_bridge".
4  For each connected component, call "dfs(starting vertex,
5  ↪ starting vertex)".
6  */
7
8 const int N = 2e5 + 10; // Careful with the constant!
9
10 vi g[N];
11 int tin[N], fup[N], timer;
12 map<pair<int, int>, bool> is_bridge;
13
14 void dfs(int v, int p){
15     tin[v] = ++timer;
16     fup[v] = tin[v];
17     for (auto u : g[v]){
18         if (!tin[u]){
19             dfs(u, v);
20             if (fup[u] > tin[v]){
21                 is_bridge[{u, v}] = is_bridge[{v, u}] = true;
22             }
23             fup[v] = min(fup[v], fup[u]);
24         }
25         else{
26             if (u != p) fup[v] = min(fup[v], tin[u]);
27         }
28     }
29 }

```

Virtual Tree

```

1 // order stores the nodes in the queried set
2 sort(all(order), [&](int u, int v){return tin[u] < tin[v]});
3 int m = sz(order);
4 for (int i = 1; i < m; i++){
5     order.pb(lca(order[i], order[i - 1]));
6 }
7 sort(all(order), [&](int u, int v){return tin[u] < tin[v]});
8 order.erase(unique(all(order)), order.end());
9 vi stk{order[0]};
10 for (int i = 1; i < sz(order); i++){
11     int v = order[i];
12     while (tout[stk.back()] < tout[v]) stk.pop_back();
13     int u = stk.back();
14     vg[u].pb({v, dep[v] - dep[u]});
15     stk.pb(v);
16 }

```

HLD on Edges DFS

```

1 void dfs1(int v, int p, int d){
2     par[v] = p;
3     for (auto e : g[v]){
4         if (e.fi == p){
5             g[v].erase(find(all(g[v]), e));
6             break;
7         }
8     }
9 }

```

```

9     dep[v] = d;
10    sz[v] = 1;
11    for (auto [u, c] : g[v]){
12        dfs1(u, v, d + 1);
13        sz[v] += sz[u];
14    }
15    if (!g[v].empty()) iter_swap(g[v].begin(),
↳ max_element(all(g[v]), comp));
16 }
17 void dfs2(int v, int rt, int c){
18     pos[v] = sz[a];
19     a.pb(c);
20     root[v] = rt;
21     forn(i, sz(g[v])){
22         auto [u, c] = g[v][i];
23         if (!i) dfs2(u, rt, c);
24         else dfs2(u, u, c);
25     }
26 }
27 int getans(int u, int v){
28     int res = 0;
29     for (; root[u] != root[v]; v = par[root[v]]){
30         if (dep[root[u]] > dep[root[v]]) swap(u, v);
31         res = max(res, rmq(0, 0, n - 1, pos[root[v]], pos[v]));
32     }
33     if (pos[u] > pos[v]) swap(u, v);
34     return max(res, rmq(0, 0, n - 1, pos[u] + 1, pos[v]));
35 }

```

Centroid Decomposition

```

1 vector<char> res(n), seen(n), sz(n);
2 function<int(int, int)> get_size = [&](int node, int fa) {
3     sz[node] = 1;
4     for (auto& ne : g[node]) {
5         if (ne == fa || seen[ne]) continue;
6         sz[node] += get_size(ne, node);
7     }
8     return sz[node];
9 };
10 function<int(int, int, int)> find_centroid = [&](int node, int
↳ fa, int t) {
11     for (auto& ne : g[node])
12         if (ne != fa && !seen[ne] && sz[ne] > t / 2) return
↳ find_centroid(ne, node, t);
13     return node;
14 };
15 function<void(int, char)> solve = [&](int node, char cur) {
16     get_size(node, -1); auto c = find_centroid(node, -1,
↳ sz[node]);
17     seen[c] = 1, res[c] = cur;
18     for (auto& ne : g[c]) {
19         if (seen[ne]) continue;
20         solve(ne, char(cur + 1)); // we can pass c here to build
↳ tree
21     }
22 };

```

Biconnected Components and Block-Cut Tree

- Biconnected components are the ones that have no articulation points.
- They are defined by edge sets that are “bounded” by articulation points in the original graph.
- The corresponding vertex sets are stored in *comps*.
- Block-Cut tree is constructed by creating a fictive node for each component, and attaching edges to its members.
- Articulation points in the original graph are the non-leaf non-fictive nodes in the BC tree.
- Complexity: $O(n)$.

```

1 // Usage: pass in adjacency list in 0-based indexation.

```

```

2 // Return: adjacency list of block-cut tree (nodes 0...n-1
↳ represent original nodes, the rest are component nodes).
3 vector<vi> biconnected_components(vector<vi> g) {
4     int n = sz(g);
5     vector<vi> comps;
6     vi stk, num(n), low(n);
7     int timer = 0;
8     // Finds the biconnected components
9     function<void(int, int)> dfs = [&](int v, int p) {
10         num[v] = low[v] = ++timer;
11         stk.pb(v);
12         for (int son : g[v]) {
13             if (son == p) continue;
14             if (num[son]) low[v] = min(low[v], num[son]);
15             else{
16                 dfs(son, v);
17                 low[v] = min(low[v], low[son]);
18                 if (low[son] >= num[v]){
19                     comps.pb({v});
20                     while (comps.back().back() != son){
21                         comps.back().pb(stk.back());
22                         stk.pop_back();
23                     }
24                 }
25             }
26         }
27     };
28     dfs(0, -1);
29     // Build the block-cut tree
30     auto build_tree = [&]() {
31         vector<vi> t(n);
32         for (auto &comp : comps){
33             t.push_back({});
34             for (int u : comp){
35                 t.back().pb(u);
36             }
37             t[u].pb(sz(t) - 1);
38         }
39         return t;
40     };
41     return build_tree();
42 }

```

Math

Binary exponentiation

```

1 ll power(ll a, ll b){
2     ll res = 1;
3     for (; b; a = a * a % MOD, b >>= 1){
4         if (b & 1) res = res * a % MOD;
5     }
6     return res;
7 }

```

Matrix Exponentiation: $O(n^3 \log b)$

```

1 const int N = 100, MOD = 1e9 + 7;
2
3 struct matrix{
4     ll m[N][N];
5     int n;
6     matrix(){
7         n = N;
8         memset(m, 0, sizeof(m));
9     };
10    matrix(int n_){
11        n = n_;
12        memset(m, 0, sizeof(m));
13    };
14    matrix(int n_, ll val){
15        n = n_;
16        memset(m, 0, sizeof(m));
17        forn(i, n) m[i][i] = val;
18    };

```

```

19
20 matrix operator* (matrix oth){
21     matrix res(n);
22     forn(i, n){
23         forn(j, n){
24             forn(k, n){
25                 res.m[i][j] = (res.m[i][j] + m[i][k] * oth.m[k][j])
↪ % MOD;
26             }
27         }
28     }
29     return res;
30 }
31 };
32
33 matrix power(matrix a, ll b){
34     matrix res(a.n, 1);
35     for (; b; a = a * a, b >>= 1){
36         if (b & 1) res = res * a;
37     }
38     return res;
39 }

```

Extended Euclidean Algorithm

- $O(\max(\log a, \log b))$
- Finds solution (x, y) to $ax + by = \gcd(a, b)$
- Can find all solutions given $(x_0, y_0) : \forall k, a(x_0 + kb/g) + b(y_0 - ka/g) = \gcd(a, b)$.

