1 Lecture 1 - Jan 10

Book: Elementary Differential Geometry

1.1 What is Geometry?

Adding more structures, start with set.

Sets	(M)	Subset, Intersection, Unions
	$\downarrow$ open set	
Topology	$(M, \mathcal{O})$	Continuity, Convergence
	$\downarrow$ atlas, connetion	
Manifolds	$(M, \mathcal{O}, \mathcal{A}, \nabla)$	Coordinates, Fields, Curvature
	$\downarrow$ metric	
Geometry	$(M, \mathcal{O}, \mathcal{A}, \nabla, g)$	Lengths and Angles

Extrinsic Geometry consider a fixed Euclidean space, and study the geometry of curves and surfaces embedded in it.

Intrinsic Geometry consider geometrical properties without reference to a larger embedding space: Riemannian geometry.

Historically, Extrinsic $\to$ Intrinsic. We follow this in class.

1.2 Topology

Definition 1.1: Topology

Consider a set M and its set of subsets, the power set $P(M)$

A topology is a set $\mathcal{O} \subset P(M)$ whose elements are called open, satisfying the following axioms:

- The set itself and the empty set are open: $\varnothing \in \mathcal{O}, M \in \mathcal{O}$
- Finitely many intersections of open sets are open: $U_i, U_j \in \mathcal{O} : U_i \cap U_j \in \mathcal{O}$
- Arbitrarily many unions of open sets are open: $U_i \in \mathcal{O}$, I is index set, $\bigcup_{i \in I} U_i \in \mathcal{O}$

Example $\mathcal{O}_c = \{\varnothing, M\}$ and $\mathcal{O}_d = P(M)$ topologies, the chaotic (smallest) and discrete (biggest) topology, respectively.

Point Given a topological space, we call $p \in M$ a point.

Neighbourhood An open set U containing a point p, $U \in \mathcal{O}, U \ni p$, is called a neighbourhood of p. A deleted neighbourhood $\hat{U} = U \backslash \{p\}$

Closed A subset $V \subset M$ whose complement is open, this is $M \backslash V \in \mathcal{O}$, is called closed. (closed $\neq$ not open)

1.3 Boundary and closure

Interior and Exterior Consider a subset $S \subset M$, and neighbourhoods $U \in \mathcal{O}$

p is called interior if $\exists U \ni p$ such that $U \cap S = U$ (U inside S). The totality of these is the interior of S.

p is called exterior if $\exists U \ni p$ such that $U \cap S = \varnothing$ (U outside S). The totality of these is the exterior of S.

Boundary One that is neither interior nor exterior is called a boundary point. The totality of these is ∂S, the boundary of S.

Closure The closure of S is $\bar{S} = S \cup \partial S$.

1.4 Connectedness and compactness

Definition 1.2: Connected

S is connected if it cannot be decomposed into disjoint open sets,

$$\forall U, V \in \mathcal{O} : U \cap V = \varnothing \to U \cup V \neq S$$

If such a $S = M$, we say M is connected.

Open Covering An open covering of S is a set of open sets $\{U_i\}$ such that $S \subset \cup_i U_i$, that is, each $p \in S$ is in at least one of the U_i .

Definition 1.3: Compact

A set $S \subset M$ is called compact if for every open covering of S, there is a finite number of open sets covering S, that is, it has a finite subcovering. If such a $S = M$, we say M is compact.

1.5 Limit points and convergence

Limit Point A point p is called limit point of S if $\forall \hat{U} \ni p, \hat{U} \cap S \neq \varnothing$, i.e. each neighbourhood of p contains at least one point in S other than p.

Definition 1.4: Converge

A sequence $S : \mathbb{N} \to M, n \to s(n) = x_n$ converge to a limit point p if, for all $U \ni p, \exists m \in \mathbb{N} : n > m \to s(n) \in U$

1.6 Mapping

Mapping Consider two sets M and N. Then a mapping (or map) is

$$f : M \to N, p \mapsto f(p)$$

such that $\forall p \in M$, $\exists$ one image point $f(p) \in N$

In general, we distinguish various kinds of mappings:

- If $f(M) = N$, then f is called surjective.
- If $f(p_1) = f(p_2) \Rightarrow p_1 = p_2$, then f is called injective. $(p_1 \neq p_2 \Rightarrow f(p_1) \neq f(p_2))$
- If f is surjective and injective, it is called bijective.

Preimage Given a map $f : M \to N$, we can define the preimage of $V \subset N$,

$$\text{preim}_f(V) = \{p \in M : f(p) \in V\}$$

Inverse If $f : M \to N$ is bijective, there is a bijective inverse,

$$f^{-1} = \text{preim}_f : N \to M$$

Isomorphic If $f : M \to N$ is bijective, then M and N are called isomorphic, $M \simeq N$.

Definition 1.5: Continuous

Consider two topological spaces $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$, then a map $f : M \to N$ is called continuous if

$$\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M$$

(preimages of open sets are open)

Note If $\mathcal{O}_M = P(M)$, the discrete (biggest) topology, then any f is continuous!

2 Lecture 2 - Jan 11

2.1 Euclidean Space

Definition 2.1: Euclidean Space

The Euclidean space $\mathbb{R}^n$ of dimension n is the set $\{p\}$ of all ordered n-tuples of real numbers, where $p = (p_1, p_2, ..., p_n), p_i \in \mathbb{R}, 1 \leq i \leq n$, is called a point of $\mathbb{R}^n$.

Euclidean Coordinate Functions We introduce the Euclidean coordinate functions as $x_i : \mathbb{R}^n \to \mathbb{R}, p \mapsto x_i$, and also operations of addition and scalar multiplication of points of $\mathbb{R}^n$ such that

$$p + q = (p_1 + q_1, p_2 + q_2, \dots, p_n + q_n),$$
$$\lambda p = (\lambda p_1, \lambda p_2, \dots, \lambda p_n), \lambda \in \mathbb{R}.$$

We may regard $\mathbb{R}^n$ also as a vector space.

Definition 2.2: Euclidean Distance

The Euclidean distance d of two points $p, q \in \mathbb{R}^n$ is the non-negative real numbers

$$d(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \dots (p_n - q_n)^2}$$

Note It is d which fundamentally defines Euclidean geometry.

2.2 Open ball topology

Definition 2.3: Open Ball Topology

$\mathcal{O} \subset \mathbb{R}$ is open if it is empty or if $\forall p \in \mathcal{O}$ there exists an ϵ-neighbourhood $\mathcal{N}_\epsilon(p) = \{q \in \mathbb{R}^n : d(p, q) < \epsilon\}, \epsilon > 0$.

Note We denote the Euclidean space thus endowed with a distance function and topology, $(\mathbb{R}^n, d, \mathcal{O})$, simply by $\mathbb{R}^n$.

2.3 Functions

C^k-differentiable Let $D \subset \mathbb{R}^n$ be the open domain of a real-valued function $f : D \to \mathbb{R}$. f is called C^k-differentiable on D if all its partial derivatives up to the kth order exist and are continuous.

Smooth C^∞-differentiable.

Algebraic Operations on $C^\infty\mathbb{R}^n$ is pointwise (inherits from $\mathbb{R}$)

$$\forall p \in \mathbb{R}^n : (f + g)(p) = f(p) + g(p), (fg)(p) = f(p)g(p)$$

2.4 Tangent spaces

Definition 2.4: Tangent Space

Each $p \in \mathbb{R}^n$ has a vector space $\mathbb{R}^n = T_p\mathbb{R}^n$ called tangent space of $\mathbb{R}^n$ at p, and any $\boldsymbol{v} = (v_1, v_2, \dots, v_n) \in T_p\mathbb{R}^n$ is called a tangent vector at a point of application p (denoted by $\boldsymbol{v}_p$)

Each p have its own tangent space and vector algebra,

$$\boldsymbol{v}_p + \boldsymbol{w}_p = (\boldsymbol{v} + \boldsymbol{w})_p \in T_p\mathbb{R}^n, \lambda(\boldsymbol{v}_p) = (\lambda \boldsymbol{v})_p \in T_p\mathbb{R}^n$$

Note Each tangent space can be identified with the underlying Euclidean space by the canonical isomorphism, $T_p\mathbb{R}^n \simeq \mathbb{R}^n$.

Equality of Tangent Vector $\boldsymbol{v}_p = \boldsymbol{w}_q \Leftrightarrow p = q$ and $\boldsymbol{v} = \boldsymbol{w}$

Tangent Bundle The union of all tangent spaces over all points of $\mathbb{R}^n$ is called its tangent bundle $T\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$.

2.5 Vector fields

Definition 2.5: Vector Field

A vector field is a map $V : \mathbb{R}^n \to T\mathbb{R}^n$ assigning to each point a tangent vector, $p \mapsto V(p) \to T_p\mathbb{R}^n$.

Algebra of vector fields is also pointwise, that is

$$\forall p \in \mathbb{R}^n : (V+W)(p) = V(p)+W(p), (fV)(p) = f(p)V(p).$$

Note Each component of a vector field is a real-valued function over Euclidean space.

Definition 2.6: Euclidean Frame Field

The Euclidean frame field of $\mathbb{R}^3$ is given by vector fields $\{U_1, U_2, U_3\}$ such that, $\forall p \in \mathbb{R}^3$

$$U_1(p) = (1, 0, 0), U_2(p) = (0, 1, 0), U_3(p) = (0, 0, 1)$$

Thus, we can write the vector field as the linear combination $V = \sum_i v_i U_i$

Note V is called C^k-differentiable if all its v_i are C^k-differentiable.

2.6 Directional derivative

Definition 2.7: Directional Derivative

Let $f \in C^\infty\mathbb{R}^3$ and $\boldsymbol{v}_p = (v_1, v_2, v_3) \in T_p\mathbb{R}^3$. Then the directional derivative of f with respect to $\boldsymbol{v}_p$ is the number

$$\boldsymbol{v}_p[f] = \frac{\mathrm{d}}{\mathrm{d}t} f(p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)|_{t=0}$$

Lemma 2.8

$$\boldsymbol{v}_p[f] = \sum_i v_i \frac{\partial f}{\partial x_i}(p), \text{in particular}, U_i[f] = \frac{\partial f}{\partial x_i}$$

Proof. Use Def. 2.7 and the chain rule. □

Theorem 2.9

Let $f, g \in C^\infty\mathbb{R}^3, \boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^3, a, b \in \mathbb{R}$ then

(1) $(a\boldsymbol{v}_p + b\boldsymbol{w}_p)[f] = a\boldsymbol{v}_p[f] + b\boldsymbol{w}_p[f]$,

(2) $\boldsymbol{v}_p[af + bg] = a\boldsymbol{v}_p[f] + b\boldsymbol{v}_p[g]$,

(3) $\boldsymbol{v}_p[fg] = \boldsymbol{v}_p[f]g(p) + f(p)\boldsymbol{v}_p[g]$.

(Three Derivations)

Proof. Use Def. 2.7 and Lem. 2.8 in (3). □

Note Property (3) is a Leibniz product rule typical of derivatives.

This pointwise property can be extended to vector fields V, W,

Theorem 2.10

$$(aV + bW)[f] = aV[f] + bW[f],$$
$$V[af + bg] = aV[f] + bV[g],$$
$$V[fg] = V[f]g + fV[g].$$

Proof. Use Def. 2.7 and Lem. 2.8 in (3). □

Note if $V[f] = W[f], \forall f \in C^\infty\mathbb{R}^3$, then $V = W$.

Proof. In general, a smooth vector field over $\mathbb{R}^3$ is written $V = \sum_i v_i U_i$, where $v_i \in C^\infty\mathbb{R}^3$ and $\{U_i\}$ is the Euclidean frame field.

Since the condition holds $\forall f$, let us choose $f = x_j$, the Euclidean coordinate function x_j. Then by Lem. 2.8, we have

$$V[x_j] = \sum_i v_i U_i [x_j] = \sum_i v_i \frac{\partial x_j}{\partial x_i} = v_j$$

whence $v_j = w_j$ and, with fixed frame field, the result follows. □

Note Likewise in general, we can write $V[f] = \sum_i v_i \frac{\partial f}{\partial x_i}$

3 Lecture 3 - Jan 12

3.1 Differential

Recall a differential $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ has a differential

$$\mathrm{d}f = \frac{\partial f}{\partial x}(p)\mathrm{d}x + \frac{\partial f}{\partial y}(p)\mathrm{d}y + \frac{\partial f}{\partial z}(p)\mathrm{d}z$$

at $p = (x, y, z) \in D$, with functional value $f(p)$.

Then for arbitrary $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$, $f(p) + \mathrm{d}f$ is the value at $(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ on the tangent plane to the graph of f at p.

Note In general, $f(p)+\mathrm{d}f$ differs from $f(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$ and is the linear approximation or linearization of the function f.

3.2 1-forms

Definition 3.1: 1-form

A 1-form on $\mathbb{R}^3$ is a map $\psi : T\mathbb{R}^3 \to \mathbb{R}$ such that, for any point $p \in \mathbb{R}^3$, tangent vector $\boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^3$, and number $a, b \in \mathbb{R}$

$$\phi_p(a\boldsymbol{v} + b\boldsymbol{w}) = a\phi_p(\boldsymbol{v}) + b\phi_p(\boldsymbol{w})$$

It can be extended to vector fields, such that for all $f, g \in C^\infty\mathbb{R}^3$ and smooth vector fields V, W,

$$\phi(fV + gW) = f\phi(V) + g\phi(W)$$

Note In other words, at every point in space, a 1-form is a linear function on tangent vectors, mapping to a real number.

Definition 3.2: Differential

Let $f \in C^\infty\mathbb{R}^3$. Then its differential is the 1-form $\mathrm{d}f$ such that

$$\mathrm{d}f(\boldsymbol{v}) = \boldsymbol{v}[f], \forall \boldsymbol{v} \in T_p\mathbb{R}^3$$

or, passing again from the pointwise definition to vector fields,

$$\mathrm{d}f(V) = V[f]$$

We can evaluate $\mathrm{d}f$ explicitly using Lemma 2.8. First, suppose $f = x_j$, that is, a coordinate function. Then for any $\boldsymbol{v} \in T_p\mathbb{R}^3$,

$$\mathrm{d}x_j(\boldsymbol{v}) = \sum_i v_i \frac{\partial x_j}{\partial x_i} = v_j$$

which means that we can decompose a 1-form to linear combination of coordinate 1-forms $\mathrm{d}x_j$

Lemma 3.3

Let ϕ be a 1-form on $\mathbb{R}^3$. Then $\phi = \sum_j \phi_j \mathrm{d}x_j$, where $\phi_j = \phi(U_j)$ are called the Euclidean coordinate functions of ϕ.

Proof. Repeating the previous computation for any $\boldsymbol{v} = \sum_i v_i U_i \in T_p\mathbb{R}^3$,

$$\phi(\boldsymbol{v}) = \sum_j \phi_j \ \mathrm{d}x_j(\boldsymbol{v}) = \sum_j \phi_j(p) v_j.$$

On the other hand, by linearity of 1-forms (Def. 3.1) and the claim,

$$\phi(\boldsymbol{v}) = \phi_p\left(\sum_i v_i U_i\right) = \sum_i v_i \phi_p(U_i) = \sum_i v_i \phi_i(p).$$

Since they agree for all tangent vectors, indeed $\phi = \sum_j \phi_j \ \mathrm{d}x_j$. □

Cotangent Space The coordinate differentials $\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$ span a cotangent space $T_p^*\mathbb{R}^3$ at any p and define the components of any 1-form.

Corollary Introducing the Kronecker delta δ_{ij} such that $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ij} = 1$ when $i = j$, and we have

$$\mathrm{d}x_i(U_j) = \delta_{ij}$$

Thus, we say 1-forms are dual to tangent vectors. Moreover, $\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$ is called Euclidean coframe field

Corollary Now apply Lem. 3.3 to f, using Def. 3.2 and Lem. 2.8,

$$\mathrm{d}f = \sum_i \mathrm{d}f(U_i)\mathrm{d}x_i = \sum_i U_i[f]\mathrm{d}x_i = \sum_i \frac{\partial f}{\partial x_i}\mathrm{d}x_i$$

which is the original definition of differential

Lemma 3.4

Let $f, g \in C^\infty\mathbb{R}^3, h : \mathbb{R} \to \mathbb{R}$ be differentiable, and $a, b \in \mathbb{R}$

$$\mathrm{d}(af + bg) = a\mathrm{d}f + b\mathrm{d}g \quad \text{(linearity)},$$
$$\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f \quad \text{(product rule)},$$
$$\mathrm{d}(h(f)) = h'(f)\mathrm{d}f \quad \text{(chain rule)}$$

Proof. Use Def. 3.2 and Thm. 2.9,

(1) Let $\boldsymbol{v} \in T_p\mathbb{R}^3$

$$\begin{aligned}\mathrm{d}(af + bg) &= \boldsymbol{v}[af + bg] \\ &= a\boldsymbol{v}[f] + b\boldsymbol{v}[g] \\ &= a\mathrm{d}f + b\mathrm{d}g\end{aligned}$$

(2) Let $\boldsymbol{v} \in T_p\mathbb{R}^3$

$$\begin{aligned}\mathrm{d}(fg)(\boldsymbol{v}) &= \boldsymbol{v}[fg] \\ &= \boldsymbol{v}[f]g(p) + f(p)\boldsymbol{v}[g] \\ &= f(p)\ \mathrm{d}g(\boldsymbol{v}) + g(p)\ \mathrm{d}f(\boldsymbol{v})\end{aligned}$$

(3) Let $\boldsymbol{v} \in T_p\mathbb{R}^3$

$$\begin{aligned}\mathrm{d}[h(f)](\boldsymbol{v}) &= \boldsymbol{v}[h(f)] \\ &= h'[f(p)]\boldsymbol{v}[f] \\ &= h'[f(p)]\ \mathrm{d}f(\boldsymbol{v})\end{aligned}$$

□

3.3 Differential Forms

Definition 3.5: k-form

A k-form $\phi^{(k)}$ on $\mathbb{R}^n$, with integer $0 \leq k \leq n$, is at every point p a linear combination of exterior products of coordinate differentials $\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \ldots \wedge \mathrm{d}x_{i_k}$ with $1 \leq i_j \leq n, 1 \leq j \leq k$, which are totally antisymmetric such that, for any pair of indices i_j, i_l, we have

$$\mathrm{d}x_{i_j} \wedge \mathrm{d}x_{i_l} = -\mathrm{d}x_{i_l} \wedge \mathrm{d}x_{i_j}$$

The set of all k-forms at $p \in \mathbb{R}^n$ is denoted by $\Omega_p^k \mathbb{R}^n$.

Note Ordering the differentials in an exterior product with P permutative exchanges yields an overall forefactor of $(-1)^P$.

Note Any repeated exterior product of the same index vanishes, $\mathrm{d}x_{i_j} \wedge \mathrm{d}x_{i_j} = -\mathrm{d}x_{i_j} \wedge \mathrm{d}x_{i_j} = 0$. So on $\mathbb{R}^n$, $k-$forms have $k \leq n$.

Example We shall now enumerate all types of k-forms on $\mathbb{R}^3$. First, recall that $0 \leq k \leq 3$ so there are 4 types, and let $f, g, h \in C^\infty \mathbb{R}^3$.

- 0-forms: $\phi^{(0)} = f$ are functions, that is, $\Omega^0 \mathbb{R}^3 = C^\infty \mathbb{R}^3$.
- 1-forms: $\phi^{(1)} = f\ \mathrm{d}x + g\ \mathrm{d}y + h\ \mathrm{d}z$, as discussed before.
- 2-forms: $\phi^{(2)} = f\ \mathrm{d}x \wedge \mathrm{d}y + g\ \mathrm{d}x \wedge \mathrm{d}z + h\ \mathrm{d}y \wedge \mathrm{d}z$. Note that any repeated factors vanish, $\mathrm{d}x \wedge \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}z = 0$.
- 3-forms: $\phi^{(3)} = f\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, also known as volume forms.

Note 0- and 3-forms are determined by 1, 1- and 2-forms by 3 coordinate functions, so there is a duality of $\Omega^k \mathbb{R}^n$ and $\Omega^{n-k} \mathbb{R}^n$.

Note $\phi^{(k)} \wedge \psi^{(l)}$ yield $(k+l)-$form, but if $k+l > n$, it is zero.

3.4 Exterior Derivative

Definition 3.6: Exterior Derivative

The exterior derivative of $\phi = \sum_i f_i\ \mathrm{d}x_i \in \Omega^1 \mathbb{R}^n$ is the 2-form

$$\mathrm{d}\phi = \sum_i \mathrm{d}f_i \wedge \mathrm{d}x_i.$$

Example if $n = 3$ with 1-form $\phi = f\ \mathrm{d}x + g\ \mathrm{d}y + h\ \mathrm{d}z$. Using the differentials of f, g, h, we get

$$\mathrm{d}\phi = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}z + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z$$

Note Given a vector field $V = (f, g, h)$, observe that components of the curl of the vector field, $\nabla \times V$, correspond to those of $\mathrm{d}\phi$. Thus, forms generalize notions of vector analysis.

Note Let $\phi = \mathrm{d}f \in \Omega^1 \mathbb{R}^3$, then $\mathrm{d}(\mathrm{d}f) = 0$

Since $\mathrm{d}f = \frac{\partial f}{\partial x}\ \mathrm{d}x + \frac{\partial f}{\partial y}\ \mathrm{d}y + \frac{\partial f}{\partial z}\ \mathrm{d}z$, applying the previous result yields

$$\mathrm{d}(\mathrm{d}f) = \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) \mathrm{d}x \wedge \mathrm{d}z + \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) \mathrm{d}y \wedge \mathrm{d}z = 0 \text{ by Clairaut-Schwarz.}$$

Note Extending the definition of exterior derivative, this is a general property of any differential k-form by axiom, $\mathrm{d}\left(\mathrm{d}\phi^{(k)}\right) = 0$. Hence, we call this nullpotency $\mathrm{d}^2 = 0$ of the exterior derivative.

For any $\phi^{(k)} \in \Omega^k \mathbb{R}^n$ and $\psi^{(l)} \in \Omega^l \mathbb{R}^n$, by axiom

$$\mathrm{d}\left(\phi^{(k)} \wedge \psi^{(l)}\right) = \mathrm{d}\phi^{(k)} \wedge \psi^{(l)} + (-1)^k \phi^{(k)} \wedge \mathrm{d}\psi^{(l)}.$$

Thus, for 0-forms f and 1-forms ϕ, ψ, we obtain the following properties (and, of course, also recover Def. 3.6):

$$\mathrm{d}(f\phi) = \mathrm{d}f \wedge \phi + f\ \mathrm{d}\phi$$
$$\mathrm{d}(\phi \wedge \psi) = \mathrm{d}\phi \wedge \psi - \phi \wedge \mathrm{d}\psi.$$

4 Lecture 4 - Jan 13

4.1 Mappings

Euclidean Coordinate Functions Consider Euclidean mapping $F : \mathbb{R}^n \to \mathbb{R}^m$

$$(x_1, x_2, \ldots, x_n) \mapsto F(p) = (f_1(p), f_2(p), \ldots, f_m(p))$$

where the $f_i, 1 \leq i \leq m$, are called Euclidean coordinate functions.

Note The case $m = 1$ covers real-valued functions $f : \mathbb{R}^n \to \mathbb{R}$.

Note The case $n = 1$ covers curves in $m-$dimensional space, $t \mapsto (x_1(t), x_2(t), \ldots, x_m(t))$, where t is the curve parameter.

Differentiable Mapping A Euclidean mapping $F : \mathbb{R}^n \to \mathbb{R}^m, F = (f_1, f_2, \ldots, f_m)$ is called differentiable if each of the Euclidean coordinate functions $f_i, 1 \leq i \leq m$ is differentiable.

4.2 Push-forward

Definition 4.1: Push-forward

Let $F = (f_1, f_2, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable and $\boldsymbol{v} \in T_p\mathbb{R}^n$ then the tangent map or push-forward at p is

$$F_* : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m, \boldsymbol{v} \mapsto (\boldsymbol{v}[f_1], \dots, \boldsymbol{v}[f_m])$$

Note So the push-forward maps domain space tangent vectors to target space tangent vectors. This can be generalized to $M \to N$.

Lemma 4.2

The push-forward is a linear mapping, that is, for any $\boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^n$, $a, b \in \mathbb{R}^n$,

$$F_*(a\boldsymbol{v} + b\boldsymbol{w}) = aF_*(\boldsymbol{v}) + bF_*(\boldsymbol{w})$$

Proof. This follows from Def. 4.1 and Thm. 2.9

$$\begin{aligned}
&F_*(a\boldsymbol{v} + b\boldsymbol{w}) \\
&= ((a\boldsymbol{v} + b\boldsymbol{w})[f_1], \dots, (a\boldsymbol{v} + b\boldsymbol{w})[f_m]) \\
&= (a\boldsymbol{v}[f_1] + b\boldsymbol{w}[f_1], \dots, a\boldsymbol{v}[f_m] + b\boldsymbol{w}[f_m]) \\
&= a(\boldsymbol{v}[f_1], \dots, \boldsymbol{v}[f_m]) + b(\boldsymbol{w}[f_1], \dots, \boldsymbol{w}[f_m]) \\
&= aF_*(\boldsymbol{v}) + bF_*(\boldsymbol{w}).
\end{aligned}$$

□

4.3 Jacobian and Regularity

Let $\{U_j^{(n)}\}, 1 \le j \le n$, and $\{U_i^{(m)}\}, 1 \le i \le m$ be the Euclidean frame fields of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then, from Lem. 2.8 and Def. 4.1 yields

$$F_*(U_j^{(n)}) = \left(\frac{\partial f_1}{\partial x_j}, \dots, \frac{\partial f_i}{\partial x_j} U_i^{(m)}\right) = \sum_i \frac{\partial f_i}{\partial x_j} U_i^{(m)}$$

Note $[\frac{\partial f_i}{\partial x_j}](p)$ is the $n \times m$ Jacobian matrix of F at p.

Definition 4.3: Regular

$F : \mathbb{R}^n \to \mathbb{R}^m$ is called regular if F_* is injective $\forall p \in \mathbb{R}^n$, that is, $\boldsymbol{v} \neq \boldsymbol{w} \Rightarrow F_*(\boldsymbol{v}) \neq F_*(\boldsymbol{w})$

Note By contraposition and linearity of F_*, this is equivalent to $F_*(\boldsymbol{v}) = \boldsymbol{0} \Rightarrow \boldsymbol{v} = \boldsymbol{0}$

4.4 Diffeomorphisms

Definition 4.4: Diffeomorphism

The differentiable mapping $F : U \to V$, where $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ are open, is called local diffeomorphism if its inverse $F^{-1} : V \to U$ exists and is differentiable.

Theorem 4.5: Inverse Function Theorem

Let a differentiable $F : \mathbb{R}^n \to \mathbb{R}^n$ have an injective F_* at p, then there exists $U \ni p$ where F is a local diffeomorphism.

Proof. See advanced real analysis, or MATH307 for complex version. □

4.5 Curves

Definition 4.6: Curve

A curve in $\mathbb{R}^3$ is a differentiable map from an open interval $I \subset \mathbb{R}$ given by $\alpha : I \to \mathbb{R}^3, t \mapsto \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

Note It is common to use α for both the map and the image $\alpha(I)$.

Example A straight line through $p = (p1, p2, p3) \in \mathbb{R}^3$ in the direction $\boldsymbol{v}_p = (v1, v2, v3) \neq 0$ has linear coordinate functions,

$$\alpha(t) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3), t \in \mathbb{R}$$

Definition 4.7: Velocity

Let α be a curve. Its velocity at t is the tangent vector at $\alpha(t)$

$$\alpha'(t) = \left(\frac{\mathrm{d}\alpha_1}{\mathrm{d}t}(t), \frac{\mathrm{d}\alpha_2}{\mathrm{d}t}(t), \frac{\mathrm{d}\alpha_3}{\mathrm{d}t}(t)\right) \in T_{\alpha(t)}\mathbb{R}^3$$

If $\alpha'(t) = \boldsymbol{0}$, the curve is called constant.

Note Using the Euclidean frame field, an equivalent notation to express this velocity vector is

$$\alpha'(t) = \sum_i \frac{\mathrm{d}\alpha_i}{\mathrm{d}t}(t) U_i(\alpha(t))$$

Definition 4.8: Reparametrization

Let $\alpha : I \to \mathbb{R}^3, t \mapsto \alpha(t)$. If $h : J \to I, s \mapsto h(s) = t$ is differentiable on the open $J \subset \mathbb{R}$, the composition

$$\beta = \underbrace{\alpha \circ h}_{\alpha(h)} : J \to \mathbb{R}^3, \quad s \mapsto \beta(s) = \alpha(h(s))$$

is called reparametrization of α by h. It is called orientation-preserving if $h' \geq 0$ and orientation-reversing if $h' \leq 0$.

Note β is a new curve, as a map, but represents the same image (trajectory) of α in $\mathbb{R}^3$, traversed with different velocity.

