Math Formula Sheet

Special Numbers

Primes Estimation:
$\pi(n) \sim n/\ln(n)$, $p_k \sim k \ln k$

Partition Function Estimation $p(n)$:
$p(n) \sim 13^{\sqrt{n}}/(7n)$

Max Highly Composites Less than Powers of 10:
$(60, 12), (840, 32), (7560, 64), (83160, 128),$
$(720720, 240), (8648640, 448), (73513440, 768),$
$(735134400, 1344), (6983776800, 2304),$
$(97772875200, 4032), (963761198400, 6720),$
$(9316358251200, 1e4), (97821761637600, 1.7e4),$
$(866421317361600, 2.7e4), (8086598962041600, 4e4),$
$(74801040398884800, 6.5e4), (897612484786617600, 1e5)$

Product of First n primes:
$(1, 2), (2, 6), (3, 30), (4, 210), (5, 2310), (6, 3e4),$
$(7, 5e5), (8, 1e7), (9, 2e8), (10, 6e9), (11, 2e11),$
$(12, 7e12), (13, 3e14), (14, 1e16), (15, 6e17)$

Bernoulli numbers

EGF of Bernoulli numbers is (FFT-able) with
$B(t) = \frac{t}{e^t - 1}$
$B[0, \ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$

Sums of powers:
$\sum_{i=1}^{n} i^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k (n+1)^{m+1-k}$

Euler-Maclaurin formula for infinite sums:
$\sum_{i=m}^{\infty} f(i) = \int_m^\infty f(x)dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(m)$
$\approx \int_m^\infty f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$

Stirling numbers of the first kind

Number of permutations on n items with k cycles.
$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k)$, $c(0, 0) = 1$
$\sum_{k=0}^{n} c(n, k)x^k = x(x+1)\ldots(x+n-1)$
$c(8, k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$
$c(n, 2) = 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \ldots$

Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j's s.t. $\pi(j) > \pi(j+1)$, $k+1$ j's s.t. $\pi(j) \geq j$, k j's s.t. $\pi(j) > j$.
$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$
$E(n, 0) = E(n, n-1) = 1$
$E(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$

Stirling numbers of the second kind

Partitions of n distinct elements into exactly k groups
$S(n, k) = S(n-1, k-1) + kS(n-1, k)$
$S(n, 1) = S(n, n) = 1$
$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$

Bell numbers

Total number of partitions of n distinct elements.
$B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$
For p prime,
$B(p^m + n) \equiv mB(n) + B(n+1) \pmod p$

Catalan numbers

$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$
$C_0 = 1,\ C_{n+1} = \frac{2(2n+1)}{n+2} C_n,\ C_{n+1} = \sum C_i C_{n-i}$
$C(x) = (1 - \sqrt{1-4x})/(2x)$
$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

- sub-diagonal monotone paths in an $n \times n$ grid
- strings with n pairs of (), correctly nested
- binary trees with $n+1$ leaves (0 or 2 children)
- ordered trees with $n+1$ vertices
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines
- perms. of $[n]$ with no 3-term increasing subseq

Fibonacci Numbers

$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2$
$F(x) = x/(1 - x - x^2)$
$F_n = (\phi^n - (1-\phi)^n)/\sqrt{5}$
$$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$$
Zeckendorf's Theorem: Let $n \gg k$ denote $n \geq k+2$. Obtained via greedy, all positive integers can be expressed uniquely as the sum
$n = F_{k_1} + F_{k_2} + F_{k_3} + \cdots + F_{k_r}$,
where $k_1 \gg k_2 \gg k_3 \gg \cdots \gg k_r \gg 0$.

Properties:

- $\forall n \in \mathbb{Z}_{\geq 0} : \sum_{j=0}^{n} F_j = F_{n+2} - 1$
- $\forall n \geq 1, \sum_{j=0}^{n} F_{2j} = F_{2n+1} - 1$
- $\forall n \geq 1, \sum_{j=0}^{n} F_{2j-1} = F_{2n}$
- $\sum_{j=1}^{2n-1} F_j F_{j+1} = F_{2n}^2$
- $\sum_{j=1}^{2n} F_j F_{j+1} = F_{2n+1}^2 - 1$
- $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$
- $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$
- $\forall m, n \in \mathbb{Z}_{>2} : m \mid n \iff F_m \mid F_n$
- $\forall m, n \in \mathbb{Z}_{>2} : \gcd(F_m, F_n) = F_{\gcd(m,n)}$

Combinatorics

$\forall r \in \mathbb{R}, k \in \mathbb{Z}, \binom{r}{k} = \prod_{j=1}^{k} \frac{r+1-j}{j}$

Identities:

- $\binom{r}{k} = (r/k)\binom{r-1}{k-1}$
- $\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$
- $\sum_k \binom{r}{m+k}\binom{s}{n+k} = \binom{r+s}{r-m+n}$
- $\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} = \binom{s}{n-r}$
- $\sum_{k=0}^{r} \binom{r-k}{m}\binom{s}{k-t}(-1)^{k-t} = \binom{r-t-s}{r-t-m}$
- For $n \geq s$, $\sum_{k=0}^{r} \binom{r-k}{m}\binom{s+k}{n} = \binom{r+s+1}{m+n+1}$

Cycles: Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S. Then
$\sum_{n=0}^{\infty} g_S(n) \frac{x^n}{n!} = \exp\left(\sum_{n \in S} \frac{x^n}{n}\right)$

Burnside's Lemma: $I(\pi)$ equals number of fixed points of group element π. Then,
$|\text{Classes}| = (1/|G|) \sum_{\pi \in G} I(\pi)$

Polya Enumeration: Each representation element can take on k distinct values. $C(\pi)$ counts the number of cycles in the permutation π.
$|\text{Classes}| = (1/|G|) \sum_{\pi \in G} k^{C(\pi)}$

Number of Labeled Unrooted Trees:

- on n vertices: n^{n-2}
- on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
- with degrees d_i : $(n-2)!/(\prod(d_i - 1)!)$

Martingale

Fair Coin Flipping: Alice has n_A pennies and Bob has n_B pennies. After each flip, Alice has probability p of winning a penny from Bob (gives a penny to Bob otherwise). Let P_A and P_B denote probabilities of players winning. Then, if $p = 1/2$,
$P_A = \frac{n_A}{n_A + n_B}$, $P_B = \frac{n_B}{n_A + n_B}$.
If $p \neq 1/2$, let $q = 1 - p$. Then,
$P_A = \frac{1 - (\frac{q}{p})^{n_A}}{1 - (\frac{q}{p})^{n_A + n_B}}$, $P_B = \frac{1 - (\frac{p}{q})^{n_B}}{1 - (\frac{p}{q})^{n_A + n_B}}$

Number Theory

CRT: For pairwise coprimes m_i with $M = \prod m_i$,

$$M_i := \prod_{i \neq j} m_j \bmod M, N_i := M_i^{-1} \bmod m_i,$$
$$a \equiv \sum a_i M_i N_i \bmod M.$$

Wilson's Theorem:

$$(n-1)! \pmod n \equiv \begin{cases} -1, & n \text{ is prime} \\ 2, & n = 4 \\ 0, & \text{otherwise} \end{cases}$$

Lucas' Theorem: For prime p and $n \geq i \in \mathbb{Z}^+$, if $n = (\overline{n_m n_{m-1} \cdots n_0})_p$ and $i = (\overline{i_m i_{m-1} \cdots i_0})_p$ then

$$\binom{n}{i} \equiv \prod_{j=0}^{m} \binom{n_j}{i_j} \pmod p$$

Mobius Inversion: For $f, g : \mathbb{Z}_+ \to \mathbb{C}$,

$$g(n) = \sum_{d|n} f(d), \ \forall n \in \mathbb{Z}_+$$
$$\iff f(n) = \sum_{d|n} \mu(d) g(n/d), \ \forall n \in \mathbb{Z}_+$$

Common Mobius Inversion Functions:

- $n = \sum_{d|n} \varphi(d)$
- $\varphi(d) = \sum_{d|n} \mu(d) n/d = \sum_{d|n} \mu(n/d) d$
- $[n == 1] = \sum_{d|n} \mu(d)$

Parametrization of Pythagorean Triples: For $m > n > 0, k > 0, \gcd(m,n) = 1$, and either m or n even, Pythagorean triples are uniquely generated by

$$a = k(m^2 - n^2), \ b = k(2mn), \ c = k(m^2 + n^2)$$

Euler's Criterion (QR): For odd prime $p \nmid a$,

$$a^{(p-1)/2} \equiv \left(\tfrac{a}{p}\right) \pmod p$$

Sums and Products of Squares:

- All positive integers can be expressed as the sum of four squares.
- For $n \in \mathbb{Z}^+, \nexists x, y \in \mathbb{Z}$ such that $n = x^2 + y^2$ iff $\exists p \equiv 3 \pmod 4$ such that $\nu_p(n)$ is odd.
- For $n \in \mathbb{Z}^+, \exists x, y, z \in \mathbb{Z}$ such that $n = x^2 + y^2 + z^2$ iff $n \neq 4^a(8b+7)$ for $a, b \in \mathbb{Z}^+$
- $(a^2 + nb^2)(c^2 + nd^2) = (ac + nbd)^2 + n(ad - bc)^2$

Number Theory Tips:

- There are $2\sqrt{n}$ distinct values of $\lfloor n/k \rfloor$
- $\sum_{k=1}^{n} n/k \approx \log n$
- Try iterating over divisors, especially in expressions with gcd
- Consider primitive root in arguments (primitive root exists for $n = 1, 2, 4, p^k, 2p^k$ for odd prime p)
- Can use probabilistic methods with prime density and with the $(p-1)/2$ quadratic residues for odd prime p.

