# Math Formula Sheet

## Special Numbers

Primes Estimation:
$\pi(n) \sim n/\ln(n)$, $p_k \sim k \ln k$

Partition Function Estimation $p(n)$:
$p(n) \sim 13^{\sqrt{n}}/(7n)$

Max Highly Composites Less than Powers of 10:
$(60, 12), (840, 32), (7560, 64), (83160, 128),$
$(720720, 240), (8648640, 448), (73513440, 768),$
$(735134400, 1344), (6983776800, 2304)$

Catalan Numbers:
$C_n = \binom{2n}{n}/(n+1)$
$C(x) = (1 - \sqrt{1-4x})/(2x)$

### Fibonacci Numbers

Definition:
$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2$
$F(x) = x/(1 - x - x^2)$

Closed Form:
$F_n = (\phi^n - (1-\phi)^n)/\sqrt{5}$

Matrix Exponentiation:
$$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$$

Zeckendorf's Theorem: Let $n \gg k$ denote $n \geq k + 2$. Using greedy, all positive integers can be expressed uniquely as the sum
$n = F_{k_1} + F_{k_2} + F_{k_3} + \cdots + F_{k_r}$,
where $k_1 \gg k_2 \gg k_3 \gg \cdots \gg k_r \gg 0$.

Summation Properties:

- $\forall n \in \mathbb{Z}_{\geq 0} : \sum_{j=0}^{n} F_j = F_{n+2} - 1$
- $\forall n \geq 1, \sum_{j=0}^{n} F_{2j} = F_{2n+1} - 1$
- $\forall n \geq 1, \sum_{j=0}^{n} F_{2j-1} = F_{2n}$
- $\sum_{j=1}^{2n-1} F_j F_{j+1} = F_{2n}^2$
- $\sum_{j=1}^{2n} F_j F_{j+1} = F_{2n+1}^2 - 1$

Additive Rule:
$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$

Cassini's Identity:
$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$

Divisibility and GCD:
$\forall m, n \in \mathbb{Z}_{>2} : m \mid n \iff F_m \mid F_n$
$\forall m, n \in \mathbb{Z}_{>2} : \gcd(F_m, F_n) = F_{\gcd(m,n)}$

## Combinatorics

Definition for Reals:
$\forall r \in \mathbb{R}, k \in \mathbb{Z}, \binom{r}{k} = \prod_{j=1}^{k} \frac{r+1-j}{j}$

Identities:

- $\binom{r}{k} = (r/k)\binom{r-1}{k-1}$
- $\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$
- $\sum_k \binom{r}{m+k}\binom{s}{n+k} = \binom{r+s}{r-m+n}$
- $\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} = \binom{s}{n-r}$
- $\sum_{k=0}^{r} \binom{r-k}{m}\binom{s}{k-t}(-1)^{k-t} = \binom{r-t-s}{r-t-m}$
- For $n \geq s$, $\sum_{k=0}^{r} \binom{r-k}{m}\binom{s+k}{n} = \binom{r+s+1}{m+n+1}$

Burnside's Lemma: Let $I(\pi)$ denote number of fixed points of a group element $\pi$. Then,
$|\text{Classes}| = (1/|G|) \sum_{\pi \in G} I(\pi)$

Polya Enumeration: Suppose each representation element can take on $k$ distinct values. Let $C(\pi)$ count the number of cycles in the permutation $\pi$. Then,
$|\text{Classes}| = (1/|G|) \sum_{\pi \in G} k^{C(\pi)}$

Number of Labeled Unrooted Trees:

- on $n$ vertices: $n^{n-2}$
- on $k$ existing trees of size $n_i$: $n_1 n_2 \cdots n_k n^{k-2}$
- with degrees $d_i : (n-2)!/(\prod(d_i - 1)!)$

## Number Theory

CRT: For pairwise coprimes $m_i$ with $M = \prod m_i$,
$M_i := \prod_{i \neq j} m_j \bmod M, N_i := M_i^{-1} \bmod m_i,$
$a \equiv \sum a_i M_i N_i \bmod M$.

Wilson's Theorem:
$$(n-1)! \pmod{n} \equiv \begin{cases} -1, & n \text{ is prime} \\ 2, & n = 4 \\ 0, & \text{otherwise} \end{cases}$$

Mobius Inversion: For $f, g : \mathbb{Z}_+ \to \mathbb{C}$,
$g(n) = \sum_{d|n} f(d), \ \forall n \in \mathbb{Z}_+$
$\iff f(n) = \sum_{d|n} \mu(d) g(n/d), \ \forall n \in \mathbb{Z}_+$

Common Mobius Inversion Functions:

- $n = \sum_{d|n} \varphi(d)$
- $\varphi(d) = \sum_{d|n} \mu(d) n/d = \sum_{d|n} \mu(n/d) d$
- $[n == 1] = \sum_{d|n} \mu(d)$

Parametrization of Pythagorean Triples: For $m > n > 0, k > 0, \gcd(m, n) = 1$, and either $m$ or $n$ even, Pythagorean triples are uniquely generated by
$a = k(m^2 - n^2)$, $b = k(2mn)$,
$c = k(m^2 + n^2)$

## Numerical and Linear Algebra

Error term $E_S$ for integration by Simpson's Rule, where $\max|f^{(IV)}| \leq K$:
$|E_S| \leq K(b-a)^5/(180n^4)$

Newton's Method for roots of $f(x) = 0$:
$x_{i+1} = x_i - f(x_i)/f'(x_i)$

Lagrange Interpolating Polynomial: For $n$ points $(x_i, y_i)$, the $n-1$ degree polynomial is
$P(x) = \sum_{j=1}^{n} y_j \prod_{k=1, k \neq j}^{n} (x - x_k)/(x_j - x_k)$

Cramer's Rule: Let $A \in \mathbb{R}^{n \times n}$ with non-zero determinant, where $A\mathbf{x} = \mathbf{b}$. Let $A_i$ be the matrix formed by replacing the $i$-th column of $A$ by the column vector $\mathbf{b}$. Then,
$x_i = \det(A_i)/\det(A)$.

## Geometry

Pick's Theorem: Consider a lattice polygon. Let $I$ be number of interior lattice points and $B$ be number of boundary lattice points. Then,
$\text{Area} = I + B/2 - 1$.

General Geometry Tips:

- Comparing by polar angle: first check the half, $[0, \pi)$ or $[\pi, 2\pi)$, then the rotation.
- When calculating $\sin^{-1}$ or $\cos^{-1}$, make sure to bound the input to $[-1, +1]$.

Checking segment intersection for $(A_1, B_1)$ and $(A_2, B_2)$: check that projections on $x$ and $y$ axis intersect; check that
$vmul(A_2 - A_1, B_1 - A_1) \cdot vmul(B_2 - A_1, B_1 - A_1) \leq 0$
and
$vmul(A_1 - A_2, B_2 - A_2) \cdot vmul(B_1 - A_2, B_2 - A_2) \leq 0$.

Counterclockwise rotation:
$(x, y) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$

## Planar Graphs

Euler's Characteristic Formula: For $n$ vertices, $m$ edges, and $f$ faces, and $k$ connected components,
$n - m + f = 1 + k$.

Properties:

- If $n \geq 3$ then we must have $m \leq 3n - 6$. Equality when each face is bounded by a triangle.
- If $n \geq 3$ then we must have $f \leq 2n - 4$.
- Every planar graph has a vertex of degree 5 or less.