```

1 ll euclid(ll a, ll b, ll &x, ll &y) {
2     if (!b) return x = 1, y = 0, a;
3     ll d = euclid(b, a % b, y, x);
4     return y -= a/b * x, d;
5 }

```

CRT

- $\text{crt}(a, m, b, n)$ computes x such that $x \equiv a \pmod{m}$, $x \equiv b \pmod{n}$
- If $|a| < m$ and $|b| < n$, x will obey $0 \leq x < \text{lcm}(m, n)$.
- Assumes $mn < 2^{62}$.
- $O(\max(\log m, \log n))$

```

1 ll crt(ll a, ll m, ll b, ll n) {
2     if (n > m) swap(a, b), swap(m, n);
3     ll x, y, g = euclid(m, n, x, y);
4     assert((a - b) % g == 0); // else no solution
5     // can replace assert with whatever needed
6     x = (b - a) % n * x % n / g * m + a;
7     return x < 0 ? x + m*n/g : x;
8 }

```

Linear Sieve

- Mobius Function

```

1 vi prime;
2 bool is_composite[MAX_N];
3 int mu[MAX_N];
4
5 void sieve(int n){
6     fill(is_composite, is_composite + n, 0);
7     mu[1] = 1;
8     for (int i = 2; i < n; i++){
9         if (!is_composite[i]){
10             prime.push_back(i);
11             mu[i] = -1; //i is prime
12         }
13         for (int j = 0; j < prime.size() && i * prime[j] < n; j++){
14             is_composite[i * prime[j]] = true;
15             if (i % prime[j] == 0){
16                 mu[i * prime[j]] = 0; //prime[j] divides i
17                 break;

```

```

18         } else {
19             mu[i * prime[j]] = -mu[i]; //prime[j] does not divide i
20         }
21     }
22 }
23 }

```

- Euler's Totient Function

```

1 vi prime;
2 bool is_composite[MAX_N];
3 int phi[MAX_N];
4
5 void sieve(int n){
6     fill(is_composite, is_composite + n, 0);
7     phi[1] = 1;
8     for (int i = 2; i < n; i++){
9         if (!is_composite[i]){
10             prime.push_back(i);
11             phi[i] = i - 1; //i is prime
12         }
13         for (int j = 0; j < prime.size() && i * prime[j] < n; j++){
14             is_composite[i * prime[j]] = true;
15             if (i % prime[j] == 0){
16                 phi[i * prime[j]] = phi[i] * prime[j]; //prime[j]
↪ divides i
17                 break;
18             } else {
19                 phi[i * prime[j]] = phi[i] * phi[prime[j]]; //prime[j]
↪ does not divide i
20             }
21         }
22     }
23 }

```

Mod Class

- For Gaussian Elimination

```

1 constexpr ll norm(ll x) { return (x % MOD + MOD) % MOD; }
2 template <typename T>
3 constexpr T power(T a, ll b, T res = 1) {
4     for (; b; b /= 2, (a *= a) %= MOD)
5         if (b & 1) (res *= a) %= MOD;
6     return res;
7 }
8 struct Z {
9     ll x;
10     constexpr Z(ll _x = 0) : x(norm(_x)) {}
11     // auto operator<=>(const Z &) const = default; // cpp20
↪ only
12     Z operator-(const Z &rhs) const { return Z(norm(MOD - x)); }
13     Z inv() const { return power(*this, MOD - 2); }
14     Z &operator*=(const Z &rhs) { return x = x * rhs.x % MOD,
↪ *this; }
15     Z &operator+=(const Z &rhs) { return x = norm(x + rhs.x),
↪ *this; }
16     Z &operator-=(const Z &rhs) { return x = norm(x - rhs.x),
↪ *this; }
17     Z &operator/=(const Z &rhs) { return *this *= rhs.inv(); }
18     Z &operator%=(const ll &rhs) { return x %= rhs, *this; }
19     friend Z operator*(Z lhs, const Z &rhs) { return lhs * rhs;
↪ }
20     friend Z operator+(Z lhs, const Z &rhs) { return lhs += rhs;
↪ }
21     friend Z operator-(Z lhs, const Z &rhs) { return lhs -= rhs;
↪ }
22     friend Z operator/(Z lhs, const Z &rhs) { return lhs /= rhs;
↪ }
23     friend Z operator%(Z lhs, const ll &rhs) { return lhs %=
↪ rhs; }
24     friend auto &operator>>(istream &i, Z &z) { return i >> z.x;
↪ }
25     friend auto &operator<<(ostream &o, const Z &z) { return o
↪ << z.x; }
26 };

```

- Fastest mod class! be careful with overflow, only use when the time limit is tight

```

1 constexpr int norm(int x) {
2     if (x < 0) x += MOD;
3     if (x >= MOD) x -= MOD;
4     return x;
5 }

```

Gaussian Elimination

```

1 bool is_0(Z v) { return v.x == 0; }
2 int abs(Z v) { return v.x; }
3 bool is_0(double v) { return abs(v) < 1e-9; }
4
5 // 1 => unique solution, 0 => no solution, -1 => multiple
6 // solutions
7 template <typename T>
8 int gaussian_elimination(vector<vector<T>> &a, int limit) {
9     if (a.empty() || a[0].empty()) return -1;
10    int h = (int)a.size(), w = (int)a[0].size(), r = 0;
11    for (int c = 0; c < limit; c++) {
12        int id = -1;
13        for (int i = r; i < h; i++) {
14            if (!is_0(a[i][c]) && (id == -1 || abs(a[id][c]) <
15                abs(a[i][c]))) {
16                id = i;
17            }
18        }
19        if (id == -1) continue;
20        if (id > r) {
21            swap(a[r], a[id]);
22            for (int j = c; j < w; j++) a[id][j] = -a[id][j];
23        }
24        vi nonzero;
25        for (int j = c; j < w; j++) {
26            if (!is_0(a[r][j])) nonzero.push_back(j);
27        }
28        T inv_a = 1 / a[r][c];
29        for (int i = r + 1; i < h; i++) {
30            if (is_0(a[i][c])) continue;
31            T coeff = -a[i][c] * inv_a;
32            for (int j : nonzero) a[i][j] += coeff * a[r][j];
33        }
34        ++r;
35    }
36    for (int row = h - 1; row >= 0; row--) {
37        for (int c = 0; c < limit; c++) {
38            if (!is_0(a[row][c])) {
39                T inv_a = 1 / a[row][c];
40                for (int i = row - 1; i >= 0; i--) {
41                    if (is_0(a[i][c])) continue;
42                    T coeff = -a[i][c] * inv_a;
43                    for (int j = c; j < w; j++) a[i][j] += coeff *
44                        a[row][j];
45                }
46                break;
47            }
48        }
49    }
50    // not-free variables: only it on its line
51    for (int i = r; i < h; i++) if (!is_0(a[i][limit])) return 0;
52    return (r == limit) ? 1 : -1;
53 }
54
55 template <typename T>
56 pair<int, vector<T>> solve_linear(vector<vector<T>> a, const
57     vector<T> &b, int w) {
58     int h = (int)a.size();
59     for (int i, h) a[i].push_back(b[i]);
60     int sol = gaussian_elimination(a, w);
61     if (!sol) return {0, vector<T>()};
62     vector<T> x(w, 0);
63     for (int i, h) {
64         for (int j, w) {
65             if (!is_0(a[i][j])) {
66                 x[j] = a[i][w] / a[i][j];
67                 break;
68             }
69         }
70     }
71 }

```

```

63     }
64 }
65 }
66 return {sol, x};
67 }

```

Pollard-Rho Factorization

- Uses Miller–Rabin primality test
- $O(n^{1/4})$ (heuristic estimation)