Lemma 4.9

If β is a reparametrization of α, then its velocity is given by

$$\beta'(s) = \alpha'(h(s))\frac{\mathrm{d}h}{\mathrm{d}s}(s)$$

Proof. According to the Def. 4.8 of reparametrization, we have

$$\beta(s) = \alpha(h(s)) = (\alpha_1(h(s)), \alpha_2(h(s)), \alpha_3(h(s))),$$

and by Def. 4.7 of velocity $\beta'(s) = \alpha(h)'(s)$. So the result follows immediately from the chain rule for compositions of differentiable functions in each coordinate, with $\alpha_i(h)'(s) = \alpha_i'(h(s))h'(s)$. □

Lemma 4.10

Suppose α is a curve and f a differentiable function. Then

$$\alpha'(t)[f] = \sum_i \frac{\mathrm{d}\alpha_i}{\mathrm{d}t}(t)\frac{\partial f}{\partial x_i}(\alpha(t)) = \frac{\mathrm{d}f(\alpha)}{\mathrm{d}t}(t)$$

Proof. This follows from Def. 4.7, Lem. 2.8 and the chain rule. □

Definition 4.11: Regularity of Curve

A curve $\alpha : I \to \mathbb{R}^3$ is called regular if $\alpha'(t) \neq \mathbf{0}\ \forall t \in I$

4.6 Push-forward revisited

Lemma 4.12: Push-forward and Velocity

Let $F : \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{v} = (v_1, \ldots, v_n) \in T_p\mathbb{R}^n$ at $p = (p_1, \ldots, p_n)$ and consider the curve $\gamma(t) = F(p_1 + tv_1, \ldots, p_n + tv_n)$. Then

$$F_*(\boldsymbol{v}) = \gamma'(0).$$

Proof. We have

$$\gamma(t) = F(p_1 + tv_1, \ldots, p_n + tv_n) = (f_1(t), \ldots, f_m(t)),$$

where

$$f_j(t) = f_j(p_1 + tv_1, \ldots, p_n + tv_n), 1 \leq j \leq m.$$

Hence, from Def. 4.7 of velocity and Def. 2.7 of the directional derivative,

$$\begin{aligned}\gamma'(0) &= \left(\frac{\mathrm{d}f_1(t)}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}f_m(t)}{\mathrm{d}t}\right)\Bigg|_{t=0} \\ &= (\boldsymbol{v}[f_1], \ldots, \boldsymbol{v}[f_m]) = F_*(\boldsymbol{v})\end{aligned}$$

by Def. 4.1, in the tangent space at $\gamma(0) = F(p)$ in $\mathbb{R}^m$. □

Corollary Consider the curve α in $\mathbb{R}^n$ mapped by F to a curve β in $\mathbb{R}^m$. Then as a consequence of Lem. 4.12, their velocities obey

$$\beta' = F_*(\alpha')$$

that is, the push-forward maps curve velocities.

To see this, we use Def. 4.1 of the push-forward and apply Lem. 8.3 on curve velocity to $\beta = F(\alpha) = (f_1(\alpha), ..., f_m(\alpha))$, whence

$$\begin{aligned}F_*(\alpha') &= (\alpha'[f_1], \ldots, \alpha'[f_m]) \\ &= \left(\frac{\mathrm{d}f_1(\alpha)}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}f_m(\alpha)}{\mathrm{d}t}\right) \\ &= \beta'\end{aligned}$$

5 Lecture 5 - Jan 17

5.1 Dot product

The tangent space of Euclidean space can be endowed with an inner product:

Definition 5.1: Dot product

Let $\boldsymbol{v} = (v_1, v_2, v_3), \boldsymbol{w} = (w_1, w_2, w_3) \in T_p\mathbb{R}^3$. Their inner product or dot product is

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3 \in \mathbb{R}$$

Note This can be extended to $T_p\mathbb{R}^n$.
Thus, for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^3, a, b \in \mathbb{R}$:

$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ (symmetry)

$\boldsymbol{u} \cdot (a\boldsymbol{v} + b\boldsymbol{w}) = a\boldsymbol{u} \cdot \boldsymbol{v} + b\boldsymbol{u} \cdot \boldsymbol{w}$ (bilinearity)

$\boldsymbol{v} \cdot \boldsymbol{v} \geq 0, \boldsymbol{v} \cdot \boldsymbol{v} = 0 \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$ (positive definiteness)

Definition 5.2: Length

Let $\boldsymbol{v} \in T_p\mathbb{R}^3$. Its length is the non-negative real number

$$|\boldsymbol{v}| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Note A tangent vector $\boldsymbol{u}$ with unit length, $|\boldsymbol{u}| = 1$, is called unit vector. Note also that, by positive definiteness, $|\boldsymbol{v}| = 0 \Leftrightarrow \boldsymbol{v} = 0$

Note Schwarz's ineqality

$$|\boldsymbol{v} \cdot \boldsymbol{w}| \leq |\boldsymbol{v}||\boldsymbol{w}|$$

Definition 5.3: Angle

Let $\boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^3$ be non-zero. The angle ϑ between them obeys

$$\cos(\vartheta) = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{|\boldsymbol{v}||\boldsymbol{w}|}$$

Note Two vectors $\boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^3$ are called orthogonal if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$. Moreover, unit vectors that are orthogonal are called orthonormal. (The vectors of the Euclidean frame field in each $T_p\mathbb{R}^3$ are orthonormal)

5.2 Isometries

Definition 5.4: Isometry

A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called an isometry of $\mathbb{R}^n$ if

$$\forall \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n : d(F(\boldsymbol{p}), F(\boldsymbol{q})) = d(\boldsymbol{p}, \boldsymbol{q})$$

Thus, isometries include translations and rotations of $\mathbb{R}^n$.

Note In general, isometries need not be differentiable but can also be discrete (e.g. reflections). But for differentiable isometries,

$$\forall \boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^n : F_*(\boldsymbol{v}) \cdot F_*(\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{w}$$

so lengths of and angles between tangent vectors also remain invariant under such isometries.

5.3 Frames

Definition 5.5: Frame

A set of orthonormal vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ in $T_p\mathbb{R}^3$, such that $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$ for all $1 \leq i, j \leq 3$, is called a frame at p.

Note In other words, we have the dot products

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1$$
$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = 0$$

and Def. 5.5 can obviously be generalized to arbitrary $T_p\mathbb{R}^n$, and also to position-dependent frame fields.

Recalling linear algebra, note that the $\{\boldsymbol{e}_i\}$ are linearly independent and so every vector $\boldsymbol{v} \in T_p\mathbb{R}^3$ can be expressed as linear combination of the orthonormal vectors of the frame thus,

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + (\boldsymbol{v} \cdot \boldsymbol{e}_2)\boldsymbol{e}_2 + (\boldsymbol{v} \cdot \boldsymbol{e}_3)\boldsymbol{e}_3$$

Attitude Matrix We may also define the attitude matrix A of our frame at p,

$$\begin{aligned} \boldsymbol{e}_1 &= (a_{11}, a_{12}, a_{13}), \\ \boldsymbol{e}_2 &= (a_{21}, a_{22}, a_{23}), \\ \boldsymbol{e}_3 &= (a_{31}, a_{32}, a_{33}), \end{aligned} \quad \to \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Note Since $\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \sum_k a_{ik}a_{jk}$, the attitude matrix is orthogonal, $AA^\top = I$, so its transpose is its inverse, $A^\top = A^{-1}$.

Note The attitude of the Euclidean frame $\{U_i\}$ is of course $A = I$.

Cross Product We also define the vector product or cross product, such that for all $\boldsymbol{v} = (v_1, v_2, w_3), \boldsymbol{w} = (w_1, w_2, w_3) \in T_p\mathbb{R}^3$ only (unlike $\cdot$), in terms of a symbolic determinant,

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} U_1 & U_2 & U_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \in T_p\mathbb{R}^3$$

Also, the cross product is antisymmetric so that

$$\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$$

and that its length is the area of the parallelogram spanned,

$$|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}| \sin \vartheta$$

5.4 Arc length and speed

Speed Given the dot product, we can now revisit curves $\alpha(t)$ and, in addition to the velocity vector

$\alpha'(t) \in T_{\alpha(t)}\mathbb{R}^3$, introduce speed,

$$v(t) = |\alpha'(t)| = \sqrt{\left(\frac{d\alpha_1}{dt}\right)^2 + \left(\frac{d\alpha_2}{dt}\right)^2 + \left(\frac{d\alpha_3}{dt}\right)^2}$$

Arc Length Function The arc length function along the curve $\alpha : I = [a, b] \to \mathbb{R}^3$ with $a \le t \le b$ is given by

$$s(t) = \int_a^t |\alpha'(u)|\, du, \quad \text{so that} \quad v(t) = |\alpha'(t)| = \frac{ds}{dt}$$

and the total curve length is $L(\alpha) = s(b)$.

Unit Speed Parametrization We will find it useful to reparametrize a curve in terms of its arc length so that it has unit speed:

$$\alpha(t) \mapsto \beta(s), |\beta'(s)| = 1$$

This is often really difficult!

Theorem 5.6

If α is a regular curve in $\mathbb{R}^3$, it has a unit speed parametrization β.

Proof. By assumption and Def. 4.11, we know that $\alpha'(t) \neq \mathbf{0}$ for all $t \in I$, so $|\alpha'| = \frac{ds}{dt} > 0$. From calculus (Thm. 4.5), we can invert $s(t)$ to $t(s)$ with $\frac{dt}{ds} = \left(\frac{ds}{dt}\right)^{-1}$.

Now suppose $\beta(s) = \alpha(t(s))$. We need to show that β has unit speed. Indeed, using Lem. 4.9, we get

$$|\beta'(s)| = \left|\alpha'(t(s))\frac{dt}{ds}(s)\right| = \left|\frac{ds}{dt}(t(s))\frac{dt}{ds}(s)\right| = 1.$$

$\square$

Note Since $|\alpha'| = ds > 0$ for a regular curve, a unit speed dt reparametrization is always orientation-preserving by Def. 4.8.

5.5 Vector fields on curves

Definition 5.7: Vector Field on Curve

A vector field on a curve $\alpha : I \to \mathbb{R}^3$ is $V : I \to T\mathbb{R}^3$ such that

$$V(t) = \sum_i v_i(\alpha(t))U_i(\alpha(t)) \in T_{\alpha(t)}\mathbb{R}^3$$

Note Velocity is a vector feild on a curve.

Definition 5.8: Parallel

A vector field on a curve in Euclidean space is called parallel if its component functions are constant, $V(t) = (c_1, c_2, c_3)_{\alpha(t)} \forall t \in I$.

Note Obviously, this is equivalent to $V'(t) = 0$. But we shall see later that in non-Euclidean e.g curved space, this is very different.

Definition 5.9: Acceleration

Acceleration of a differentiable curve $\alpha : I \to \mathbb{R}^3$ is the vector field

$$\alpha''(t) = \left(\frac{d^2\alpha_1}{dt^2}, \frac{d^2\alpha_2}{dt^2}, \frac{d^2\alpha_3}{dt^2}\right)_{\alpha(t)}$$

6 Lecture 6 - Jan 18

6.1 Fernet frame

Unit Tangent Let $\beta : I \to \mathbb{R}^3$ be an (arclength-parametrized) unit speed curve, $|\beta'| = 1$ for all $s \in I$. Then the unit tangent vector field on β is

$$T = \beta'$$

Curvature The curvature is defined as the non-negative real-valued function on β such that at any point

$$\kappa(s) = |T'(s)|$$

whose inverse is the radius of the osculating circle, $r_{\text{osc}} = \kappa^{-1}$

Principal Normal Now since $T \cdot T = 1$ for all $s \in I$, upon differentiation $T \cdot T' = 0$, so T' is a vector field on β which is orthogonal to T or normal to β. Hence, we define the unit or principal normal vector field on β

$$N = \frac{T'}{\kappa}$$

At every point $\beta(s)$, the vectors $\{T(s), N(s)\}$ span a plane containing the osculating circle, the osculating plane.

Binormal Next, we can define another unit vector field on β orthogonal to both T and N called the binormal,

$$B = T \times N$$

Frenet Frame Therefore, at every point $\beta(s)$, $\{T(s), N(s), B(s)\}$ are three orthonormal vectors in $T_{\beta(s)}\mathbb{R}^3$ with

$$T(s) \cdot T(s) = N(s) \cdot N(s) = B(s) \cdot B(s) = 1$$
$$T(s) \cdot N(s) = T(s) \cdot B(s) = N(s) \cdot B(s) = 0$$

and so, by Def. 5.5, they form a frame called Frenet frame.

Torsion To quantify the change of Frenet frame to the fixed Euclidean frame, we need to find the derivatives of the frame vectors. We already know $T' = \kappa N$. Next, notice that

$$B \cdot T = 0 \Rightarrow B' \cdot T = -B \cdot T' = -\kappa B \cdot N = 0$$

Moreover, $B \cdot B = 1 \Rightarrow B' \cdot B = 0$. Hence, B' is orthogonal to both T and B. Therefore, it must be parallel to N, and so we define torsion as the real-valued function τ on β such that

$$B' = -\tau N$$

Theorem 6.1: Frenet-Serret

Let $\beta : I \to \mathbb{R}^3$ be a unit speed curve with Frenet-Serret apparatus, that is, curvature $\kappa > 0$, torsion τ, and frame $\{T, N, B\}$, then
(1) $T' = \kappa N$,
(2) $N' = -\kappa T + \tau B$,
(3) $B' = -\tau N$.

Proof. (1) and (3) follow directly from the definitions of κ and τ. To show (2), expand N' in terms of the orthonormal $\{T, N, B\}$,

$$N' = (N' \cdot T) T + (N' \cdot N) N + (N' \cdot B) B.$$

Since $N \cdot N = 1$, $N' \cdot N = 0$. Moreover, from $N \cdot T = 0$,

$$N' \cdot T = -N \cdot T' = -\kappa N \cdot N = -\kappa,$$

and likewise from $N \cdot B = 0$ we obtain, as required,

$$N' \cdot B = -N \cdot B' = \tau N \cdot N = \tau$$

□

Note For a planar circle of radius a and $b = 0$, the curvature is $\kappa = \frac{1}{a}$ so, of course, $r_{\text{osc}} = a$. Also, the torsion vanishes, $\tau = 0$.

Note Crucially, this gives the derivatives $\{T', N', B'\}$ in terms of $\{T, N, B\}$ itself, paving the way for intrinsic geometry later!

6.2 Frenet approximation

To gain a better understanding of the influence of κ, τ on curve shapes in general, let us develop the Frenet approximation of $\beta = (\beta_1, \beta_2, \beta_3)$ through Taylor expansion of the β_i about $s = 0$:

$$\beta_i(s) = \beta_i(0) + \frac{\mathrm{d}\beta_i}{\mathrm{d}s}(0)s + \frac{1}{2!}\frac{\mathrm{d}^2\beta_i}{\mathrm{d}s^2}(0)s^2 + \frac{1}{3!}\frac{\mathrm{d}^3\beta_i}{\mathrm{d}s^3}(0)s^3 + \mathcal{O}\left(s^4\right).$$

In this discussion, it is helpful to employ the canonical isomorphism $\mathbb{R}^3 \simeq T_{\beta(0)}\mathbb{R}^3$ so that, up to third order,

$$\beta(s) - \beta(0) \simeq s\beta'(0) + \frac{s^2}{2}\beta''(0) + \frac{s^3}{6}\beta'''(0)$$

This curve approximation yields upon differentiation $T = \beta'$,

$$T(s) \simeq \beta'(0) + s\beta''(0) + \frac{s^2}{2}\beta'''(0), T'(s) \simeq \beta''(0) + s\beta'''(0)$$

so $\beta'(0) = T(0)$ and $\beta''(0) = T'(0) = \kappa(0)N(0)$. Similarly,

$$\beta'''(0) = T''(0) = (\kappa N)'(0) = \kappa(0)N'(0) + \kappa'(0)N(0)$$

, so using statement (2) of the Frenet-Serret Thm. 6.1, we obtain

$$\beta'''(0) = -\kappa(0)^2 T(0) + \kappa'(0)N(0) + \kappa(0)\tau(0)B(0)$$

expressing the vector $\beta'''(0)$ in terms of the Frenet frame at $\beta(0)$.

Retaining only leading terms in s for each of $\{T(0), N(0), B(0)\}$, our curve approximation becomes

$$\beta(s) \simeq \beta(0) + sT(0) + \kappa(0)\frac{s^2}{2}N(0) + \kappa(0)\tau(0)\frac{s^3}{6}B(0) = \hat{\beta}(s)$$

called the Frenet approximation $\hat{\beta}$ of β. Notice, then:

- The linear approximation of the curve, $\beta(0) + sT(0)$, represents its straight tangent line at $\beta(0)$.
- The quadratic approximation, $\beta(0) + sT(0) + \kappa(0)\frac{s^2}{2}N(0)$, is a parabola in the osculating plane spanned by $\{T(0), N(0)\}$ and defined by curvature $\kappa(0)$.
- The cubic term in the $B(0)$ direction proportional to torsion $\tau(0)$ yields the curve motion out of the osculating plane.

6.3 Planar curves

So curvature seems to measure deviation from straightness. Also, it seems that torsion measures the deviation of a curve from being planar, that is, remaining in a plane $(\beta(s) - p) \cdot \boldsymbol{n} = 0$ for all $s \in I$ and fixed $p, \boldsymbol{n} \neq \mathbf{0}$ (with canonical isomorphism). Indeed,

Lemma 6.2

Let $\beta : I \to \mathbb{R}^3$ be a unit speed curve with $\kappa > 0$. Then β is planar if, and only if, $\tau = 0$.

Proof. Suppose, firstly, that β is a planar curve. Hence $(\beta(s) - p) \cdot \boldsymbol{n} = 0$ and upon differentiating, $\beta'(s) \cdot \boldsymbol{n} = \beta''(s) \cdot \boldsymbol{n} = 0$ for all s.

So both $T = \beta'$ and $N = \beta''/\kappa$ are orthogonal to $\boldsymbol{n}$ and therefore $B = \pm\boldsymbol{n}/|\boldsymbol{n}|$, but then $B' = 0$ and so by definition, indeed, $\tau = 0$.

Secondly, suppose $\tau = 0$. Then by definition, $B' = 0$ so B is parallel i.e. a constant vector. We claim that β is in the plane $(\beta(s) - \beta(0)) \cdot B = 0$. Indeed, consider the real-valued function $f(s) = (\beta(s) - \beta(0)) \cdot B$ for all $s \in I$. Then $f(0) = 0$ and $f'(s) = \beta'(s) \cdot B = T \cdot B = 0$ so f vanishes identically. □

6.4 Osculating circle

Lemma 6.3: Osculating Circle

Let $\beta : I \to \mathbb{R}^3$ be a unit speed curve with constant $\kappa > 0$ and $\tau = 0$. Then β is part of a circle of radius $1/\kappa$.

Proof. Since $\tau = 0$ by assumption, we know from Lem. 6.2 that β is planar. Hence, we must show that β is equidistant from some fixed point, that is, the centre of a circle.

Indeed, consider the curve $\gamma = \beta + \frac{1}{\kappa}N$. By Frenet-Serret Thm. 6.1

$$\gamma' = \beta' + \frac{1}{\kappa}N' = T - T = 0$$

so γ is constant for all s, that is, a point. And since

$$d(\gamma, \beta(s)) = \left|\frac{1}{\kappa}N\right| = \frac{1}{\kappa}$$

we see that it is the circle centre with the required radius. □

Example Suppose the unit speed curve β remains on a sphere of radius a. Show that its curvature satisfies $\kappa \geq \frac{1}{a}$.

Proof. Again using the canonical isomorphism, we have $\beta \cdot \beta = a^2$ and so by differentiating once $\beta \cdot T = 0$ and again, $T \cdot T + \beta \cdot T' = 0$.

Now by Thm. 6.1, this simplifies to $\kappa\beta \cdot N = -1$. Taking the absolute value, and noting that $\kappa \geq 0$ and $|\beta \cdot N| \leq |\beta||N| = a$ by Schwarz's inequality, the result follows. □

7 Lecture 7 - Jan 19

7.1 Arbitrary speed curve

Let $\alpha : I \to \mathbb{R}^3$ be a regular curve with an arbitrary parametrization $t \mapsto \alpha(t)$. Also consider an arc length reparametrization such that

$$\alpha(t) = \beta(s(t)).$$

In other words, t and $s(t)$ refer to the same point in $\mathbb{R}^3$, and the Frenet apparatus of α and β agree so that we write, for short, $\kappa(t) = \kappa(s), \tau(t) = \tau(s), T(t) = T(s), N(t) = N(s), B(t) = B(s)$

Lemma 7.1

Let $\alpha : I \to \mathbb{R}^3$ be a regular curve with speed $v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}$ and Frenet-Serret apparatus $\kappa > 0, \tau$, and $\{T, N, B\}$, then

(1) $T' = \kappa v N$,

(2) $N' = -\kappa v T + \tau v B$,

(3) $B' = -\tau v N$.

Proof. Remark: Let V be a vector field on α. Given a reparametrization h, then $V(h)$ is a vector field on $\alpha(h)$ with $V(h)' = h'V'(h)$.

To show that $V(h)$ is a vector field on $\alpha(h)$, we need to show $V(h(t)) \in T_{\alpha(h(t))}\mathbb{R}^3$. This is true since V is a vector field on α, and $V(s) \in T_{\alpha(s)}\mathbb{R}^3$. We also have

$$V(h(t)) = \sum_i v_i(\alpha(h(t)))U_i(\alpha(h(t)))$$

Then,

$$\begin{aligned} V(h(t))' &= \sum_i v_i(\alpha(h(t)))'U_i(\alpha(h(t))) \\ &= \sum_i h'(t)v_i'(\alpha(h(t)))U_i(\alpha(h(t))) \\ &= h'(t)\sum_i v_i'(\alpha(h(t)))U_i(\alpha(h(t))) \\ &= h'(t)V'(h(t)) \end{aligned}$$

As shown in the remark, an arc length reparametrization changes the derivatives of vector fields along curves according to

$$T'(t) = T'(s)\frac{\mathrm{d}s}{\mathrm{d}t}, N'(t) = N'(s)\frac{\mathrm{d}s}{\mathrm{d}t}, B'(t) = B'(s)\frac{\mathrm{d}s}{\mathrm{d}t}$$

so the result follows directly from the Frenet-Serret Thm. 6.1. □

Lemma 7.2

The velocity and acceleration of a regular curve α are

$$\alpha' = vT, \quad \alpha'' = v'T + \kappa v^2 N.$$

Proof. By the same token as before, or using Lem. 4.9, we have

$$\alpha'(t) = \alpha'(s)\frac{\mathrm{d}s}{\mathrm{d}t} = v(t)T(t)$$

as required. Differentiating again and using Lem. 7.2,

$$\alpha''(t) = v'(t)T(t)+v(t)T'(t) = v'(t)T(t)+\kappa(t)v(t)^2N(t)$$

□

Note Although the acceleration is, of course, a vector in $T_{\alpha(t)}\mathbb{R}^3$ according to Def. 5.8, it has no binormal component but lies in the osculating plane of each $\alpha(t)$. Writing

$$\alpha'' = a_T T + a_N N, \qquad \text{we call } a_T = v' \text{ and } a_N = \kappa v^2$$

the tangential and normal components of α'', respectively.

Theorem 7.3: General Apparatus

Let α be a regular curve in $\mathbb{R}^3$. Then $N = B\times T$ and

$$T = \frac{\alpha'}{|\alpha'|}, B = \frac{\alpha'\times\alpha''}{|\alpha'\times\alpha''|},$$
$$\kappa = \frac{|\alpha'\times\alpha''|}{|\alpha'|^3}, \tau = \frac{(\alpha'\times\alpha'')\cdot\alpha'''}{|\alpha'\times\alpha''|^2}$$

Proof. By regularity, $v>0$. The statements for N and T follow from definition and the previous result. Moreover, using Lem. 7.2,

$$\alpha'\times\alpha'' = vv'T\times T + \kappa v^3 T\times N = \kappa v^3 B,$$

yielding the claims for B and κ, since $|B|=1$ and $\kappa\geq 0$.

To find $(\alpha'\times\alpha'')\cdot\alpha'''$ we need only the B-component of $\alpha''' = \left(v'T+\kappa v^2 N\right)'$ which comes from N' by Lem. 7.1 and is $\tau\kappa v^3$. Hence, the claim for τ follows from

$$(\alpha'\times\alpha'')\cdot\alpha''' = \left(\kappa v^3 B\right)\cdot\left(\tau\kappa v^3 B\right) = \tau\kappa^2 v^6.$$

□

Definition 7.4: Cylindrical Helix

A cylindrical helix is a regular curve α in $\mathbb{R}^3$ whose unit tangent T has a constant angle ϑ with some non-zero fixed unit vector U, $T(t)\cdot U(\alpha(t)) = \cos\vartheta = \text{const.} \neq 0$ for all t.

Note Rulings of the cylinder are parallel to U. A circular cylinder gives rise to the special case of circular helix encountered earlier.

Theorem 7.5

A regular curve α with $\kappa>0$ is a cylindrical helix if, and only if,

$$\frac{\tau}{\kappa} = \text{const.} \neq 0.$$

Proof. For convenience and without loss of generality, we may assume that α has unit speed parametrization.

Suppose, firstly, that α is a cylindrical helix. Now by Def. 7.4

$$0 = (T\cdot U)' = T'\cdot U = \kappa N\cdot U$$

but then $\kappa>0$ implies $N\cdot U = 0$, so in each $T_{\alpha(s)}\mathbb{R}^3$ we have $U = \cos\vartheta T + \sin\vartheta B$. Upon differentation and using Thm. 6.1,

$$0 = (\kappa\cos\vartheta - \tau\sin\vartheta)N \Rightarrow \frac{\tau}{\kappa} = \cot\vartheta = \text{ const.}$$

Conversely, suppose $\frac{\tau}{\kappa} = \text{const.} = \cot\vartheta$ for some ϑ. Now if U is chosen as above, then by the same token $U'=0$ so $U = \text{const.}$ is a parallel vector field and α a cylindrical helix by Def. 7.4. □

Finally, let us summarize a some of the earlier results to give a classification of certain special regular curves in $\mathbb{R}^3$:

$$\begin{aligned}
\text{straight line } &\Leftrightarrow \kappa = 0,\\
\text{planar curve } &\Leftrightarrow \tau = 0,\\
\text{circle } &\Leftrightarrow \tau = 0 \text{ and } \kappa = \text{const.} > 0,\\
\text{circular helix } &\Leftrightarrow \tau = \text{const.} > 0 \text{ and } \kappa = \text{const.} > 0,\\
\text{cylindrical helix } &\Leftrightarrow \frac{\tau}{\kappa} = \text{const.} \neq 0.
\end{aligned}$$

7.2 Covariant derivative

Definition 7.6: Covariant Derivative

Let W be a differentiable vector field on $\mathbb{R}^3$ and $\boldsymbol{v} = (v_1,v_2,v_3)\in T_p\mathbb{R}^3$, with $W(t)$ on $\alpha(t) = (p_1+tv_1, p_2+tv_2, p_3+tv_3)$, then the covariant derivative of W with respect to $\boldsymbol{v}$ at p is the vector

$$\nabla_{\boldsymbol{v}} W = W'(0)\in T_p\mathbb{R}^3$$

Note Thus, $\nabla_{\boldsymbol{v}}W$ yields the initial rate of change of W when moving from p in the $\boldsymbol{v}$-direction (with canonical isomorphism).

Lemma 7.7

Let $W = (w_1,w_2,w_3)$ and $\boldsymbol{v}\in T_p\mathbb{R}^3$ as above, then

$$\nabla_{\boldsymbol{v}}W = \sum_i \boldsymbol{v}\left[w_i\right]U_i(p)$$

Proof. Now $W(t) = \sum_i w_i(\alpha(t))U_i(\alpha(t))$ by Def. 5.7 of a vector field on a curve, so by Def. 7.6 of the covariant derivative,

$$\nabla_{\boldsymbol{v}} W = \sum_i \frac{\mathrm{d}w_i(\alpha(t))}{\mathrm{d}t}(0)U_i(p),$$
$$\text{and } \frac{\mathrm{d}w_i(\alpha(t))}{\mathrm{d}t}(0) = \boldsymbol{v}\,[w_i]$$

by Def. 2.7 of the directional derivative, so the result follows. □

Theorem 7.8

Let X, Y be differentiable vector fields on $\mathbb{R}^3$, $\boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^3$, $a, b \in \mathbb{R}$, and $f \in C^\infty\mathbb{R}^3$. Then for all $p \in \mathbb{R}^3$, we have
(1) $\nabla_{a\boldsymbol{v}+b\boldsymbol{w}} X = a\nabla_{\boldsymbol{v}} X + b\nabla_{\boldsymbol{w}} X$,
(2) $\nabla_{\boldsymbol{v}}(aX + bY) = a\nabla_{\boldsymbol{v}} X + b\nabla_{\boldsymbol{v}} Y$,
(3) $\nabla_{\boldsymbol{v}}(fX) = \boldsymbol{v}[f]X(p) + f(p)\nabla_{\boldsymbol{v}} X$,
(4) $v[X \cdot Y] = \nabla_{\boldsymbol{v}} X \cdot Y(p) + X(p) \cdot \nabla_{\boldsymbol{v}} Y$.

