Welcome to Discrete

- Peer tutor sessions
  MTW 7-9pm, F 1-3pm
- Discussion section
  TTh, TBD
- PSETs
  Assigned Th, due next Th

WHAT IS DISCRETE?

- Enumerative Combinatorics
- Graph Theory

- Deals with finite - not continuous - things

MIDTERM 1

- Ch. 1
  - Induction
  - Sets, functions, etc
  - Relations: equivalence, partial order
- Ch. 2
  - Posets
  - Min/max elements
  - (anti) chains
  - linear extensions, order-preserving, embeddings
- Ch. 3
  - Counting, power sets, injective functions
  - Binomial coefficients
  - Permutations

Numbers

$\mathbb{N} = \{1, 2, 3 \cdots\}$ or $\{0, 1, 2, 3 \cdots\}$ natural numbers

$\mathbb{Z}$ integers

$\mathbb{Q}$ rational numbers $= \left\{ \frac{a}{b} \,\middle|\, \substack{a, b \in \mathbb{Z} \\ b \neq 0} \right\}$

$\mathbb{R}$ real #s

$\lfloor x \rfloor$ Floor function

$\lceil x \rceil$ ceil function

$A \cup B$ union

$A \cap B$ intersection

$A \setminus B$ or $A - B$ difference

$A \times B$ cartesian product: put elements into vectors

$$A \times B = \{(a_1, b_1), (a_2, b_1), \cdots (a_n, b_1) \cdots (a_1, b_2), (a_2, b_2) \cdots (a_n, b_n)\}$$

$|A|$ or $\#A$ count of elements in a finite set

$P(A)$ or 2^A power set of A: set of all subsets of A, including $\emptyset$ empty set

Induction

To prove statement P:

1. prove $P(1)$
2. Assuming $P(n)$ is true, prove $P(n+1)$
3. P is true for all $n \geq 1$

Sets

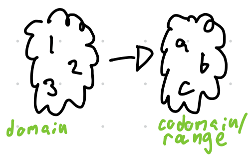
▷ RELATIONS

- If A and B are sets, a <u>relation</u> from A to B is a subset of $A \times B$.
- If (a, b) is a part of relation R, "a is related to b", i.e. aRb.

Ex. equality. For a set A, $R = \{(a, a) \mid a \in A\}$, i.e. aRb means $a = b$

▷ FUNCTIONS

- A function $A \to B$ is a relation where
  for each $a \in A$ there is exactly one $b \in B$ with $(a, b) \in R$
  i.e. only one



$\{(1, a), (2, b), (3, c)\}$

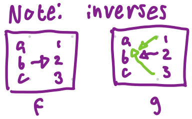
  - There cannot be multiple entries with the same a, since A is a set and has no duplicates. But multiple a's can map to the same b.

- A function is injective, i.e. one-to-one, if the statement
  $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$
  holds true. Contrapositively,
  $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ i.e. all a's have different outputs than one another

  horizontal line test

  $f^{-1}(x)$ exists iff $f(x)$ is injective

Note: inverses



the inverse g exists, and to make it, you map unmapped elements willy-nilly, it is an inverse because $g(f(x)) = x$

- A function is surjective iff for every $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$.

  i.e. the entire specified range is hit

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = \sin(x)$

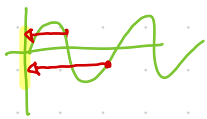
⊗ not surjective

$f: \mathbb{R} \to [-1, 1]$
$f(x) = \sin(x)$

✓ surjective

- A function is bijective iff it is injective and surjective.

crazy notes



surjective iff f projected onto y axis equals range of f
bijective iff such projection is inversible

- Composition
  - $\boxed{a}$ If f and g are both injective, then so is $g \circ f$.
  - $\boxed{b}$ If f and g are both surjective, then so is $g \circ f$.
  - $\boxed{c}$ If f and g are both bijective, then so is $g \circ f$.
  - $\boxed{d}$ There exists a set D and functions $f_1: A \to D$, $f_2: D \to B$ where f_1 is surjective, f_2 is injective, and $f = f_2 \circ f_1$.

Proof for $\boxed{a}$

Assume $(g \circ f)(a_1) = (g \circ f)(a_2)$

$\therefore g(f(a_1)) = g(f(a_2))$

let $f(a_1) = b$, $f(a_2) = c$

$g(b) = g(c)$, but g is injective, so $b = c$, so $f(a_1) = f(a_2)$

but f is injective so $a_1 = a_2$.

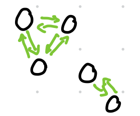
$\therefore g \circ f$ is injective

▷ RELATIONS OF GRAPHS

- A relation between A and B can be drawn as a graph with directed edges from "A" nodes to "B" nodes

▷ EQUIVALENCE RELATIONS

- A relation is an equivalence relation if it satisfies 3 conditions:
  $\boxed{1}$ reflexive property: aRa (any element is related to itself)
  $\boxed{2}$ symmetric property: if aRb, bRa
  $\boxed{3}$ transitive property: if aRb and bRc, aRc



it's like an infection

- For example, aRb where $a=b$.

- For an element a in a set, given an equivalence relation, the equivalence class of A, $[a]$, is the set of elements related to a
  - If aRb, $[a]=[b]$ ✱
  - If $a\cancel{R}b$, $[a]\cap[b]=\emptyset$ ✱
- The # of ways you can break a set into disjoint subsets is the same as the # of equivalence relations you can find.

✱ Proof

To show $[a]=[b]$, show $[a]\subseteq[b]$ and $[b]\subseteq[a]$ (they are subsets of each other)

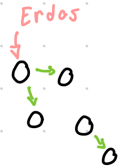
Let $c\in[a]$. ∴ cRa ∴ aRc
Let's assume aRb. By transitivity, cRa + $aRb \Rightarrow cRb$
So $c\in[b]$, so $[a]\subseteq[b]$. (repeat for other case)

✱ Proof

Prove the contrapositive: if $[a]\cap[b]\neq\emptyset$, aRb

Assume $c\in[a]\cap[b]$
So, $c\in[a]$ and $c\in[b]$ ∴ cRa and cRb
by transitivity, aRb

▷ PARTIAL ORDER RELATIONS



this time, the infection is directed

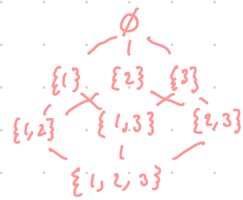
- A relation is an equivalence relation if it satisfies 3 conditions:
  $\boxed{1}$ reflexive property: aRa (any element is related to itself)
  $\boxed{2}$ anti-symmetric property: if aRb and bRa, $a=b$
  $\boxed{3}$ transitive property: if aRb and bRc, aRc

- Example: aRb where $a \leq b$

- If $x \leq y$ or $y \leq x$, x and y are comparable.
  - If all elements are comparable to one another, the relation is a total order
- If $x \leq y$ and there is no z such that $x \leq z \leq y$, "y covers x"

- A set P is a partially-ordered set ("poset") if P is related to P by a partial order relation
  - Posets are notated as $(P, \leq)$

Ex. $2^{\{1,2,3\}}$

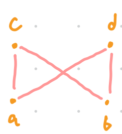


A Hasse diagram is a graph with lines drawn only for covering relations.