```

1 typedef __int128_t i128;
2
3 i128 power(i128 a, i128 b, i128 MOD = 1, i128 res = 1) {
4     for (; b; b /= 2, (a *= a) %= MOD)
5         if (b & 1) (res *= a) %= MOD;
6     return res;
7 }
8
9 bool is_prime(ll n) {
10    if (n < 2) return false;
11    static constexpr int A[] = {2, 3, 5, 7, 11, 13, 17, 19, 23};
12    int s = __builtin_ctzll(n - 1);
13    ll d = (n - 1) >> s;
14    for (auto a : A) {
15        if (a == n) return true;
16        ll x = (ll)power(a, d, n);
17        if (x == 1 || x == n - 1) continue;
18        bool ok = false;
19        for (int i = 0; i < s - 1; ++i) {
20            x = ll((i128)x * x % n); // potential overflow!
21            if (x == n - 1) {
22                ok = true;
23                break;
24            }
25        }
26        if (!ok) return false;
27    }
28    return true;
29 }
30
31 ll pollard_rho(ll x) {
32    ll s = 0, t = 0, c = rng() % (x - 1) + 1;
33    ll stp = 0, goal = 1, val = 1;
34    for (goal = 1;; goal *= 2, s = t, val = 1) {
35        for (stp = 1; stp <= goal; ++stp) {
36            t = ll(((i128)t * t + c) % x);
37            val = ll(((i128)val * abs(t - s) % x);
38            if ((stp % 127) == 0) {
39                ll d = gcd(val, x);
40                if (d > 1) return d;
41            }
42        }
43        ll d = gcd(val, x);
44        if (d > 1) return d;
45    }
46 }
47
48 ll get_max_factor(ll _x) {
49     ll max_factor = 0;
50     function<void(ll)> fac = [&](ll x) {
51         if (x <= max_factor || x < 2) return;
52         if (is_prime(x)) {
53             max_factor = max_factor > x ? max_factor : x;
54             return;
55         }
56         ll p = x;
57         while (p >= x) p = pollard_rho(x);
58         while ((x % p) == 0) x /= p;
59         fac(x), fac(p);
60     };
61     fac(_x);
62     return max_factor;
63 }

```

Modular Square Root

- $O(\log^2 p)$ in worst case, typically $O(\log p)$ for most p

```

1 ll sqrt(ll a, ll p) {
2     a %= p; if (a < 0) a += p;
3     if (a == 0) return 0;
4     assert(pow(a, (p-1)/2, p) == 1); // else no solution
5     if (p % 4 == 3) return pow(a, (p+1)/4, p);
6     // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if p % 8 == 5
7     ll s = p - 1, n = 2;
8     int r = 0, m;
9     while (s % 2 == 0)
10         ++r, s /= 2;
11     // find a non-square mod p
12     while (pow(n, (p - 1) / 2, p) != p - 1) ++n;
13     ll x = pow(a, (s + 1) / 2, p);
14     ll b = pow(a, s, p), g = pow(n, s, p);
15     for (; r = m) {
16         ll t = b;
17         for (m = 0; m < r && t != 1; ++m)
18             t = t * t % p;
19         if (m == 0) return x;
20         ll gs = pow(g, 1LL << (r - m - 1), p);
21         g = gs * gs % p;
22         x = x * gs % p;
23         b = b * g % p;
24     }
25 }

```

Berlekamp-Massey

- Recovers any n -order linear recurrence relation from the first $2n$ terms of the sequence.
- Input s is the sequence to be analyzed.
- Output c is the shortest sequence c_1, \dots, c_n , such that

$$s_m = \sum_{i=1}^n c_i \cdot s_{m-i}, \text{ for all } m \geq n.$$

- Be careful since c is returned in 0-based indexation.
- Complexity: $O(N^2)$

```

1 vll berlekamp_massey(vll s) {
2     int n = sz(s), l = 0, m = 1;
3     vll b(n), c(n);
4     ll ldd = b[0] = c[0] = 1;
5     for (int i = 0; i < n; i++, m++) {
6         ll d = s[i];
7         for (int j = 1; j <= l; j++) d = (d + c[j] * s[i - j]) %
8             MOD;
9         if (d == 0) continue;
10        vll temp = c;
11        ll coef = d * power(ldd, MOD - 2) % MOD;
12        for (int j = m; j < n; j++){
13            c[j] = (c[j] + MOD - coef * b[j - m]) % MOD;
14            if (c[j] < 0) c[j] += MOD;
15        }
16        if (2 * l <= i) {
17            l = i + 1 - l;
18            b = temp;
19            ldd = d;
20            m = 0;
21        }
22    }
23    c.resize(l + 1);
24    c.erase(c.begin());
25    for (ll &x : c)
26        x = (MOD - x) % MOD;
27    return c;
28 }

```

Calculating k-th term of a linear recurrence

- Given the first n terms s_0, s_1, \dots, s_{n-1} and the sequence c_1, c_2, \dots, c_n such that

$$s_m = \sum_{i=1}^n c_i \cdot s_{m-i}, \text{ for all } m \geq n,$$

the function `calc_kth` computes s_k .

- Complexity: $O(n^2 \log k)$

```

1 vll poly_mult_mod(vll p, vll q, vll& c){
2     vll ans(sz(p) + sz(q) - 1);
3     forn(i, sz(p)){
4         forn(j, sz(q)){
5             ans[i + j] = (ans[i + j] + p[i] * q[j]) % MOD;
6         }
7     }
8     int n = sz(ans), m = sz(c);
9     for (int i = n - 1; i >= m; i--){
10        forn(j, m){
11            ans[i - 1 - j] = (ans[i - 1 - j] + c[j] * ans[i]) % MOD;
12        }
13    }
14    ans.resize(m);
15    return ans;
16 }
17
18 ll calc_kth(vll s, vll c, ll k){
19     assert(sz(s) >= sz(c)); // size of s can be greater than c,
20     // but not less
21     if (k < sz(s)) return s[k];
22     vll res{1};
23     for (vll poly = {0, 1}; k; poly = poly_mult_mod(poly, poly,
24         // c, k >= 1){
25         if (k & 1) res = poly_mult_mod(res, poly, c);
26     }
27     ll ans = 0;
28     forn(i, min(sz(res), sz(c))) ans = (ans + s[i] * res[i]) %
29         MOD;
30     return ans;
31 }

```

Partition Function

- Returns number of partitions of n in $O(n^{1.5})$

```

1 int partition(int n) {
2     int dp[n + 1];
3     dp[0] = 1;
4     for (int i = 1; i <= n; i++) {
5         dp[i] = 0;
6         for (int j = 1, r = 1; i - (3 * j * j - j) / 2 >= 0; ++j,
7             // r **=-1) {
8             dp[i] += dp[i - (3 * j * j - j) / 2] * r;
9             if (i - (3 * j * j + j) / 2 >= 0) dp[i] += dp[i - (3 * j
10                 // * j + j) / 2] * r;
11         }
12     }
13     return dp[n];
14 }

```

NTT

- large mod (for NTT to do FFT in ll range without modulo)

```
1 constexpr i128 MOD = 9223372036737335297;
```

- Otherwise, use below

```

1 const int MOD = 998244353;
2 void ntt(vll& a, int f) {
3     int n = int(a.size());
4     vll w(n);

```



```

5   vi rev(n);
6   forn(i, n) rev[i] = (rev[i / 2] / 2) | ((i & 1) * (n / 2));
7   forn(i, n) {
8       if (i < rev[i]) swap(a[i], a[rev[i]]);
9   }
10  ll wn = power(f ? (MOD + 1) / 3 : 3, (MOD - 1) / n);
11  w[0] = 1;
12  for (int i = 1; i < n; i++) w[i] = w[i - 1] * wn % MOD;
13  for (int mid = 1; mid < n; mid *= 2) {
14      for (int i = 0; i < n; i += 2 * mid) {
15          forn(j, mid) {
16              ll x = a[i + j], y = a[i + j + mid] * w[n / (2 * mid)
↪ * j] % MOD;
17              a[i + j] = (x + y) % MOD, a[i + j + mid] = (x + MOD -
↪ y) % MOD;
18          }
19      }
20  }
21  if (f) {
22      ll iv = power(n, MOD - 2);
23      for (auto& x : a) x = x * iv % MOD;
24  }
25  }
26  vll mul(vll a, vll b) {
27      int n = 1, m = (int)a.size() + (int)b.size() - 1;
28      while (n < m) n *= 2;
29      a.resize(n), b.resize(n);
30      ntt(a, 0), ntt(b, 0); // if squaring, you can save one NTT
↪ here
31      forn(i, n) a[i] = a[i] * b[i] % MOD;
32      ntt(a, 1);
33      a.resize(m);
34      return a;
35  }

```

FFT