Note Claims (1), (2) show linearity and (3), (4) are product rules.

Proof. These follow mostly from Lem. 7.7 and Thm. 2.9.

(1) With x_i as the component functions of X (not Euclidean coordinates):

$$\begin{aligned}
\nabla_{a\boldsymbol{v}+b\boldsymbol{w}} X &= \sum_i (a\boldsymbol{v} + b\boldsymbol{w})\,[x_i]\,U_i(p) \\
&= \sum_i (a\boldsymbol{v}\,[x_i] + b\boldsymbol{w}\,[x_i])\,U_i(p) \\
&= a\sum_i \boldsymbol{v}\,[x_i]\,U_i(p) + b\sum_i \boldsymbol{w}\,[x_i]\,U_i(p) \\
&= a\nabla_{\boldsymbol{v}} X + b\nabla_{\boldsymbol{w}} X
\end{aligned}$$

(2)

$$\begin{aligned}
\nabla_{\boldsymbol{v}}(aX + bY) &= \sum_i \boldsymbol{v}\,[ax_i + by_i]\,U_i(p) \\
&= a\sum_i \boldsymbol{v}\,[x_i]\,U_i(p) + v\sum_i \boldsymbol{v}\,[y_i]\,U_i(p) \\
&= a\nabla_{\boldsymbol{v}} X + b\nabla_{\boldsymbol{v}} Y
\end{aligned}$$

(3)

$$\begin{aligned}
\nabla_{\boldsymbol{v}}(fX) &= \sum_i \boldsymbol{v}[fx_i]U_i(p) \\
&= \sum_i (\boldsymbol{v}[f]x_i(p) + f(p)\boldsymbol{v}[x_i])U_i(p) \\
&= \sum_i \boldsymbol{v}[f]x_i(p)U_i(p) + \sum_i f(p)\boldsymbol{v}[x_i]U_i(p) \\
&= \boldsymbol{v}[f]X(p) + f(p)\nabla_{\boldsymbol{v}} X
\end{aligned}$$

(4) We have $X \cdot Y = \sum_i x_i y_i$ and,

$$\begin{aligned}
\boldsymbol{v}[X \cdot Y] &= \boldsymbol{v}\left[\sum_i x_i y_i\right] \\
&= \sum_i \boldsymbol{v}[x_i]y_i(p) + \sum_i \boldsymbol{v}[y_i]x_i(p) \\
&= \nabla_{\boldsymbol{v}} X \cdot Y(p) + X(p) \cdot \nabla_{\boldsymbol{v}} Y
\end{aligned}$$

□

Note Our Def. 7.6 of the covariant derivative and its properties as stated in Thm. 7.8 are pointwise. As before, can extend them to vector fields such that $V(p) = \boldsymbol{v} \in T_p\mathbb{R}^3$ with covariant derivative

$$\nabla_V W = \sum_i V\,[w_i]\,U_i(p) \quad \forall p \in \mathbb{R}^3$$

For computations, recall from Lem. 2.8 that $U_i\,[w_j] = \frac{\partial w_j}{\partial x_i}$.

Theorem 7.9

Let X, Y, V, W be differentiable vector fields on $\mathbb{R}^3$, $a, b \in \mathbb{R}$, and $f, g \in C^\infty\mathbb{R}^3$. Then
(1) $\nabla_{fV+gW} X = f\nabla_V X + g\nabla_W X$,
(2) $\nabla_V(aX + bY) = a\nabla_V X + b\nabla_V Y$,
(3) $\nabla_V(fX) = V[f]X + f\nabla_V X$,
(4) $V[X \cdot Y] = \nabla_V X \cdot Y + X \cdot \nabla_V Y$.

Proof. This follows directly from Thm. 7.8. □

Note the linearities in (1) and (2) are different, cf. also (3).

8 Lecture 8 - Jan 20

8.1 Frame fields

Definition 8.1: Frame Fields

A frame field on $\mathbb{R}^3$ consists of a set of differentiable vector fields $\{E_1, E_2, E_3\}$ such that

$$E_i \cdot E_j = \delta_{ij}, \quad 1 \le i, j \le 3$$

with Kronecker delta $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Note The definition in $\mathbb{R}^3$ can obviously be generalized to $\mathbb{R}^n$.

Note : In other words, the vectors constituting a frame field on $\mathbb{R}^3$ are orthonormal in $T_p\mathbb{R}^3$ for each $p \in \mathbb{R}^3$.

Lemma 8.2

Suppose $\{E_1, E_2, E_3\}$ is a frame field on $\mathbb{R}^3$. Then
(1) Any differentiable vector field V on $\mathbb{R}^3$ can be written

$$V = \sum_i v_i E_i \quad \text{with} \quad v_i = V \cdot E_i \in C^\infty \mathbb{R}^3,$$

called coordinate functions of V with respect to the frame field.
(2) If $V = \sum_i v_i E_i$ and $W = \sum_i w_i E_i$ then

$$V \cdot W = \sum_i v_i w_i \in C^\infty \mathbb{R}^3, \quad |V|^2 = \sum_i v_i^2.$$

Proof. Use linear conbination and orthonormality. □

8.2 Cylindrical & Spherical frame fields

Cylindrical Coordinate The cylindrical coordinate (r, ϑ, z) have

$$0 < r < \infty, 0 \leq \vartheta < 2\pi, -\infty < z < \infty$$

Cylindrical Frame Field We have

$$\begin{aligned} r: &\quad E_1^{\text{cyl}} = \cos\vartheta U_1 + \sin\vartheta U_2 \\ \vartheta: &\quad E_2^{\text{cyl}} = -\sin\vartheta U_1 + \cos\vartheta U_2 \\ z: &\quad E_3^{\text{cyl}} = U_3 \end{aligned}$$

Spherical Coordinate Consider spherical coordinate $(\rho, \vartheta, \varphi)$ with

$$0 < \rho < \infty, 0 \leq \vartheta < 2\pi, -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

Spherical Frame Field We have

$$\begin{aligned} \rho: &\quad E_1^{\text{sph}} = \cos\varphi(\cos\vartheta U_1 + \sin\vartheta U_2) + \sin\varphi U_3, \\ \vartheta: &\quad E_2^{\text{sph}} = -\sin\vartheta U_1 + \cos\vartheta U_2, \\ \varphi: &\quad E_3^{\text{sph}} = -\sin\varphi(\cos\vartheta U_1 + \sin\vartheta U_2) + \cos\varphi U_3. \end{aligned}$$

8.3 Connection forms

Ultimately, we would like to understand the intrinsic properties of a frame field $\{E_1, E_2, E_3\}$, without reference to the fixed Euclidean frame field $\{U_1, U_2, U_3\}$.

Covariant Derivatives of Frame Fields To this end, we begin by fixing a $\boldsymbol{v} \in T_p\mathbb{R}^3$ and express the covariant derivatives of $\{E_1, E_2, E_3\}$ in terms of $\{E_1, E_2, E_3\}$ itself - which will be similar to the Frenet-Serret Thm. 6.1! Thus,

$$\begin{aligned} \nabla_{\boldsymbol{v}} E_1 &= c_{11}E_1(p) + c_{12}E_2(p) + c_{13}E_3(p) \\ \nabla_{\boldsymbol{v}} E_2 &= c_{21}E_1(p) + c_{22}E_2(p) + c_{23}E_3(p) \\ \nabla_{\boldsymbol{v}} E_3 &= c_{31}E_1(p) + c_{32}E_2(p) + c_{33}E_3(p) \end{aligned}$$

with coefficients $c_{ij} = \nabla_v E_i \cdot E_j(p)$.

Lemma 8.3: Connection Forms

Suppose $\{E_1, E_2, E_3\}$ is a frame field on $\mathbb{R}^3$ and, for any $\boldsymbol{v} \in T_p\mathbb{R}^3$ at a point $p \in \mathbb{R}^3$, let the real number

$$\omega_{ij}(\boldsymbol{v}) = \nabla_{\boldsymbol{v}} E_i \cdot E_j(p), \quad 1 \leq i, j \leq 3$$

Then each ω_{ij} is an antisymmetric 1-form, $\omega_{ij} = -\omega_{ji}$, called connection form of the frame field, at p.

Proof. Firstly, we need to establish that each ω_{ij} is a 1-form according to Def. 3.1. Now $\omega_{ij}(\boldsymbol{v})$ for each $\boldsymbol{v} \in T_p\mathbb{R}^3$ is a number, and we also need to check linearity, that is, for all $\boldsymbol{v}, \boldsymbol{w} \in T_p\mathbb{R}^3, a, b \in \mathbb{R}$,

$$\omega_{ij}(a\boldsymbol{v} + b\boldsymbol{w}) = a\omega_{ij}(\boldsymbol{v}) + b\omega_{ij}(\boldsymbol{w})$$

Indeed, by Thm. 7.8 on covariant derivatives, we have

$$\begin{aligned} \omega_{ij}(a\boldsymbol{v} + b\boldsymbol{w}) &= \nabla_{a\boldsymbol{v}+b_{\boldsymbol{w}}} E_i \cdot E_j(p) \\ &= (a\nabla_{\boldsymbol{v}} E_i + b\nabla_{\boldsymbol{w}} E_i) \cdot E_j(p) \\ &= a\nabla_{\boldsymbol{v}} E_i \cdot E_j(p) + b\nabla_{\boldsymbol{w}} E_i \cdot E_j(p) \\ &= a\omega_{ij}(\boldsymbol{v}) + b\omega_{ij}(\boldsymbol{w}) \end{aligned}$$

Secondly, we need to prove antisymmetry. Again from Thm. 7.8, also orthonormality of the frame field, and symmetry of the dot product, we obtain the required statement,

$$0 = \boldsymbol{v}[E_i \cdot E_j] = \nabla_{\boldsymbol{v}} E_i \cdot E_j(p) + E_i(p) \cdot \nabla_{\boldsymbol{v}} E_j = \omega_{ij}(\boldsymbol{v}) + \omega_{ji}(\boldsymbol{v})$$

□

Theorem 8.4: Connection Equations

Let $\omega_{ij}, 1 \leq i, j \leq 3$ be the connection 1-forms of the frame $\{E_1, E_2, E_3\}$ on $\mathbb{R}^3$. Then for any vector field V on $\mathbb{R}^3$,

$$\nabla_V E_i = \sum_j \omega_{ij}(V) E_j, \quad 1 \leq i \leq 3,$$

called the connection equations of $\{E_1, E_2, E_3\}$.

Note Each of the three $\nabla_V E_i$ is a vector field on $\mathbb{R}^3$.

Proof. This follows immediately from orthonormal expansion at each p,

$$\begin{aligned}\nabla_{V(p)} E_i &= \sum_j \left(\nabla_{V(p)} E_i \cdot E_j(p)\right) E_j(p) \\ &= \sum_j \omega_{ij}(V(p)) E_j(p)\end{aligned}$$

□

Note There are only 3 independent connection forms in $\mathbb{R}^3$. Others can be generated by antisymmetry.

Connection Form Matrix It is helpful to employ a skew-symmetric matrix to represent the ω_{ij} as components of the connection form matrix

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}$$

Note Of course, connection forms can be generalized to $\mathbb{R}^n$ (and far beyond!). In n dimensions, there are $\frac{1}{2}n(n-1)$ components.

Let us now return to our starting point, the covariant derivatives of the frame field. In terms of the connection forms, we obtain

$$\begin{aligned} \nabla_V E_1 &= && \omega_{12}(V) E_2 && +\omega_{13}(V) E_3 \\ \nabla_V E_2 &= -\omega_{12}(V) E_1 && && +\omega_{23}(V) E_3 \\ \nabla_V E_3 &= -\omega_{13}(V) E_1 && -\omega_{23}(V) E_2 && \end{aligned}$$

Recall from Lecture 5 that we define the attitude matrix $A = [a_{ij}]$ of a frame with respect to the fixed Euclidean frame such that $a_{ij} = E_i \cdot U_j$ at each p. Let us also define the matrix of differentials $\mathrm{d}A = [\mathrm{d}a_{ij}]$. Then the following is useful to obtain the ω_{ij}:

Theorem 8.5

Given the attitude matrix A of a frame field $\{E_1, E_2, E_3\}$, its connection form matrix is the matrix product

$$\omega = \mathrm{d}A\ A^\top$$

Proof. Note that the i-th row vector $\boldsymbol{a}_i$ of A is E_i written in terms of the standard Euclidean frame field $\{U_1, U_2, U_3\}$. By beaking down the matrix multiplication, we have that for any $\boldsymbol{v} \in T_p\mathbb{R}^3$ at a point $p \in \mathbb{R}^3$,

$$\begin{aligned}\omega_{ij}(\boldsymbol{v}) &= \sum_k a_{jk}(p)\ \mathrm{d}a_{ik}(\boldsymbol{v}) = \boldsymbol{a}_j(p) \cdot \mathrm{d}\boldsymbol{a}_i(\boldsymbol{v}) \\ &= E_j(p) \cdot \mathrm{d}E_i(\boldsymbol{v}) = E_j(p) \cdot \nabla_{\boldsymbol{v}} E_i\end{aligned}$$

□

8.4 Coframes

Definition 8.6: Coframe Field

Let $\{E_1, E_2, E_3\}$ be a frame field on $\mathbb{R}^3$. Then its dual or coframe field $\{\theta_1, \theta_2, \theta_3\}$ comprises 1-forms such that, at any point p,

$$\theta_i(\boldsymbol{v}) = \boldsymbol{v} \cdot E_i(p) \quad \forall \boldsymbol{v} \in T_p\mathbb{R}^3, 1 \le i \le 3.$$

Note This implies the duality condition $\theta_j(E_i) = E_i \cdot E_j = \delta_{ij}$.

Example Given the Euclidean frame field $\{U_1, U_2, U_3\}$, we have

$$\mathrm{d}x_i(\boldsymbol{v}) = v_i = \boldsymbol{v} \cdot U_i$$

so $\{\mathrm{d}x_1,\ \mathrm{d}x_2,\ \mathrm{d}x_3\}$ is indeed its dual coframe field.

Lemma 8.7

Let $\{\theta_1, \theta_2, \theta_3\}$ be a coframe field of $\{E_1 E_2, E_3\}$ on $\mathbb{R}^3$. Then any 1-form ϕ on $\mathbb{R}^3$ has the unique expansion

$$\phi = \sum_i \phi(E_i)\, \theta_i.$$

Proof. We check this by applying any differentiable vector field V,

$$\begin{aligned}\left(\sum_i \phi(E_i)\, \theta_i\right)(V) &= \sum_i \phi(E_i)\, \theta_i(V) \\ &= \phi\left(\sum_i \theta_i(V) E_i\right) = \phi(V)\end{aligned}$$

□

Note Given the attitude matrix $A = [a_{ij}]$ with $E_i = \sum_j a_{ij} U_j$, Lem. 8.7 implies that each form of the coframe can be written

$$\begin{aligned}\theta_i &= \sum_j \theta_i(U_j)\, \mathrm{d}x_j = \sum_j (U_j \cdot E_i)\, \mathrm{d}x_j \\ &= \sum_j \left(U_j \cdot \sum_k a_{ik} U_k\right) \mathrm{d}x_j \\ &= \sum_j \left(\sum_k a_{ik}\delta_{jk}\right) \mathrm{d}x_j = \sum_j a_{ij}\ \mathrm{d}x_j \quad (*)\end{aligned}$$

Hence, the expansions of the $\{E_i\}$ and $\{\theta_i\}$ with respect to the Euclidean frame and coframe fields have the same coefficients. Using vector and matrix notation, we can also write $(*)$ as

$$\theta = A\ \mathrm{d}x$$

Now the derivatives of the $\{\theta_i\}$ obey a fundamentally relationship with the connection forms ω_{ij} called structural equations:

Theorem 8.8: (Cartan)

Let $\{E_i\}$ be a frame field on $\mathbb{R}^3$ with coframe $\{\theta_i\}$ and connection forms $\omega_{ij}, 1 \le i, j \le 3$. Then the structural equations are
(1) $\mathrm{d}\theta_i = \sum_j \omega_{ij} \wedge \theta_j$
(2) $\mathrm{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$.

Proof. We apply matrix notation, recalling that $A^T = A^{-1}$. Then (1) is obtained using the nil-potency of the exterior derivative, $d^2x = 0$,

$$\mathrm{d}\theta = \mathrm{d}(A\ \mathrm{d}x) = \mathrm{d}A\ \mathrm{d}x = \mathrm{d}A \underbrace{A^T A}_{I}\ \mathrm{d}x = \omega\theta.$$

For (2), we use antisymmetry, $\omega = -\omega^T$, and recall a general property of the transposition of matrix products, $(AB)^T = B^T A^T$:

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}\left(\mathrm{d}AA^T\right) = -\mathrm{d}A\ \mathrm{d}A^T \\ &= -\mathrm{d}A \underbrace{A^T A}_{I}\ \mathrm{d}A^T = -\omega\omega^T = \omega\omega \end{aligned}$$

□

Note Thus, the structural equations of Thm. 8.8 are the dual counterpart of the connection equations Thm. 8.4.

Example The coframe and connection forms of the spherical frame field are

$$\begin{aligned} &\theta_1^{\text{sph}} = \mathrm{d}\varphi, && \omega_{12}^{\text{sph}} = \cos\varphi \mathrm{d}\vartheta \\ &\theta_2^{\text{sph}} = \rho\cos\varphi \mathrm{d}\vartheta, && \omega_{13}^{\text{sph}} = \mathrm{d}\varphi \\ &\theta_3^{\text{sph}} = \rho \mathrm{d}\varphi, && \omega_{23}^{\text{sph}} = \sin\varphi \mathrm{d}\vartheta \end{aligned}$$

9 Lecture 8 - Jan 24

9.1 Surfaces

Definition 9.1: Patch and Chart

A 2-dimensional coordinate patch $(\boldsymbol{x}, D)$ is given by an injective regular mapping $\mathrm{x} : D \to \mathbb{R}^3$ on an open $D \subset \mathbb{R}^2$. If $\boldsymbol{x}$ has a continuous inverse, we call it proper patch or chart.

Note We will see the generalization to manifolds later. A chart gives a homeomorphism to Euclidean coordinates. In differential geometry, in fact, we often have a diffeomorphism.

Definition 9.2: Surfaces

A surface in $\mathbb{R}^3$ is a non-empty subset $M \subset \mathbb{R}^3$ such that for each $p \in M$ there is a chart $(\boldsymbol{x}, D)$ so that $\boldsymbol{x}(D) \subset M$ contains a neighbourhood of p. If $M = \boldsymbol{x}(D)$, we call it a simple surface.

9.2 Monge patch

Definition 9.3: Monge Patch

A patch $(\boldsymbol{x}, D)$ with differentiable $f : D \to \mathbb{R}$ such that

$$\boldsymbol{x}(u, v) = (u, v, f(u, v)), (u, v) \in D \subset \mathbb{R}^2,$$

is a proper patch or chart, called Monge patch.

Note This patch is again proper, by the same token as above.

Note Thus, the graph of f is the image of the Monge patch, that is $x(D) \subset \mathbb{R}^3$, and hence a simple surface by Def. 9.2.

Theorem 9.4: Level Surfaces

Let $g : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and $c \in \mathbb{R}$. $M : g(x, y, z) = c$ is a surface if the differential $\mathrm{d}g \neq 0$ at any point of M.

Proof. By assumption, for an arbitrary point $p \in M$,

$$\mathrm{d}g(p) = \frac{\partial g}{\partial x}(p)\mathrm{d}x + \mathrm{d}g + \frac{\partial g}{\partial y}(p)\mathrm{d}y + \frac{\partial g}{\partial z}(p)\mathrm{d}z \neq 0.$$

Say $\frac{\partial g}{\partial z}(p) \neq 0$. Then, by the Implicit Function Theorem, in a neighbourhood of (u, v) there is a differentiable h such that $(u, v, h(u, v))$ is in M. i.e. there is a Monge patch, and M is a surface, as required. □

9.3 Parametrization

Parameter Curves Let $\boldsymbol{x}(u, v) = (x_1, x_2, x_3)$ on D be a patch of M. Now by fixing either $v = v_0$ or $u = u_0$, $\boldsymbol{x}$ gives rise to $u-$ and $v-$parameter curves, respectively, through the point $p = \boldsymbol{x}(u_0, v_0) \in M$,

$$u \mapsto \boldsymbol{x}(u, v_0), \quad v \mapsto \boldsymbol{x}(u_0, v).$$

Partial Velocities The partial velocities are tangent to these curves at p,

$$\boldsymbol{x}_u(u_0, v_0) = \left(\frac{\partial x_1(u, v_0)}{\partial u}, \frac{\partial x_2(u, v_0)}{\partial u}, \frac{\partial x_3(u, v_0)}{\partial u}\right)_{u=u_0} \in T_p\mathbb{R}^3$$

$$\boldsymbol{x}_v(u_0, v_0) = \left(\frac{\partial x_1(u_0, v)}{\partial v}, \frac{\partial x_2(u_0, v)}{\partial v}, \frac{\partial x_3(u_0, v)}{\partial v}\right)_{v=v_0} \in T_p\mathbb{R}^3$$

Definition 9.5: Parametrization

A regular mapping $\boldsymbol{x} : D \to \mathbb{R}^3$ whose images lies in a surface, $\boldsymbol{x}(D) \subset M$ is called parametrization of $\boldsymbol{x}(D)$ in M.

Note If $x(D) = M$, we have a parametrization of the surface M. Given M, its parametrization is, of course, not unique.

Note By Def. 9.1, a patch is an injective parametrization.

Regular Revisit Recall that, by Def. 4.3, $\boldsymbol{x}(u,v) = (x_1, x_2, x_3)$ is regular if its push-forward $\boldsymbol{x}_*$ is injective, that is, for $\boldsymbol{v} = (v_1, v_2) \in T_{(u,v)}D$,

$$\boldsymbol{x}_*(\boldsymbol{v}) = v_1 \underbrace{\left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right)}_{\boldsymbol{x}_u} + v_2 \underbrace{\left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right)}_{\boldsymbol{x}_v} = \mathbf{0}$$

$$\Rightarrow \boldsymbol{v} = \mathbf{0}.$$

Thus, $\boldsymbol{x}$ is regular if its partial velocities are linearly independent.

In other words, $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ span a plane in $\mathbb{R}^3$, the tangent plane to M at $x(u,v)$ (more later). So here, regularity is also equivalent to

$$\boldsymbol{x}_u \times \boldsymbol{x}_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix} \neq \mathbf{0}$$

In fact, $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is a normal vector to the tangent plane.

Example A standard example is again the unit sphere. So, using spherical coordinates, we have $\rho = 1, u = \vartheta, v = \varphi$ and

$$\boldsymbol{x}(\vartheta, \varphi) = (\cos\vartheta\cos\varphi, \sin\vartheta\cos\varphi, \sin\varphi),$$
$$D = (-\pi, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

with longitude ϑ and latitude φ.

As in geography, a semicircle $\vartheta = \vartheta_0$ is called meridian of longitude, and a circle $\varphi = \varphi_0$ parallel of latitude.

Note D needs to be open and we exclude the meridian from the north to the south pole at $\vartheta = \pi$ (i.e. the international date line).

Moreover, at every (ϑ, φ) in D, the corresponding partial velocity $\boldsymbol{x}_\vartheta = (-\sin\vartheta\cos\varphi, \cos\vartheta\cos\varphi, 0)$ points due east, and $\boldsymbol{x}_\varphi = (-\cos\vartheta\sin\varphi, -\sin\vartheta\sin\varphi, \cos\varphi)$ points due north.

To check regularity, compute the partial velocity cross product,

$$\boldsymbol{x}_\vartheta \times \boldsymbol{x}_\varphi = \left(\cos\vartheta\cos^2\varphi, \sin\vartheta\cos^2\varphi, \sin\varphi\cos\varphi\right),$$

so $|\boldsymbol{x}_\vartheta \times \boldsymbol{x}_\varphi| = \cos^2\varphi > 0$ for all $(\vartheta, \varphi) \in D$,

since the poles are excluded. Hence, $\boldsymbol{x}_*$ is injective and therefore $\boldsymbol{x}$ is regular and thus a parametrization by Def. 9.5.

For it to be a patch by Def. 9.1, we need $\boldsymbol{x}$ itself to be injective in addition, that is $\boldsymbol{x}(\vartheta, \varphi) = \boldsymbol{x}(\vartheta_0, \varphi_0) \Rightarrow (\vartheta, \varphi) = (\vartheta_0, \varphi_0)$:

$$\left.\begin{aligned} \cos\vartheta\cos\varphi &= \cos\vartheta_0\cos\varphi_0 \\ \sin\vartheta\cos\varphi &= \sin\vartheta_0\cos\varphi_0 \\ \sin\varphi &= \sin\varphi_0 \end{aligned}\right\} \text{ so in } D, \vartheta = \vartheta_0, \varphi = \varphi_0$$

Moreover, $\boldsymbol{x}^{-1}$ is clearly continuous (even differentiable), so the patch $(\boldsymbol{x}, D)$ is in fact a chart called geographical chart.

Definition 9.6: Surface of Revolution

C be a curve called profile curve in the plane $P \subset \mathbb{R}^3$ and A be a line called axis in P such that $A \cap C = \emptyset$. Then revolving C around A yields a surface of revolution $M \subset \mathbb{R}^3$.

Example Suppose P is the xz-plane and A the z-axis, and

$$C : \alpha(u) = (g(u), 0, h(u)), \quad u \in I,$$

such that it does not intersect the z-axis. Then the corresponding surface of revolution M is parametrized by

$$x(u,v) = (g(u)\cos v, g(u)\sin v, h(u)), \quad v \in (0, 2\pi)$$

As for the sphere, the revolved C at each angle v are called meridians of M, and circles of revolution for each point of C are called parallels of M.

Definition 9.7: Ruled Surface

A ruled surface consists of lines called rulings along straight line segments from a base curve $\beta(u)$ along a director curve $\delta(u)$. Obviously, such a surface has a ruled parametrization

$$\boldsymbol{x}(u,v) = \beta(u) + v\delta(u).$$

Example

$\delta(u) = (c_1, c_2, c_3) = \text{const}$, yields a generalized cylinder.

$\beta(u) = (p_1, p_2, p_3) = \text{const}$. yields a generalized cone, with apex excluded (not injective at singular point).

10 Lecture 10 - Jan 25

10.1 Functions on Surfaces

Definition 10.1: Coordinate Expression

Given a real-valued function on the surface $f : M \to \mathbb{R}$ with chart $\boldsymbol{x} : D \to M$, then $f \circ \boldsymbol{x} : D \subset \mathbb{R}^2 \to \mathbb{R}$ is called its coordinate expression, and f is called differentiable if its coordinate expression is differentiable as a Euclidean mapping (Lecture 4). The set of differentiable functions on M is denoted by $C^\infty M$.

Of course, this generalizes also to mappings $F : \mathbb{R}^n \to M$, now with a coordinate expression given by $x^{-1} \circ F$: F is differentiable if this Euclidean mapping is, on a suitable open subset.

Transition Maps This applies to charts, too: changing coordinates from $(\boldsymbol{x}, D)$ to another chart $(\boldsymbol{y}, E)$ which overlaps, i.e. $\boldsymbol{x}(D) \cap \boldsymbol{y}(E)$ non-empty, then there is a (diffeomorphic) Euclidean transition map $\boldsymbol{y}^{-1} \circ \boldsymbol{x}$ on an open subset (more later: cf. manifolds):

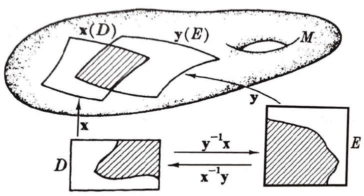


Figure 1: O'Neill Fig. 4.23

So there are unique differentiable functions $\tilde{u}, \tilde{v}$ such that

$$\boldsymbol{y}(u, v) = \boldsymbol{x}(\tilde{u}(u, v), \tilde{v}(u, v))$$

Lemma 10.2

Given a differentiable curve $\alpha : I \to M$ such that its trajectory $\alpha(I) \subset \boldsymbol{x}(D)$ of a single patch, then there exist unique differentiable functions $a_1, a_2 : I \to \mathbb{R}$ such that, for all $t \in I$,

$$\alpha(t) = \boldsymbol{x}(a_1(t), a_2(t))$$

Proof. By definition, $\boldsymbol{x}^{-1} \circ \alpha : I \to D, t \mapsto (a_1(t), a_2(t))$ is differentiable,

$$\alpha = \underbrace{\boldsymbol{x} \circ \boldsymbol{x}^{-1}}_{\text{id.}} \circ \alpha = \boldsymbol{x}(a_1, a_2),$$

and uniqueness follows since, supposing $\alpha = \boldsymbol{x}(b_1, b_2)$,

$$(a_1, a_2) = \boldsymbol{x}^{-1} \circ \alpha = \boldsymbol{x}^{-1} \circ \boldsymbol{x}(b_1, b_2) = (b_1, b_2)$$

□

Consider the relationships of I, D and M:

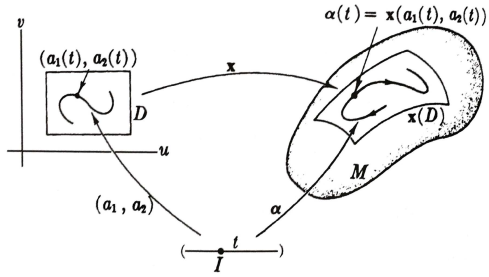


Figure 2: O'Neill Fig. 4.22

10.2 Tangent space

Definition 10.3: Tangent Space

Let $p \in M$. Then the vector $\boldsymbol{v} \in T_p\mathbb{R}^3$ is called tangent to M at p if it is the velocity of some curve in M. The set of all such tangent vectors is the tangent space T_pM of M at p.