▷ MINIMAL/MAXIMAL & MINIMUM/MAXIMUM

- x is a minimal element if there is no $y \in P$ such that $y < x$.
  - Similarly, maximal elements ... $x < y$
  - It is not guaranteed that a minimal/maximal exists, or that it is unique.

  Ex. non-unique minimal elements Hasse diagram



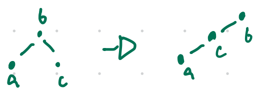
- x is a minimum element if $x \leq y$ for all $y \in P$
  - Similarly, maximum element ... $x \geq y$

Proof: every finite poset has at least one minimal element.

Suppose there's no minimal element. Take any $x \in P$ and there will be some $y < x$. Repeat.
At some point, there will be no $y < x$, since the set is finite.
This is a contradiction of the assumption.

▷ LINEAR EXTENSIONS

- Poset A is a *linear extension* of poset P such that
  1. If $x \leq y$ in P, $x \leq y$ in A (preserve relations)
  2. A's partial order relation is a total order



Proof: every finite poset has at least one linear extension.

Base case:

$|P| = 1$ this is already a total order

Inductive step:

Assume we know all posets of size $< n$ have linear extensions.

Let $x \in P$ be minimal.

Then, remove x, to get $P \setminus \{x\}$ with size $n-1$.

Thus, you can find a linear extension of $P \setminus \{x\}$ $y_1 < y_2 < \cdots < y_{n-1}$

Take $x < y_1 < \cdots < y_{n-1}$ as a linear extension of P.

▷ ORDER-PRESERVING FUNCTIONS

- Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets.
  A function $f: P_1 \to P_2$ is *order-preserving* if $x \leq_1 y \Rightarrow f(x) \leq_2 f(y)$.
- For any finite poset P, there exists an order-preserving injection $P \to \mathbb{N}$.
  (equivalent to previous proof)

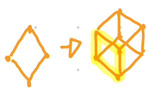
▷ EMBEDDINGS

- Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets.
  A function $f: P_1 \to P_2$ is an embedding if
  1. It is injective
  2. It is order-preserving
  3. $f(x) \leq_2 f(y) \Rightarrow x \leq_1 y$

  Technically, f^{-1} doesn't necessarily exist, but this is essentially saying f^{-1} is order-preserving.

- Graphically, you must be able to find the source Hasse diagram in the resultant diagram.

Ex. $2^{\{1,2\}} \to 2^{\{1,2,3\}}$



Hasse links preserved

▷ ORDER IDEALS

- Let $(P, \leq)$ be a poset, and I be a subset of P.
  I is an order ideal if for any $x \in I$, and any $y \in P$, $y \leq x$ implies $y \in I$
  aka "down-set"

order ideals of this poset:
a, b, a & b, a & b & c, $\emptyset$

Ex. For $x \in P$, let $\langle x \rangle = \{y \in P \mid y \leq x\}$ be the "order ideal generated by x".
☆ It is ideal since if $y \in \langle x \rangle$ and $z \leq y$, by transitivity, $z \leq x$, so $z \in \langle x \rangle$

Theorem: For any finite poset $(P, \leq)$, there exists an embedding $P \to (2^P, \leq)$

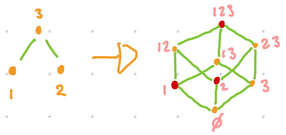
i.e. embedding f maps $x \in P \mapsto \langle x \rangle$

Domain is the power set, but not all elements in the power set actually show up (like $\sin x$ is $\mathbb{R} \to \mathbb{R}$ but only $[-1, 1]$ shows up)

$\{x, y, z\} \mapsto \{\langle x \rangle, \langle y \rangle, \langle z \rangle\}$

type: poset P — type: Poset of all subsets of P

"For every poset, there's an embedding into the power set, given by the order ideals of each element"
And the resultant set of downsets is a poset with the same order as the source poset



Red dots are the downsets generated by elements

1. Injectivity

Assume $f(x) = f(y)$. Let's prove $x = y$

This means $\langle x \rangle = \langle y \rangle$; $x \in \langle x \rangle = \langle y \rangle$

So $x \leq y$, and $y \leq x$

By anti-symmetry, $x = y$

$\therefore f$ is injective

2. order preserving

Suppose $x \leq y$. Let's prove $f(x) \subseteq f(y)$ i.e. $\langle x \rangle \subseteq \langle y \rangle$

Take any $z \in \langle x \rangle$. By definition, $z \leq x$.
Since $x \leq y$, $z \leq y$. So $z \in \langle y \rangle$
Thus, $\langle x \rangle \subseteq \langle y \rangle$, so f is order-preserving.

3. $f(x) \subseteq f(y) \Rightarrow x < y$

Start with $f(x) \subseteq f(y)$, i.e. $\langle x \rangle \subseteq \langle y \rangle$
Since $x \in \langle x \rangle \subseteq \langle y \rangle$, then $x \in \langle y \rangle$, so $x < y$.

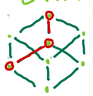
All 3 conditions of an embedding were met, so it exists.

In short: make downsets from every element in P. If you order these downsets by what is a subset of what, the order is the same as the order of P.
(this is obvious if you look at the Hesse graph)
Thus, you can define an embedding from P to downsets.

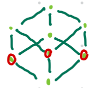
▷ CHAINS

- Let $(P, \leq)$ be a poset, and $A \subseteq P$.
    - A is a chain if any two elements in A are comparable
    - A is an anti-chain if any two elements of A are incomparable

chain



anti-chain



- $\alpha(P)$ is the max size of an antichain $A \in P$
- $w(P)$ is the max size of a chain $A \in P$

Theorem: $\alpha(P) \cdot w(P) \geq |P|$

Write P as a disjoint union of antichains

$P = A_1 \cup A_2 \cup \cdots \cup A_K$

Let A_1 = set of minimal elements of P

Since $P \setminus A_1$ is a poset, let A_2 = set of min elements of new poset $P \setminus A_1$

Recurse...