```

1  const ld PI = acos(-1);
2  auto mul = [&](const vector<ld>& aa, const vector<ld>& bb) {
3      int n = (int)aa.size(), m = (int)bb.size(), bit = 1;
4      while ((1 << bit) < n + m - 1) bit++;
5      int len = 1 << bit;
6      vector<complex<ld>> a(len), b(len);
7      vi rev(len);
8      forn(i, n) a[i].real(aa[i]);
9      forn(i, m) b[i].real(bb[i]);
10     forn(i, len) rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << (bit
↪ - 1));
11     auto fft = [&](vector<complex<ld>>& p, int inv) {
12         forn(i, len)
13             if (i < rev[i]) swap(p[i], p[rev[i]]);
14         for (int mid = 1; mid < len; mid *= 2) {
15             auto w1 = complex<ld>(cos(PI / mid), (inv ? -1 : 1) *
↪ sin(PI / mid));
16             for (int i = 0; i < len; i += mid * 2) {
17                 auto wk = complex<ld>(1, 0);
18                 for (int j = 0; j < mid; j++, wk = wk * w1) {
19                     auto x = p[i + j], y = wk * p[i + j + mid];
20                     p[i + j] = x + y, p[i + j + mid] = x - y;
21                 }
22             }
23         }
24         if (inv == 1) {
25             forn(i, len) p[i].real(p[i].real() / len);
26         }
27     };
28     fft(a, 0), fft(b, 0);
29     forn(i, len) a[i] = a[i] * b[i];
30     fft(a, 1);
31     a.resize(n + m - 1);
32     vector<ld> res(n + m - 1);
33     forn(i, n + m - 1) res[i] = a[i].real();
34     return res;
35 };

```

Poly mod, log, exp, multipoint, interpolation

- $\frac{1}{P(x)}$ in $O(n \log n)$, $e^{P(x)}$ in $O(n \log n)$, $\ln(P(x))$ in $O(n \log n)$, $P(x)^k$ in $O(n \log n)$, Evaluates $P(x_1), \dots, P(x_n)$ in $O(n \log^2 n)$, Lagrange Interpolation in $O(n \log^2 n)$

```

1  // Examples:
2  // poly a(n+1); // constructs degree n poly
3  // a[0].v = 10; // assigns constant term a_0 = 10
4  // poly b = exp(a);
5  // poly is vector<num>
6  // for NTT, num stores just one int named v
7
8  #define sz(x) ((int)x.size())
9  #define rep(i, j, k) for (int i = int(j); i < int(k); i++)
10 #define per(i, a, b) for (int i = (b)-1; i >= (a); --i)
11 using vi = vi;
12
13 const int MOD = 998244353, g = 3;
14
15 // NTT
16 // For p < 2^30 there is also (5 << 25, 3), (7 << 26, 3),
17 // (479 << 21, 3) and (483 << 21, 5). Last two are > 10^9.
18 struct num {
19     int v;
20     num(ll v_ = 0): v(int(v_ % MOD)) {
21         if (v < 0) v += MOD;
22     }
23     explicit operator int() const { return v; }
24 };
25 inline num operator+(num a, num b) { return num(a.v + b.v); }
26 inline num operator-(num a, num b) { return num(a.v + MOD -
↪ b.v); }
27 inline num operator*(num a, num b) { return num(1ll * a.v *
↪ b.v); }
28 inline num pow(num a, int b) {
29     num r = 1;
30     do {
31         if (b & 1) r = r * a;
32         a = a * a;
33     } while (b >>= 1);
34     return r;
35 }
36 inline num inv(num a) { return pow(a, MOD - 2); }
37 using vn = vector<num>;
38 vi rev({0, 1});
39 vn rt(2, num(1)), fa, fb;
40 inline void init(int n) {
41     if (n <= sz(rt)) return;
42     rev.resize(n);
43     rep(i, 0, n) rev[i] = (rev[i >> 1] | ((i & 1) * n)) >> 1;
44     rt.reserve(n);
45     for (int k = sz(rt); k < n; k *= 2) {
46         rt.resize(2 * k);
47         num z = pow(num(g), (MOD - 1) / (2 * k)); // NTT
48         rep(i, k / 2, k) rt[2 * i] = rt[i], rt[2 * i + 1] = rt[i]
↪ * z;
49     }
50 }
51 inline void fft(vector<num>& a, int n) {
52     init(n);
53     int s = __builtin_ctz(sz(rev) / n);
54     rep(i, 0, n) if (i < rev[i] >> s) swap(a[i], a[rev[i] >>
↪ s]);
55     for (int k = 1; k < n; k *= 2)
56         for (int i = 0; i < n; i += 2 * k) rep(j, 0, k) {
57             num t = rt[j + k] * a[i + j + k];
58             a[i + j + k] = a[i + j] - t;
59             a[i + j] = a[i + j] + t;
60         }
61 }
62 // NTT
63 vn multiply(vn a, vn b) {
64     int s = sz(a) + sz(b) - 1;
65     if (s <= 0) return {};

```