Note This notion of tangent space obviously generalizes beyond surfaces in $\mathbb{R}^3$, to higher dimensions and to intrinsic geometry.

Here, of course, we would expect T_pM to be a 2-dimensional vector space and thus a plane. Unsurprisingly, this is the case:

Lemma 10.4

Let $p \in M$ with a chart such that $\boldsymbol{x}(u_0, v_0) = p$. Then $\boldsymbol{v} \in T_pM$ if, and only if, it is a linear combination of $\boldsymbol{x}_u(u_0, v_0)$ and $\boldsymbol{x}_v(u_0, v_0)$.

Proof. Suppose $\boldsymbol{v}$ is tangent at $p \in M$, then by Def. 10.3 there exists a curve α such that $p = \alpha(0)$ and $\boldsymbol{v} = \alpha'(0)$, say.

Now by Lem. 10.2, $\alpha = \boldsymbol{x}(a_1, a_2)$, so using the chain rule,

$$\alpha' = \boldsymbol{x}_u(a_1, a_2) \frac{\mathrm{d}a_1}{\mathrm{d}t} + \boldsymbol{x}_v(a_1, a_2) \frac{\mathrm{d}a_2}{\mathrm{d}t}.$$

Evaluating this at $p = \alpha(0) = \boldsymbol{x}(a_1(0), a_2(0)) = \boldsymbol{x}(u_0, v_0)$,

$$\boldsymbol{v} = \alpha'(0) = \boldsymbol{x}_u(u_0, v_0) \frac{\mathrm{d}a_1}{\mathrm{d}t}(0) + \boldsymbol{x}_v(u_0, v_0) \frac{\mathrm{d}a_2}{\mathrm{d}t}(0),$$

as required.

Conversely, given some $c_1, c_2 \in \mathbb{R}$, consider the vector

$$\boldsymbol{v} = c_1 \boldsymbol{x}_u(u_0, v_0) + c_2 \boldsymbol{x}_v(u_0, v_0) \in T_{\boldsymbol{x}(u_0,v_0)}\mathbb{R}^3$$

Then $\boldsymbol{v} = \beta'(0)$ of the curve $\beta(t) = \boldsymbol{x}(u_0 + c_1 t, v_0 + c_2 t)$, and thus it is tangent to M at p by Def. 10.3. $\square$

Basis The partial velocities $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ of the parameter curves are always tangent and linearly indepedent, forming a basis of T_pM for all p.

Tangent Bundle Collecting all tangent planes of M, we obtain its tangent bundle

$$TM = \bigcup_{p \in M} T_pM$$

and so $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ provides a frame field for the tangent bundle.

Definition 10.5

(1) A Euclidean vector field X on a surface M is a differentiable mapping $p \mapsto X(p) \in T_p\mathbb{R}^3$ for each $p \in M$.
(2) A Euclidean vector field V such that $V(p) \in T_pM$ for all $p \in M$ is called tangent vector field.
(3) A Euclidean vector N such that $N(p) \cdot V(p) = 0$ for all $p \in M$ is called normal vector field.

Note A tangent vector field is defined on TM and thus belongs to the intrinsic geometry of M, a normal vector field is extrinsic.

Lemma 10.6

Let $M : g = c$ be a surface in $\mathbb{R}^3$. Then its gradient vector field $\nabla g = \sum_i \frac{\partial g}{\partial x_i} U_i$ is a non-vanishing normal vector field of M.

Proof. M is defined implicitly by $g = c$ so not all $\frac{\partial g}{\partial x_i}$ vanish, thus ∇g is a non-vanishing vector field.

Suppose α is a curve in $M \subset \mathbb{R}^3$, and use Euclidean coordinates in $\mathbb{R}^3$. Then $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ with $g(\alpha(t)) = c = \text{const}$. Hence by the chain rule,

$$\sum_i \frac{\partial g}{\partial x_i}(\alpha(t)) \frac{\mathrm{d}\alpha_i}{\mathrm{d}t}(t) = 0$$

Now given any $\boldsymbol{v} \in T_pM$ for any $p \in M$, choose an α such that $\alpha(0) = p$ and $\alpha'(0) = \boldsymbol{v}$. Thus,

$$\nabla g(p) \cdot \boldsymbol{v} = 0$$

so ∇g is indeed normal. $\square$

Definition 10.7: Directional Derivative

Suppose $\boldsymbol{v} \in T_pM$ and $f \in C^\infty M$. Then the directional derivative of f with respect to $\boldsymbol{v}$ is the real number

$$\boldsymbol{v}[f] = \frac{\mathrm{d} f(\alpha)}{\mathrm{d}t}(0)$$

the common value of for all curves α with $\alpha(0) = p$ and $\alpha'(0) = \boldsymbol{v}$.

10.3 Forms on surfaces

Definition 10.8: Forms

Let M be a surface, and $\Omega^k M$ be the space of k-forms on M with $0 \le k \le 2$. Also, let $f \in C^\infty M$ be differentiable functions on M, and X, Y be tangent vector fields in TM.
(1) A 0-form on M is a differentiable function on M, that is, $\Omega^0 M = C^\infty M$.
(2) A 1-form $\phi \in \Omega^1 M$ is a real-valued function on tangent vectors, $\phi(X) \in C^\infty M$, which is linear, $\phi(fX + Y) = f\phi(X) + \phi(Y)$.
(3) A 2-form $\eta \in \Omega^2 M$ is a real-valued function pairs of tangent vectors, $\eta(X, Y) \in C^\infty M$ which is linear in X and Y as above, and also antisymmetric, $\eta(X, Y) = -\eta(Y, X)$.

Note By the antisymmetry property of 2-forms, for any tangent vector $\boldsymbol{v} \in T_pM$ at any $p \in M$, we have

$$\eta(\boldsymbol{v}, \boldsymbol{v}) = -\eta(\boldsymbol{v}, \boldsymbol{v}) = 0$$

Lemma 10.9

Let $\eta \in \Omega^2 M$, and let $\boldsymbol{v}, \boldsymbol{w} \in T_pM$ be linearly independent tangent vectors at some point $p \in M$, and $a, b, c, d \in \mathbb{R}$. Then

$$\eta(a\boldsymbol{v} + b\boldsymbol{w}, c\boldsymbol{v} + d\boldsymbol{w}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \eta(\boldsymbol{v}, \boldsymbol{w})$$

Proof. Using linearity of the first then second entry, then antisymmetry, we obtain

$$\begin{aligned} &\eta(a\boldsymbol{v} + b\boldsymbol{w}, c\boldsymbol{v} + d\boldsymbol{w}) \\ &= a\eta(\boldsymbol{v}, c\boldsymbol{v} + d\boldsymbol{w}) + b\eta(\boldsymbol{w}, c\boldsymbol{v} + d\boldsymbol{w}) \\ &= ac\underbrace{\eta(\boldsymbol{v}, \boldsymbol{v})}_{0} + ad\eta(\boldsymbol{v}, \boldsymbol{w}) + bc\underbrace{\eta(\boldsymbol{w}, \boldsymbol{v})}_{-\eta(\boldsymbol{v}, \boldsymbol{w})} + bd\underbrace{\eta(\boldsymbol{w}, \boldsymbol{w})}_{0} \\ &= (ad - bc)\eta(\boldsymbol{v}, \boldsymbol{w}), \end{aligned}$$

and the result follows. ∎

Definition 10.10: Exterior Product

Suppose $\phi, \psi \in \Omega^1 M$ and X, Y be any vector fields in TM. Then exterior product or wedge product of the two 1-forms is a 2-form $\phi \wedge \psi \in \Omega^2 M$ such that

$$(\phi \wedge \psi)(X, Y) = \phi(X)\psi(Y) - \phi(Y)\psi(X)$$

Example Confirm that $\phi \wedge \psi$ is indeed a 2-form according to Def. 10.8: antisymmetry is clear by Def. 10.10. For linearity, note that

$$\begin{aligned}
&(\phi \wedge \psi)(fX + Y, Z) \\
&= \phi(fX + Y)\psi(Z) - \phi(Z)\psi(fX + Y) \\
&= f\phi(X)\psi(Z) + \phi(Y)\psi(Z) - f\phi(Z)\psi(X) - \phi(Z)\psi(Y) \\
&= f(\phi \wedge \psi)(X, Z) + (\phi \wedge \psi)(Y, Z)
\end{aligned}$$

The exterior derivative for differential forms $\phi \in \Omega^k M, \psi \in \Omega^l M$ on a surface obeys the same axiomatic properties as differential forms on $\mathbb{R}^n$, as noted at the end on Seminar 3,

$$\mathrm{d}^2\phi = 0 \quad \text{(nil-potency)}$$
$$\mathrm{d}(\phi \wedge \psi) = \mathrm{d}\phi \wedge \psi + (-1)^k \phi \wedge \mathrm{d}\psi$$

Next, consider forms in local coordinates on M. We have already seen that $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ provides basis vectors for T_pM (though in general not orthonormal) induced by the (u, v)-coordinates.

As shown in Homework 3, this parametrization also provides a corresponding dual basis of forms given by $\{\mathrm{d}u, \ \mathrm{d}v\}$ with

$$\begin{aligned}
\mathrm{d}u\,(\boldsymbol{x}_u) = 1, \quad \mathrm{d}u\,(\boldsymbol{x}_v) = 0 \\
\mathrm{d}v\,(\boldsymbol{x}_u) = 0, \quad \mathrm{d}v\,(\boldsymbol{x}_v) = 1
\end{aligned}$$

Lemma 10.11

Let ϕ be a 1-form on a surface M parametrized by $\boldsymbol{x}(u, v)$. Then its exterior derviative is the 2-form satisfying

$$\mathrm{d}\phi\,(\boldsymbol{x}_u, \boldsymbol{x}_v) = \frac{\partial}{\partial u}(\phi\,(\boldsymbol{x}_v)) - \frac{\partial}{\partial v}(\phi\,(\boldsymbol{x}_u)).$$

Proof. Proof. We can expand ϕ in local coordinates as $\phi = \phi\,(\boldsymbol{x}_u)\,\mathrm{d}u + \phi\,(\boldsymbol{x}_v)\,\mathrm{d}v$. Then by definition of the exterior derivative,

$$\mathrm{d}\phi = \frac{\partial}{\partial v}(\phi\,(\boldsymbol{x}_u))\underbrace{\mathrm{d}v \wedge \mathrm{d}u}_{-\mathrm{d}u\wedge \mathrm{d}v} + \frac{\partial}{\partial u}(\phi\,(\boldsymbol{x}_v))\,\mathrm{d}u \wedge \mathrm{d}v.$$

The result follows by evaluating this on $\boldsymbol{x}_u, \boldsymbol{x}_v$ with Def. 10.10. ∎

Definition 10.12: Closed & Exact

A differential form ϕ is closed if its exterior derivative is zero, and ϕ is called exact if it is the exterior derivative of some form, $\phi = \mathrm{d}\psi$.

Note Therefore, no 0-form is exact, and every k-form of highest order (e.g. $k = 2$ for a surface) is closed.

Note By nil-potency of the exterior derivative, every exact form is closed, $\mathrm{d}\phi = \mathrm{d}^2\psi = 0$. But which closed forms are exact?

This is, in fact, a fundamental topological question, related to homotopy and homology within the framework of differential geometry (cf. Lecture 12, and cohomology in graduate school).

11 Lecture 11 - Jan 26

11.1 Mappings of surfaces

Definition 11.1: Mapping of Surfaces

Let M be a surface with chart $(\boldsymbol{x}, D)$ and N be a surface with chart $(\boldsymbol{y}, E)$. Then $F : M \to N$ is called a differentiable mapping of surfaces if $\boldsymbol{y}^{-1} \circ F \circ \boldsymbol{x}$ is differentiable on an open set of $\mathbb{R}^2$.

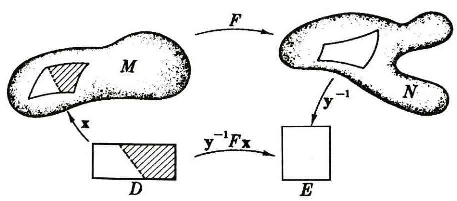


Figure 3: O'Neill Fig. 4.27

Example A classic example is the stereographic projection of the sphere: Let Σ be the unit sphere centered at $(0, 0, 1)$ in $\mathbb{R}^3$ punctured by deleting the north pole. Then a ray from the north pole through $p = (p_1, p_2, p_3) \in \Sigma$ intersects the xy-plane in P.

Considering similar triangles of the north pole, p, P, and their projections on the z-axis, and identifying the xy-plane with $\mathbb{R}^2$,

$$P\,(p_1, p_2, p_3) = \left(\frac{2p_1}{2 - p_3}, \frac{2p_2}{2 - p_3}\right)$$

so $P : \Sigma \to \mathbb{R}^2$ is a differentiable mapping of surfaces by Def. 11.1.

11.2 Push-forward revisited

Definition 11.2: Push-forward on Surfaces

Let $F : M \to N$ be a differentiable mapping of surfaces. Then its tangent map or push-forward maps each tangent vector of M to a tangent vector of N such that if $\boldsymbol{v} = \alpha'(p) \in T_pM$ then

$$F_* : T_pM \to T_{F(p)}N, \quad F_*\boldsymbol{v} = (F(\alpha))'(F(p)).$$

In other words, the push-forward maps velocities of a curve to the velocities of its image under F.

So supposing that $\boldsymbol{x}(u, v) : D \to M$ is a parametrization and $\boldsymbol{y} = F \circ \boldsymbol{x} : D \to N$, which need not be a parametrization, we can still have the useful property that

$$F_*(\boldsymbol{x}_u) = \boldsymbol{y}_u, \quad F_*(\boldsymbol{x}_v) = \boldsymbol{y}_v$$

Also, F_* is a linear map, and properties of regularity and the existence of local diffeomorphisms apply as discussed in Lecture 4.

Example Let us revisit the stereographic projection of the punctured sphere Σ described previously. Now adapting the geographical parametrization as discussed in Lecture 9,

$$\boldsymbol{x}(\vartheta, \varphi) = (\cos\varphi\cos\vartheta, \cos\varphi\sin\vartheta, 1 + \sin\varphi)$$

Now our explicit formula for the stereographic projection becomes

$$\boldsymbol{y}(\vartheta, \varphi) = P(\boldsymbol{x}(\vartheta, \varphi)) = \frac{2\cos\varphi}{1 - \sin\varphi}(\cos\vartheta, \sin\vartheta).$$

Hence, computing the push-forward,

$$\boldsymbol{y}_\vartheta = P_*(\boldsymbol{x}_\vartheta) \propto (-\sin\vartheta, \cos\vartheta)$$
$$\boldsymbol{y}_\varphi = P_*(\boldsymbol{x}_\varphi) \propto (-\cos\vartheta, -\sin\vartheta)$$

which are non-zero and orthogonal, $\boldsymbol{y}_\vartheta \cdot \boldsymbol{y}_\varphi = 0$, thus linearly independent, so P_* is injective and P is a regular mapping.

Note P preserves angles, cf. conformal mappings in MATH307!

11.3 Pull-back

Definition 11.3: Pull-back

Let $F : M \to N$ be a differentiable mapping of surfaces, and $F_* : T_pM \to T_{F(p)}N$ its push-forward.
(1) If $\xi \in \Omega^1 N$, then $F^*(\xi) \in \Omega^1 M$ such that for all $\boldsymbol{v} \in T_pM$,

$$(F^*\xi)(\boldsymbol{v}) = \xi(F_*\boldsymbol{v}).$$

(2) If $\eta \in \Omega^2 N$, then $F^*(\eta) \in \Omega^2 M$ such that for all $\boldsymbol{v}, \boldsymbol{w} \in T_pM$,

$$(F^*\eta)(\boldsymbol{v}, \boldsymbol{w}) = \eta(F_*\boldsymbol{v}, F_*\boldsymbol{w}).$$

Note The pull-back goes in the opposite direction of the pushforward. So we cannot push-forward forms, nor pull-back vectors.

Note Since real-valued functions can also be regarded as 0-forms, one may extend the definition by setting $F^*f = f \circ F$ for $f \in \Omega^0 N$.

Basic properties of the pull-back can be summarized as follows:

Theorem 11.4

Let $F : M \to N$ be a differentiable mapping of surfaces, and $\xi \in \Omega^k N, \eta \in \Omega^l N$ be forms on N. Then their pull-back to M,
(1) $F^*(\xi + \eta) = F^*\xi + F^*\eta$, for $k = l \leq 2$,
(2) $F^*(\xi \wedge \eta) = F^*\xi \wedge F^*\eta$, for $k + l \leq 2$,
(3) $F^*(\mathrm{d}\xi) = \mathrm{d}(F^*\xi)$, for $k \leq 1$.

Proof. (1) and (2) follow directly from Def. 11.3 and properties of forms and push-forwards as discussed before, so see Homework 3.

For (3), let us here take the case $\xi \in \Omega^1 N$ and recall that we can expand any $\boldsymbol{v}, \boldsymbol{w} \in T_pM$ in terms of the parametrization-induced basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$. Hence, we only need to show that

$$F^*(\mathrm{d}\xi)(\boldsymbol{x}_u, \boldsymbol{x}_v) = (\mathrm{d}(F^*\xi))(\boldsymbol{x}_u, \boldsymbol{x}_v).$$

Letting, as before, $\boldsymbol{y} = F(\boldsymbol{x})$ with the correspondinng push-fowards $F_*\boldsymbol{x}_u = \boldsymbol{y}_u, F_*\boldsymbol{x}_v = \boldsymbol{y}_v$, and using Lem. 10.11 for the exterior derivative and also Def. 11.3 of the pull-back, we obtain:

$$\begin{aligned}(\mathrm{d}(F^*\xi))(\boldsymbol{x}_u,\boldsymbol{x}_v) &= \frac{\partial}{\partial u}((F^*\xi)(\boldsymbol{x}_v)) - \frac{\partial}{\partial v}((F^*\xi)(\boldsymbol{x}_u)) \\ &= \frac{\partial}{\partial u}(\xi(F_*\boldsymbol{x}_v)) - \frac{\partial}{\partial v}(\xi(F_*\boldsymbol{x}_u)) \\ &= \frac{\partial}{\partial u}(\xi(\boldsymbol{y}_v)) - \frac{\partial}{\partial v}(\xi(\boldsymbol{y}_u)) \\ &= \mathrm{d}\xi(\boldsymbol{y}_u,\boldsymbol{y}_v) \\ &= \mathrm{d}\xi(F_*\boldsymbol{x}_u, F_*\boldsymbol{x}_v) \\ &= (F^*(\mathrm{d}\xi))(\boldsymbol{x}_u,\boldsymbol{x}_v).\end{aligned}$$

□

Note Statement (3) of Thm. 11.4 is the important property that the pull-back commutes with the exterior derivative.

11.4 Integration of forms

Recall that any 1-form on $\mathbb{R}$ is $\psi = \psi(U_1)\,\mathrm{d}t \in \Omega^1\mathbb{R}$ and so can be integrated. Now given $\phi \in \Omega^1 M$ along α in $M, \alpha^*\phi \in \Omega^1\mathbb{R}$ with

$$\alpha^*\phi = (\alpha^*\phi)(U_1)\,\mathrm{d}t = \phi(\alpha_*(U_1))\,\mathrm{d}t = \phi(\alpha')\,\mathrm{d}t,$$

which suggests the following definition for integration of 1-forms:

Definition 11.5: Integral of 1-form

Suppose $\phi \in \Omega^1 M$ and $\alpha : [a,b] \subset \mathbb{R} \to M$. Then the integral of ϕ over the (image of the) curve α is the real number

$$\int_{\alpha(I)} \phi = \int_I \alpha^*\phi = \int_a^b \phi(\alpha'(t))\,\mathrm{d}t.$$

Note As is customary, we often use the same symbol for the map α of the curve and its image $\alpha(I)$ in M, writing $\int_\alpha$ for short.

Note Def. 11.5 is intrinsic to the curve, surface and form, independent of any parametrization in terms of coordinates!

Hence, we immediately deduce the corresponding generalization of the fundamental theorem line integrals:

Theorem 11.6

Consider a differentiable $f : M \to \mathbb{R}$ and the curve $\alpha : [a,b] \to M$ with $\alpha(a) = p, \alpha(b) = q$, then

$$\int_\alpha \mathrm{d}f = f(q) - f(p).$$

Proof. This follows readily from the definition and the standard fundamental theorem of calculus:

$$\begin{aligned}\int_\alpha \mathrm{d}f &= \int_a^b \mathrm{d}f(\alpha')\,\mathrm{d}t \quad \text{by Def. 11.5} \\ &= \int_a^b \alpha'[f]\mathrm{d}t = \int_a^b \frac{\mathrm{d}f(\alpha)}{\mathrm{d}t}\,\mathrm{d}t \\ &= f(\alpha(b)) - f(\alpha(a)) = f(q) - f(p)\end{aligned}$$

□

Note The integral of $\mathrm{d}f$ along the curve equals f evaluated on its boundary, the endpoints p, q : so this integral is path-indpendent, and we shall see the generalization of this (Stokes) shortly.

2-segment But first, let us define the integration of 2-forms η over regions of M. To this end, we start with a simple region called a 2-segment, defined by the differentiable $\boldsymbol{x} : R \to M$ on the closed rectangle

$$R \subset \mathbb{R}^2 : \quad a \leq u \leq b, c \leq v \leq d$$

Definition 11.7: Integral of 2-Form

Let $\eta \in \Omega^2 M$ and $\boldsymbol{x} : R \to M$ a 2-segment, then the integral of η over the (image of) x is the real number

$$\iint_{\boldsymbol{x}(R)} \eta = \iint_R \boldsymbol{x}^*\eta = \int_a^b \int_c^d \eta(\boldsymbol{x}_u,\boldsymbol{x}_v)\,\mathrm{d}u\,\mathrm{d}v$$

Note Similar to 1-form integrals, we write $\iint_{\boldsymbol{x}}$ for short.

Note A 2-segment is not a parametrization or chart: R is not open, $\boldsymbol{x}$ is just differentiable, not necessarily regular or invertible. It will also be useful for later to define the edges of $\boldsymbol{x}$ expicitly:

Definition 11.8: Edge Curve

Let $\boldsymbol{x} : R \to M$ be a 2 -segment in M. Then the edge curves are

$$\begin{aligned}\alpha(u) = \boldsymbol{x}(u,c), \beta(v) = \boldsymbol{x}(b,v), \\ \gamma(u) = \boldsymbol{x}(u,d), \delta(v) = \boldsymbol{x}(a,v),\end{aligned}$$

and the positively oriented (counterclockwise) boundary is denoted formally by $\partial\boldsymbol{x}$ (cf. also next time!).

Thus, the integral of a 1-form $\phi \in \Omega^1 M$ along the boundary of $\boldsymbol{x}$ is

$$\int_{\partial x} \phi = \int_\alpha \phi + \int_\beta \phi - \int_\gamma \phi - \int_\delta \phi$$

Theorem 11.9: (Stokes)

Let $\phi \in \Omega^1 M$ and $\boldsymbol{x} : R \to M$ a 2-segment, then

$$\iint_{\boldsymbol{x}} \mathrm{d}\phi = \int_{\partial \boldsymbol{x}} \phi.$$

Proof. Now by the Def. 11.7, we can recast the left-hand side thus,

$$\iint_{\boldsymbol{x}} \mathrm{d}\phi = \iint_R \mathrm{d}\phi\left(\boldsymbol{x}_u, \boldsymbol{x}_v\right) \mathrm{d}u\ \mathrm{d}v$$

which, applying Lem. 10.11, can be written as

$$\begin{aligned} &\iint_R \mathrm{d}\phi\left(\boldsymbol{x}_u, \boldsymbol{x}_v\right) \mathrm{d}u\ \mathrm{d}v \\ = &\iint_R \left(\frac{\partial}{\partial u}\left(\phi\left(\boldsymbol{x}_v\right)\right) - \frac{\partial}{\partial v}\left(\phi\left(\boldsymbol{x}_u\right)\right)\right) \mathrm{d}u\ \mathrm{d}v. \end{aligned}$$

Given our Def. 11.8 of the edge curves of the 2-segment, the first integrand of the right-hand side yields

$$\begin{aligned} &\iint_R \frac{\partial}{\partial u}\left(\phi\left(\boldsymbol{x}_v\right)\right) \mathrm{d}u\ \mathrm{d}v \\ &= \int_c^d \phi\left(\boldsymbol{x}_v\right)(b, v) \mathrm{d}v - \int_c^d \phi\left(\boldsymbol{x}_v\right)(a, v) \mathrm{d}v \\ &= \int_c^d \phi\left(\beta'(v)\right) \mathrm{d}v - \int_c^d \phi\left(\delta'(v)\right) \mathrm{d}v \\ &= \int_\beta \phi - \int_\delta \phi \end{aligned}$$

The second term is dealt with similarly, noting that the order of u, v-integration may be reversed on R by Fubini's theorem. Thus,

$$\iint_R \frac{\partial}{\partial v}\left(\phi\left(\boldsymbol{x}_u\right)\right) \mathrm{d}u\ \mathrm{d}v = \iint_R \frac{\partial}{\partial v}\left(\phi\left(\boldsymbol{x}_u\right)\right) \mathrm{d}v\ \mathrm{d}u = \int_\gamma \phi - \int_\alpha \phi.$$

Subtracting, we obtain the required overall result,

$$\iint_{\boldsymbol{x}} \mathrm{d}\phi = \int_\beta \phi - \int_\delta \phi - \left(\int_\gamma \phi - \int_\alpha \phi\right) = \int_{\partial \boldsymbol{x}} \phi.$$

□

Note the similarities and differences of Thm. 11.9 and Thm. 11.6.

Note For integration, it suffices that $\boldsymbol{x}$ be merely a 2-segment. A stronger $\boldsymbol{x}$, e.g. as provided by a chart, may also work, but we need to be careful about improper integrals (more later: pavings).

12 Lecture 12 - Jan 27

12.1 Curve orientation

Curves have an orientation induced by the parametrization. Just like for line integrals, changing this may impact integrals of forms:

Lemma 12.1

Consider the curve $\alpha : [a, b] \to M$ and let $h : [a, b] \to [c, d]$ be a reparametrization. For any $\phi \in \Omega^1 M$,
(1) if h is orientation-preserving, $h(a) = c, h(b) = d$, then $\int_{\alpha(h)} \phi = \int_\alpha \phi$,
(2) if h is orientation-reversing, $h(a) = d, h(b) = c$, then $\int_{\alpha(h)} \phi = -\int_\alpha \phi$.

Proof. This follows straight from Def. 11.5, Lem. 4.9 on the velocity of reparametrized curves, and change of variables in integrals on $\mathbb{R}$:

$$\begin{aligned} \int_{\alpha(h)} \phi &= \int_a^b \phi\left(\alpha(h)'\right) \mathrm{d}t = \int_a^b \phi\left(\alpha'(h) h'(t)\right) \mathrm{d}t \\ &= \int_a^b \phi\left(\alpha'(h)\right) \frac{\mathrm{d}h}{\mathrm{d}t}\ \mathrm{d}t \end{aligned}$$

So, indeed, if h is orientation-preserving, with value $h(t) = u$,

$$\int_{\alpha(h)} \phi = \int_{c=h(a)}^{d=h(b)} \phi\left(\alpha'(u)\right) \mathrm{d}u = \int_\alpha \phi$$

and if h is orientation-reversing,

$$\int_{\alpha(h)} \phi = \int_d^c \phi\left(\alpha'(u)\right) \mathrm{d}u = -\int_c^d \phi\left(\alpha'(u)\right) \mathrm{d}u = -\int_\alpha \phi$$

□

Suppose we let $-\alpha$ denote an orientation-reversing reparametrization of the curve α then, by this Lem. 12.1,

$$\int_{-\alpha} \phi = -\int_\alpha \phi.$$

Note, then, that we can recast the integral of the 1-form around the boundary of a 2-segment (cf. Def. 11.8) as follows,

$$\int_{\partial \boldsymbol{x}} \phi = \int_\alpha \phi + \int_\beta \phi - \int_\gamma \phi - \int_\delta \phi = \int_\alpha \phi + \int_\beta \phi + \int_{-\gamma} \phi + \int_{-\delta} \phi,$$

in other words, simply as the sum of oriented curve segments, $\partial \boldsymbol{x} = \alpha \cup \beta \cup (-\gamma) \cup (-\delta)$.