Then $|P| = |A_1| + |A_2| + \cdots + |A_k|$

Each $|A_i| \leq \alpha(P)$, so $|P| \leq K \cdot \alpha(P)$ for some integer K.

Next, find a chain of size K, to prove K $w(P)$.

$x_K \in A_K$. Since $x_K \notin A_{K-1}$ (since we would've thrown it away), x_K is not minimal in $A_{K-1} \cup A_K$

So, there is some $x_{K-1} \in A_{K-1}$ with $x_{K-1} \leq x_K$.

$$\begin{array}{ccccccccc} x_K & > & x_{K-1} & > & x_{K-2} & > & \cdots & > & x_1 \\ \cap & & \cap & & \cap & & & & \cap \\ A_K & & A_{K-1} & & A_{K-2} & & & & A_1 \end{array}$$

This is a chain of size K, so $w(P) \geq k$.

Application of this theorem:

For every sequence $(x_1, x_2, \dots, x_{n^2+1}) \in \mathbb{R}^{n^2+1}$ has a <u>monotone subsequence</u> of length $n+1$

- monotone: either increasing or decreasing ($\leq$ or $\geq$)
- subsequence: subset, but preserving order

Ex. $n=2$: $(8, 4, 5, 2, 7)$: $8 \geq 5 \geq 2$

Funny symbol, not $\leq$

Proof. Fix a sequence $(x_1, x_2, \dots, x_{n^2+1})$. Define order $\preceq$ on the set $\{1, 2, 3, \dots, n^2+1\}$ by $i \preceq j$, if:

1. $i \leq j$ (by normal integer order)
2. $x_i \leq x_j$ (by normal integer order)

Using the result $\alpha(P) \cdot \omega(P) \geq |P|$, and the fact that $|P| = n^2+1$, we know at least one of $\alpha(P)$ or $\omega(P)$ must be $> n$.

Case 1:

If we had a chain $i_1 \preceq i_2 \preceq \cdots \preceq i_n$, this means $i_1 \leq i_2 \leq \cdots \leq i_n$ by ① and $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$

If it happens to be $\omega(P)$ that is $> n$, $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$ is a monotone subsequence, so QED.

Case 2:

Otherwise, if $\alpha(P) > n$, $\{i_1, i_2, \dots, i_{n+1}\}$ but $i_j \cancel{\preceq} i_K$ for all j, K.
Suppose $i_j \leq i_K$, i.e. put them in normal integer order
For $i_j \cancel{\preceq} i_K$ to hold, it must be that ① holds but ② doesn't. So, $x_{i_j} > x_{i_j}$.
Thus, a monotone subsequence, QED.

Counting

- $\mathrm{Fun}(X, Y)$ or Y^X is the set of all functions from finite series X to Y

- Theorem: if $|X| = K$, and $|Y| = n$, then $|Y^X| = n^K$

Proof: induction

Base case $(K=1)$:

There are n functions

Inductive step:

Suppose we know $|Y^{X'}| = n^{|X'|}$ for all sets with $|X'| < K$.
Now let X have size n'.
Take any $x \in X$. Specify a function $f: X \to Y$. Choose one relation (i.e. draw one line).
Now, choosing the rest of the function is the same as choosing a function $X \setminus \{x\} \to Y$
By induction, there are n^{K-1} choices.
In total, $n \cdot n^{K-1} = n^K$ choices.

- Similarly, the # of injective functions $X \to Y = n(n-1)(n-2)\cdots(n-K+1)$

- Theorem: for a set X with $|X| = n$, $|2^X| = 2^n$

We know $2^n = |\{0,1\}^n| = |[2]^n|$
We can find a bijection $2^X \to \{0,1\}^n$.
For $A \subseteq X$, $b_1 = 0$ if $x_1 \in A$, else 1.

▷ PERMUTATIONS

- A permutation of a set X is a bijection $f: X \to X$
- The set of all permutations on X is $\mathrm{Sym}(X)$
  - If $X = [n]$, $S_n = \mathrm{Sym}([n])$
- Theorem: $|S_n| = n!$

  proof: if X is finite, a function $X \to X$ is injective $\Leftrightarrow$ it is bijective
  we already know # injections $X \to X$ is $n!$ by proof above

Note: one-line notation

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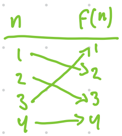

Note: cycle notation

start with 1. Draw an arrow from 1 to $f(1)$.
Find $f(f(1))$, and draw the next arrow to that.

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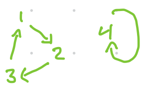


in cycle notation: $(1\,2\,3)(4)$

▷ BINOMIAL COEFFICIENTS

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
  - $\binom{n}{0} = 1$, $\binom{0}{k} = 0$, $\binom{0}{0} = 1$
- $\binom{X}{k}$ is the set of subsets of X with size k

Proof: $\left|\binom{X}{k}\right| = \binom{|X|}{k}$

Falling Factorial

To choose an ordered list of k distinct elements, there are $n^{\underline{k}} = \frac{n^k}{(n-k)!}$ choices

There are $k!$ ways to order the list

- Some identities hold
  - $\binom{n}{k} = \binom{n}{n-k}$
  - $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
  - $\sum_{k=1}^{n} \binom{n}{k} = 2^n$

▷ PRINCIPAL OF INCLUSION/EXCLUSION

- Let S be a set, and $A_1, A_2, \cdots A_n \subseteq S$.
- $|A_1 \cup A_2 \cup \cdots \cup A_n| = \left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|$
- $\left|S \setminus \bigcup_{i=1}^{n} A_i\right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right|$

Ex. $n=2$

essentially, we are preventing double-counting

$|A \cup B| = |A| + |B| - |A \cap B|$

$|S \setminus (A \cup B)| = |S| - |A| - |B| + |A \cap B|$

Ex. $n=3$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

Proof for the PIE part (b)

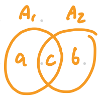
We are going to replace $\left|\bigcap_{i \in I} A_i\right|$ with repeated $1+1+1+\cdots$ since $|x| = 1+1+1+\cdots$ $|x|$ times

Starting with the RHS...

$$\sum_{I \in [n]} (-1)^{|I|} (1+1+1+\cdots+1)$$

For each $x \in S$, which 1's and -1's correspond to x?