```

66     int L = s > 1 ? 32 - __builtin_clz(s - 1) : 0, n = 1 << L;
67     a.resize(n), b.resize(n);
68     fft(a, n);
69     fft(b, n);
70     num d = inv(num(n));
71     rep(i, 0, n) a[i] = a[i] * b[i] * d;
72     reverse(a.begin() + 1, a.end());
73     fft(a, n);
74     a.resize(s);
75     return a;
76 }
77 // NTT power-series inverse
78 // Doubles b as b[:n] = (2 - a[:n] * b[:n/2]) * b[:n/2]
79 vn inverse(const vn& a) {
80     if (a.empty()) return {};
81     vn b({inv(a[0])});
82     b.reserve(2 * a.size());
83     while (sz(b) < sz(a)) {
84         int n = 2 * sz(b);
85         b.resize(2 * n, 0);
86         if (sz(fa) < 2 * n) fa.resize(2 * n);
87         fill(fa.begin(), fa.begin() + 2 * n, 0);
88         copy(a.begin(), a.begin() + min(n, sz(a)), fa.begin());
89         fft(b, 2 * n);
90         fft(fa, 2 * n);
91         num d = inv(num(2 * n));
92         rep(i, 0, 2 * n) b[i] = b[i] * (2 - fa[i] * b[i]) * d;
93         reverse(b.begin() + 1, b.end());
94         fft(b, 2 * n);
95         b.resize(n);
96     }
97     b.resize(a.size());
98     return b;
99 }
100
101 using poly = vn;
102
103 poly operator+(const poly& a, const poly& b) {
104     poly r = a;
105     if (sz(r) < sz(b)) r.resize(b.size());
106     rep(i, 0, sz(b)) r[i] = r[i] + b[i];
107     return r;
108 }
109
110 poly operator-(const poly& a, const poly& b) {
111     poly r = a;
112     if (sz(r) < sz(b)) r.resize(b.size());
113     rep(i, 0, sz(b)) r[i] = r[i] - b[i];
114     return r;
115 }
116
117 poly operator*(const poly& a, const poly& b) {
118     return multiply(a, b);
119 }
120
121 // Polynomial floor division; no leading 0's please
122 poly operator/(poly a, poly b) {
123     if (sz(a) < sz(b)) return {};
124     int s = sz(a) - sz(b) + 1;
125     reverse(a.begin(), a.end());
126     reverse(b.begin(), b.end());
127     a.resize(s);
128     b.resize(s);
129     a = a * inverse(move(b));
130     a.resize(s);
131     reverse(a.begin(), a.end());
132     return a;
133 }
134
135 poly operator%(const poly& a, const poly& b) {
136     poly r = a;
137     if (sz(r) >= sz(b)) {
138         poly c = (r / b) * b;
139         r.resize(sz(b) - 1);
140         rep(i, 0, sz(r)) r[i] = r[i] - c[i];
141     }
142     return r;
143 }
144
145 // Log/exp/pow
146 poly deriv(const poly& a) {
147     if (a.empty()) return {};
148     poly b(sz(a) - 1);
149     rep(i, 1, sz(a)) b[i - 1] = a[i] * i;
150     return b;
151 }
152
153 poly integ(const poly& a) {
154     poly b(sz(a) + 1);
155     b[1] = 1; // mod p
156     rep(i, 2, sz(b)) b[i] =
157         b[MOD % i] * (-MOD / i); // mod p
158     rep(i, 1, sz(b)) b[i] = a[i - 1] * b[i]; // mod p
159     //rep(i, 1, sz(b)) b[i] = a[i - 1] * inv(num(i)); // else
160     return b;
161 }
162
163 poly log(const poly& a) { // MUST have a[0] == 1
164     poly b = integ(deriv(a) * inverse(a));
165     b.resize(a.size());
166     return b;
167 }
168
169 poly exp(const poly& a) { // MUST have a[0] == 0
170     poly b(1, num(1));
171     if (a.empty()) return b;
172     while (sz(b) < sz(a)) {
173         int n = min(sz(b) * 2, sz(a));
174         b.resize(n);
175         poly v = poly(a.begin(), a.begin() + n) - log(b);
176         v[0] = v[0] + num(1);
177         b = b * v;
178         b.resize(n);
179     }
180     return b;
181 }
182
183 poly pow(const poly& a, int m) { // m >= 0
184     poly b(a.size());
185     if (!m) {
186         b[0] = 1;
187         return b;
188     }
189     int p = 0;
190     while (p < sz(a) && a[p].v == 0) ++p;
191     if (1ll * m * p >= sz(a)) return b;
192     num mu = pow(a[p], m), di = inv(a[p]);
193     poly c(sz(a) - m * p);
194     rep(i, 0, sz(c)) c[i] = a[i + p] * di;
195     c = log(c);
196     for(auto &v : c) v = v * m;
197     c = exp(c);
198     rep(i, 0, sz(c)) b[i + m * p] = c[i] * mu;
199     return b;
200 }
201
202 // Multipoint evaluation/interpolation
203
204 vector<num> eval(const poly& a, const vector<num>& x) {
205     int n = sz(x);
206     if (!n) return {};
207     vector<poly> up(2 * n);
208     rep(i, 0, n) up[i + n] = poly({0 - x[i], 1});
209     per(i, 1, n) up[i] = up[2 * i] * up[2 * i + 1];
210     vector<poly> down(2 * n);
211     down[1] = a % up[1];
212     rep(i, 2, 2 * n) down[i] = down[i / 2] % up[i];
213     vector<num> y(n);
214     rep(i, 0, n) y[i] = down[i + n][0];
215     return y;
216 }
217
218 poly interp(const vector<num>& x, const vector<num>& y) {
219     int n = sz(x);
220     assert(n);
221     vector<poly> up(n * 2);
222     rep(i, 0, n) up[i + n] = poly({0 - x[i], 1});
223     per(i, 1, n) up[i] = up[2 * i] * up[2 * i + 1];
224     vector<num> a = eval(deriv(up[1]), x);
225     vector<poly> down(2 * n);
226     rep(i, 0, n) down[i + n] = poly({y[i] * inv(a[i])});
227     per(i, 1, n) down[i] =

```



```

220     down[i * 2] * up[i * 2 + 1] + down[i * 2 + 1] * up[i * 2];
221     return down[1];
222 }

```

Simplex method for linear programs

- Maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$.
- Returns $-\infty$ if there is no solution, $+\infty$ if there are arbitrarily good solutions, or the maximum value of $c^T x$ otherwise. The (arbitrary) input vector is set to an optimal x (or in the unbounded case, an arbitrary solution fulfilling the constraints). Numerical stability is not guaranteed. For better performance, define variables such that $x = 0$ is viable.
- Complexity: $O(NM \cdot \text{pivots})$. $O(2^n)$ in general (very hard to achieve).

```

1  typedef double T; // might be much slower with long doubles
2  typedef vector<T> vd;
3  typedef vector<vd> vvd;
4  const T eps = 1e-8, inf = 1/.0;
5  #define MP make_pair
6  #define ltj(X) if(s == -1 || MP(X[j], N[j]) < MP(X[s], N[s]))
7  ↪ s = j
8  #define rep(i, a, b) for(int i = a; i < (b); ++i)
9
10 struct LPSolver {
11     int m, n;
12     vi N, B;
13     vvd D;
14     LPSolver(const vvd& A, const vd& b, const vd& c) : m(sz(b)),
15     ↪ n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)){
16         rep(i, 0, m) rep(j, 0, n) D[i][j] = A[i][j];
17         rep(i, 0, m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; }
18         ↪ rep(j, 0, n) { N[j] = j; D[m][j] = -c[j]; }
19         N[n] = -1; D[m+1][n] = 1;
20     };
21     void pivot(int r, int s){
22         T *a = D[r].data(), inv = 1 / a[s];
23         rep(i, 0, m+2) if (i != r && abs(D[i][s]) > eps) {
24             T *b = D[i].data(), inv2 = b[s] * inv;
25             rep(j, 0, n+2) b[j] -= a[j] * inv2;
26             b[s] = a[s] * inv2;
27         }
28         rep(j, 0, n+2) if (j != s) D[r][j] *= inv;
29         rep(i, 0, m+2) if (i != r) D[i][s] *= -inv;
30         D[r][s] = inv;
31         swap(B[r], N[s]);
32     }
33     bool simplex(int phase){
34         int x = m + phase - 1;
35         for (;;) {
36             int s = -1;
37             rep(j, 0, n+1) if (N[j] != -phase) ltj(D[x]); if (D[x][s]
38             ↪ >= -eps) return true;
39             int r = -1;
40             rep(i, 0, m) {
41                 if (D[i][s] <= eps) continue;
42                 if (r == -1 || MP(D[i][n+1] / D[i][s], B[i]) <
43                 ↪ MP(D[r][n+1] / D[r][s], B[r])) r = i;
44             }
45             if (r == -1) return false;
46             pivot(r, s);
47         }
48     }
49     T solve(vd &x){
50         int r = 0;
51         rep(i, 1, m) if (D[i][n+1] < D[r][n+1]) r = i;
52         if (D[r][n+1] < -eps) {
53             pivot(r, n);
54             if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
55             rep(i, 0, m) if (B[i] == -1) {
56                 int s = 0;
57                 rep(j, 1, n+1) ltj(D[i]);

```

```

53     pivot(i, s);
54 }
55 }
56 bool ok = simplex(1); x = vd(n);
57 rep(i, 0, m) if (B[i] < n) x[B[i]] = D[i][n+1];
58 return ok ? D[m][n+1] : inf;
59 }
60 };

```

Matroid Intersection

- Matroid is a pair $\langle X, I \rangle$, where X is a finite set and I is a family of subsets of X satisfying:
 1. $\emptyset \in I$.
 2. If $A \in I$ and $B \subseteq A$, then $B \in I$.
 3. If $A, B \in I$ and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\} \in I$.
- Set S is called **independent** if $S \in I$.
- **Common matroids:** uniform (sets of bounded size); colorful (sets of colored elements where each color only appears once); graphic (acyclic sets of edges in a graph); linear-algebraic (sets of linearly independent vectors).
- **Matroid Intersection Problem:** Given two matroids, find the largest common independent set.
- A matroid has 3 functions:
 - *check(int x)*: returns if current matroid can add x without becoming dependent.
 - *add(int x)*: adds an element to the matroid (guaranteed to never make it dependent).
 - *clear()*: sets the matroid to the empty matroid.
- The matroid is given an *int* representing the element, and is expected to convert it (e.g: color or edge endpoints)
- Pass the matroid with more expensive add/clear operations to M1.
- **Complexity:** $R^2 \cdot N \cdot (M2.add + M1.check + M2.check) + R^3 \cdot (M1.add) + R^2 \cdot (M1.clear) + R \cdot N \cdot (M2.clear)$, where $R = \text{answer}$.