12.2 Topological properties

Definition 12.2: Connected

A surface M is called connected if for any two $p, q \in M$ there is a curve in M from p to q.

Example Suppose $f : M \to \mathbb{R}$ is differentiable and non-zero on a connected M. Show that it cannot change sign.

Proof. Since $f \neq 0$ by assumption, $\operatorname{sgn} f = \frac{f}{|f|}$ is differentiable and $\mathrm{d}(\operatorname{sgn} f) = 0$ so $\operatorname{sgn} f = \text{const}$. □

Lemma 12.3

A surface M is compact if, and only if, it can be covered by the images of a finite number of 2-segments.

Proof. Firstly, suppose M is compact. For a every point $p \in M$, given a chart, let the 2-segment image $\boldsymbol{x}(R) \ni U \ni p$ with open U.

Then, by Def. 1.3 of compactness, there is finite number of such U covering M, and hence a finite number of such 2-segments.

Conversely, suppose M is covered by a finite number of 2-segments $\boldsymbol{x}_i : R_i \to M$, and let $\{U_j\}$ be an open covering of M.

Since each $\boldsymbol{x}_i$ is differentiable and hence continuous so, by Def. 1.5, preimages $x_i^{-1}(U_j)$ of open sets $U_j \subset M$ are also open in $\mathbb{R}^2$.

But since closed intervals $R_i \subset \mathbb{R}^2$ are compact, a finite number of $x_i^{-1}(U_j)$ suffices to cover R_i, so finite number of U_j covers M. □

Corollary The sphere $\mathbb{S}^2$ is compact, since we can extend the geographical chart to include both poles and the international date line by using only one 2-segment $x : R \to \mathbb{S}^2$ covering all of $\mathbb{S}^2$,

$$R: \quad -\pi \leq \vartheta \leq \pi, \quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

Lemma 12.4

Let M be a compact surface. Then any continuous non-constant real-valued $f : M \to \mathbb{R}$ attains a maximum at some point in M.

Proof. By the previous Lem. 12.3, we have a finite number of 2-segments $\mathrm{x}_i : R_i \to M$ covering M with compositions $f \circ \mathrm{x}_i : R_i \subset \mathbb{R}^2 \to \mathbb{R}$. But compositions of continuous functions are continuous and so, by the standard maximum principle (cf. MATH201), each $f \circ \boldsymbol{x}_i$ attains its maximum at some $(u_i, v_i) \in R_i$.

Of this finite number of maxima, let $f(\boldsymbol{x}_k(u_k, v_k))$ be the global maximum, so indeed we have $f(p) \leq f(\boldsymbol{x}_k(u_k, v_k))\, \forall p \in M$. □

We can use the contraposition of Lem. 12.4 as a criterion to easily prove non-compactness of a surface:

Example Consider the cylinder $C : x^2 + y^2 = r^2$ in $\mathbb{R}^3$. Then the continuous $f(x, y, z) = z$ has no maximum (diverges) on C, so C is not compact.

Example Next, consider the open disk $D : x^2 + y^2 < 1$ in $\mathbb{R}^2$. Note the continuous $f(x, y) = (1 - x^2 - y^2)^{-1}$ has no maximum on D, so again D is not compact.

Note The last example shows that a finite size (area) of a surface does not by itself imply compactness!

Definition 12.5: Orientable

A surface M is called orientable if there exists a $\mu \in \Omega^2 M$ such that $\mu_p \neq 0$ for all $p \in M$.

Considering surfaces in $\mathbb{R}^3$, this is in fact equivalent to a perhaps more familiar notion of orientability:

Lemma 12.6

A surface $M \subset \mathbb{R}^3$ is orientable if, and only if, there is a unit normal vector field on M.

Note Def. 12.5 is more general and intrinsic to M, whereas Lem. 12.6 refers to an extrinsic property of M, having a normal in $\mathbb{R}^3$.

Proof. Firstly, suppose there is a unit normal U. Then for any linearly independent $\boldsymbol{v}, \boldsymbol{w} \in T_p M$, define

$$\mu_p(\boldsymbol{v}, \boldsymbol{w}) = U(p) \cdot \boldsymbol{v} \times \boldsymbol{w}$$

so μ is a non-zero 2-form and M is orientable by Def. 12.5.

Conversely, let M be orientable with non-zero 2-form μ such that $\mu_p(\boldsymbol{v}, \boldsymbol{w}) \neq 0$. Now we can define a vector

$$X(p) = \frac{\boldsymbol{v} \times \boldsymbol{w}}{\mu_p(\boldsymbol{v}, \boldsymbol{w})}$$

which is normal to $T_p M$ and non-zero. Moreover, it is independent of the choice of $\boldsymbol{v}, \boldsymbol{w}$ by Lem. 10.9. Hence, we get a differentiable normal vector field $U = \frac{X}{|X|}$ as required. □

Corollary If M is orientable and connected, there are exactly two such normals, $\pm U$: if V is any other unit normal, $V \cdot U = \pm 1$ but this cannot change sign by the earlier example, cf. Def. 12.2.

Note: There are non-orientable surfaces in $\mathbb{R}^3$, e.g. Möbius strip.

Definition 12.7: Homotopic

Let $\alpha : [a,b] \to M$ be a closed curve or loop at p whereby $p = \alpha(a) = \alpha(b)$. Then α is called homotopic to p if is there a 2-segment $\boldsymbol{x} : R \to M$ on $R : a \le u \le b, 0 \le v \le 1$, so that

$$\boldsymbol{x}(u,0) = \alpha(u), \boldsymbol{x}(u,1) = p, \boldsymbol{x}(a,v) = p, \boldsymbol{x}(b,v) = p.$$

Note In other words, all fixed v yield loops at p, shrinking from $\alpha(u)$ for $v = 0$ to the 'constant loop', p itself, for $v = 1$.

Example Any loop in the Euclidean plane, $\alpha : [a,b] \to \mathbb{R}^2$, is homotopic to the constant p, by defining the 2 -segment

$$x(u,v) = v\alpha(a) + (1-v)\alpha(u)$$

Definition 12.8: Simply Connected

A surface M is called simply connected if it is connected and every loop in M is homotopic to a point.

Example Thus, the previous example implies that $\mathbb{R}^2$ is simply connected.

Lemma 12.9

Suppose $\phi \in \Omega^1 M$ is closed and α a loop homotopic to a point. Then

$$\int_\alpha \phi = 0.$$

Proof. Since $\mathrm{d}\phi = 0$ by assumption, this follows directly from Stokes' Thm. 11.9 applied to a homotopy $\boldsymbol{x}$ as in Def 12.7,

$$0 = \iint_{\boldsymbol{x}} \mathrm{d}\phi = \int_\alpha \phi$$

□

Example In the punctured plane $\hat{P} = \mathbb{R}^2 \backslash \{(0,0)\}$, let α be a loop encircling the (missing) origin. Now $\int_\alpha \phi \neq 0$ for the closed form

$$\phi = \frac{xdy - ydx}{x^2 + y^2}$$

as computed in Homework 3 ,therefore P is not simply connected.

Recall from Lecture 10 that exact forms are always closed though the converse, whether closed forms are exact, is a topological question. We have the following famous result for 1-forms:

Lemma 12.10: (Poincaré)

On a simply connected surface M, every closed 1-form is exact.

Proof. Let $\phi \in \Omega^1 M$, $\mathrm{d}\phi = 0$. Also, fix arbitrary $q, p \in M, q \neq p$, and let α, β be any two curves, each from q to p. Then $\alpha \cup (-\beta)$ is a loop, and since M is simply connected by assumption, it is homotopic to a point and so, by Lem. 12.1 and Lem. 12.9, the integral of the closed form is path-independent,

$$0 = \int_{\alpha \cup (-\beta)} \phi = \int_\alpha \phi - \int_\beta \phi,$$

so fixing q only, $f(p) = \int_\alpha \phi$ is a real-valued function on M.

Now to show exactness as claimed, we need $\phi = \mathrm{d}f$. Thus, we need to confirm that $\phi(\boldsymbol{v}) = \mathrm{d}f(\boldsymbol{v}) = \boldsymbol{v}[f]$ for all $\boldsymbol{v} \in T_pM$. So let $\gamma : [a,b] \to M$ with $\gamma(a) = p$ and $\gamma'(a) = \boldsymbol{v}$. Then by Def. 12.5,

$$f(\gamma(t)) = f(p) + \int_a^t \phi\left(\gamma'(u)\right) \mathrm{d}u$$

so $\gamma'(t)[f] = (f(\gamma))'(t) = \phi\left(\gamma'(t)\right)$ using Def. 10.7 of the directional derivative, and the required property follows for $t = 0$. □

Theorem 12.11

A compact surface in $\mathbb{R}^3$ *is orientable.*

Beyond present scope. Cf. Jordan curve theorem (MATH307).

13 Lecture 13 - Feb 7

13.1 Shape operator

Definition 13.1: Shape Operator

Suppose $p \in M$ and U is a unit normal vector field on a neighbourhood of p. For any $\boldsymbol{v} \in T_pM$, let

$$S_p(\boldsymbol{v}) = -\nabla_{\boldsymbol{v}} U$$

then S_p is called shape operator of M at p with respect to U.

Lemma 13.2

For each $p \in M$, the shape operator is linear map on tangent space,

$$S_p : T_pM \to T_pM$$

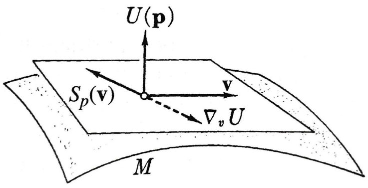


Figure 4: O'Neill Fig. 5.1

Proof. Since U is a unit vector field, $U \cdot U = 1$. Thus, using Thm. 7.8 (4) for any $\boldsymbol{v} \in T_pM$ and applying the Def. 13.1 of the shape operator,

$$0 = \boldsymbol{v}[U \cdot U] = 2\nabla_{\boldsymbol{v}} U \cdot U(p) = -2S_p(\boldsymbol{v}) \cdot U(p),$$

so $S_p(\boldsymbol{v})$ is perpendicular to U and hence in T_pM.

Next, we must establish that S_p acts linearly on tangent vectors. To see this, suppose that $\boldsymbol{v}, \boldsymbol{w} \in T_pM$ and $a, b \in \mathbb{R}$. Then by Def. 13.1 and linearity of covariant derivatives, Thm. 7.8(1),

$$\begin{aligned} S_p(a\boldsymbol{v} + b\boldsymbol{w}) &= -\nabla_{a\boldsymbol{v}+b_{\boldsymbol{w}}} U \\ &= -\left(a\nabla_{\boldsymbol{v}} U + b\nabla_{\boldsymbol{w}} U\right) \\ &= aS_p(\boldsymbol{v}) + bS_p(\boldsymbol{w}) \end{aligned}$$

□

Lemma 13.3

For each $p \in M$, the shape operator is symmetric, that is, for any $\boldsymbol{v}, \boldsymbol{w} \in T_pM$,

$$S_p(\boldsymbol{v}) \cdot \boldsymbol{w} = S_p(\boldsymbol{w}) \cdot \boldsymbol{v}$$

Proof. By Lem. 10.4, any $v \in T_pM$ is a linear combination of $\{x_u, x_v\}$, so it suffices to consider these basis vectors. Now with $U = \sum_i u_i U_i$ and $U \cdot \boldsymbol{x}_u = 0$

$$0 = \frac{\partial}{\partial v}\left(U \cdot \boldsymbol{x}_u\right) = \frac{\partial}{\partial v} \sum_i u_i \frac{\partial x_i}{\partial u} = \frac{\partial U}{\partial v} \cdot \boldsymbol{x}_u + U \cdot \boldsymbol{x}_{uv}.$$

Next, from Def. 13.1 of the shape operator as well as Lem. 7.7 and Lem. 2.8 on the covariant and the directional derivative,

$$\begin{aligned} -S\left(\boldsymbol{x}_v\right) &= \nabla_{\boldsymbol{x}_v} U = \sum_i \boldsymbol{x}_v\left[u_i\right] U_i \\ &= \sum_i \underbrace{\sum_j \frac{\partial x_j}{\partial v} \frac{\partial u_i}{\partial x_j}}_{\frac{\partial u_i}{\partial v}} U_i = \frac{\partial U}{\partial v}. \end{aligned}$$

Then swapping $u \leftrightarrow v$ and using the commutativity of second partial derivatives (Clairaut-Schwarz) we obtain, as required,

$$S\left(\boldsymbol{x}_v\right) \cdot \boldsymbol{x}_u = U \cdot \boldsymbol{x}_{uv} = U \cdot \boldsymbol{x}_{vu} = S\left(\boldsymbol{x}_u\right) \cdot \boldsymbol{x}_v$$

□

$\boxed{\textbf{Example}}$ On any plane in $\mathbb{R}^3, U$ is parallel vector field (constant Euclidean components) and therefore $S_p(\boldsymbol{v}) = -\nabla_{\boldsymbol{v}} U = 0$, so the shape operator vanishes identically.

$\boxed{\textbf{Example}}$ For sphere in $\mathbb{R}^3$ with $r\left(x_1, x_2, x_3\right) =$ const., recall from Lem. 10.6 that ∇r provides a normal such that the outwardpointing (say) unit normal vector field becomes

$$U = \frac{1}{r} \sum_i x_i U_i$$

Computing the covariant derivative, the shape operator is

$$S_p(\boldsymbol{v}) = -\nabla_{\boldsymbol{v}} U = -\frac{1}{r} \sum_i \boldsymbol{v}\left[x_i\right] U_i(p) = -\frac{1}{r} \sum_i v_i U_i = -\frac{1}{r} \boldsymbol{v}$$

which is just constant scalar multiplication by $-\frac{1}{r}$, reflecting the uniform roundness of the sphere.

More generally, since S is a linear and symmetric operator on T_pM by Lem. 13.2 and Lem. 13.3, it can be represented by a symmetric 2×2 matrix.

Its eigenvectors, eigenvalues, and matrix invariants, that is trace and determinant, will thus play an important rôle and carry geometric information, e.g. about curvature, as discussed later.

Lemma 13.4

Let $\alpha : I \to M \subset \mathbb{R}^3$ be a curve in a surface with unit normal vector field U, then

$$\alpha'' \cdot U = S\left(\alpha'\right) \cdot \alpha'$$

Proof. Since the velocity is $\alpha'(t) \in T_{\alpha(t)}M$, it is always orthogonal to U, $\alpha'(t) \cdot U(\alpha(t)) = 0$, upon differentiating we have

$$\alpha'' \cdot U + \alpha' \cdot U' = 0$$

So, using the result of the exercise, we obtain as required

$$\alpha'' \cdot U = \left(-U'\right) \cdot \alpha' = S\left(\alpha'\right) \cdot \alpha'.$$

□

Hence, every curve with the same velocity at a point p will have the same normal component of the acceleration $\alpha''(p) \cdot U(p)$, that is, the component orthogonal to the tangent plane T_pM there.

13.2 Normal curvature

Definition 13.5: Normal Curvature

Let $\boldsymbol{u} \in T_pM$ be a unit tangent vector to M at p. Then the real number $k(\boldsymbol{u}) = S(\boldsymbol{u}) \cdot \boldsymbol{u}$ is called normal curvature of M with respect to $\boldsymbol{u}$.

Note Since $k(\boldsymbol{u}) = k(-\boldsymbol{u})$ (why?), the normal curvature is really defined with respect to an undirected line in T_pM parallel to $\boldsymbol{u}$. In order to interpret k, it is helpful to introduce the following:

Definition 13.6: Normal Section

The normal section $\sigma : I \to M$ of $\boldsymbol{u} \in T_pM$ is the unit speed curve whose image is the intersection of M with the plane through the point $p = \sigma(0)$ spanned by $\boldsymbol{u}$ and $U(p)$. (I.e. it is a trace curve!)

Since the normal section is a planar curve with $|\sigma'| = 1$ by Def. 13.6, we know $\sigma''(0) = \kappa_\sigma(0)N(0)$ with $N(0) = \pm U(p)$. Hence,

$$k(\boldsymbol{u}) = \pm\kappa_\sigma(0)$$

Thus, by rotating $\boldsymbol{u} \in T_pM$ we can find all values of the normal curvature $k(\boldsymbol{u})$ and define:

Definition 13.7: Principal Curvatures and Directions

Let $p \in M \subset \mathbb{R}^3$ and $\boldsymbol{u} \in T_pM$ be a unit vector. The maximum and minimum values of $k(\boldsymbol{u})$ (if they exist) are called principal curvatures $k_{\max}$ and $k_{\min}$ of M at p, with corresponding principal directions $\boldsymbol{u}_{\max}$ and $\boldsymbol{u}_{\min}$, respectively.

Definition 13.8: Umbilic

A point $p \in M \subset \mathbb{R}^3$ is called umbilic if $k(\boldsymbol{u}) =$ const. for all unit vectors $\boldsymbol{u} \in T_pM$, and $k_{\max} = k_{\min}$.

Example For the sphere of radius $r, k = -\frac{1}{r} =$ const. as seen earlier, so every point of the sphere is umbilic.

Theorem 13.9

Let $p \in M \subset \mathbb{R}^3$ be not umbilic. Then there are exactly two orthogonal principal directions with

$$S(\boldsymbol{u}_{\max}) = k_{\max}\boldsymbol{u}_{\max}, \quad S(\boldsymbol{u}_{\min}) = k_{\min}\boldsymbol{u}_{\min}.$$

Proof. Note $k(\vartheta)$ is continuous (why?) on a compact interval $[0, 2\pi]$ and so attains a maximum (cf. Lem. 12.4) at some $\boldsymbol{e}_1 = \boldsymbol{u}_{\max}$, say, with $k_{\max} = S(\boldsymbol{u}_{\max}) \cdot \boldsymbol{u}_{\max}$. Let $\boldsymbol{e}_2 \in T_pM$ with $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$, then $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ form an orthonormal basis of T_pM, so any $\boldsymbol{u} \in T_pM$ is

$$\boldsymbol{u}(\vartheta) = \cos\vartheta \boldsymbol{e}_1 + \sin\vartheta \boldsymbol{e}_2.$$

Now let $S_{ij} = S(\boldsymbol{e}_i) \cdot \boldsymbol{e}_j, 1 \le i, j \le 2$, and so $S_{11} = k_{\max}$ and, by Lem. 13.3 on symmetry of the shape operator, $S_{12} = S_{21}$. Then using Lem. 13.2 on the linearity of the shape operator, we get

$$\begin{aligned} k(\vartheta) &= S(\boldsymbol{u}(\vartheta)) \cdot \boldsymbol{u}(\vartheta) \\ &= S(\cos\vartheta \boldsymbol{e}_1 + \sin\vartheta \boldsymbol{e}_2) \cdot (\cos\vartheta \boldsymbol{e}_1 + \sin\vartheta \boldsymbol{e}_2) \\ &= \cos^2\vartheta S_{11} + 2\sin\vartheta\cos\vartheta S_{12} + \sin^2\vartheta S_{22} \end{aligned}$$

Note that the S_{ij} are fixed at p and, upon differentiating, we obtain

$$\frac{\mathrm{d}k}{\mathrm{d}\vartheta} = 2\sin\vartheta\cos\vartheta(S_{22} - S_{11}) + 2\left(\cos^2\vartheta - \sin^2\vartheta\right)S_{12}.$$

By assumption, $k = S_{11}$ is a maximum at $\vartheta = 0$, so $S_{12} = 0$. Moreover, since p is not umbilic, $S_{11} = S_{22}$ is excluded. Now k is defined on an undirected line (cf. above), it suffices to consider $\vartheta \in [0, \pi)$. Then the only other extremum is a minimum at $\vartheta = \frac{\pi}{2}$, that is, in the $\boldsymbol{e}_2 = \boldsymbol{u}_{\min}$ direction, as required. □

Corollary (Euler's formula). With a basis as in the proof, we get

$$k(\vartheta) = k_{\max}\cos^2\vartheta + k_{\min}\sin^2\vartheta.$$

Note In other words, regarding the shape operator as a matrix, $k_{\max}, k_{\min}$ are its eigenvalues and $\boldsymbol{u}_{\max}, \boldsymbol{u}_{\min}$ are its eigenvectors.

Thus, writing the shape operator as a matrix with respect to the basis $\{\boldsymbol{u}_{\max}, \boldsymbol{u}_{\min}\}$ of T_pM, we have

$$[S_{ij}(p)] = \begin{bmatrix} k_{\max} & 0 \\ 0 & k_{\min} \end{bmatrix}$$

Notice, in particular, that can read off its matrix invariants, the trace and determinant, as follows,

$$\operatorname{tr}[S] = k_{\max} + k_{\min}, \quad \det[S] = k_{\max}k_{\min}$$

In fact, we shall see later that these carry some important extrinsic and intrinsic geometrical information about M.

14 Lecture 14 - Feb 8

14.1 Gaussian and mean curvature

Definition 14.1: Gaussian and Mean Curvature

Let $M \subset \mathbb{R}^3$ and $[S_p]$ be the matrix of its shape operator at p. Then the Gaussian curavature of M at p is $K(p) = \det[S_p]$ and the mean curvature of M at p is $H(p) = \frac{1}{2}\operatorname{tr}[S_p]$.

Note Recall from Linear Algebra that trace and determinant are matrix invariants, that is, they are independent of the chosen vector space basis.

As discussed last time, it is convenient to take $\{\boldsymbol{u}_{\max}, \boldsymbol{u}_{\min}\}$ as the orthonormal basis of T_pM and so we can conclude immediately,

$$K(p) = k_{\max}k_{\min}$$
$$H(p) = \frac{1}{2}(k_{\max} + k_{\min})$$

Example For the sphere, $k_{\max} = k_{\min} = -\frac{1}{r}$ so $K = \frac{1}{r^2}$.

Note Recall that k is defined with respect to the chosen unit normal $\pm U$. Notice, then, that the sign of K is independent of the sign choice of U, but the sign of H does depend on that choice.

We also saw last time that the sign of k depends on whether the normal section curves in the same or opposite direction of U. Thus, we can determine the sign of K by inspection, as follows.

It is helpful to consider the quadratic approximation of a surface as the graph of $z = f(x,y)$, assuming that $p = x(0,0)$ corresponds to a critical point such that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at $(0,0)$ and aligning $\boldsymbol{x}_x, \boldsymbol{x}_y$ with $\boldsymbol{u}_{\max}, \boldsymbol{u}_{\min}$, respectively. Then (cf. Homework 4),

$$z = \frac{1}{2}k_{\max}(p)x^2 + \frac{1}{2}k_{\min}(p)y^2$$

Positive Gaussian Curvature $K(p) > 0$ obtains if if the principal curvatures are

$$k_{\max} > 0, k_{\min} > 0, \text{ or } k_{\max} < 0, k_{\min} < 0$$

that is, p is either at a local minimum or at a local maximum, as shown below.

Negative Gaussian Curvature $K(p) < 0$ obtains if the principal curvatures are $k_{\max} > 0, k_{\min} < 0$, or $k_{\max} < 0, k_{\min} > 0$, so p is at a saddle point, as shown below.

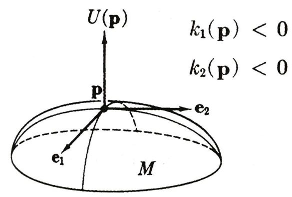


Figure 5: O'Neill Fig. 5.15

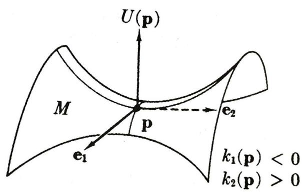


Figure 6: O'Neill Fig. 5.16

Zero Gaussian Curvature $K(p) = 0$ obtains if the principal curvatures are

$$k_{\max} = 0, \text{ or } k_{\min} = 0,$$

or both zero, that is, the surface near p is cylindrical, as shown below, or planar.

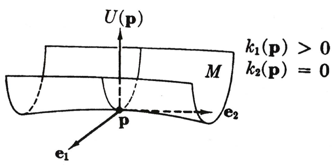


Figure 7: O'Neill Fig. 5.17

The following result will help us to compute the Gaussian and mean curvatures in practice:

Lemma 14.2

Let $\boldsymbol{v}, \boldsymbol{w} \in T_pM$ be linearly independent, S_p be the shape operator, $K(p)$ the Gaussian curvature and $H(p)$ the mean curvature at $p \in M \subset \mathbb{R}^3$. Then

$$S_p(\boldsymbol{v}) \times S_p(\boldsymbol{w}) = K(p)\boldsymbol{v} \times \boldsymbol{w}$$
$$S_p(\boldsymbol{v}) \times \boldsymbol{w} + \boldsymbol{v} \times S_p(\boldsymbol{w}) = 2H(p)\boldsymbol{v} \times \boldsymbol{w}$$

Proof. By assumption, $\{\boldsymbol{v}, \boldsymbol{w}\}$ form a basis for T_pM, though not necessarily othonormal.

Now writing the action of S_p on this basis with some $a, b, c, d \in \mathbb{R}$ as

$$S_p(\boldsymbol{v}) = a\boldsymbol{v} + b\boldsymbol{w}, \quad S_p(\boldsymbol{w}) = c\boldsymbol{v} + d\boldsymbol{w},$$

then the matrix representing S_p with respect to it is simply (why?)

$$[S_p] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

and therefore we can read off, by Def. 14.1,

$$K(p) = \det [S_p] = ad-bc, \quad H(p) = \frac{1}{2}\operatorname{tr}[S_p] = \frac{1}{2}(a+d).$$

Now direct computation reveals that

$$\begin{aligned} S_p(\boldsymbol{v}) \times S_p(\boldsymbol{w}) &= (a\boldsymbol{v} + b\boldsymbol{w}) \times (c\boldsymbol{v} + d\boldsymbol{w}) \\ &= (ad-bc)\boldsymbol{v} \times \boldsymbol{w} \\ &= K(p)\boldsymbol{v} \times \boldsymbol{w}, \end{aligned}$$

establishing the first equation, and similarly for the second one,

$$\begin{aligned} & S_p(\boldsymbol{v}) \times \boldsymbol{w} + \boldsymbol{v} \times S_p(\boldsymbol{w}) \\ =& (a\boldsymbol{v} + b\boldsymbol{w}) \times \boldsymbol{w} + \boldsymbol{v} \times (c\boldsymbol{v} + d\boldsymbol{w}) \\ =& (a+d)\boldsymbol{v} \times \boldsymbol{w} \\ =& 2H(p)\boldsymbol{v} \times \boldsymbol{w} \end{aligned}$$

□

Lem. 14.2 is, of course, a pointwise statement but, as before, we can extend it to vector fields V, W so that $V(p) = \boldsymbol{v}, W(p) = \boldsymbol{w}$ and real-valued functions K and H over M, and thus we have

$$S(V) \times S(W) = KV \times W$$
$$S(V) \times W + V \times S(W) = 2HV \times W$$

Recasting this using the so-called Lagrange identity in Homework 4, we can also solve explicitly for the Gaussian curvature,

$$K = \frac{\begin{vmatrix} S(V)\cdot V & S(V)\cdot W \\ S(W)\cdot V & S(W)\cdot W \end{vmatrix}}{\begin{vmatrix} V\cdot V & V\cdot W \\ W\cdot V & W\cdot W \end{vmatrix}}, (*)$$

and similarly the mean curvature can be expressed as

$$H = \frac{\begin{vmatrix} S(V)\cdot V & S(V)\cdot W \\ W\cdot V & W\cdot W \end{vmatrix} + \begin{vmatrix} V\cdot V & V\cdot W \\ S(W)\cdot V & S(W)\cdot W \end{vmatrix}}{2\begin{vmatrix} V\cdot V & V\cdot W \\ W\cdot V & W\cdot W \end{vmatrix}}. (\dagger)$$

Obviously, these equations are more useful for computer algebra than for computations by hand (not really cheat sheet material!).

Lemma 14.3

Let K be the Gaussian curvature function and H be the mean curvature function of M. Then its principal curvature functions are

$$k_{\max} = H + \sqrt{H^2 - K}, \quad k_{\min} = H - \sqrt{H^2-K}.$$

Proof. Recall that by Def. 14.1, we have

$$K = k_{\max}k_{\min}, \quad 2H = k_{\max} + k_{\min}$$

so the result follows by directly solving the corresponding quadratic equations, noting that

$$H^2 - K = \frac{1}{4}\left(k_{\max} - k_{\min}\right)^2.$$

□

Note Because of the square root, $k_{\max}, k_{\min}$ are continuous but need not be differentiable in general. Note also that the square root vanishes for umbilic points by Def. 13.8.

We also use Gaussian and mean curvatures to define the following two important kinds of special surfaces (more later):

Definition 14.4: Flat Surface

A surface $M \subset \mathbb{R}^3$ is called flat if its Gaussian curvature vanishes throughout, $K(p) = 0$ for all $p \in M$.

Example Planes and cylinders are flat (cf. discussion above).

Definition 14.5: Minimal Surface

A surface $M \subset \mathbb{R}^3$ is called minimal if its mean curvature vanishes throughout, $H(p) = 0$ for all $p \in M$.