Numerical and Linear Algebra

Error term E_S for integration by Simpson's Rule, where $\max |f^{(IV)}| \leq K$:

$$|E_S| \leq K(b-a)^5/(180n^4)$$

Newton's Method for roots of $f(x) = 0$:

$$x_{i+1} = x_i - f(x_i)/f'(x_i)$$

Lagrange Interpolating Polynomial: For n points (x_i, y_i), the $n-1$ degree polynomial is

$$P(x) = \sum_{j=1}^{n} y_j \prod_{k=1, k \neq j}^{n} (x - x_k)/(x_j - x_k)$$

Cramer's Rule: Let $A \in \mathbb{R}^{n \times n}$ with non-zero determinant, where $A\mathbf{x} = \mathbf{b}$. Let A_i be the matrix formed by replacing the i-th column of A by the column vector $\mathbf{b}$. Then,

$$x_i = \det(A_i)/\det(A).$$

Diagonalization: Let $\lambda_1, \cdots, \lambda_n$ be associated with eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$ of a matrix A. Then, $A = PDP^{-1}$, where P has $\vec{v}_1, \cdots, \vec{v}_n$ as column vectors and $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$.

Principle of LP Duality: Suppose we wish to maximize $\vec{c}^T \vec{x}$ over all $\vec{x}$, subject to constraints $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$. Taking the dual yields the equivalent optimization problem where we wish to minimize $\vec{b}^T \vec{\lambda}$ over all $\vec{\lambda}$, subject to constraints $\vec{\lambda} \geq 0$ and $A^T \vec{\lambda} \geq \vec{c}$.

Game Theory

Sprague-Grundy: The nimber of the sum of independent games is the XOR of their values. The nimber of the state of one game is the MEX of nimbers of all reachable positions.

Hackenbush Value: A stalk $BWxxx$ has value $0.xxx1_2$, where B is 1 and W is 0

General Game Theory Tips:

- Abuse symmetry! Mirroring opponent moves can result in wins.
- Break single games into independent games.
- Moves that make one lose can be made "forbidden". Try to play with these new rules.

Geometry

Pick's Theorem: Consider a lattice polygon. I equals number of interior lattice points, B equals number of boundary lattice points.

$$\text{Area} = I + B/2 - 1.$$

Incenter: $A = rs$. For $\triangle ABC$, where $a = |BC|$, etc., the coordinates of the incenter are:

$$I = \frac{1}{a+b+c}(aA + bB + cC)$$

Heron's Formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \frac{1}{4}\sqrt{2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)}$$

General Geometry Tips:

- When calculating $\sin^{-1}$ or $\cos^{-1}$, make sure to bound the input to $[-1, +1]$.

Counterclockwise rotation:

$$(x, y) \mapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

Stokes' Theorem: When calculating something such as area or another function evaluated on the interior of for instance a union of shapes, consider using Stokes' Theorem and working with the boundary.

$$\iiint_V (\nabla \cdot \vec{F}) \mathrm{d}V = \oiint_S (\vec{F} \cdot \hat{n}) \mathrm{d}S, \ \int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega$$

Graphs

Planar: Euler's Characteristic Formula: For n vertices, m edges, and f faces, and k connected components, $n - m + f = 1 + k$.

Planar Graph Properties:

- If $n \geq 3$ then we must have $m \leq 3n - 6$. Equality when each face is bounded by a triangle.
- If $n \geq 3$ then we must have $f \leq 2n - 4$.
- Every planar graph has a vertex of degree 5 or less.

Determinant of Adjacency Matrix: Equals sum over all graph decompositions of vertices into directed cycles of $(-1)^{\text{Number of Even Length Cycles}}$.

Flows with Demands: Let $d(u, v)$ be the demanded flow in edge from u to v. Change the network, add a new source s' and a new sink t', a new edge from the source s' to every other vertex, a new edge for every vertex to the sink t', and one edge from t to s. Define the new capacity function c' as below. If the new graph is saturated from s' to t', there exists a solution, where the total flow in the original networok is the flow from t to s.

- $c'((s', v)) = \sum_{u \in V} d((u, v)), \forall$ edges (s', v).
- $c'((v, t')) = \sum_{w \in V} d((v, w)), \forall$ edges (v, t').
- $c'((u, v)) = c((u, v)) - d((u, v)), \forall$ previous edges (u, v).
- $c'((t, s)) = \infty$ (can reduce to bound net flow)