For each x, let its J be the list of the indices of the sets $A_1 \cdots A_n$ it's in.



a's J: $\{1\}$
b's J: $\{2\}$
c's J: $\{1,2\}$

Note: $x \in \bigcap_{i \in I} A_i$ iff $I \subseteq J$

Case 1: $J \neq \emptyset$ (x in at least one A_i)

Counting the contribution of +1's and -1's from x...

$$\sum_{I \subseteq J} (-1)^{|I|} = \sum_{\ell=0}^{K} \binom{K}{\ell} (-1)^{\ell} = 0$$

choose all contributions from subsets of length ℓ

$= 0$ due to binomial theorem (symmetry of Pascal's triangle)

case 2: $J = \emptyset$

$I + J = \emptyset$

The only contribution from x is a single $+1$, from $I = \emptyset$.

Thus, RHS = $1 + 1 + 1 + \cdots$, $S \setminus U_A$ times

Ex. How many surjective functions from $[n] \to [k]$?

Let $A_i = \{f : [n] \to [k] \mid f(x) \neq i \ \forall x\}$ for $1 \leq i \leq k$ AKA. all functions with an element in the range that's <u>not</u> covered

Thus, $\bigcup_{i=1}^{k} A_i = \{$ <u>not</u> surjective functions $\}$

By PIE, # surj. functions $= \left| \{\text{all fn's}\} \setminus \bigcup_{i=1}^{k} A_i \right|$

Equivalently, $\bigcap_{i \in I} A_i = \{\text{fn's } f : [n] \to [k] \setminus I\}$ i.e. set I has all the elements that aren't covered

so $\left| \bigcap_{i \in I} A_i \right| = (k - |I|)^n$ since $|Y^X| = |n^k|$

plug into PIE...

$$\# \text{ surj. fns} = \left| \{\text{all fn's}\} \setminus \bigcup_{i=1}^{k} A_i \right| = \sum_{I \subseteq [k]} (-1)^{|I|} (k - |I|)^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{k-j} (k-j)^n = \boxed{\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n}$$

Ex. Hat-check problem

For a permutation what's the probability the permutation is a derangement — that is, no number maps to itself?

Let $D_n = \#$ derangements in S_n

A_i is the set of permutations where i maps to itself, i.e. $A_i = \{\sigma \in S_n \mid \sigma(i) = i\}$

$\bigcap_{i \in I} A_i$ is the set of permutations where all i in I map to themselves.

Choosing such a permutation is equivalent to choosing a permutation of $[n] \setminus I$

So $\left| \bigcap_{i \in I} A_i \right| = (n - |I|)!$

By PIE, $D_n = \sum_{I \subseteq [n]} (-1)^{|I|} (n - |I|)! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$

The final probability is $\frac{D_n}{n!} = \cancel{\frac{1}{n!}} \sum_{k=0}^{n} (-1)^k \frac{\cancel{n!}}{k! \cancel{(n-k)!}} \cancel{(n-k)!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{(-1)^n}{n!}$

By calculus, as $n \to \infty$, the probability approaches $e^{-1} \approx 37\%$

The number of derangements of an n-element set (sequence A000166 in the OEIS) for small n

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$!n$	1	0	1	2	9	44	265	1,854	14,833	133,496	1,334,961	14,684,570	176,214,841	2,290,792,932

Graphs

- Graphically, graphs are vertices connected by edges
- A graph is a pair $G = (V, E)$ where

  V is any set

  $E \subseteq \binom{V}{2}$ pairs of vertices, like $\{a, b\}$

  - E is a set (no duplicates), so no vertex is connected to itself

Terminology

- If there is an edge connecting A & B, A & B are adjacent
- An edge and one of its vertices are said to be incident

Common Graphs

- A path of length n:  $n=4$
- A cycle of length n:  $n=4$
- A complete graph is one where every vertex is connected to every other vertex
  i.e. complete graph $K_n = \left([n], \binom{[n]}{2}\right)$

- A bipartite graph is one where you can color vertices black & white, every edge connects two vertices of opposite color

  - The complete bipartite graph $K_{m,n}$ is where all m white vertices are connected to all n black vertices

Homomorphisms

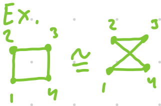
- For two graphs $G=(V,E)$ & $G'=(V',E')$, function $f: V \to V'$ is "graph homomorphic" if
  $\{v,w\} \in E \Rightarrow \{f(v), f(w)\} \in E'$
  i.e. adjacent vertices are mapped to adjacent vertices

Ex. 

abc	abc	abc	abc
123	321	121	323

4 homomorphisms

- A homomorphism f is an isomorphism if f is a bijection and f^{-1} is a homomorphism
- If there is an isomorphism from G's vertices to G''s vertices,
  $G \cong G'$ AKA they are isomorphic

Ex. 

Walks

- A walk is a sequence of vertices $\{v_0, \cdots, v_n\}$ where $\{v_i, v_{i+1}\} \in E$
- A closed walk is when you end where you started
- A path is a walk where no vertices/edges are repeated
- A cycle is a closed walk without repeats (except for start/end)

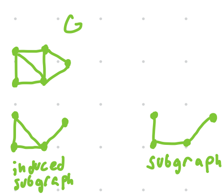
Distance

- The distance between vertices v & w, $d(v,w)$, is the length of the shortest path from v to w.
  - If v & w are not in the same connected component $d(v,w)$ is undefined or ∞.

Subgraph

- Graph $G'=(V',E')$ is a subgraph of $G=(V,E)$ iff
  $V' \subseteq V$ and $E' \subseteq E$.

- An induced subgraph is one where you preserve all edges involved with the vertices
  $G'=(V', E \cap \binom{V'}{2})$



Proof: if G has an odd-length closed walk, then it has an odd-length cycle.

Suppose we have an odd-length closed walk that's not a cycle.

case 1. It's not a cycle because of a repeated vertex
The set of vertices in the walk is the union of two closed walks.

By induction, one subset has an even length and one has an odd length.

case 2.
Same logic but w/ edges

Proof: a graph is bipartite iff it has no odd-length cycles

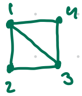
Forwards implication: $\Rightarrow$

Suppose G is bipartite, and there is an odd-length cycle.
The colors must alternate on the cycle.
But then, the first & last vertices are the same color. This is a contradiction.

Reverse implication: $\Leftarrow$

Suppose G has no odd-length cycles.
Pick any vertex v.
Color all w for even $d(d, w)$ white.
Color all w for odd $d(d, w)$ black.
Now, prove the validity of the coloring.
Suppose there was a white-white edge.
Distance from both vertices to d is even, so the shortest path has an even # of edges.
From $A \to d \to B \to A$, you have (even + even + 1) edges.
This contradicts the fact that there are no odd-length cycles.