Exercise Show that minimal sufaces must have negative Gaussian curvature, $K < 0$.

14.2 Curvatures and coordinates

Next, we need to understand the Gaussian and mean curvature of a surface in terms of its coordinate representations.

So suppose that $x : D \to M$ is a proper patch for $M \subset \mathbb{R}^3$ with coordinates $(u, v) \in D \subset \mathbb{R}^2$, and we already know from Lem. 10.4 that any $\boldsymbol{v} \in T_pM$ is a linear combination of $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$.

Definition 14.6: Warping and Shape Functions

Given a proper patch with $\mathrm{x} : D \to M \subset \mathbb{R}^3$ and shape operator S, the warping functions are real-valued functions on D such that

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v$$

and the shape functions are real-valued functions on D such that

$$L = S(\boldsymbol{x}_u)\cdot\boldsymbol{x}_u, \quad M = S(\boldsymbol{x}_u)\cdot\boldsymbol{x}_v, \quad N = S(\boldsymbol{x}_v)\cdot\boldsymbol{x}_v.$$

Note Historically, these define the symmetric matrices of the first and second fundamental forms I, II (not differential forms!),

$$\mathrm{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}, \quad \mathrm{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

Later in intrinsic geometry, we shall meet I again as metric matrix.

Note If ϑ is the angle between partial velocities, we have

$$F = |\boldsymbol{x}_u|\,|\boldsymbol{x}_v|\cos\vartheta = \sqrt{EG}\cos\vartheta$$

and also $|\boldsymbol{x}_u \times \boldsymbol{x}_v|^2 = |\boldsymbol{x}_u|^2\,|\boldsymbol{x}_v|^2 \sin^2\vartheta = EG - F^2$

The latter formula occurs when using the unit normal,

$$U = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{|\boldsymbol{x}_u \times \boldsymbol{x}_v|}$$

Lemma 14.7

Let $M \subset \mathbb{R}^3$ be a surface with unit normal $U, \boldsymbol{x} : D \to M$, and warping functions E, F, G, then its shape functions are given by

$$L = U \cdot \boldsymbol{x}_{uu}, \quad M = U \cdot \boldsymbol{x}_{uv}, \quad N = U \cdot \boldsymbol{x}_{vv}$$

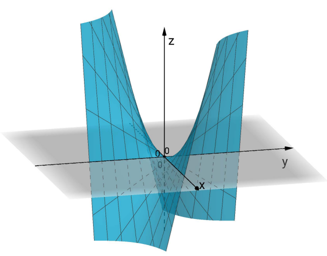
and the Gaussian and mean curvatures are

$$K = \frac{LN - M^2}{EG - F^2} = \frac{\det II}{\det I},$$
$$H = \frac{GL + EN - 2FM}{2\,(EG - F^2)}$$

Proof. The proof for M was already given in the proof of Lem. 13.3 on the symmetry of S, and the proofs for L, N proceed analogously. The statements for K, H follow directly from Def. 14.6 and the equations $(*)$ and $(\dagger)$ above, by setting $V = \boldsymbol{x}_u$ and $W = \boldsymbol{x}_v$. $\square$

Example Consider a saddle surface given by the graph of $z = xy$. In this case, a convenient patch is provided by the Monge patch,

$$\boldsymbol{x}(u, v) = (u, v, uv), \quad (u, v) \in D \subset \mathbb{R}^2$$



$$\begin{aligned} \boldsymbol{x}_u &= (1, 0, v) \\ \boldsymbol{x}_v &= (0, 1, u) \\ \boldsymbol{x}_{uu} &= (0, 0, 0) \\ \boldsymbol{x}_{uv} &= (0, 0, 1) \\ \boldsymbol{x}_{vv} &= (0, 0, 0) \end{aligned}$$

Figure 8: Saddle

and so by applying Def. 14.6,

$$E = 1 + v^2, F = uv, G = 1 + u^2, U = \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}$$

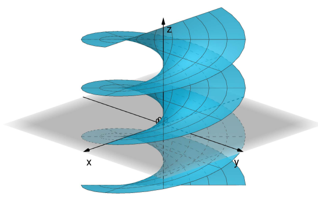
Then from Lem. 14.7, one can easily compute the shape functions $L = U \cdot \boldsymbol{x}_{uu} = 0$, $M = U \cdot \boldsymbol{x}_{uv} = \frac{1}{\sqrt{1+u^2+v^2}}$, $N = U \cdot \boldsymbol{x}_{vv} = 0$ and also the Gaussian and mean curvatures,

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{1}{(1 + u^2 + v^2)^2}$$
$$H = \frac{GL + EN - 2FM}{2\,(EG - F^2)} = -\frac{uv}{(1 + u^2 + v^2)^{\frac{3}{2}}}$$

Note The saddle surface has negative Gaussian curvature, as expected from our earlier discussion.

Example The helicoid is a surface described by the single patch

$$\boldsymbol{x}(u, v) = (u\cos v, u\sin v, bv), \quad b \neq 0, (u, v) \in D \subset \mathbb{R}^2.$$



$$\begin{aligned} \boldsymbol{x}_u &= (\cos v, \sin v, 0), \\ \boldsymbol{x}_v &= (-u\sin v, u\cos v, b), \\ \boldsymbol{x}_{uu} &= (0, 0, 0), \\ \boldsymbol{x}_{uv} &= (-\sin v, \cos v, 0) \\ \boldsymbol{x}_{vv} &= (-u\cos v, -u\sin v, 0), \end{aligned}$$

Figure 9: Helicoid

so again applying Def. 14.6

$$E = 1, F = 0, G = u^2 + b^2, U = \frac{(b\sin v, -b\cos v, u)}{\sqrt{u^2 + b^2}}.$$

The shape functions can again be found easily from Lem. 14.7, $L = U \cdot \boldsymbol{x}_{uu} = 0$, $M = U \cdot \boldsymbol{x}_{uv} = -\frac{b}{\sqrt{u^2+b^2}}$, $N = U \cdot \boldsymbol{x}_{vv} = 0$ yielding the Gaussian and mean curvatures thus,

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{b^2}{(u^2 + b^2)^2}$$
$$H = \frac{GL + EN - 2FM}{2\,(EG - F^2)} = 0$$

Note The helicoid is thus a minimal surface by Def. 14.5, with negative Gaussian curvature as expected.

Note Both the saddle and the helicoid are ruled surfaces according to Def. 9.7: identify base and director curves in each case.

15 Lecture 15 - Feb 15

15.1 Curvature: surface of revolution

Example For the surface of revolution $\boldsymbol{x}(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$, we have

$$\begin{aligned}
\boldsymbol{x}_u &= (g', h'\cos v, h'\sin v)\\
\boldsymbol{x}_v &= (0, -h\sin v, h\cos v)\\
\boldsymbol{x}_u \times \boldsymbol{x}_v &= (hh', -g'h\cos v, -g'h\sin v)\\
|\boldsymbol{x}_u \times \boldsymbol{x}_v| &= h\sqrt{(g')^2 + (h')^2}\\
\boldsymbol{x}_{uu} &= (g'', h''\cos v, h''\sin v)\\
\boldsymbol{x}_{uv} &= (0, -h'\sin v, h'\cos v)\\
\boldsymbol{x}_{vv} &= (0, -h\cos v, -h\sin v)
\end{aligned}$$

Then,

$$\begin{aligned}
E &= \boldsymbol{x}_u \cdot \boldsymbol{x}_u = (g')^2 + (h')^2\\
F &= \boldsymbol{x}_u \cdot \boldsymbol{x}_v = 0\\
G &= \boldsymbol{x}_v \cdot \boldsymbol{x}_v = h^2\\
U &= \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{|\boldsymbol{x}_u \times \boldsymbol{x}_v|} = \frac{(h', -g'\cos v, -g'\sin v)}{\sqrt{(g')^2 + (h')^2}}\\
L &= U \cdot \boldsymbol{x}_{uu} = \frac{g''h' - h''g'}{\sqrt{(g')^2 + (h')^2}}\\
M &= U \cdot \boldsymbol{x}_{uv} = 0\\
N &= U \cdot \boldsymbol{x}_{vv} = \frac{g'h}{\sqrt{(g')^2 + (h')^2}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
K &= \frac{LN - M^2}{EG - F^2} = \frac{g'(g''h' - g'h'')}{h((g')^2 + (h')^2)^2} \qquad (**)\\
H &= \frac{GL + EN - 2FM}{2(EG - F^2)}\\
&= \frac{1}{2}\left(\frac{g''h' - h''g'}{((g')^2 + (h')^2)^{\frac{3}{2}}} + \frac{g'h}{h^2\sqrt{(g')^2 + (h')^2}}\right)
\end{aligned}$$

Example A torus such that $g(u) = r\sin u$, $h(u) = R + r\cos u$ has Gaussian curvature

$$K = \frac{\cos u}{r(R + r\cos u)}$$

Catenoid The surface of revolution with catenary meridians, e.g. with profile curve $(g(u), h(u)) = (u, \cosh u)$, is called catenoid. It $H = 0$.

Theorem 15.1

If a surface of revolution is a minimal surface, then it is contained either in a plane or catenoid.

Proof. See O'Neill Thm. 5.7.2, p. 255. □

Example A tractrix is a planar curve $\alpha(u) = (u, h(u)), u > 0$, starting at fixed $(0, r)$ such that the distance to the u-axis along tangent lines is always r. This condition means that

$$h' = -\frac{h}{\sqrt{r^2 - h^2}}$$

Taking the tractrix as profile curve, the corresponding surface of revolution is called tractroid, bugle surface, or pseudosphere:

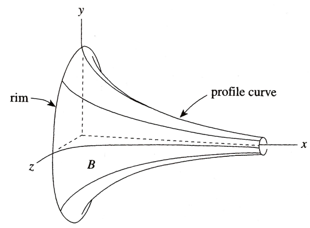


Figure 10: O'Neill Fig. 5.41

The Gaussian curvature of pseudosphere is constant negative (hence the name!), $K = -\frac{1}{r^2}$.

Lemma 15.2

Consider a surface of revolution as before but with canonical, or unit speed, parametrization of its profile curve such that $E = g'^2 + h'^2 \equiv 1$, then its Gaussian curvature is

$$K = -\frac{h''}{h}$$

Proof. Differentiating the condition of canonical parametrization, that is $g'^2 + h'^2 \equiv 1$, we obtain $g'g'' = -h'h''$. Now substituting this into $(**)$ yields the required result,

$$K = \frac{g'\left(g''h' - g'h''\right)}{h\left(g'^2 + h'^2\right)^2} = -\frac{h''}{h}.$$

□

15.2 Curvature: implicit case

So consider $M : g = 0$ and recall from Lem. 10.6 that a unit normal vector field is provided by the gradient according to

$$U = \frac{\nabla g}{|\nabla g|}$$

Now let V, W be tangent vector fields with $V \times W \| U$, then

$$\begin{aligned}
S(V) &= -\nabla_V U = -\frac{\nabla_V \nabla g}{|\nabla g|} - V\left[\frac{1}{|\nabla g|}\right]\nabla g\\
S(W) &= -\nabla_W U = -\frac{\nabla_W \nabla g}{|\nabla g|} - W\left[\frac{1}{|\nabla g|}\right]\nabla g
\end{aligned}$$

by the product rule, and since $\nabla_V \nabla g \times \nabla g \perp \nabla g$ we obtain

$$\nabla g \cdot S(V) \times S(W) = \nabla g \cdot \frac{\nabla_V \nabla g}{|\nabla g|} \times \frac{\nabla_W \nabla g}{|\nabla g|}.$$

On the other hand, Lem. 14.2 implies that

$$\nabla g \cdot S(V) \times S(W) = K \nabla g \cdot V \times W = K |\nabla g| |V \times W|$$

yielding an explicit formula for the Gaussian and mean curvature,

$$K = \frac{\nabla g \cdot \nabla_V \nabla g \times \nabla_W \nabla g}{|\nabla g|^3 |V \times W|} \qquad (\ddagger)$$

$$H = -\nabla g \cdot \frac{\nabla_V \nabla g \times W + V \times \nabla_W \nabla g}{2|\nabla g|^2 |V \times W|}$$

Example Compute the Gaussian curvature of the ellipsoid,

$$M : g(x_1, x_2, x_3) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

The gradient is $\nabla g = 2 \sum_i \frac{x_i}{a_i^2} U_i$. Writing $V = \sum_i v_i U_i$, we have $\nabla_V \nabla g = 2 \sum_i \frac{1}{a_i^2} V[x_i] U_i = 2 \sum_i \frac{v_i}{a_i^2} U_i$, and similarly for W, so

$$\nabla g \cdot \nabla v \nabla_g \times \nabla w \nabla g = \begin{vmatrix} 2\frac{x_1}{a_1^2} & 2\frac{x_2}{a_2^2} & 2\frac{x_3}{a_3^2} \\ 2\frac{v_1}{a_1^2} & 2\frac{v_2}{a_2^2} & 2\frac{v_3}{a_3^2} \\ 2\frac{w_1}{a_1^2} & 2\frac{w_2}{a_2^2} & 2\frac{w_3}{a_3^2} \end{vmatrix}$$

$$= \frac{8}{a_1^2 a_2^2 a_3^2} |V \times W| \frac{2g}{|\nabla g|}$$

whence $K = \frac{16}{a_1^2 a_2^2 a_3^2} |\nabla g|^{-4} = \frac{1}{a_1^2 a_2^2 a_3^2} \left(\frac{x_1^2}{a_1^4} + \frac{x_2^2}{a_2^4} + \frac{x_3^2}{a_3^4} \right)^{-2}$ by ($\ddagger$).

15.3 Special curves in surfaces

Definition 15.3: Principal Curve

A regular curve α in $M \subset \mathbb{R}^3$ is called principal curve if α' is always in a principal direction.

Lemma 15.4

A regular curve α in a surface $M \subset \mathbb{R}^3$ with unit normal U is principal if, and only if, α' and U' are collinear.

Proof. As shown in Homework 4, $S(\alpha') = -U'$. Now on a principal curve, $S(\alpha')$ is collinear with α' by Thm. 13.9 and linearity of S, Lem. 13.2, so α' and U' are also collinear, as required. □

Note Using also Lem. 13.4, we see that its principal curvature is

$$k\left(\frac{\alpha'}{|\alpha'|}\right) = S\left(\frac{\alpha'}{|\alpha'|}\right) \cdot \frac{\alpha'}{|\alpha'|} = \frac{S(\alpha') \cdot \alpha'}{|\alpha'|^2} = \frac{\alpha'' \cdot U}{|\alpha'|^2}.$$

Lemma 15.5

Let M, P be a surface and a plane in $\mathbb{R}^3$ with unit normals U, V, respectively, and $\alpha = M \cap P$ a curve where $\cos\vartheta = U \cdot V \neq \pm 1$. If the intersection angle ϑ is constant along α, then α is principal.

Proof. U is a unit vector field, so $U' \cdot U = 0$ along α. Moreover, $V' = 0$ since P is a plane, so by differentiating $U \cdot V = \text{const.}$,

$$0 = U' \cdot V + U \cdot V' = U' \cdot V.$$

Thus, U' is orthogonal to both U, V. But so is α' since α is the intersection of M and P. Hence, α' and U' are collinear and so α is a principal curve by Lem. 15.4. □

Example It is clear from this Lemma that meridians and parallels of a surface of revolution are, in fact, principal curves.

Definition 15.6: Asymptotic Curve

A direction $\boldsymbol{v} \in T_pM$ such that the normal curvature $k(\boldsymbol{v}) = 0$ is called asymptotic at $p \in M \subset \mathbb{R}^3$. A regular curve α in $M \subset \mathbb{R}^3$ is called asymptotic curve if α' is asymptotic at all points of α.

Lemma 15.7

α is asymptotic if, and only if, its acceleration α'' is tangent to M.

Proof. By Def. 15.6, $0 = k(\alpha') = S(\alpha') \cdot \alpha'$. Thus, given the unit normal U of M as before, $U' \cdot \alpha' = 0$. Since $U \cdot \alpha' = 0$, by differentiating

$$0 = U' \cdot \alpha' + U \cdot \alpha'' = U \cdot \alpha''$$

so the normal component of the acceleration indeed vanishes. □

Asymptotic directions and Gaussian curvature are related thus:

Lemma 15.8

Let $p \in M \subset \mathbb{R}^3$. Then
(1) if $K(p) > 0$, there are no asymptotic directions,
(2) if $K(p) < 0$, there are two asymptotic directions,
(3) if $K(p) = 0$, every direction is asymptotic if p is planar, otherwise there is one which is also principal.

Proof. We use $K = k_{\max} k_{\min}$ and Euler's formula (cf. Thm. 13.9),

$$k(\vartheta) = k_{\max} \cos^2 \vartheta + k_{\min} \sin^2 \vartheta.$$

(1) $K > 0 \Rightarrow k_{\max}, k_{\min}$ have the same sign so k is never zero and thus there are no asymptotic directions,
(2) $K < 0 \Rightarrow k_{\max}, k_{\min}$ have opposite sign so $k = 0$ is obtained for two solutions of $\tan^2 \vartheta = -\frac{k_{\max}}{k_{\min}}$,
(3) $K = 0 \Rightarrow k_{\max} = 0$ or $k_{\min} = 0$. If p is planar both vanish so $k = 0$ for all directions, or else $k = 0$ in a principal direction. □

Note : In Lecture 14, saddle and helicoid were found to be ruled surfaces with $K < 0$. Now we see that this is no coincidence:

A ruled surfaces M contains a line α through each point by Def. 9.7, but acceleration $\alpha'' = 0$ for lines whence $k(\alpha') = 0$ as we showed earlier. So there exist asymptotic directions at every point of M, and thus, by Lem. 15.8, we immediately obtain that

Corollary A ruled surface has $K \leq 0$.

Definition 15.9: Geodesic

A regular curve α in $M \subset \mathbb{R}^3$ is called geodesic if its acceleration α'' is always normal to M.

Note Since $\alpha'' \cdot \alpha' = 0$ by Def. 15.9, the speed along geodesics is constant, that is, $|\alpha'|' = 0$.

Example The geodesics of a plane $P \subset \mathbb{R}^3$ are straight lines:

Any curve α in P with constant normal $\boldsymbol{n}$ has $\alpha' \cdot \boldsymbol{n} = 0$ whence $\alpha'' \cdot \boldsymbol{n} = 0$. So if α is geodesic, α'' collinear with $\boldsymbol{n}$ implies $\alpha'' = 0$ and hence a straight line. Conversely, a straight line $\alpha \subset P$ has $\alpha'' = 0$ which is trivially normal to P, thus α is a geodesic.

Example The geodesics of a sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ are great circles:

Their acceleration α'' is radial and thus normal to the surface.

Example On a circular cylinder $M \subset \mathbb{R}^3 : x^2 + y^2 = r^2$,

$$\alpha(t) = (r \cos \vartheta(t), r \sin \vartheta(t), z(t))$$

so a geodesic has $z''(t) = 0$ (why?) and its constant speed $|\alpha'|^2 = r^2 \vartheta'^2 + z'^2$ implies $\vartheta' =$ const., yielding helices and circles.

To summarize the classification of special curves in M :

Principal: $k(\alpha') = k_{\max}$ or $k_{\min}, S(\alpha')$ collinear with α'.

Asymptotic: $k(\alpha') = 0, S(\alpha')$ orthogonal to α', α'' tangent to M.

Geodesic: α'' normal to M.

16 Lecture 16 - Feb 16

16.1 Adapted frame field

Definition 16.1: Adapted Frame Field

An adapted frame field $\{E_1, E_2, E_3\}$ on a region $S \subset M \subset \mathbb{R}^3$ is a Euclidean frame field such that E_3 is always normal to M.

Note Hence, E_1, E_2 are necessarily always tangent M.

Lemma 16.2

An adapted frame field exists if, and only if, S is orientable and there exists a non-vanishing tangent vector field on S.

Proof. An adapted frame field provides orientation and such a vector field. Conversely, suppose there is a tangent $V \neq 0$ and S is orientable. Then by Lem. 12.6, there is a unit normal U, so we may put

$$E_1 = \frac{V}{|V|}, \quad E_2 = U \times E_1, \quad E_3 = U \left(= E_1 \times E_2\right).$$

□

Example The normal of a circular cylinder $M = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ g(x, y, z) = x^2 + y^2 = r^2\right\}$ is ∇g by Lem. 10.6, so the result yields

$$E_1 = U_3, \quad E_2 = -\frac{y}{r} U_1 + \frac{x}{r} U_2, \quad E_3 = \frac{x}{r} U_1 + \frac{y}{r} U_2.$$

Theorem 16.3

Let $\omega_{ij} = -\omega_{ji}$, $1 \le i, j \le 3$, be the connection 1-forms of the adapted frame $\{E_1, E_2, E_3\}$ on $M \subset \mathbb{R}^3$. Then for any tangent vector field V on M, the connection equations of M are

$$\nabla_V E_i = \sum_j \omega_{ij}(V) E_j, \quad 1 \le i \le 3$$

Proof. As for Thm. 8.4 but adapted to M. □

Note As before, $\omega_{ij}(\boldsymbol{v}) = \nabla_{\boldsymbol{v}} E_i \cdot E_j(p)$, but now for any $\boldsymbol{v} \in T_pM$ giving the corresponding initial rotation rate of E_i towards E_j.

Corollary The shape operator can now be expressed thus,

$$S(\boldsymbol{v}) = -\nabla_{\boldsymbol{v}} E_3(p) = \omega_{13}(\boldsymbol{v}) E_1(p) + \omega_{23}(\boldsymbol{v}) E_2(p).$$

Note The shape operator is depends on ω_{13}, ω_{23} only and, in addition, ω_{12} gives the rotation of E_1, E_2 in the tangent plane.

Moreover, notice that in addition to the adapted frame of Def. 16.1, we can also write down the adapted coframe $\{\theta_1, \theta_2, \theta_3\}$ such that it is dual, with the usual requirement (Def. 8.6) that

$$\forall \boldsymbol{v} \in T_pM: \quad \theta_i(\boldsymbol{v}) = \boldsymbol{v} \cdot E_i(p).$$

But any $\boldsymbol{v}$ is orthogonal to E_3 by setup, so in the adapted coframe $\theta_3 \equiv 0$ and, effectively, we have $\{\theta_1, \theta_2\}$ only.

Example To find the adapted frame field of the sphere of radius r in $\mathbb{R}^3$, we simply use the spherical frame field of $\mathbb{R}^3$ (cf. Homework 2), set $\rho = r$ and suitably reorder coordinates. Thus, (check!)

$$\begin{aligned} \theta_1^{\text{sph}} &= r\cos\varphi \, \mathrm{d}\vartheta, & \omega_{12}^{\text{sph}} &= \sin\varphi \, \mathrm{d}\vartheta \\ \theta_2^{\text{sph}} &= r \, \mathrm{d}\varphi, & \omega_{13}^{\text{sph}} &= -\cos\varphi \, \mathrm{d}\vartheta \\ & & \omega_{23}^{\text{sph}} &= -\mathrm{d}\varphi \end{aligned}$$

We are now ready to meet one of the central results of this course, the set of so-called fundamental equations of surface geometry:

Theorem 16.4: Fundamental Equations of Surface Geometry

Let $\{E_1, E_2, E_3\}$ be an adapted frame field of $M \subset \mathbb{R}^3$, then its dual forms θ_1, θ_2 and connection forms $\omega_{12}, \omega_{13}, \omega_{23}$ satisfy:

(1) First structural equations:

$$\mathrm{d}\theta_1 = \omega_{12} \wedge \theta_2, \qquad \mathrm{d}\theta_2 = \omega_{21} \wedge \theta_1.$$

(2) Symmetry equation:

$$\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0.$$

(3) Gauss equation:

$$\mathrm{d}\omega_{12} = \omega_{13} \wedge \omega_{32}$$

(4) Codazzi equations:

$$\mathrm{d}\omega_{13} = \omega_{12} \wedge \omega_{23}, \qquad \mathrm{d}\omega_{23} = \omega_{21} \wedge \omega_{13}.$$

Proof. These follow by adapting Cartan's structural equations of Thm. 8.8 and the above observation that $\theta_3 \equiv 0$.

Then (1) and (2) are obtained immediately from the first structural equation, $\mathrm{d}\theta_i = \sum_j \omega_{ij} \wedge \theta_j$, and similarly (3) and (4) follow from the second structural equation, $\mathrm{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$. □

Lemma 16.5

(1) $\omega_{13} \wedge \omega_{23} = K\theta_1 \wedge \theta_2$,
(2) $\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\theta_1 \wedge \theta_2$.

Note Before proceeding with the proof, it may be helpful to recall that a coframe provides an orthonormal expansion for forms just like a frame does for vectors. So, in our current case with $\theta_3 \equiv 0$, any 1-forms and 2-forms on M may be expressed uniquely as

$$\begin{aligned} \phi &= \phi(E_1)\,\theta_1 + \phi(E_2)\,\theta_2 \in \Omega^1 M \\ \mu &= \mu(E_1, E_2)\,\theta_1 \wedge \theta_2 \in \Omega^2 M \end{aligned}$$

Proof. Now computing the shape operator for E_1, E_2, one obtains

$$\begin{aligned} S(E_1) &= -\nabla_{E_1} E_3 = -\omega_{31}(E_1)\,E_1 - \omega_{32}(E_1)\,E_2 \\ S(E_2) &= -\nabla_{E_2} E_3 = -\omega_{31}(E_2)\,E_1 - \omega_{32}(E_2)\,E_2 \end{aligned}$$

from which we can read off the matrix $[S]$ representing the shape operator as discussed in Lecture 14.

Now to establish statement (1) of the Lemma, given the previous comment, it suffices to evaluate the equation on E_1, E_2, whence the left-hand side yields, as required by the right-hand side,

$$\begin{aligned}&\left(\omega_{13}\wedge\omega_{23}\right)\left(E_1,E_2\right)\\=&\,\omega_{13}\left(E_1\right)\omega_{23}\left(E_2\right)-\omega_{13}\left(E_2\right)\omega_{23}\left(E_1\right)\\=&\det[S]=K\end{aligned}$$

and similarly for statement (2). □

Corollary Now the Gauss equation (3) of Thm. 16.4 yields

$$\mathrm{d}\omega_{12}=-K\theta_1\wedge\theta_2,$$

called the second structural equation of M. Note its intrinsic nature!

Example Applying it to the sphere of radius r in $\mathbb{R}^3$ as before,

$$\mathrm{d}\omega_{12}^{\text{sph}}=\mathrm{d}(\sin\varphi\mathrm{d}\vartheta)=\cos\varphi\mathrm{d}\varphi\wedge\mathrm{d}\vartheta=-\cos\varphi\mathrm{d}\vartheta\wedge\mathrm{d}\varphi,$$
$$\theta_1^{\text{sph}}\wedge\theta_2^{\text{sph}}=(r\cos\phi\mathrm{d}\vartheta)\wedge(r\ \mathrm{d}\varphi)=r^2\cos\varphi\mathrm{d}\vartheta\wedge\mathrm{d}\varphi,$$

reading off the Gaussian curvature of the sphere, as expected,

$$K=\frac{1}{r^2}.$$

16.2 Principal frame field

Definition 16.6: Principal Frame Field

A principal frame field on $M\subset\mathbb{R}^3$ is an adapted frame field $\{E_1,E_2,E_3\}$ such that, at each point, E_1,E_2 are principal.

Then as discussed in Lecture 13, E_1,E_2 are eigenvectors of the shape operator with normal curvature eigenvalues, by Thm. 13.9,

$$S\left(E_1\right)=k_1E_1,\quad S\left(E_2\right)=k_2E_2$$

Note Given a chosen orientation (normal) of M and $\{E_1,E_2,E_3\}$ right-handed, normal curvatures k_1,k_2 may be either $k_{\max},k_{\min}$.

Note A principal frame field exists at least locally on a surface and is unique, except at umbilic points (1cf. Def. 13.8).

Note Comparing the previous equations with the corollary of Thm. 16.3, the orthonormal expansion of the connection forms is simply

$$\omega_{13}=k_1\theta_1,\quad\omega_{23}=k_2\theta_2$$

Now given a principal frame field, the Codazzi equations (4) of Thm. 16.4 may be recast in the following useful form:

Theorem 16.7

Suppose $\{E_1,E_2,E_3\}$ is a principal frame field of $M\subset\mathbb{R}^3$, then
(1) $E_1\left[k_2\right]=\left(k_1-k_2\right)\omega_{12}\left(E_2\right)$,
(2) $E_2\left[k_1\right]=\left(k_1-k_2\right)\omega_{12}\left(E_1\right)$.

Proof. The first Codazzi equation is $\mathrm{d}\omega_{13}=\omega_{12}\wedge\omega_{23}$ so by the previous remark we obtain $\mathrm{d}k_1\wedge\theta_1+k_1\ \mathrm{d}\theta_1=k_2\omega_{12}\wedge\theta_2$.