```

1
2 // Example matroid
3 struct GraphicMatroid{
4     vector<pair<int, int>> e;
5     int n;
6     DSU dsu;
7
8     GraphicMatroid(vector<pair<int, int>> edges, int vertices){
9         e = edges, n = vertices;
10        dsu = DSU(n);
11    };
12    bool check(int idx){
13        return !dsu.same(e[idx].fi, e[idx].se);
14    }
15    void add(int idx){
16        dsu.unite(e[idx].fi, e[idx].se);
17    }
18    void clear(){
19        dsu = DSU(n);
20    }
21 };
22
23 template <class M1, class M2> struct MatroidIsect {
24     int n;
25     vector<char> iset;
26     M1 m1; M2 m2;
27     MatroidIsect(M1 m1, M2 m2, int n) : n(n), iset(n + 1),
28     ↪ m1(m1), m2(m2) {}
29     vi solve() {
30         forn(i, n) if (m1.check(i) && m2.check(i))
31             iset[i] = true, m1.add(i), m2.add(i);
32         while (augment());

```

```

32     vi ans;
33     forn(i, n) if (iset[i]) ans.push_back(i);
34     return ans;
35 }
36 bool augment() {
37     vi frm(n, -1);
38     queue<int> q({n}); // starts at dummy node
39     auto fwdE = [&](int a) {
40         vi ans;
41         m1.clear();
42         for (int v = 0; v < n; v++) if (iset[v] && v != a)
43             m1.add(v);
44         for (int b = 0; b < n; b++) if (!iset[b] && frm[b]
45             == -1 && m1.check(b))
46             ans.push_back(b), frm[b] = a;
47         return ans;
48     };
49     auto backE = [&](int b) {
50         m2.clear();
51         for (int cas = 0; cas < 2; cas++) for (int v = 0;
52             v < n; v++){
53             if ((v == b || iset[v]) && (frm[v] == -1) ==
54             cas) {
55                 if (!m2.check(v))
56                     return cas ? q.push(v), frm[v] = b, v
57                     : -1;
58                 m2.add(v);
59             }
60         }
61         return n;
62     };
63     while (!q.empty()) {
64         int a = q.front(), c; q.pop();
65         for (int b : fwdE(a))
66             while((c = backE(b)) >= 0) if (c == n) {
67                 while (b != n) iset[b] ^= 1, b = frm[b];
68                 return true;
69             }
70     }
71     return false;
72 }
73
74 /*
75 Usage:
76 MatroidIsect<GraphicMatroid, ColorfulMatroid> solver(matroid1,
77     matroid2, n);
78 vi answer = solver.solve();
79 */

```

Data Structures

Fenwick Tree

```

1 ll sum(int r) {
2     ll ret = 0;
3     for (; r >= 0; r = (r & r + 1) - 1) ret += bit[r];
4     return ret;
5 }
6 void add(int idx, ll delta) {
7     for (; idx < n; idx |= idx + 1) bit[idx] += delta;
8 }

```

Lazy Propagation SegTree

```

1 // Clear: clear() or build()
2 const int N = 2e5 + 10; // Change the constant!
3 template<typename T>
4 struct LazySegTree{
5     T t[4 * N];
6     T lazy[4 * N];
7     int n;
8
9     // Change these functions, default return, and lazy mark.
10    T default_return = 0, lazy_mark = numeric_limits<T>::min();

```

```

11    // Lazy mark is how the algorithm will identify that no
12    // propagation is needed.
13    function<T(T, T)> f = [&] (T a, T b){
14        return a + b;
15    };
16    // f_on_seg calculates the function f, knowing the lazy
17    // value on segment,
18    // segment's size and the previous value.
19    // The default is segment modification for RSQ. For
20    // increments change to:
21    // return cur_seg_val + seg_size * lazy_val;
22    // For RMQ. Modification: return lazy_val; Increments:
23    // return cur_seg_val + lazy_val;
24    function<T(T, int, T)> f_on_seg = [&] (T cur_seg_val, int
25    seg_size, T lazy_val){
26        return seg_size * lazy_val;
27    };
28    // upd_lazy updates the value to be propagated to child
29    // segments.
30    // Default: modification. For increments change to:
31    // lazy[v] = (lazy[v] == lazy_mark? val : lazy[v] +
32    // val);
33    function<void(int, T)> upd_lazy = [&] (int v, T val){
34        lazy[v] = val;
35    };
36    // Tip: for "get element on single index" queries, use max()
37    // on segment: no overflows.
38
39    LazySegTree(int n_) : n(n_) {
40        clear(n);
41    }
42
43    void build(int v, int tl, int tr, vector<T>& a){
44        if (tl == tr) {
45            t[v] = a[tl];
46            return;
47        }
48        int tm = (tl + tr) / 2;
49        // left child: [tl, tm]
50        // right child: [tm + 1, tr]
51        build(2 * v + 1, tl, tm, a);
52        build(2 * v + 2, tm + 1, tr, a);
53        t[v] = f(t[2 * v + 1], t[2 * v + 2]);
54    }
55
56    LazySegTree(vector<T>& a){
57        build(a);
58    }
59
60    void push(int v, int tl, int tr){
61        if (lazy[v] == lazy_mark) return;
62        int tm = (tl + tr) / 2;
63        t[2 * v + 1] = f_on_seg(t[2 * v + 1], tm - tl + 1,
64        lazy[v]);
65        t[2 * v + 2] = f_on_seg(t[2 * v + 2], tr - tm, lazy[v]);
66        upd_lazy(2 * v + 1, lazy[v]), upd_lazy(2 * v + 2,
67        lazy[v]);
68        lazy[v] = lazy_mark;
69    }
70
71    void modify(int v, int tl, int tr, int l, int r, T val){
72        if (l > r) return;
73        if (tl == l && tr == r){
74            t[v] = f_on_seg(t[v], tr - tl + 1, val);
75            upd_lazy(v, val);
76            return;
77        }
78        push(v, tl, tr);
79        int tm = (tl + tr) / 2;
80        modify(2 * v + 1, tl, tm, l, min(r, tm), val);
81        modify(2 * v + 2, tm + 1, tr, max(l, tm + 1), r, val);
82        t[v] = f(t[2 * v + 1], t[2 * v + 2]);
83    }
84
85    T query(int v, int tl, int tr, int l, int r) {
86        if (l > r) return default_return;
87        if (tl == l && tr == r) return t[v];

```

```

78     push(v, tl, tr);
79     int tm = (tl + tr) / 2;
80     return f(
81         query(2 * v + 1, tl, tm, l, min(r, tm)),
82         query(2 * v + 2, tm + 1, tr, max(l, tm + 1), r)
83     );
84 }
85
86 void modify(int l, int r, T val){
87     modify(0, 0, n - 1, l, r, val);
88 }
89
90 T query(int l, int r){
91     return query(0, 0, n - 1, l, r);
92 }
93
94 T get(int pos){
95     return query(pos, pos);
96 }
97
98 // Change clear() function to t.clear() if using
99 ↪ unordered_map for SegTree!!!
100 void clear(int n_){
101     n = n_;
102     forn(i, 4 * n) t[i] = 0, lazy[i] = lazy_mark;
103 }
104
105 void build(vector<T>& a){
106     n = sz(a);
107     clear(n);
108     build(0, 0, n - 1, a);
109 }

```

Sparse Table

```

1  const int N = 2e5 + 10, LOG = 20; // Change the constant!
2  template<typename T>
3  struct SparseTable{
4      int lg[N];
5      T st[N][LOG];
6      int n;
7
8      // Change this function
9      function<T(T, T)> f = [&] (T a, T b){
10         return min(a, b);
11     };
12
13     void build(vector<T>& a){
14         n = sz(a);
15         lg[1] = 0;
16         for (int i = 2; i <= n; i++) lg[i] = lg[i / 2] + 1;
17
18         for (int k = 0; k < LOG; k++){
19             forn(i, n){
20                 if (!k) st[i][k] = a[i];
21                 else st[i][k] = f(st[i][k - 1], st[min(n - 1, i + (1 <<
22 ↪ (k - 1)))][k - 1]);
23             }
24         }
25
26         T query(int l, int r){
27             int sz = r - l + 1;
28             return f(st[l][lg[sz]], st[r - (1 << lg[sz]) + 1][lg[sz]]);
29         }
30     };

```

Suffix Array and LCP array

- (uses SparseTable above)

```

1  struct SuffixArray{
2      vi p, c, h;
3      SparseTable<int> st;
4      /*

```

In the end, array c gives the position of each suffix in p using 1-based indexation!

