Math Formula Sheet

Special Numbers

Primes Estimation:
$$\pi(n) \sim n/\ln(n),\ p_k \sim k \ln k$$

Partition Function Estimation $p(n)$:
$$p(n) \sim 13^{\sqrt{n}}/(7n)$$

Max Highly Composites Less than Powers of 10:
$(60, 12), (840, 32), (7560, 64), (83160, 128),$
$(720720, 240), (8648640, 448), (73513440, 768),$
$(735134400, 1344), (6983776800, 2304)$

Catalan Numbers:
$$C_n = \binom{2n}{n}/(n+1)$$
$$C(x) = (1 - \sqrt{1-4x})/(2x)$$

Fibonacci Numbers

Definition:
$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2$$
$$F(x) = x/(1 - x - x^2)$$

Closed Form:
$$F_n = (\phi^n - (1-\phi)^n)/\sqrt{5}$$

Matrix Exponentiation:
$$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$$

Zeckendorf's Theorem: Let $n \gg k$ denote $n \geq k + 2$. Using greedy, all positive integers can be expressed uniquely as the sum
$$n = F_{k_1} + F_{k_2} + F_{k_3} + \cdots + F_{k_r},$$
where $k_1 \gg k_2 \gg k_3 \gg \cdots \gg k_r \gg 0$.

Summation Properties:

- $\forall n \in \mathbb{Z}_{\geq 0} : \sum_{j=0}^{n} F_j = F_{n+2} - 1$
- $\forall n \geq 1, \sum_{j=0}^{n} F_{2j} = F_{2n+1} - 1$
- $\forall n \geq 1, \sum_{j=0}^{n} F_{2j-1} = F_{2n}$
- $\sum_{j=1}^{2n-1} F_j F_{j+1} = F_{2n}^2$
- $\sum_{j=1}^{2n} F_j F_{j+1} = F_{2n+1}^2 - 1$

Additive Rule:
$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

Cassini's Identity:
$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$$

Divisibility and GCD:
$$\forall m, n \in \mathbb{Z}_{>2} : m \mid n \iff F_m \mid F_n$$
$$\forall m, n \in \mathbb{Z}_{>2} : \gcd(F_m, F_n) = F_{\gcd(m,n)}$$

Combinatorics

Definition for Reals:
$$\forall r \in \mathbb{R}, k \in \mathbb{Z}, \binom{r}{k} = \prod_{j=1}^{k} \frac{r+1-j}{j}$$

Identities:

- $\binom{r}{k} = (r/k)\binom{r-1}{k-1}$
- $\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$
- $\sum_k \binom{r}{m+k}\binom{s}{n+k} = \binom{r+s}{r-m+n}$
- $\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} = \binom{s}{n-r}$
- $\sum_{k=0}^{r} \binom{r-k}{m}\binom{s}{k-t}(-1)^{k-t} = \binom{r-t-s}{r-t-m}$
- For $n \geq s$, $\sum_{k=0}^{r}\binom{r-k}{m}\binom{s+k}{n} = \binom{r+s+1}{m+n+1}$

Burnside's Lemma: Let $I(\pi)$ denote number of fixed points of a group element π. Then,
$$|\text{Classes}| = (1/|G|)\sum_{\pi \in G} I(\pi)$$

Polya Enumeration: Suppose each representation element can take on k distinct values. Let $C(\pi)$ count the number of cycles in the permutation π. Then,
$$|\text{Classes}| = (1/|G|)\sum_{\pi \in G} k^{C(\pi)}$$

Number of Labeled Unrooted Trees:

- on n vertices: n^{n-2}
- on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
- with degrees d_i : $(n-2)!/(\prod(d_i - 1)!)$

Number Theory

CRT: For pairwise coprimes m_i with $M = \prod m_i$,
$$M_i := \prod_{i \neq j} m_j \bmod M, N_i := M_i^{-1} \bmod m_i,$$
$$a \equiv \sum a_i M_i N_i \bmod M.$$

Wilson's Theorem:
$$(n-1)! \pmod n \equiv \begin{cases} -1, & n \text{ is prime} \\ 2, & n = 4 \\ 0, & \text{otherwise} \end{cases}$$

Mobius Inversion: For $f, g : \mathbb{Z}_+ \to \mathbb{C}$,
$$g(n) = \sum_{d|n} f(d),\ \forall n \in \mathbb{Z}_+$$
$$\iff f(n) = \sum_{d|n} \mu(d)g(n/d),\ \forall n \in \mathbb{Z}_+$$

Common Mobius Inversion Functions:

- $n = \sum_{d|n} \varphi(d)$
- $\varphi(d) = \sum_{d|n} \mu(d)n/d = \sum_{d|n} \mu(n/d)d$
- $[n == 1] = \sum_{d|n} \mu(d)$

Parametrization of Pythagorean Triples: For $m > n > 0, k > 0, \gcd(m,n) = 1$, and either m or n even, Pythagorean triples are uniquely generated by
$$a = k(m^2 - n^2), b = k(2mn),$$
$$c = k(m^2 + n^2)$$

Numerical and Linear Algebra

Error term E_S for integration by Simpson's Rule, where $\max|f^{(IV)}| \leq K$:
$$|E_S| \leq K(b-a)^5/(180n^4)$$

Newton's Method for roots of $f(x) = 0$:
$$x_{i+1} = x_i - f(x_i)/f'(x_i)$$

Lagrange Interpolating Polynomial: For n points (x_i, y_i), the $n-1$ degree polynomial is
$$P(x) = \sum_{j=1}^{n} y_j \prod_{k=1, k \neq j}^{n} (x - x_k)/(x_j - x_k)$$

Cramer's Rule: Let $A \in \mathbb{R}^{n \times n}$ with non-zero determinant, where $A\mathbf{x} = \mathbf{b}$. Let A_i be the matrix formed by replacing the i-th column of A by the column vector $\mathbf{b}$. Then,
$$x_i = \det(A_i)/\det(A).$$

Geometry

Pick's Theorem: Consider a lattice polygon. Let I be number of interior lattice points and B be number of boundary lattice points. Then,
$$\text{Area} = I + B/2 - 1.$$

General Geometry Tips:

- Comparing by polar angle: first check the half, $[0, \pi)$ or $[\pi, 2\pi)$, then the rotation.
- When calculating $\sin^{-1}$ or $\cos^{-1}$, make sure to bound the input to $[-1, +1]$.

Checking segment intersection for (A_1, B_1) and (A_2, B_2): check that projections on x and y axis intersect; check that
$$vmul(A_2 - A_1, B_1 - A_1) \cdot vmul(B_2 - A_1, B_1 - A_1) \leq 0$$
and
$$vmul(A_1 - A_2, B_2 - A_2) \cdot vmul(B_1 - A_2, B_2 - A_2) \leq 0.$$

Counterclockwise rotation:
$$(x, y) \mapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

Planar Graphs

Euler's Characteristic Formula: For n vertices, m edges, and f faces, and k connected components,
$$n - m + f = 1 + k.$$

Properties:

- If $n \geq 3$ then we must have $m \leq 3n - 6$. Equality when each face is bounded by a triangle.
- If $n \geq 3$ then we must have $f \leq 2n - 4$.
- Every planar graph has a vertex of degree 5 or less.