Then substituting the structural equation $\mathrm{d}\theta_1=\omega_{12}\wedge\theta_2$ and evaluating the 2-form on E_1,E_2

$$\left(\mathrm{d}k_1\wedge\theta_1\right)\left(E_1,E_2\right)=\left(k_2-k_1\right)\left(\omega_{12}\wedge\theta_2\right)\left(E_1,E_2\right)$$

whence using Def. 10.10, Def. 3.2 and orthonormality,

$$\begin{aligned}&\mathrm{d}k_1\left(E_1\right)\underbrace{\theta_1\left(E_2\right)}_{0}-\underbrace{\mathrm{d}k_1\left(E_2\right)}_{E_2[k_1]}\underbrace{\theta_1\left(E_1\right)}_{1}\\&=\left(k_2-k_1\right)\left(\omega_{12}\left(E_1\right)\underbrace{\theta_2\left(E_2\right)}_{1}-0\right)\end{aligned}$$

so statement (1) follows, and (2) by similar computation. □

16.3 All-umbilic surfaces

All-umbilic Surfaces This version of Codazzi is now applied to establish a fundamental result about all-umbilic surfaces, whereby each point is umbilic:

$M\subset\mathbb{R}^3$ is all-umbilic if, and only if, it is part of a plane or sphere.

If M is part of a plane or sphere, we know of course that it is all-umbilic. The converse is less obvious and requires first the

Lemma 16.8

If $M\subset\mathbb{R}^3$ is all-umbilic then K is a non-negative constant.

Proof. Since M is all-umbilic, $k_1=k_2=k$ and an adapted frame field $\{E_1,E_2,E_3\}$ may be regarded as principal.

Thus by Thm. 16.7, $E_1[k]=E_2[k]=0$ or equivalently $\mathrm{d}k\left(E_1\right)=\mathrm{d}k\left(E_2\right)=0$ whence $\mathrm{d}k=0$.

But $K=k_1k_2=k_2\geq 0$ and $\mathrm{d}K=2k\mathrm{d}k=0$ so $K=$ const., as required. □

Then the fundamental result is established by considering the two possibilities, $K=0$ and $K>0$, given here as theorems:

Theorem 16.9

If $M \subset \mathbb{R}^3$ is all-umbilic with $K = 0$, then M is part of a plane.

Proof. For any curve α in M through $p = \alpha(0)$, we have (cf. Homework 4), $E_3' = -S(\alpha') = k\alpha' = 0$, so E_3 is Euclidean parallel, that is, a constant normal vector field on M.

As always, we assume M to be connected (cf. Lecture 13), so by Def. 12.2 there exists a curve from p to any $q \in M$, say $\alpha(1) = q$.

Then using the canonical isomorphism and introducing the function $f(t) = (\alpha(t) - p) \cdot E_3$, note that $f(0) = 0$ and $f'(t) = \alpha' \cdot E_3 = 0$ so $f \equiv 0$ whence, in particular, $f(1) = (q - p) \cdot E_3 = 0$, which is the standard scalar equation of a plane. $\square$

Theorem 16.10

If $M \subset \mathbb{R}^3$ is all-umbilic with $K > 0$, then M is part of a sphere.

Proof. Now $k_1 = k_2 = k$ and $K = k^2$ is constant by Lem. 16.8, so k is constant too. Again, by connectedness, there is a curve α from a given $p = \alpha(0)$ to any $q = \alpha(1)$ in M. Now consider a curve $\gamma = \alpha + \frac{1}{k}E_3$ in $\mathbb{R}^3$ (with canonical isomorphism). Then

$$\gamma' = \alpha' + \frac{1}{k}E_3' = \alpha' + \frac{1}{k}(-S(\alpha')) = \alpha' + \frac{1}{k}(-k\alpha') = 0.$$

Hence, γ is constant with $c = \gamma(0) = \gamma(1) = q + \frac{1}{k}E_3$ so that $d(c, q) = \frac{1}{|k|}$ so M is a sphere centered at c with radius $\frac{1}{\sqrt{K}}$. $\square$

17 Lecture 17 - Feb 17

17.1 Global theorems

Theorem 17.1

On every compact $M \subset \mathbb{R}^3$, there is a point with $K > 0$.

Proof. Let $f(p) = d^2(0, p)$ be the squared distance of $p \in M$ from the origin in $\mathbb{R}^3$. Since M is compact, by the maximum principle Lem. 12.4, f attains its maximum at some $m \in M$ with $d(0, m) = r$ say.

Now given any unit $\boldsymbol{u} \in T_m M$ and unit-speed curve α so that $\alpha(0) = m$ and $\alpha'(0) = \boldsymbol{u}$, the maximum condition reads (why?)

$$\frac{\mathrm{d}(f \circ \alpha)}{\mathrm{d}t}(0) = 0, \quad \frac{\mathrm{d}^2(f \circ \alpha)}{\mathrm{d}t^2}(0) \leq 0.$$

Using for convenience the canonical isomorphism, we may write $f \circ \alpha = \alpha \cdot \alpha$, and the previous two equations may be recast thus,

$$m \cdot \boldsymbol{u} = 0, \quad 1 + m \cdot \alpha''(0) \leq 0.$$

So $\frac{m}{r}$ is unit normal, and by the note to Lem. 15.4 we have $k(\boldsymbol{u}) = \frac{m}{r} \cdot \alpha''(0) \leq -\frac{1}{r}$, for both principal curvatures. Hence,

$$K(m) \geq \frac{1}{r^2} > 0.$$

$\square$

Now adding to compactness the condition that K be constant, we can obtain the remarkable result that the surface must be a sphere. In order to establish this result, we first need the

Lemma 17.2: (Hilbert)

Let $m \in M \subset \mathbb{R}^3$ such that $k_1(m)$ is a local maximum and $k_2(m)$ a local minimum, with $k_1(m) > k_2(m)$. Then

$$K(m) \leq 0.$$

Note Since $k_1 - k_2 > 0, m$ is by assumption not umbilic. Also, the ordering implies a suitable choice of orientation.

Proof. Given a suitable principal frame field $\{E_1, E_2, E_3\}$, the conditions of k_1 being a maximum and k_2 a minimum at m imply that

$$E_2[k_1] = E_1[k_2] = 0, \quad E_2[E_2[k_1]] \leq 0, E_1[E_1[k_2]] \geq 0.$$

Then using $k_1 - k_2 > 0$, Thm. 16.7 yields $\omega_{12}(E_1) = \omega_{12}(E_2) = 0$, and so a result of Homework 4 implies that, at m,

$$K = E_2[\omega_{12}(E_1)] - E_1[\omega_{12}(E_2)].$$

Finally, applying directional derivatives to the Codazzi equation, we get $E_1[E_1[k_2]] = (k_1 - k_2) E_1[\omega_{12}(E_2)] \geq 0 \Rightarrow E_1[\omega_{12}(E_2)] \geq 0$, and similarly $E_2[\omega_{12}(E_1)] \leq 0$, at m. Hence, $K(m) \leq 0$, as required. $\square$

Theorem 17.3

If $M \subset \mathbb{R}^3$ is compact with constant K, then it is a sphere.

Proof. By compactness and Thm. 12.11, M is orientable and we can arrange $k_1 \geq k_2$. By Lem. 12.4, k_1 attains its maximum at m, say. Then since $K = k_1 k_2 =$ const., k_2 must attain its minimum at m.

Now by compactness and Thm. 17.1, there is a point where $K > 0$ but since $K =$ const. on M by assumption, $K > 0$ throughout M. Then Hilbert's Lemma 17.2 implies $k_1(m) = k_2(m)$ (why?), but all k are between the maximum $k_1(m)$ and minimum $k_2(m)$ so M is all-umbilic. Thus, the result follows from Thm. 16.10. □

Note Thus, there can be no compact $M \subset \mathbb{R}^3$ with constant negative K! Recall the pseudosphere of Lecture 15 as example.

17.2 Intrinsic distance

Definition 17.4: Intrinsic Distance

Let $p, q \in M \subset \mathbb{R}^3$ and Σ be the set of all curve segments α in M from p to q. Then the intrinsic distance from p to q is

$$\rho(p, q) = \inf_{\alpha \in \Sigma} L(\alpha)$$

Note Euclidean distance c.f. Def. 2.2. Curve length $L(\alpha)$ is defined in Lecture 5.

Note As always with the infimum, or greatest lower bound, there need not be an α whose length actually equals $\rho(p, q)$.

17.3 Isometries of surfaces

Definition 17.5: Local Isometry

A local isometry $F : M \to M$ of surfaces M, M in $\mathbb{R}^3$ is a differentiable map that preserves tangent vector dot products,

$$\forall \boldsymbol{v}, \boldsymbol{w} \in T_p M : \quad F_*(\boldsymbol{v}) \cdot F_*(\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{w}.$$

If F is also bijective, it is called isometry.

Note By the above condition, $|F_*(\boldsymbol{v})| = |\boldsymbol{v}|$ so $F_*(\boldsymbol{v}) = \mathbf{0} \Rightarrow \boldsymbol{v} = \mathbf{0}$ whence F_* is injective, so (local) isometries are regular by Def. 4.3.

Note By regularity, a local isometry is locally diffeomorphic, of. Inverse Function Theorem 4.5, and thus bijective. In other words, a local isometry is, in fact, locally an isometry!

Note So inverse F^{-1} is differentiable and also an isometry.

Theorem 17.6

Isometries preserve intrinsic distance. That is, if $F : M \to \bar{M}$ is an isometry of surfaces M and $\bar{M}$ with intrinsic distance functions ρ and $\bar{\rho}$, respectively, then

$$\forall p, q \in M : \quad \rho(p, q) = \bar{\rho}(F(p), F(q))$$

Proof. By Def. 17.5, an isometry F provides a bijection between points, thus also between curve segments α in M and $\bar{\alpha} = F(\alpha)$ in $\bar{M}$.

Moreover, we know by Def. 11.2 of the push-forward for surface maps that $\bar{\alpha}' = F_*(\alpha')$. Therefore, again by Def. 17.5 of isometries, F preserves speeds and hence also lengths,

$$|\bar{\alpha}'| = |F_*(\alpha')| = |\alpha'| \Rightarrow L(\bar{\alpha}) = L(\alpha)$$

and the result follows from Def. 17.4 of intrinsic distance. □

A useful practical criterion for isometries is given by

Lemma 17.7

Let $F : M \to \bar{M}$ be a differentiable map of surfaces, $\boldsymbol{x} : D \to M$ be a proper patch of M, and $\bar{\boldsymbol{x}} = F \circ \boldsymbol{x} : D \to \bar{M}$. Then F is a local isometry if, and only if, the corresponding warping functions satisfy

$$E = \bar{E}, \quad F = \bar{F}, \quad G = \bar{G}$$

Proof. Again by Def. 11.2 of the push-forward, we know that

$$F_*(\boldsymbol{x}_u) = \bar{x}_u, \quad F_*(\boldsymbol{x}_v) = \bar{x}_v.$$

Also, by Def. 14.6, the warping functions for $\boldsymbol{x}$ are given by

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v$$

and similarly for $\bar{\boldsymbol{x}}$. But since the push-forward of an isometry preserves dot products by Def. 17.5, the statement holds. □

Note F is only a local isometry and $\bar{\boldsymbol{x}}$ need not be a patch but may be merely a parametrization (e.g., not injective).

Next, observe that an isometry $F : M \to \bar{M}$ maps, point by point, a frame field $\{E_1, E_2\}$ of the tangent bundle of surface M to tangent vector fields on the surface $\bar{M}$ by push-forward,

$$F_*(E_1) = \bar{E}_1, \quad F_*(E_2) = \bar{E}_2$$

Crucially, since by definition a frame field satisfies $E_i \cdot E_j = \delta_{ij}$, and isometries preserve dot products,

$$\bar{E}_i \cdot \bar{E}_j = \delta_{ij}, \quad 1 \leq i, j \leq 2$$

Hence, $\{\bar{E}_1, \bar{E}_2\}$ is in fact a frame field of the tangent bundle of $\bar{M}$. In other words, isometries map tangent bundle frame fields: this will be useful in our further explorations of intrinsic geometry.

17.4 Conformal mapping

Definition 17.8: Conformal Mapping

A conformal mapping $F : M \to \bar{M}$ of surfaces $M, \bar{M}$ is a differentiable map such that there is a positive $h \in C^\infty M$ and

$$\forall \boldsymbol{v}, \boldsymbol{w} \in T_p M : \quad F_*(\boldsymbol{v}) \cdot F_*(\boldsymbol{w}) = \frac{1}{h^2(p)} \boldsymbol{v} \cdot \boldsymbol{w}.$$

Thus, rather than preserving all dot products like an isometry, a conformal mapping scales the length of each tangent vector at $p \in M$ by $h(p)^{-1}$ but preserves angles between tangent vectors.

18 Lecture 18 - Feb 21

18.1 Intrinsic geometry

Last time we introduced the notion of intrinsic distance within a surface and saw that this is invariant under isometries.

Thus generalizing, let us agree that a geometrical property be regarded as instrinsic if it is invariant under isometries.

We have also seen that an isometry of surfaces $F : M \to \bar{M}$ maps tangent frame fields to tangent frame fields, according to

$$F_*(E_1) = \bar{E}_1, \quad F_*(E_2) = \bar{E}_2. \quad (*)$$

Now in order to understand the effect of F on the fundamental equations, we also need to derive its effect on the surface coframe and connection form.

Unsurprisingly, given a surface $M \in \mathbb{R}^3$ with tangent frame field $\{E_1, E_2\}$, it is $\omega_{12}-$ without the normal coordinate 3!$-$ which plays an important role in intrinsic geometry.

In fact, it has the uniqueness property

Lemma 18.1

The connection form ω_{12} is the unique 1-form satisfying the first structural equations, Thm. 16.4 (1).

Proof. Consider $\omega_{12} = -\omega_{21}$ satisfying the first structural equations

$$\mathrm{d}\theta_1 = \omega_{12} \wedge \theta_2, \quad \mathrm{d}\theta_2 = \omega_{21} \wedge \theta_1,$$

and likewise for some $\tilde{\omega}_{12} = -\tilde{\omega}_{21}$.

Then applying the given tangent frame field $\{E_1, E_2\}$,

$$\omega_{12}(E_1) = \mathrm{d}\theta_1(E_1, E_2) = \tilde{\omega}_{12}(E_1)$$
$$\omega_{12}(E_2) = \mathrm{d}\theta_2(E_1, E_2) = \tilde{\omega}_{12}(E_2)$$

and since component functions agree, $\omega = \tilde{\omega}$ as promised. □

In other words, given M with tangent frame field $\{E_1, E_2\}$ and dual such that $\theta_i(E_j) = \delta_{ij}$, then the first structural equations uniquely define the connection form ω_{12}.

Note No reference to an embedding space $\mathbb{R}^3$ needed!

Lemma 18.2

Let $F : M \to \bar{M}$ be an isometry, then coframe and connection form of $\bar{M}$ are pulled back to M according to

(1) $\theta_1 = F^*(\bar{\theta}_1), \quad \theta_2 = F^*(\bar{\theta}_2)$,

(2) $\omega_{12} = F^*(\bar{\omega}_{12})$.

Proof. To confirm (1), we again confirm that component-functions agree, using the Def. 11.3 of the pull-back and the mapping of frame fields $(*)$ under F, since for $1 \leq i, j \leq 2$ we have

$$F^*(\bar{\theta}_i)(E_j) = \bar{\theta}_i(F_*(E_j)) = \bar{\theta}_i(\bar{E}_j) = \delta_{ij} = \theta_i(E_j).$$

To confirm (2), we use (1), the first structural equation, and also properties of the pull-back from Thm. 11.4, such as commutativity with the exterior derivative, whence

$$\mathrm{d}(F^*\bar{\theta}_1) = F^*(\mathrm{d}\bar{\theta}_1) = F^*(\bar{\omega}_{12}) \wedge F^*(\bar{\theta}_2)$$
$$\Rightarrow \mathrm{d}\theta_1 = F^*(\bar{\omega}_{12}) \wedge \theta_2,$$

and similarly for the other equation, $\mathrm{d}\theta_2 = F^*(\bar{\omega}_{21}) \wedge \theta_1$.

Hence, by the uniqueness result of Lem. 18.1, and by comparison with the first structural equation for M, we can read off, as required, $F^*(\bar{\omega}_{12}) = \omega_{12}$. □

Thus, at this point, we reach one of the central results of the course and, indeed, of Mathematics: the theorema egregium:

18.2 Theorem egregium

Theorem 18.3: (Gauss)

Gaussian curvature is invariant under isometries. That is, if $F : M \to \bar{M}$ is an isometry, then for every $p \in M$,

$$K(p) = \bar{K}(F(p)).$$

Proof. By the corollary of Thm. 16.4, the second structural equation of $\bar{M}$ with its Gaussian curvature function is given by

$$\mathrm{d}\bar{\omega}_{12} = -\bar{K}\bar{\theta}_1 \wedge \bar{\theta}_2.$$

Now pulling this back to M, and recalling again the properties of the pull-back from Thm. 11.4 and also the pull-back of functions as 0-forms, $F^*(\bar{K}) = \bar{K}(F)$, we obtain

$$\mathrm{d}\left(F^*\bar{\omega}_{12}\right) = -\bar{K}(F)F^*\left(\bar{\theta}_1\right) \wedge F^*\left(\bar{\theta}_2\right)$$

Using Lem. 18.2, this results in

$$\mathrm{d}\omega_{12} = -\bar{K}(F)\theta_1 \wedge \theta_2,$$

so the result follows by the second structural equation of M. □

Note The principal curvatures k_1, k_2 defined as normal curvatures in terms of the shape operator are manifestly extrinsic and not invariant under isometries, but their product the Gaussian curvature $K = k_1 k_2$ is invariant under isometries (and thus also locally under local isometries!) by Thm. 18.3 and hence intrinsic.

Example Consider the local isometry of plane and cylinder of Lecture 17 and what the wrapping does to k_1, k_2 and K.

Note Contraposing the statement of Thm. 18.3, we gain a criterion to show that a surface mapping is not isometric: if F changes K, F cannot be an isometry.

Example This gives rise to the famous dilemma of the mapmaker: since the sphere has $K = \frac{1}{r^2} > 0$ and the plane has $K = 0$, it is impossible to construct an isometric map. Hence, geographical maps are necessarily always distorted in some way.

18.3 Total curvature

Unsurprisingly, the integral of K over all M, or total curvature, will be an important property of M that is both intrinsically geometrical and global. To get there, we need to revisit integration first.

Thus let us first recall the surface area element from MATH201,

$$\begin{aligned} & \left|\boldsymbol{x}_u \Delta u \times \boldsymbol{x}_v \Delta v\right| \\ = & \left|\boldsymbol{x}_u \times \boldsymbol{x}_v\right| \Delta u \Delta v \\ = & \sqrt{\left|\left(\boldsymbol{x}_u \cdot \boldsymbol{x}_u\right)\left(\boldsymbol{x}_v \cdot \boldsymbol{x}_v\right)\right| - \left(\boldsymbol{x}_u \cdot \boldsymbol{x}_v\right)^2} \Delta u \Delta v \\ = & \sqrt{EG - F^2} \Delta u \Delta v \end{aligned}$$

Hence, this suggests introducing the following

Definition 18.4: Area Form

A form $\mathrm{d}M \in \Omega^2 M$ such that for all $\boldsymbol{v}, \boldsymbol{w} \in T_p M$,

$$\mathrm{d}M(\boldsymbol{v}, \boldsymbol{w}) = \pm\sqrt{(\boldsymbol{v} \cdot \boldsymbol{v})(\boldsymbol{w} \cdot \boldsymbol{w}) - (\boldsymbol{v} \cdot \boldsymbol{w})^2}$$

is called area form of M, and the sign its orientation.

Note By comparison with Def. 12.5, orientability is equivalent to having an area form. Also, if we regard $M \subset \mathbb{R}^3$, we may put $\mathrm{d}M(\boldsymbol{v}, \boldsymbol{w}) = \pm U \cdot \boldsymbol{v} \times \boldsymbol{w}$, but Def. 18.4 works intrinsically as well.

However, recall that $(D, \boldsymbol{x})$ has an open D, so in order to avoid diverging improper integrals, we modify 2-segments R_i to define

Definition 18.5: Paving

A paving $\{(R_i, \boldsymbol{x}_i)\}$ of M is a finite number of 2-segments that are patches on int R_i so that each $p \in M$ is in at most one x_i (int R_i).

Note The existence of a paving depends on the compactness of M.

Then using Def. 18.4, Def. 18.5, and Def. 11.7 on the integration of 2 -forms, the area of a pavable surface M becomes

$$\begin{aligned} A(M) = \iint_M \mathrm{d}M = \sum_i \iint_{\boldsymbol{x}_i(R_i)} \mathrm{d}M = \sum_i \iint_{R_i} \boldsymbol{x}_i^*(\mathrm{d}M) \\ = \iint_{R_i} \mathrm{d}M\left(\boldsymbol{x}_u, \boldsymbol{x}_v\right) \mathrm{d}u \; \mathrm{d}v \\ = \sum_i \iint_{R_i} \sqrt{EG - F^2} \; \mathrm{d}u \; \mathrm{d}v \end{aligned}$$

Definition 18.6: Total Curvature

Let M be a compact surface oriented by the area form $\mathrm{d}M$ with Gaussian curvature K, then its total curvature

$$\iint_M K \, \mathrm{d}M.$$

Example The plane clearly has zero total curvature, and since the sphere of radius r has constant Gaussian curvature,

$$\iint_M K \, \mathrm{d}M = KA(M) = \frac{1}{r^2} 4\pi r^2 = 4\pi$$

and see also O'Neill, p. 305f, for other examples.

In fact, the total curvature is an integer multiple of 2π, related to the topology of the surface (its Euler characteristic: more later).

18.4 Jacobian revisited

Definition 18.7: Jacobian

Let M, N be surfaces oriented by $\mathrm{d}M$, $\mathrm{d}N$. Then the Jacobian $J_F \in C^\infty M$ of a differentiable $F : M \to N$ is the function such that

$$F^*(\mathrm{d}N) = J_F \, \mathrm{d}M.$$

To understand its meaning, observe that for any $\boldsymbol{v}, \boldsymbol{w} \in T_pM$,

$$J_F(p)\mathrm{d}M(\boldsymbol{v}, \boldsymbol{w}) = F^*(\mathrm{~d}N)(\boldsymbol{v}, \boldsymbol{w}) = \mathrm{d}N\left(F_*(\boldsymbol{v}), F_*(\boldsymbol{w})\right),$$

then we know that $J_F \neq 0$ for a regular F (why?).

Also, F is preserves/reverses orientation at p if $J_F(p) > 0 / < 0$.

Moreover, note that $|J_F(p)|$ relates area elements $|\mathrm{d}M|$ and $|\mathrm{d}N|$.

18.5 Gauss map

Definition 18.8: Gauss map

Let $M \subset \mathbb{R}^3$ with unit normal $U = \sum_i g_i U_i$. Then the Gauss map is $G : M \to \mathbb{S}^2$ with $p \mapsto (g_1(p), g_2(p), g_3(p))$, $\sum_i \left|g_i^2(p)\right| = 1$.

Theorem 18.9

The Gaussian curvature K of an oriented surface $M \subset \mathbb{R}^3$ is the Jacobian of its Gauss map. That is, if $\mathrm{d}\Sigma$ is the area form on $\mathbb{S}^2$,

$$K \, \mathrm{d}M = G^*(\mathrm{d}\Sigma).$$

Corollary The total curvature of M is its area under Gauss map,

$$\iint_M K \, \mathrm{d}M = \iint_M G^*(\mathrm{d}\Sigma) = \iint_{G(M)} \mathrm{d}\Sigma.$$

Proof. $U(p)$ and the unit normal $\bar{U}(G(p))$ of $\mathbb{S}^2$ are Euclidean parallel. Also, the push-forward and shape operator for any $\boldsymbol{v} \in T_pM \subset T_p\mathbb{R}^3$ are Euclidean parallel since, by Def. 4.1 and Def. 13.1, $G_*(\boldsymbol{v}) = \sum_i \boldsymbol{v}\left[g_i\right] U_i(G(p))$, $\quad -S(\boldsymbol{v}) = \nabla_{\boldsymbol{v}} U = \sum_i \boldsymbol{v}\left[g_i\right] U_i(p)$

Hence, using the extrinsic area form of Def. 18.4 and also Lem. 14.2 for the Gaussian curvature, we obtain as required

$$\begin{aligned}
(K \, \mathrm{d}M)(\boldsymbol{v}, \boldsymbol{w}) &= K(p)\mathrm{d}M(\boldsymbol{v}, \boldsymbol{w}) = K(p)U(p) \cdot \boldsymbol{v} \times \boldsymbol{w} \\
&= U(p) \cdot S(\boldsymbol{v}) \times S(\boldsymbol{w}) \\
&= \bar{U}(G(p)) \cdot G_*(\boldsymbol{v}) \times G_*(\boldsymbol{w}) \\
&= \mathrm{d}\Sigma\left(G_*(\boldsymbol{v}), G_*(\boldsymbol{w})\right) \\
&= \left(G^*(\mathrm{~d}\Sigma)\right)(\boldsymbol{v}, \boldsymbol{w})
\end{aligned}$$

□

Theorem 18.10: (Fenchel)

A simple closed curve α in $\mathbb{R}^3$ has total curvature $\int_\alpha \kappa \mathrm{d}s \geq 2\pi$.

Proof. See O'Neill, p. 309f, and cf. Gauss-Bonnet Theorem (later). □

19 Lecture 19 - Feb 22

19.1 Manifolds

Definition 19.1: Manifold

A smooth n-dimensional manifold is a set M endowed with charts $\{(D_i, \boldsymbol{x}_i)\}$, whereby $D_i \subset \mathbb{R}^n$ are open and $\boldsymbol{x}_i : D_i \to M$ bijective continuous with continuous inverse (homeomorphisms), and the
- Covering axiom: $M \subset \bigcup_i \boldsymbol{x}_i(D_i)$;
- Smoothness axiom: for any overlapping charts, the transition maps $\boldsymbol{x}_j^{-1} \circ \boldsymbol{x}_i$ are Euclidean diffeomorphisms;
- Hausdorff axiom: for any $p \neq q \in M$, there exist $\boldsymbol{x}_i(D_i) \ni p$ and $\boldsymbol{x}_j(D_j) \ni q$ that are disjoint, that is, $\boldsymbol{x}_i(D_i) \cap \boldsymbol{x}_j(D_j) = \emptyset$.

Note Charts generalize to manifolds our earlier notion of proper patches for M. Also, since it is no longer in $\mathbb{R}^3$, there is no longer any immediate notion of regularity of $\boldsymbol{x}$. Smoothness now enters our definition via the transition maps on open subsets of $\mathbb{R}^n \to \mathbb{R}^n$.

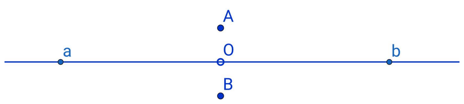
Note The Hausdorff property clearly applies to $\mathbb{R}^n$ with standard open ball topology. So surfaces $M \subset \mathbb{R}^3$ in the sense of Def. 9.2 are indeed also 2-dimensional manifolds in the sense of Def. 19.1.

But why do we need the Hausdorff property in general? Consider:

Exercise On a Hausdorff $(M, \mathcal{O})$, every convergent sequence has a unique limit point, as on Euclidean $\mathbb{R}^n$ (cf. Lecture 1, Homework 5).

Note M with a biggest topology $\mathcal{O}_d$ is Hausdorff, but with the smallest topology $\mathcal{O}_c$ it is not (why?), so there is a choice. Does the locally Euclidean choice not already imply that M is Hausdorff?

Example Open sets on the real line with two origins A, B comprise $(a, 0) \cup \{A\} \cup (0, b)$ and $(a, 0) \cup \{B\} \cup (0, b)$.



Despite being locally Euclidean, it is not Hausdorff! So to better model M on Euclidean space, Hausdorff needs to be an axiom.

Conversely, we may ask whether every 2-dimensional manifold M can also be embedded as a surface in $\mathbb{R}^3$. The answer is no!

Example The projective plane $\mathbb{RP}^2$ is the set of all unoriented rays through the origin in $\mathbb{R}^3 \backslash \{(0,0,0)\}$. Each ray is identified uniquely by an unordered pair of antipodal points $\{p, -p\}$ in $\mathbb{S}^2$, a point of the 2-dimensional manifold $\mathbb{RP}^2$ (cf. O'Neill p. 194f).

This gives rise to the antipodal map A (already met in Homework 3!) and the projection map P with the following properties:

$A : \mathbb{S}^2 \to \mathbb{S}^2, \quad A(p) = -p, \quad \text{with} \quad A \circ A = I, A_*(\boldsymbol{v}_p) = -\boldsymbol{v}_{-p},$

$P : \mathbb{S}^2 \to \mathbb{RP}^2, \quad P(p) = \{p, -p\}, \quad \text{with} \quad P \circ A = P.$

Suppose that $\mathbb{RP}^2$ is orientable, then Def. 12.5 guarantees that there is non-zero $\mu \in \Omega^2\mathbb{RP}^2$ and $\sigma = P^*\mu \in \Omega^2\mathbb{S}^2$. However, $P^* = A^* \circ P^*$ (why?), so $A^*\sigma = \sigma$ and so A preserves orientation, which is a contradiction (why?). Hence, $\mathbb{RP}^2$ is not orientable.