```

7      */
8
9      SuffixArray() {}
10
11     SuffixArray(string s){
12         buildArray(s);
13         buildLCP(s);
14         buildSparse();
15     }
16
17     void buildArray(string s){
18         int n = sz(s) + 1;
19         p.resize(n), c.resize(n);
20         forn(i, n) p[i] = i;
21         sort(all(p), [&] (int a, int b){return s[a] < s[b];});
22         c[p[0]] = 0;
23         for (int i = 1; i < n; i++){
24             c[p[i]] = c[p[i - 1]] + (s[p[i]] != s[p[i - 1]]);
25         }
26         vi p2(n), c2(n);
27         // w is half-length of each string.
28         for (int w = 1; w < n; w <= 1){
29             forn(i, n){
30                 p2[i] = (p[i] - w + n) % n;
31             }
32             vi cnt(n);
33             for (auto i : c) cnt[i]++;
34             for (int i = 1; i < n; i++) cnt[i] += cnt[i - 1];
35             for (int i = n - 1; i >= 0; i--){
36                 p[--cnt[c[p2[i]]]] = p2[i];
37             }
38             c2[p[0]] = 0;
39             for (int i = 1; i < n; i++){
40                 c2[p[i]] = c2[p[i - 1]] +
41                     (c[p[i]] != c[p[i - 1]] ||
42                     c[(p[i] + w) % n] != c[(p[i - 1] + w) % n]);
43             }
44             c.swap(c2);
45         }
46         p.erase(p.begin());
47     }
48
49     void buildLCP(string s){
50         // The algorithm assumes that suffix array is already
51 ↪ built on the same string.
52         int n = sz(s);
53         h.resize(n - 1);
54         int k = 0;
55         forn(i, n){
56             if (c[i] == n){
57                 k = 0;
58                 continue;
59             }
60             int j = p[c[i]];
61             while (i + k < n && j + k < n && s[i + k] == s[j + k])
62 ↪ k++;
63             h[c[i] - 1] = k;
64             if (k) k--;
65         }
66         /*
67         Then an RMQ Sparse Table can be built on array h
68         to calculate LCP of 2 non-consecutive suffixes.
69         */
70     }
71
72     void buildSparse(){
73         st.build(h);
74     }
75
76     // l and r must be in 0-BASED INDEXATION
77     int lcp(int l, int r){
78         l = c[l] - 1, r = c[r] - 1;
79         if (l > r) swap(l, r);
80         return st.query(l, r - 1);
81     }

```

```
80 };
```

Aho Corasick Trie

- For each node in the trie, the suffix link points to the longest proper suffix of the represented string. The terminal-link tree has square-root height (can be constructed by DFS).

```
1  const int S = 26;
2
3  // Function converting char to int.
4  int ctoi(char c){
5      return c - 'a';
6  }
7
8  // To add terminal links, use DFS
9  struct Node{
10     vi nxt;
11     int link;
12     bool terminal;
13
14     Node() {
15         nxt.assign(S, -1), link = 0, terminal = 0;
16     }
17 };
18
19 vector<Node> trie(1);
20
21 // add_string returns the terminal vertex.
22 int add_string(string& s){
23     int v = 0;
24     for (auto c : s){
25         int cur = ctoi(c);
26         if (trie[v].nxt[cur] == -1){
27             trie[v].nxt[cur] = sz(trie);
28             trie.emplace_back();
29         }
30         v = trie[v].nxt[cur];
31     }
32     trie[v].terminal = 1;
33     return v;
34 }
35
36 /*
37 Suffix links are compressed.
38 This means that:
39 If vertex v has a child by letter x, then:
40     trie[v].nxt[x] points to that child.
41 If vertex v doesn't have such child, then:
42     trie[v].nxt[x] points to the suffix link of that child
43     if we would actually have it.
44 */
45 void add_links(){
46     queue<int> q;
47     q.push(0);
48     while (!q.empty()){
49         auto v = q.front();
50         int u = trie[v].link;
51         q.pop();
52         for(i, S){
53             int& ch = trie[v].nxt[i];
54             if (ch == -1){
55                 ch = v? trie[u].nxt[i] : 0;
56             }
57             else{
58                 trie[ch].link = v? trie[u].nxt[i] : 0;
59                 q.push(ch);
60             }
61         }
62     }
63 }
64
65 bool is_terminal(int v){
66     return trie[v].terminal;
67 }
```

```
68
69 int get_link(int v){
70     return trie[v].link;
71 }
72
73 int go(int v, char c){
74     return trie[v].nxt[ctoi(c)];
75 }
```

Convex Hull Trick

- Allows to insert a linear function to the hull in (1) and get the minimum/maximum value of the stored function at a point in $O(\log n)$.
- NOTE: The lines must be added in the order of decreasing/increasing gradients. CAREFULLY CHECK THE SETUP BEFORE USING!
- IMPORTANT: DO NOT MODIFY TO QUERY MAX, IT WILL BREAK

```
1  struct line{
2      ll k, b;
3      ll f(ll x){
4          return k * x + b;
5      };
6  };
7
8  vector<line> hull;
9
10 void add_line(line nl){
11     if (!hull.empty() && hull.back().k == nl.k){
12         nl.b = min(nl.b, hull.back().b);
13         hull.pop_back();
14     }
15     while (sz(hull) > 1){
16         auto& l1 = hull.end()[-2], l2 = hull.back();
17         if ((nl.b - l1.b) * (l2.k - nl.k) >= (nl.b - l2.b) * (l1.k
18         ↪ - nl.k)) hull.pop_back();
19         else break;
20     }
21     hull.pb(nl);
22 }
23
24 ll get(ll x){
25     int l = 0, r = sz(hull);
26     while (r - l > 1){
27         int mid = (l + r) / 2;
28         if (hull[mid - 1].f(x) >= hull[mid].f(x)) l = mid;
29         else r = mid;
30     }
31     return hull[l].f(x);
32 }
```

Li-Chao Segment Tree

- allows to add linear functions in any order and query minimum/maximum value of those at a point, all in $O(\log n)$.
- Clear: clear()

```
1  const ll INF = 1e18; // Change the constant!
2  struct LiChaoTree{
3      struct line{
4          ll k, b;
5          line(){
6              k = b = 0;
7          };
8          line(ll k_, ll b_){
9              k = k_, b = b_;
10         };
11         ll f(ll x){
12             return k * x + b;
13         };
14     };
15 }
```