Adjacency Matrix



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- Theorem: The # of walks of length n from v_i to v_j is $(A^n)_{ij}$
  row $\uparrow$ (i), column $\uparrow$ (j)

Degree & Neighborhood

- The degree of vertex v, $\deg(v)$, is the # of edges connected to v
- The neighborhood $N(v)$ of a vertex is the set of all adjacent vertices.
- $\sum_{v \in V} \deg(v) = 2|E|$ ("handshake lemma")
- $\sum_{\{a,b\} \in E} (\deg(A) + \deg(B)) = \sum_{v \in V} \deg(v)^2$

Proof: If $|E| > \frac{1}{4}|V|^2$, the graph contains a triangle

Prereq: Cauchy-Schwartz inequality

For $a_1, a_2 \cdots a_n \in \mathbb{R}$, $(a_1 + a_2 + \cdots + a_n)^2 \leq n \cdot (a_1^2 + a_2^2 + \cdots + a_n^2)$

Prove the contrapositive: if G has no triangle, $|E| \leq \frac{1}{4}|V|^2$

Choose any edge $\{x,y\} \in E$. There are no triangles, so $N(x) \cap N(y) = \phi$

$\therefore \deg(x) + \deg(y) = |N(x)| + |N(y)| = |N(x) \cup N(y)| \leq |V|$

Taking this inequality $\deg(x) + \deg(y) \leq |V|$, summing them over all edges

$\sum_{\{x,y\} \in E} (\deg(x) + \deg(y)) \leq |E||V|$

LHS $= \sum_{v} \deg(v)^2$ by prev. theorem $\rightarrow \sum_{v \in V} \deg(v)^2 \leq |V||E|$

Using clever algebra: $|V| \sum_{v \in V} \deg(v)^2 \leq |V|^2 |E|$

Use Cauchy-Schwartz with $a_i : \deg(v)$

$\left(\sum_{v} \deg(v)\right)^2 \leq |V| \sum_{v} \deg(v)^2 \leq |V|^2 |E|$

By handshake lemma,

$\sum_{v} \deg(v) = 2|E|$ so $4|E|^{\cancel{2}} \leq |V|^2 \cancel{|E|} \rightarrow |E| \leq \frac{1}{4}|V|^2$

Eulerian Graphs

- Tour: a walk where no edge is repeated
- Circuit: a closed tour
- Eulerian circuit: a circuit that visits every edge
- Eulerian graph: a graph with an eulerian circuit

Proof: For a connected finite graph G, G is eulerian iff $\deg(v)$ is even for all $v \in V$

$\Rightarrow$ proof

Suppose there exists a Eulerian circuit.

For vertex v, every time the circuit passes through v, it uses 2 edges.

$\Leftarrow$ proof

Assume all vertices have an even degree

Let T be a tour of maximum length.

We will prove T is a Euler circuit.

Claim 1: T is closed, i.e. a circuit

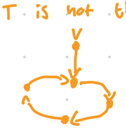
suppose T is not closed.

Since we assume all vertizes have even degree, the starting vertex has one edge that was not traversed. You can add this edge to T to make it longer, so T is not of maximum length.

Claim 2: T must use all vertices

If not, since G is connected, there is some vertex $v \notin T$ that is adjacent to some $w \in T$.

T is not the longest since you can form a new, longer tour starting at v.



Claim 3: T uses all edges

If T does not use all edges, there is some $\{v, w\} \in E$ with $\{v, w\} \notin T$

You can make a longer tour by adding $\vec{vw}$.

Thus, T is a Eulerian circuit.

Directed Graphs

- Directed adjacency matrix

  $$A_{ij} = \begin{cases} 1 & \text{if } V_i \to V_j \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

- For directed adjacency matrix A, $(A^m)_{ij}$ is the number of directed walks of length m from V_i to V_j

- $\deg^+(v)$ is # outgoing arrows; $\deg^-(v)$ is # incoming

- An eulerian digraph is a graph which has a directed circuit that visits all edges
  - A connected digraph is eulerian iff $\deg^+(v) = \deg^-(v)$ for all $v \in V$

Connectivity

- A graph is K-vertex connected (i.e. K-connected) if
  1. G has at least K+1 vertices
  2. G remains connected whenever K-1 vertices are removed

1-connected:
cannot remove any vertices

2-connected:
stays connected even if you remove 1 vertex

- A graph is K-edge connected if it remains connected upon removing K-1 edges

Proof: if G is 2-vertex connected, it is also 2-edge connected

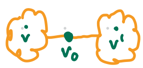
Take an isthmus — an edge where removal disconnects G

Remove one of the endpoints of the isthmus. This also removes the isthmus

Proof: G is 2-connected iff for every pair of vertices vv', there is a cycle containing both

1. If G is not 2-connected, no such cycle exists

   For a pinch point v_0 — where removing v_0 disconnects the graph — there must be a v on the left & a v' on the right



   There is no cycle vv' since it would go through v_0 twice.

2. if G is 2-connected, we can find a cycle including v & v'

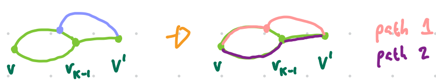
   Base case: $d(v,v')=1$

   Since G is 2-connected, you can remove the direct link from v to v' and there is still a path, the direct link + long path make a cycle

   Inductive step: $d(v,v') < K$

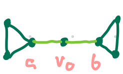
   let v, v' be at a distance K. Assume any 2 vertices of distance $< K$ are contained in a cycle.

   We find a path from $v \to v'$ that does not go through v_{K-1}.
   If this path doesn't intersect the original path, we're good.
   Otherwise, construct 2 paths like so.



GRAPH OPERATIONS

- Edge deletion: for $e \in E$, $G - e = (V, E \setminus \{e\})$
- Edge addition: for $e \notin E$, $G + e = (V, E \cup \{e\})$
- Vertex deletion: for $v \in V$, $G - v = (V \setminus \{v\}, E \setminus \{e \mid e = uv\})$
- Edge subdivision: for $e \in E$, $G \% e = (V \cup \{v_0\}, E \setminus \{ab\} \cup \{av_0, v_0b\})$



Planar Graphs

- A graph is planar if it can be drawn in $\mathbb{R}^2$ without crossing edges.

Trees

- All of these statements define a tree
  - For any $v_1, v_2 \in V$, there is a unique path from v_1 to v_2
  - G is connected, but becomes disconnected upon removal of any edge
  - No cycles; adding any edge makes a cycle
  - $|V| = |E| + 1$

Note: spanning subgraph
- $G' = (V, E')$ is a spanning subgraph of $G = (V, E)$ if $E' \in E$
- i.e. same vertices, subset of edges

▷ SPANNING TREE

- Every connected graph has a spanning tree — a tree with a subset of G's edges that hits all of G's vertices

Proof: Any connected graph has a spanning tree

Induct on $n = |V|$

Base case:

$n = 1$; already a tree

Inductive step:

Assume any graph $|V| < n$ has a spanning tree
Consider the spanning tree T of the graph, but without new vertex v_i
Add v_i to T, connecting with any incident edge of v_i

- Algorithm to generate spanning tree

  While there are vertices not in the tree:
    Pick an edge $\{x, y\}$ where x is in the tree but y is not.
    Add the edge to the tree

▷ SPANNING TREES ON WEIGHTED GRAPHS

- A minimum spanning tree is a spanning tree that minimizes the sum of weights of its edges

- Algorithm to generate minimum weight spanning tree for weighted graph (Kruskal's Algorithm):

  Order the edges with ascending weights
  Start with $E_0 = \emptyset$
  While there are vertices left:
    If the smallest remaining edge doesn't create a cycle:
      Add it
    Else:
      Continue

  - This is a greedy algorithm — chooses best option at each stage

Traversal

▷ DIJKSTRA'S ALGORITHM

- Dijkstra's Algorithm finds the best path from A to B on a weighted graph

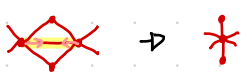
- Let the distance of a vertex be its distance from the start node

```
While there are unvisited vertices:
    Pick the vertex with the smallest distance
    For each neighbor:
        The neighbor's new distance is current vertex's distance + weight of edge to neighbor
        If this new distance is less than the neighbor's old distance, update neighbor's distance
    Mark current vertex as visited
```

Coloring

- A coloring is a map from each vertex to one of K colors
  - A proper coloring satisfies for every edge $\{x,y\}$, $c(x) \neq c(y)$ (no touching same color)
- The chromatic number of a graph is the minimum K for which a proper coloring exists
- The chromatic polynomial of a graph, $\chi_G(K)$, is the number of proper colorings if you use K colors.
  - Coefficient on x^n is 1
  - Coefficient on x^{n-1} is $-|E|$
  - coefficients are nonzero and alternate in sign (for a fully-connected graph)
  - If you plug in $x=0$, you must get 0

Contraction (G/e)



Proof: $\chi_G(K) = \underbrace{\chi_{G-e}(K)}_{\text{deletion}} - \underbrace{\chi_{G/e}(K)}_{\text{contraction}}$ for any edge $e = \{u, v\}$

For each edge:

Case 1: u & v are different colors
  e is included in χ_G and χ_{G-e}.

Case 2: u & v are same color
  e is not included in χ, but is included in $\chi_{G/e}$.
  (the merged vertex has color of u & v)

Additional Matrices

▷ LAPLACIAN

- Let $D(G)$ to be the diagonal matrix with the degree of every node:

$$D = \begin{bmatrix} \deg(v_1) & 0 & 0 & 0 \\ 0 & \deg(v_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \deg(v_n) \end{bmatrix}$$

- The Laplacian matrix $L(G)$ is $D(G) - \underbrace{A(G)}_{\text{adjacency matrix}}$

Ex.

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

- The sum on each row/column is zero $\therefore$ determinant is 0 $\therefore$ $L(G)$ is not invertible

▷ INCIDENCE MATRIX

- For a graph with n vertices and m edges, the incidence matrix is an $n \times m$ matrix where $b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is an endpoint of edge } e_j \\ 0 & \text{otherwise} \end{cases}$
  - For a directed graph, -1 if start point, 1 if end point

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

- The directed incidence matrix is more helpful, so let $\vec{G}$ be any oriented version of G.
- For directed incidence matrix B of $\vec{G}$, $B \cdot B^T = L(G)$

Proof:

By matrix multiplication, $(B \cdot B^T)_{ij} = \sum_k B_{ik} \cdot B^T_{kj} = \sum_k B_{ik} B_{jk}$

If $i = j$ (diagonal):

$\sum_k B_{ik}^2 = \#$ 1's in the row, i.e. # incident edges, i.e. the degree

If $i \neq j$:

$B_{ik} B_{jk}$ is $(-1 \cdot 1) = -1$ if both are endpoints of edge e_k, else 0

DETERMINANT

- $\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma)\, a_{1,\sigma_1} \cdot a_{2,\sigma_2} \cdots a_{n,\sigma_n}$
- The sign of a permutation is +1 if the # of swaps needed to get to it is even, -1 otherwise
- Properties of the determinant
  - If some linear combination of columns is $\vec{0}$, det $A = 0$
  - Multiplying one row/column by a constant also scales the determinant by that constant
  - Swapping rows/columns only affects sign of determinant

Proof.

Let $\bar{L}(G)$ be the Laplacian of a graph, but you delete one row & one column.

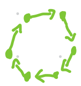
The # of spanning trees of G is $\det(\bar{L}(G))$.

Prerequisite. Let T be a connected graph with n vertices & $n-1$ edges. For some orientation $\vec{T}$,

$$\det \bar{B} = \begin{cases} \pm 1 & \text{if } T \text{ is a tree} \\ 0 & \text{otherwise} \end{cases}$$

incidence matrix with only one row deleted

If T is not a tree, it must have a cycle. Suppose it does.



$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The sum of the first k columns is $\vec{0}$ $\therefore$ $\det \bar{B} = 0$

Thus, suppose T is a tree. Induct on $n = |V|$

Base case: $n = 2$

$B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\bar{B} = [-1]$ $\det \bar{B} = -1$

Inductive step:

Since every tree has at least one leaf node,
number the vertices so that V_n is a leaf node, and edge e_1 connects V_n to V_1

Thus $B = \begin{bmatrix} 1 & \cdots & - \\ 0 & & \\ 0 & & \\ 0 & & \\ \vdots & & \\ -1 & & \end{bmatrix}$

Remove last row to get $\bar{B}$

$$\bar{B} = \left[\begin{array}{c|c} 1 & \cdots \\ \hline 0 & \\ 0 & \\ 0 & A \\ 0 & \\ \vdots & \end{array}\right]$$

by determinant properties, $\det \bar{B} = \det A$ for above sub-matrix A

The incidence matrix for $G - V_n$ is $\begin{bmatrix} \cdots \\ A \end{bmatrix}$ so $\bar{B}$ for $G - V_n$ is A

By induction, $\det A = 1$.