```

15     int n;
16     bool minimum, on_points;
17     vll pts;
18     vector<line> t;
19
20     void clear(){
21         for (auto& l : t) l.k = 0, l.b = minimum? INF : -INF;
22     }
23
24     LiChaoTree(int n_, bool min_){ // This is a default
↳ constructor for numbers in range [0, n - 1].
        n = n_, minimum = min_, on_points = false;
25         t.resize(4 * n);
26         clear();
27     };
28
29     LiChaoTree(vll pts_, bool min_){ // This constructor will
↳ build LCT on the set of points you pass. The points may be
↳ in any order and contain duplicates.
        pts = pts_, minimum = min_;
30         sort(all(pts));
31         pts.erase(unique(all(pts)), pts.end());
32         on_points = true;
33         n = sz(pts);
34         t.resize(4 * n);
35         clear();
36     };
37
38     void add_line(int v, int l, int r, line nl){
39         // Adding on segment [l, r)
40         int m = (l + r) / 2;
41         ll lval = on_points? pts[l] : 1, mval = on_points? pts[m]
↳ : m;
42         if ((minimum && nl.f(mval) < t[v].f(mval)) || (!minimum &&
↳ nl.f(mval) > t[v].f(mval))) swap(t[v], nl);
43         if (r - l == 1) return;
44         if ((minimum && nl.f(lval) < t[v].f(lval)) || (!minimum &&
↳ nl.f(lval) > t[v].f(lval))) add_line(2 * v + 1, l, m, nl);
45         else add_line(2 * v + 2, m, r, nl);
46     }
47
48     ll get(int v, int l, int r, int x){
49         int m = (l + r) / 2;
50         if (r - l == 1) return t[v].f(on_points? pts[x] : x);
51         else{
52             if (minimum) return min(t[v].f(on_points? pts[x] : x), x
↳ < m? get(2 * v + 1, l, m, x) : get(2 * v + 2, m, r, x));
53             else return max(t[v].f(on_points? pts[x] : x), x < m?
↳ get(2 * v + 1, l, m, x) : get(2 * v + 2, m, r, x));
54         }
55     }
56
57     void add_line(ll k, ll b){
58         add_line(0, 0, n, line(k, b));
59     }
60
61     ll get(ll x){
62         return get(0, 0, n, on_points? lower_bound(all(pts), x) -
↳ pts.begin() : x);
63     }; // Always pass the actual value of x, even if LCT is on
↳ points.
64 };
```

Persistent Segment Tree

- for RSQ

```

1 struct Node {
2     ll val;
3     Node *l, *r;
4
5     Node(ll x) : val(x), l(nullptr), r(nullptr) {}
6     Node(Node *ll, Node *rr) {
7         l = ll, r = rr;
8         val = 0;
9         if (l) val += l->val;
10        if (r) val += r->val;
```

```

11    }
12    Node(Node *cp) : val(cp->val), l(cp->l), r(cp->r) {}
13 };
14 const int N = 2e5 + 20;
15 ll a[N];
16 Node *roots[N];
17 int n, cnt = 1;
18 Node *build(int l = 1, int r = n) {
19     if (l == r) return new Node(a[l]);
20     int mid = (l + r) / 2;
21     return new Node(build(l, mid), build(mid + 1, r));
22 }
23 Node *update(Node *node, int val, int pos, int l = 1, int r =
↳ n) {
24     if (l == r) return new Node(val);
25     int mid = (l + r) / 2;
26     if (pos > mid)
27         return new Node(node->l, update(node->r, val, pos, mid +
↳ 1, r));
28     else return new Node(update(node->l, val, pos, l, mid),
↳ node->r);
29 }
30 ll query(Node *node, int a, int b, int l = 1, int r = n) {
31     if (l > b || r < a) return 0;
32     if (l >= a && r <= b) return node->val;
33     int mid = (l + r) / 2;
34     return query(node->l, a, b, l, mid) + query(node->r, a, b,
↳ mid + 1, r);
35 }
```

Dynamic Programming

Sum over Subset DP

- Computes $f[A] = \sum_{B \subseteq A} a[B]$.
- Complexity: $O(2^n \cdot n)$.

```

1 forn(i, (1 << n)) f[i] = a[i];
2 forn(i, n) for (int mask = 0; mask < (1 << n); mask++) if
↳ ((mask >> i) & 1){
3     f[mask] += f[mask ^ (1 << i)];
4 }
```

Divide and Conquer DP

- Helps to compute 2D DP of the form:
- $dp[i][j] = \min_{0 \leq k \leq j-1} (dp[i-1][k] + cost(k+1, j))$
- **Necessary condition:** let $opt(i, j)$ be the optimal k for the state (i, j) . Then, $opt(i, j) \leq opt(i, j+1)$.
- **Sufficient condition:** $cost(a, d) + cost(b, c) \geq cost(a, c) + cost(b, d)$ where $a < b < c < d$.
- Complexity: $O(M \cdot N \cdot \log N)$ for computing $dp[M][N]$.

```

1 vll dp_old(N), dp_new(N);
2
3 void rec(int l, int r, int optl, int optr){
4     if (l > r) return;
5     int mid = (l + r) / 2;
6     pair<ll, int> best = {INF, optl};
7     for (int i = optl; i <= min(mid - 1, optr); i++){ // If k
↳ can be j, change to "i <= min(mid, optr)".
8         ll cur = dp_old[i] + cost(i + 1, mid);
9         if (cur < best.fi) best = {cur, i};
10    }
11    dp_new[mid] = best.fi;
12
13    rec(l, mid - 1, optl, best.se);
14    rec(mid + 1, r, best.se, optr);
15 }
16
17 // Computes the DP "by layers"
18 fill(all(dp_old), INF);
19 dp_old[0] = 0;
```

```

20 while (layers--){
21     rec(0, n, 0, n);
22     dp_old = dp_new;
23 }

```

- Do something instead of nothing, stay organized
- Write stuff down!
- Don't get stuck on one approach!

Knuth's DP Optimization

- Computes DP of the form
- $dp[i][j] = \min_{i \leq k \leq j-1} (dp[i][k] + dp[k+1][j] + cost(i, j))$
- **Necessary Condition:** $opt(i, j-1) \leq opt(i, j) \leq opt(i+1, j)$
- **Sufficient Condition:** For $a \leq b \leq c \leq d$, $cost(b, c) \leq cost(a, d)$ AND $cost(a, d) + cost(b, c) \geq cost(a, c) + cost(b, d)$
- Complexity: $O(n^2)$

```

1 int N;
2 int dp[N][N], opt[N][N];
3 auto C = [&](int i, int j) {
4     // Implement cost function C.
5 };
6 forn(i, N) {
7     opt[i][i] = i;
8     // Initialize dp[i][i] according to the problem
9 }
10 for (int i = N-2; i >= 0; i--) {
11     for (int j = i+1; j < N; j++) {
12         int mn = INT_MAX;
13         int cost = C(i, j);
14         for (int k = opt[i][j-1]; k <= min(j-1, opt[i+1][j]); k++)
15             ↪ {
16                 if (mn >= dp[i][k] + dp[k+1][j] + cost) {
17                     opt[i][j] = k;
18                     mn = dp[i][k] + dp[k+1][j] + cost;
19                 }
20             }
21         dp[i][j] = mn;
22     }
23 }

```

Miscellaneous

Ordered Set

```

1 #include <ext/pb_ds/assoc_container.hpp>
2 #include <ext/pb_ds/tree_policy.hpp>
3 using namespace __gnu_pbds;
4 typedef tree<int, null_type, less<int>, rb_tree_tag,
5     ↪ tree_order_statistics_node_update> ordered_set;

```

Measuring Execution Time

```

1 ld tic = clock();
2 // execute algo...
3 ld tac = clock();
4 // Time in milliseconds
5 cerr << (tac - tic) / CLOCKS_PER_SEC * 1000 << endl;
6 // No need to comment out the print because it's done to cerr.

```

Setting Fixed D.P. Precision

```

1 cout << setprecision(d) << fixed;
2 // Each number is rounded to d digits after the decimal point,
3     ↪ and truncated.

```

Common Bugs and General Advice

- Check overflow, array bounds
- Check variable overloading
- Check special cases (n=1?)