Prerequisite. Cauchy-Binet theorem

Let A be an $n \times m$ matrix, B be an $m \times n$ matrix.

$$\det(A \cdot B) = \sum_{\substack{I \in [m] \\ |I| = n}} \Delta^I(A)\, \Delta^I(B)$$

where $\Delta^I(A)$ is the determinant of the square matrix of A using rows from I

The proof.

Recall $\bar{L} = \bar{B}\bar{B}^T$. By Cauchy-Binet,

$\det(\bar{L}) = \det(\bar{B}\bar{B}^T) = \sum_I \Delta^I(\bar{B})\, \Delta^I(\bar{B}^T) = \sum_I \Delta^I(\bar{B})^2$

By prereq #1, $\Delta^I(\bar{B})^2 = (\pm 1)^2 = 1$ if edges I form a spanning tree, 0 otherwise

So $\det(\bar{L}) = \#$ spanning trees

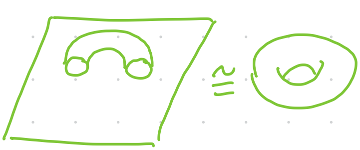
Planar Graphs

- A graph is planar if it can be drawn in the plane with no edge crossings.
- Genus: min. # of handles/topological holes we need to attach to the plane/sphere in order to draw G with no edge crossings

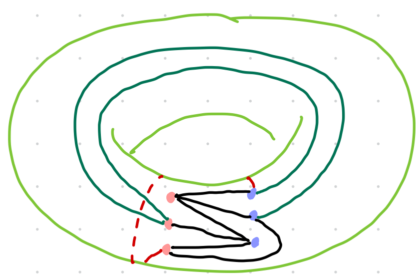
▷ KURATOWSKI'S THEOREM

- G is planar iff G has no subgraph isomorphic to a subdivision of $K_{3,3}$ or K_5.

A handle



$K_{3,3}$ on a torus



▷ EULER'S FORMULA

- For a planar drawing of a planar graph, a maximally connected region of $\mathbb{R}^2 \setminus G$ is called a face of G.
  i.e. take away all shaded-in points on the plane, a region is a "paint bucket tool" region

- For a planar graph with faces F,

  $|V| - |E| + |F| = 2$

  $|F|$ includes the infinite face outside the graph.

- For a graph with genus g,
  $|V| - |E| + |F| = 2 - 2g$

Proof for Euler's theorem

Base case: $|F| = 1$

This is a tree which has $|V| = |E| + 1$;
Only one face, so $|V| - |E| + |F| = 1$

Inductive step: $|F| > 1$

choose an edge e that seperates two faces.

For $G' = G - e$,

$|V'| = |V|$

$|E'| = |E| - 1$

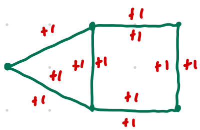
$|F'| = |F| - 1$

Thus, $|V'| - |E'| - |F'| = 2.$

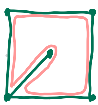
- For a planar graph G, let F_i be the # of faces with i sides.

$$\sum_i i \cdot F_i = 2|E|$$

Proof: inside each face, each edge is counted once.
But each edge touches two faces.



- Caution: something like this still is considered to have 2 faces with 5 sides



BOUND ON # EDGES

- For a planar graph $G=(V,E,F)$ with $|V| \geq 3$,
  a) $|E| \leq 3|V| - 6 = |V|-2$
  b) If G has no triangles (if G is bipartite),
     $|E| \leq 2|V|-4$

planar $\Rightarrow$ upper bound
but upper bound $\not\Rightarrow$ planar

Proof of (a)

By lemma, $2|E| = \sum_i i \cdot f_i$
$= \sum_{i \geq 3} i \cdot f_i$ since every face has ≥ 3 sides

Since $3 \leq i$ for all i, $2|E| \geq \sum_{i \geq 3} 3f_i = 3\sum_{i \geq 3} f_i = 3|F|$
So $2|E| \geq 3|F|$
By Euler's formula, $|E| = 3(|V|-2)$

- This can be used to show $K_{3,3}$ & K_5 are not planar.

Generating Functions

- Recall: a convergent infinite power series can be expressed as a closed-form sum equation.

  Ex. $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$ for $x \in (-1,1)$.
- From a sum function, you can get the series using Taylor series.

- For a sequence $a_0, a_1, a_2, \cdots, a_n$, the ordinary generating function $F(x)=\sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$
  - Ex. For $a_n = n$, OGF $F(x) = x + 2x^2 + 3x^3 + \cdots$
  - Ex. for $a_n = \binom{r}{n}$, OGF $F(x) = \binom{r}{0}x^0 + \binom{r}{1}x^1 + \binom{r}{2}x^2 + \cdots = (1+x)^r$ by binomial theorem

- You can use recurrence relations to find OGFs.

Ex. Fibonacci sequence $a_0=0$, $a_1=1$, $a_{n+2}=a_{n+1}+a_n$

OGF $F(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

Through a clever trick, let's shift the sequences:

$x \cdot F(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$

$x^2 \cdot F(x) = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$

Now, subtract the OGFs:

$F(x) - x\cdot F(x) - x^2 \cdot F(x) = a_0 + [a_1 - a_0]x + [a_2 - a_1 - a_0]x^2 + [a_3 - a_2 - a_1]x^3 + \cdots$

but, $a_n - a_{n-1} - a_{n-2} = 0$. So this becomes

$= 0 + x + 0x^2 + 0x^3 + \cdots = x$

Solving for $F(x)$:

$F(x) - x \cdot F(x) - x^2 \cdot F(x) = x$

$F(x) \cdot [1 - x - x^2] = x$

$F(x) = \frac{x}{1-x-x^2}$

You can then use quadratic formula to factor denominator, and do partial fraction decomposition. Use the convergent geometric series formula and extract the coefficient.

- Remember the geometric series formula:

  $a + ax + ax^2 + \cdots = \frac{a}{1-x}$

▷ OPERATIONS ON GENERATING FUNCTIONS

- Addition: $(a_0+b_0,\ a_1+b_1,\ \cdots)$ has GF $a(x)+b(x)$
- Multiplication by constant: $(ca_0,\ ca_1,\ \cdots)$ has GF $ca(x)$
- Shift right: $(\underbrace{0, 0, \cdots, 0,}_{n \text{ times}}\ a_0,\ a_1,\ \cdots)$ has GF $x^n a(x)$

- To shift left, subtract first n terms & divide by x^n:
  ex. $(a_3, a_4, a_5, \cdots)$ has GF
  $$\frac{a(x) - a_0 - a_1 x - a_2 x^2}{x^3}$$

- Coefficient on x: $(a_0,\ ca_1,\ c^2 a_2,\ \cdots)$ has GF $a(cx)$
  - Ex. $\frac{1}{1-x}$ is $1+x+x^2+\cdots$, so $\frac{1}{1-2x}$ is $1+2x+4x^2+8x^3+\cdots$
  - Trick: only odds
    $$\frac{1}{2}\big(a(x)+a(-x)\big) = a_0 + \cancel{0x} + a_2x^2 + \cancel{0x^3} + a_4x^4 + \cancel{0x^5} + \cdots$$

- Substitute x: $(a_0, 0, 0, a_1, 0, 0, a_2, 0, 0, \cdots)$ has OGF $a(x^3)$

- Differentiation: $(a_1,\ 2a_2,\ 3a_3,\ \cdots)$ has GF $a'(x)$
  - Quick reference: $(1, 2, 3, 4, \cdots) \to \frac{1}{(1-x)^2}$

- Integration: $(0,\ a_0,\ \frac{1}{2}a_1,\ \frac{1}{3}a_2,\ \frac{1}{4}a_3,\ \cdots)$ has GF $\int_0^x a(t)\,dt$

- Multiplication of two GFs:
  $a(x)b(x)$ is the GF of $(c_0, c_1, c_2, \cdots)$ where

  $$c_0 = a_0 b_0$$
  $$c_1 = a_0 b_1 + a_1 b_0$$
  $$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$
  $$\vdots$$
  $$c_k = \sum_{i,j \geq 0;\ i+j=k} a_i b_j$$
  i.e. all the ways subscripts can add to k

▷ WEIGHT GENERATING FUNCTION

- For a set A, let the weight function $wt: A \to \mathbb{N}$
- The weight generating series is $\sum_{a \in A} x^{wt(a)} = \sum_{k \geq 0} c_k x^k$ # elems of weight k

Ex. on $A = 2^{[n]}$, $wt(S) = |S|$

Weight GF $= \sum_{S \in 2^{[n]}} x^{|S|} = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$

▷ COUNTING ON GENERATING FUNCTIONS

- If $A \cap B = \emptyset$, then $F_{A \cup B}(x) = F_A(x) + F_B(x)$

- On $A \times B$, $wt(a,b) = wt_A(a) + wt_B(b)$.
  Then $F_{A \times B}(x) = F_A(x) + F_B(x)$

- If there is a bijection $f: A \to B$ such that $wt_B(f(x)) = wt_A(a)$ for all $a \in A$,
  then $F_A(x) = F_B(x)$

▷ FALLING FACTORIAL

- $x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1) = x(x-1)^{\underline{k-1}}$
  - $x^{\underline{0}} = 1$
  - $x^{\underline{1}} = x$

- Falling Factorial Choose

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

- Falling Factorial Binomial Theorem

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}$$

- Chu-Vandermonde Identity

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$$

▷ SQUARING POWER SERIES

- $$\left(\sum_{n=0}^{\infty} \left[c_n x^n\right]\right)^2 = \sum_{n=0}^{\infty} \left[\left[\sum_{k=0}^{n} c_k c_{n-k}\right] x^n\right]$$

Special result:

- $$\left(\sum_{n=0}^{\infty} \binom{1/2}{n} x^n\right)^2 = 1 + x$$ also notated as $\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$

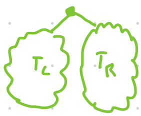
  for all x where the series converges

▷ PLANE BINARY TREE

- A plane binary tree has a root vertex, and every non-leaf vertex has 2 children, left & right
  - Left & right matter, i.e. these are not the same PBT

- The # of PBTs with $(n+1)$ leaves is C_n, the Catalan numbers

There's a recursion: call T_L & T_R on root → left & root → right



There are $n-1$ leaf nodes to distribute between T_L & T_R.
Let k be the number of leaf nodes in T_L.
The remaining $n-1-k$ leaf nodes are in T_R.
And, we can choose any $k < n-1$.

From this, $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$

We found a recurrence relation. Let's use this to find a formula.

- $\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$

Proof:

$$f(x) = \sum_{n=0}^{\infty} C_n x^n = 1 + \sum_{n=1}^{\infty} C_n x^n$$

$$= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} C_k C_{n-k-1} \right) x^n \quad \text{by above justification}$$

$$= 1 + x \cdot \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} C_k C_{n-k-1} \right) x^{n-1}$$

Re-indexing...

$$= 1 + x \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n}_{f(x)^2}$$

$$= 1 + x \cdot f(x)^2$$

Given $x \cdot f(x)^2 - f(x) + 1 = 0$, $f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$ by quadratic formula

The first Catalan numbers for $n = 0, 1, 2, 3, \ldots$ are

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, ...

- $C_n = \frac{1}{n+1}\binom{2n}{n}$

Proof:

Let's use the previous result: $\sum_{n=0}^{\infty} C_n x^n = \frac{\boxed{1-\sqrt{1-4x}}}{\boxed{2x}}$

Also, proved earlier, $\sqrt{1+x} = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n$

$\therefore \sqrt{1-4x} = \sum_{n=0}^{\infty}\binom{1/2}{n}(-4x)^n$

Since the coefficient of x_0 is 1, the power series $\boxed{1-\sqrt{1-4x}}$ has no constant term
Thus, divide it by $\boxed{2x}$ (shift left, divide coefficients by 2)

$C_n = -\frac{1}{2}(-4)^{n+1}\binom{1/2}{n+1} = \frac{1}{n+1}\binom{2n}{n}$

Review

How many triples (a,b,c) s.t. $a+b+c=n$?

$wt(a,b,c) = a+b+c$

$$\sum_{(a,b,c)\in\mathbb{N}^3} x^{a+b+c} = \sum_{n=0}^{\infty} r_n x^n$$

LHS is factorable

$$\sum_{(a,b,c)\in\mathbb{N}^3} x^{a+b+c} = \left(\sum_{a=0}^{\infty} x^a\right)\left(\sum_{b=0}^{\infty} x^b\right)\left(\sum_{c=0}^{\infty} x^c\right)$$

$$= \frac{1}{(1-x)^3}$$

$$= \frac{1}{x^2}\cdot\frac{x^2}{(1-x)^3} = \frac{1}{x^2}\sum_{n\geq 2}\binom{n}{2}x^n \quad \text{by HW9}$$

$$= \sum_{n\geq 0}\binom{n+2}{2}x^n$$ we only care abt coeff of x^n term

$\therefore$ answer $= \binom{n+2}{2}$