Now $\mathbb{RP}^2$ is compact (why?) but then, being also non-orientable, Thm. 12.11 implies that $\mathbb{RP}^2$ cannot be embedded in $\mathbb{R}^3$.

Note In other words, here we find a topological obstruction to embedding a manifold in Euclidean space (more later).

19.2 Riemannian geometry

Definition 19.2: Riemannian Metric

On a smooth manifold M, a Riemannian metric g is a map such that for all $p \in M, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in T_pM$, and $a, b \in \mathbb{R}$, we have $g_p : T_pM \times T_pM \to \mathbb{R}$ with the following properties,
- Bilinearity: $g_p(\boldsymbol{u}, a\boldsymbol{v} + b\boldsymbol{w}) = a g_p(\boldsymbol{u}, \boldsymbol{v}) + b g(\boldsymbol{u}, \boldsymbol{w})$,
- Symmetry: $g_p(\boldsymbol{v}, \boldsymbol{w}) = g_p(\boldsymbol{w}, \boldsymbol{v})$,
- Positive definiteness: $g_p(\boldsymbol{v}, \boldsymbol{v}) \geq 0, g_p(\boldsymbol{v}, \boldsymbol{v}) = 0 \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$.

Lengths of vectors and angles between them are defined as before,

$$|\boldsymbol{v}| = \sqrt{g(\boldsymbol{v}, \boldsymbol{v})}, \quad \cos\vartheta = \frac{g(\boldsymbol{v}, \boldsymbol{w})}{|\boldsymbol{v}||\boldsymbol{w}|}$$

Passing from the pointwise Def. 19.2 to fields, a Riemannian metric g is a bilinear real-valued function on vector fields like a 2-form, though symmetric instead of antisymmtric. Both metric and 2-form are special cases of tensors (more later).

Analogously to forms, we can also define the pullback of the metric, such that for smooth manifolds with

$F: M \to \bar{M}$ and metric $\bar{g}$ on $\bar{M}$, the pull-back metric is $g = F^*\bar{g}$ such that

$$\forall \boldsymbol{v}, \boldsymbol{w} \in T_pM: \quad (F^*\bar{g})(\boldsymbol{v}, \boldsymbol{w}) = \bar{g}(F_*\boldsymbol{v}, F_*\boldsymbol{w})$$

But this corresponds precisely to Def. 17.5 of local isometries (there with respect to the Euclidean dot product). Likewise here we also regard F as local isometry with respect to the metric, thus defining intrinsic geometry in the sense discussed last time.

Now, finally, we can announce our new intrinsic notion of surface:

Definition 19.3: Geometric Surface

A geometric surface (M, g) is a smooth 2-dimensional manifold with Riemannian metric g.

Naturally, many definitions and intrinsic results encountered before for surfaces will now simply carry over to geometric surfaces with a small change of notation and proofs, e.g. replacing g for dot.

In order to work in (M, g), it is useful to introduce suitable coordinate systems and frame fields, so let us revisit this topic next.

19.3 Metric components

Given (M, g) with a chart $(D, \boldsymbol{x})$, we have partial velocities $\boldsymbol{x}_u, \boldsymbol{x}_v$ as before, but let us adapt the warping functions of Def. 14.6 to

$$E = g(\boldsymbol{x}_u, \boldsymbol{x}_u), \quad F = g(\boldsymbol{x}_u, \boldsymbol{x}_v), \quad G = g(\boldsymbol{x}_v, \boldsymbol{x}_v)$$

We already know $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ forms a basis for T_pM, cf. Lem. 10.4. Thus, our intrinsic interpretation of the warping functions E, F, G is simply the components of g with respect to this basis,

$$E = g_{uu}, \quad F = g_{uv}, \quad G = g_{vv}$$

Thus, g can be represented by a symmetric 2×2 matrix,

$$[g_{ij}] = \begin{bmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^T = [g_{ji}]$$

Note Recall the first fundamental form mentioned in Lecture 14.

Note Extending pointwise definition to fields, of course each metric component becomes a real-valued function $g_{ij} \in C^\infty M$. So when encountering the notation g_{ij}, check carefully whether the metric itself, its component functions or its matrix is intended!

Considering a curve $\alpha(t) = \boldsymbol{x}(u(t), v(t))$ in M, it is customary to write the metric line element of arc-length s along $\alpha(t)$ as

$$\mathrm{d}s^2 = E\ \mathrm{d}u^2 + 2F\ \mathrm{d}u\ \mathrm{d}v + G\ \mathrm{d}v^2 = \sum_{i,j} g_{ij}\ \mathrm{d}x_i\ \mathrm{d}x_j$$

where $x_1 = u, x_2 = v$ (confirm this in Homework 5).

Note So, unsurprisingly, the metric is used to measure lengths within M along curves and thus between points (more later).

Now it is most convenient to choose an orthogonal coordinate system such that $F \equiv 0$, that is, an orthogonal chart yielding Definition 19.4

Definition 19.4: Associated Frame Field

The associated frame field $\{E_1, E_2\}$ of an orthogonal chart $(D, \boldsymbol{x})$ on (M, g) has orthonormal vectors

$$E_1 = \frac{\boldsymbol{x}_u(u, v)}{\sqrt{E}(u, v)}, \quad E_2 = \frac{\boldsymbol{x}_v(u, v)}{\sqrt{G}(u, v)}.$$

Exercise Confirm that $|E_1| = |E_2| = 1$ and $g(E_1, E_2) = 0$.

Coframes Next consider coframes. The coordinate chart-induced basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ gives rise to a dual $\{\mathrm{d}u,\ \mathrm{d}v\}$ such that (cf. Homework 3)

$$\mathrm{d}u(\boldsymbol{x}_u) = 1, dv(\boldsymbol{x}_u) = 0, du(\boldsymbol{x}_v) = 0, dv(\boldsymbol{x}_v) = 1.$$

Now if $\{\theta_1, \theta_2\}$ be the coframe dual to the associated frame field of Def. 19.4 such taht $\theta_i(E_j) = \delta_{ij}$, we have (check!)

$$\theta_1 = \sqrt{E}\ \mathrm{d}u, \quad \theta_2 = \sqrt{G}\ \mathrm{d}v$$

Exercise Now compute the exterior derivatives of θ_1, θ_2.

$$\mathrm{d}\theta_1 = d(\sqrt{E}) \wedge \mathrm{d}u = \frac{\partial\sqrt{E}}{\partial v}\ \mathrm{d}v \wedge \mathrm{d}u = \frac{-(\sqrt{E})_v}{\sqrt{G}}\ \mathrm{d}u \wedge \theta_2$$

$$\mathrm{d}\theta_2 = d(\sqrt{G}) \wedge \mathrm{d}v = \frac{\partial\sqrt{G}}{\partial u}\ \mathrm{d}u \wedge \mathrm{d}v = \frac{-(\sqrt{G})_u}{\sqrt{E}}\ \mathrm{d}v \wedge \theta_1$$

Connection Form Thus, comparison with the first structural equations from Thm. 16.4, $\mathrm{d}\theta_1 = \omega_{12} \wedge \theta_2$, $\mathrm{d}\theta_2 = \omega_{21} \wedge \theta_1$, and the uniqueness result from Lem. 18.1, which are intrinsic and thus also applicable here, the connection form of the associated frame field can be read off,

$$\omega_{12} = -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\ \mathrm{d}u + \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\ \mathrm{d}v.$$

Angle Function We need to understand how ω_{12} changes under a change of frame. So suppose $\{\bar{E}_1, \bar{E}_2\}$

is another orthonormal frame with the same orientation (right-handed), related with $\{E_1, , E_2\}$ by a differentiable angle function $\varphi : M \to [0, 2\pi)$ such that

$$\bar{E}_1 = \cos\varphi E_1 + \sin\varphi E_2,$$
$$\bar{E}_2 = -\sin\varphi E_1 + \cos\varphi E_2.$$

Lemma 19.5

Connection forms and coframes of $\{\bar{E}_1, \bar{E}_2\}$ and $\{E_1, E_2\}$ obey
(1) $\bar{\omega}_{12} = \omega_{12} + \mathrm{d}\varphi$, (2) $\bar{\theta}_1 \wedge \bar{\theta}_2 = \theta_1 \wedge \theta_2$.

Proof. By orthonormal expansion and duality, we have

$$\theta_1 = \theta_1\left(\bar{E}_1\right)\bar{\theta}_1 + \theta_1\left(\bar{E}_2\right)\bar{\theta}_2 = \cos\varphi\bar{\theta}_1 - \sin\varphi\bar{\theta}_2$$
$$\theta_2 = \theta_2\left(\bar{E}_1\right)\bar{\theta}_1 + \theta_2\left(\bar{E}_2\right)\bar{\theta}_2 = \sin\varphi\bar{\theta}_1 + \cos\varphi\bar{\theta}_2$$

Taking $\wedge$, (2) follows. For (1), use the first structural equation,

$$\begin{aligned} \mathrm{d}\theta_1 &= -\sin\varphi \mathrm{d}\varphi \wedge \bar{\theta}_1 + \cos\varphi \mathrm{d}\bar{\theta}_1 - \cos\varphi \mathrm{d}\varphi \wedge \bar{\theta}_2 - \sin\varphi \mathrm{d}\bar{\theta}_2 \\ &= (\bar{\omega}_{12} - \mathrm{d}\varphi) \wedge \left(\sin\varphi\bar{\theta}_1 + \cos\varphi\bar{\theta}_2\right) = (\bar{\omega}_{12} - \mathrm{d}\varphi) \wedge \theta_2 \end{aligned}$$

□

20 Lecture 20 - Feb 23

20.1 Gaussian curvature revisited

We will define Gaussian curvature intrinsically.

Theorem 20.1

On (M, g), there is a unique real-valued function K called Gaussian curvature such that, for every frame field, the second structural equation holds,

$$\mathrm{d}\omega_{12} = -K\theta_1 \wedge \theta_2.$$

Proof. We need to show uniqueness of K under a frame change: given $\{E_1, E_2\}$ and its coframe, suppose $\{\bar{E}_1, \bar{E}_2\}$ is another frame with

$$\mathrm{d}\bar{\omega}_{12} = -\bar{K}\bar{\theta}_1 \wedge \bar{\theta}_2.$$

Then by Lem. 19.5, we have the property that $\bar{\theta}_1 \wedge \bar{\theta}_2 = \theta_1 \wedge \theta_2$. Moreover, using nil-potency of the exterior derivative,

$$\mathrm{d}\bar{\omega}_{12} = \mathrm{d}\left(\omega_{12} + \mathrm{d}\varphi\right) = \mathrm{d}\omega_{12},$$

since $d^2\varphi = 0$. Hence, the result $\bar{K} = K$ follows. □

Note Recall that orientation of M is a choice, and we have taken the frame change to keep orientation as in Lem. 19.5. In fact, the result holds even under an orientation change (cf. O'Neill, p. 329).

So in order to obtain the metric Gaussian curvature, we may indeed read it off from the second structural equation.

For simplicity, suppose we have an orthogonal chart so that $F = g\left(\boldsymbol{x}_u, \boldsymbol{x}_v\right) = 0$, and its associated frame field of Def. 19.4 has the connection form ω_{12} computed last time. Then the corrresponding Gaussian curvature turns out to be

$$K = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right] \quad (*)$$

Note Recall from last time how the E, F, G can be regarded as metric components. Thus, our formula above yields the Gaussian curvature of a geometric surface (M, g) straight from the metric!

Torus revisited We met the torus with its standard parametrization in Lecture 9,

$$x(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u).$$

This is a compact 2-dimensional manifold T (without geometry!).

Taking $T \subset \mathbb{R}^3$ with metric induced by the Euclidean dot product, that is metric δ_{ij}, we get the geometric surface (T, δ) whose K was computed in Lecture 15, with domains of $K < 0, K = 0, K > 0$.

Now we can, of course, endow T with other metrics g to yield different geometric surfaces (still the same manifold!). E.g., letting

$$g\left(\boldsymbol{x}_u, \boldsymbol{x}_u\right) = 1, \quad g\left(\boldsymbol{x}_u, \boldsymbol{x}_v\right) = 0, \quad g\left(\boldsymbol{x}_v, \boldsymbol{x}_v\right) = 1$$

and recalling that $\boldsymbol{x}_*\left(U_1\right) = \boldsymbol{x}_u, \boldsymbol{x}_*\left(U_2\right) = \boldsymbol{x}_v$ (cf. Homework 3), we see x is now a local isometry of the Euclidean plane (why?), hence the Gaussian curvature of (T, g), called the flat torus, is $K = 0$.

Note: Since a compact surface in $\mathbb{R}^3$ must have $K > 0$ somewhere, by Thm. 17.1, the compact flat torus with $K = 0$ cannot be embedded in $\mathbb{R}^3$! In other words, here we have a geometrical obstruction to a manifold being a surface in $\mathbb{R}^3$ (compare the topological obstruction for the projective plane in Lecture 19).

20.2 Conformal plane

Next, we consider geometric surfaces constructed from the Euclidean plane by means of conformal mappings, cf. Def. 17.8.

Thus, given a positive-valued conformal factor $h \in$

$C^\infty\mathbb{R}^2$, we define the geometric surface $(\mathbb{R}^2, g_h)$ with metric such that

$$g_h(\boldsymbol{v}, \boldsymbol{w}) = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{h^2(p)}$$

Example The stereographic projection P, cf. Lecture 11. In fact, the pull-back of δ yields a flat stereographic sphere with $K = 0$. Likewise, the pull-back with P^{-1} gives the stereographic plane with $K = 1$ like the round sphere (see O'Neill p. 331f for details).

Lemma 20.2

The Gaussian curvature of the conformal plane $(\mathbb{R}^2, g_h)$ is

$$K = h\left(\frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2}\right) - \left(\frac{\partial h}{\partial u}\right)^2 - \left(\frac{\partial h}{\partial v}\right)^2$$

Proof. This follows directly from our earlier formula $(*)$ for K, with

$$E = G = \frac{1}{h^2}.$$

□

20.3 Hyperbolic plane

Example An important example of conformal plane is the Poincaré disk of the hyperbolic plane $H \subset \mathbb{R}^2 : u^2 + v^2 < 4$, with

$$h(u, v) = 1 - \frac{u^2 + v^2}{4}$$

Now applying Lem. 20.2 to (H, g_h), one finds that it has (check!)

$$K = -1$$

i.e. constant negative curvature, cf. pseudosphere (Lecture 15).

Note As (u, v) approaches the rim of $H, h \to 0$. But since vector lengths scale as $\frac{1}{h}$ by definition, (H, g_h) gets 'bigger' towards the rim. This property is useful in Data Science (more on H later).

20.4 Covariant derivative revisited

Next, we adapt the notion of covariant derivative to geometric surfaces. This is done by simply requiring axiomatically that the properties already familiar from Thm. 7.9 hold, as well as the familiar relation with frame and connection form, Lem. 18.1.

Definition 20.3: Covariant Derivative

On (M, g) with frame $\{E_1, E_2\}$ and connection from ω_{12}, let $\nabla : TM \times TM \to TM$ such that for any differentiable X, Y, V, W on TM, and $a, b \in \mathbb{R}$, and $f, g \in C^\infty M$,
(1) $\nabla_{fV+gW}X = f\nabla_V X + g\nabla_W X$,
(2) $\nabla_V(aX + bY) = a\nabla_V X + b\nabla_V Y$,
(3) $\nabla_V(fX) = V[f]X + f\nabla_V X$,
(4) $V[g(X, Y)] = g(\nabla_V X, Y) + g(X, \nabla_V Y)$,
(5) $\omega_{12}(V) = g(\nabla_V E_1, E_2)$.

Note Recall from Lem. 18.1 that the connection form ω_{12} for a given frame $\{E_1, E_2\}$ is unique, so it is not additonal information. Just like in the Euclidean case, cf. Thm. 8.4, our Def. 20.3 suffices to fix ∇ for a geometric surface. Now supposing it exists,

Lemma 20.4

On (M, g), a covariant derivative given by ∇ of Def. 20.3 satisfies the connection equations

$$\nabla_V E_1 = \omega_{12}(V)E_2, \quad \nabla_V E_2 = \omega_{21}(V)E_1,$$

and, for any vector field $W = w_1E_1 + w_2E_2$ in TM,

$$\begin{aligned}\nabla_V W = &\left(V[w_1] + w_2\omega_{21}(V)\right)E_1\\ &+ \left(V[w_2] + w_1\omega_{12}(V)\right)E_2,\end{aligned}$$

called the covariant derivative formula.

Proof. As usual, we perform an orthonormal expansion whence

$$\nabla_V E_1 = \underbrace{g(\nabla_V E_1, E_1)}_{\frac{1}{2}V[g(E_1,E_1)]=0} E_1 + \underbrace{g(\nabla_V E_1, E_2)}_{(5)} E_2 = \omega_{12}(V)E_2,$$

similarly for the second connection equation. Using (1) and (3),

$$\begin{aligned}\nabla_V W &= \nabla_V(w_1E_1 + w_2E_2)\\ &= V[w_1]E_1 + w_1\nabla_V E_1 + V[w_2]E_2 + w_2\nabla_V E_2\\ &= V[w_1]E_1 + w_1\omega_{12}(V)E_2 + V[w_2]E_2 + w_2\omega_{21}(V)E_1\end{aligned}$$

by substitution of the connection equations, yielding the covariant derivative formula as required. □

We shall now prove existence and uniqueness of ∇, which is fundamental result for Riemannian geometric surfaces:

Theorem 20.5

On a geometric surface (M, g) there is a unique covariant derivative with ∇ satisfying Def. 20.3.

Proof. We start constructively: given a particular frame $\{E_1, E_2\}$ and vector fields V, W, use the covariant derivative formula of Lem. 20.4 to define $\nabla_V W$. Then the properties of Def. 20.3 are satisfied: (1), (2), (5) apply, check (3) and (4) in Homework 5.

Therefore, given a frame there exists a ∇. But for uniqueness, we also need to establish its independence of the frame.

So given another frame $\{\bar{E}_1, \bar{E}_2\}$ (with same orientation, say) and corresponding $\bar{\nabla}$, we need to show that, for any smooth V on TM,

$$\nabla_v \bar{E}_1 = \bar{\nabla}_v \bar{E}_1, \quad \nabla_v \bar{E}_2 = \bar{\nabla}_v \bar{E}_2$$

Using Lem. 19.5 with angle function φ and $V[\varphi] = \mathrm{d}\varphi(V)$

$$\begin{aligned}
\nabla_V \bar{E}_1 &= \nabla_V (\cos\varphi E_1 + \sin\varphi E_2) \\
&= \sin\varphi\,(-V[\varphi] + \omega_{21}(V))\,E_1 + \cos\varphi\,(V[\varphi] + \omega_{12}(V))\,E_2 \\
&= (\omega_{12}(V) + \mathrm{d}\varphi(V))\,(-\sin\varphi E_1 + \cos\varphi E_2) \\
&= \bar{\omega}_{12}(V)\bar{E}_2 = \bar{\nabla}_V \bar{E}_1
\end{aligned}$$

and similarly for the second equation. □

Note It turns out that there is a unique ∇ for metric manifolds in general, not just geometric surfaces (more later: Levi-Civita).

Now consider $V = v_1 E_1 + v_2 E_2$ and curves α in (M, g). Then the covariant derivative formula of Lem. 20.4 also defines the covariant derivative of V along α (note same notation as Euclidean),

$$\begin{aligned}
V' &= \nabla_{\alpha'} V \\
&= (\alpha'[v_1] + v_2\omega_{21}(\alpha'))\,E_1 + (\alpha'[v_2] + v_1\omega_{12}(\alpha'))\,E_2 \\
&= (v_1' + v_2\omega_{21}(\alpha'))\,E_1 + (v_2' + v_1\omega_{12}(\alpha'))\,E_2
\end{aligned}$$

This generalizes parallel transport from Euclidean, cf. Lecture 5:

Definition 20.6: Parallel

A vector field V on a curve α in (M, g) is called parallel if $V' = 0$.

An important special case concerns V of constant length, say, $|V| = v = \text{const.} > 0$. Let us also introduce the operator J for vector rotation through $+\frac{\pi}{2}$ in T_pM. In particular, $E_2 = J(E_1)$. (For orientation)

Lemma 20.7

Let V be a vector field of constant length v along a curve α in (M, g) with $\{E_1, E_2\}$, and ψ be the angle from E_1 to V, then

$$V' = (\psi' + \omega_{12}(\alpha'))\,J(V).$$

Proof. Now we have the unit vector field $\frac{V}{v} = \cos\psi E_1 + \sin\psi E_2$ by assumption. So applying the covariant derivative formula,

$$\begin{aligned}
\frac{V'}{v} &= \sin\psi\,(-\psi' - \omega_{12}(\alpha'))\,E_1 + \cos\psi\,(\psi' + \omega_{12}(\alpha'))\,E_2 \\
&= (\psi' + \omega_{12}(\alpha'))\,(\cos\psi E_2 - \sin\psi E_1) \\
&= (\psi' + \omega_{12}(\alpha'))\,J\left(\frac{V}{v}\right)
\end{aligned}$$

□

So by Def. 20.6 and Lem. 20.7, V is parallelly transported along α if, and only if, $\psi' + \omega_{12}(\alpha') = 0$.

20.5 Holonomy

Suppose $\alpha : [a, b] \to M$ is a loop at $p = \alpha(a) = \alpha(b)$. Then the difference of angles of V over that closed curve is

$$\Delta\psi = \psi(b) - \psi(a) = -\int_a^b \omega_{12}(\alpha')\,\mathrm{d}t = -\int_\alpha \omega_{12}$$

Clearly, this is independent of V but path-dependent in general, and e.g. zero in a planar surface with $\omega_{12} = 0$, thus related to curvature. This is called the holonomy angle of α.

Example On the sphere, $\omega_{12}^{\text{sph}} = \sin\varphi \mathrm{d}\vartheta$ (Lecture 16). Hence, for a φ-parallel circle, the holomy angle is $\Delta\psi = -2\pi\sin\varphi$. Note that a parallelly transported V returns into itself only on the equator ($\varphi = 0$), a geodesic: we return to this special type of curves next.

21 Lecture 21 - Feb 24

21.1 Geodesics

Originally, we encountered geodesics in extrinsic geometry as curves in $M \in \mathbb{R}^3$ with normal acceleration, Def. 15.9. Now in intrinsic geometry, we shall use Def. 20.6 of parallel transport:

Definition 21.1: Geodesics

Curves α in (M, g) such that covariant acceleration vanishes, $\alpha'' = \nabla_{\alpha'}\alpha' = 0$, are called geodesics.

Note This means that the velocity of a geodesic curve is parallelly transported along itself. Hence, α is also called autoparallel.

Note Again, we use the same notation as for Euclidean acceleration, however, here it denotes covariant acceleration.

Also, technically, covariant derivatives are defined for vector fields on open sets, hence we use plural here to indicate a family or congruence of α. However, we need not dwell on this point here.

Note As before, geodesics are curves of constant speed since

$$\left(|\alpha'|^2\right)' = \nabla_{\alpha'} g(\alpha', \alpha') = 2g(\nabla_{\alpha'}\alpha', \alpha') = 0$$

Note Moreover, if a geometric surface M is embedded in $\mathbb{R}^3$ with unit normal U, then intrinsic geodesics according to Def. 21.1 are, in fact, identical to the extrinsic geodesics of Def. 15.9:

if $\ddot{\alpha}$ is the Euclidean acceleration, then as shown in Homework 5 ,

$$\ddot{\alpha} = \alpha'' + (S(\alpha') \cdot \alpha') U$$

so $\ddot{\alpha}$ is normal to M if, and only if, $\alpha'' = 0$ (why?), as required.

Moreover, geodesics are preserved under isometries and, thus, are clearly features of the intrinsic geometry of a geometric surface.

To see this, suppose that F is an isometry from (M, g) to $(\bar{M}, \bar{g})$ with $F^*\bar{g} = g$, as described in Lecture 20 (cf. also Thm. 17.6).

Note that isometries map unit tangent vectors, $F_*T = \bar{T}$, since $1 = g(T,T) = (F^*\bar{g})(T,T) = \bar{g}(F_*T, F_*T) = \bar{g}(\bar{T}, \bar{T})$

Then using a result of Homework 4 and Lem. 18.2, which is intrinsic and also applicable here, we thus obtain for geodesics

$$0 = \omega_{12}(T) = (F^*\bar{\omega}_{12})(T) = \bar{\omega}_{12}(F_*T) = \bar{\omega}_{12}(\bar{T})$$

so F maps geodesics of (M, g) to geodesics of $(\bar{M}, \bar{g})$, as claimed.

Next, we seek coordinate expressions for geodesic curves. Suppose, then, that $\{E_1, E_2\}$ provides a frame field for (M, g), and write velocity and (covariant) acceleration of a curve α as follows,

$$\alpha' = v_1 E_1 + v_2 E_2$$
$$\alpha'' = a_1 E_1 + a_2 E_2$$

then α is geodesic if, and only if, $a_1 = a_2 = 0$ by Def. 21.1 .

Moreover, we can use the covariant derviative formula as discussed in Lecture 20, and write these acceleration components as

$$a_1 = v_1' + v_2\omega_{21}(\alpha')$$
$$a_2 = v_2' + v_1\omega_{12}(\alpha')$$

Theorem 21.2

Let α be a curve on (M, g) with orthogonal chart $\boldsymbol{x}$ such that $\alpha(t) = \boldsymbol{x}(x_1(t), x_2(t))$. Then α is a geodesic if, and only if,

$$a_1 = x_1'' + \frac{1}{2E}\left(E_1 x_1'^2 + 2E_2 x_1' x_2' - G_1 x_2'^2\right) = 0$$
$$a_2 = x_2'' + \frac{1}{2G}\left(-E_2 x_1'^2 + 2G_1 x_1' x_2' + G_2 x_2'^2\right) = 0$$

with subscripts 1,2 denoting $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ except for components.

Proof. By Def. 19.4, an orthogonal chart has $F = 0$ with associated frame field $\left\{E_1 = \frac{\boldsymbol{x}_1}{\sqrt{E}}, E_2 = \frac{\boldsymbol{x}_2}{\sqrt{G}}\right\}$, and $\alpha' = x_1'\boldsymbol{x}_1 + x_2'\boldsymbol{x}_2$ so

$$\alpha' = \left(x_1'\sqrt{E}\right) E_1 + \left(x_2'\sqrt{G}\right) E_2 = v_1 E_1 + v_2 E_2.$$

Hence, the corresponding acceleration components become

$$a_1 = v_1' + v_2\omega_{21}(\alpha') = \left(x_1'\sqrt{E}\right)' + \left(x_2'\sqrt{G}\right)\omega_{21}(\alpha') = 0$$
$$a_2 = v_2' + v_1\omega_{12}(\alpha') = \left(x_2'\sqrt{G}\right)' + \left(x_1'\sqrt{E}\right)\omega_{12}(\alpha') = 0$$

Now constructing the coframe $\{\mathrm{d}x_1, \mathrm{d}x_2\}$ dual to $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ (cf. Homework 3), then the connection form discussed in Lecture 19,

$$\omega_{12}(\alpha') = \omega_{12}(x_1'\boldsymbol{x}_1 + x_2'\boldsymbol{x}_2) = -\frac{(\sqrt{E})_2}{\sqrt{G}} x_1' + \frac{(\sqrt{G})_1}{\sqrt{E}} x_2'.$$

Substituting this into the previous set of equations for the acceleration components, differentiating, applying the chain rule, and simplifying, the result follows (check!). □

Theorem 21.3

Given a point p in (M, g) with $\boldsymbol{v} \in T_pM$, there is a unique geodesic $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = \boldsymbol{v}$.

Proof. Existence and uniqueness of solutions of differential equations in a parameter interval given initial conitions. Cf. MATH303. □

In fact, this interval can be quite large! Given initial conditions $p, \boldsymbol{v}$ and the properties of (M, g), the largest interval $I \subset \mathbb{R}$ for which the solution is defined yields the maximal geodesic $\gamma : I \to M$.

Definition 21.4

A geometric surface (M, g) for which every maximal geodesic is defined on the whole real line, $I = \mathbb{R}$, is called complete.

Example The Euclidean plane and sphere are complete surfaces.

Note It is not necessary to solve both 2nd order ODEs of Thm. 21.2, since we can also utilize the geodesic unit speed condition,

$$g\left(\alpha', \alpha'\right) = E\left(x_1'\right)^2 + G\left(x_2'\right)^2 = \text{ const.}$$

This 1st order ODE is also known as first integral. In Physics, it corresponds to an energy equation in classical mechanics.

Next, let us briefly consider the analogue of the Frenet-Serret formalism in intrinsic geometry for a unit speed curve β in an oriented surface (M, g) : with $T = \beta'$ as before, let $N = J(T)$ by rotation through $+\frac{\pi}{2}$ so that $\{T, N\}$ span T_pM, and there is now no B, of course. Then we define geodesic curvature κ_g by

$$T' = \kappa_g N$$

Corollary (Liouville's formula) now follows from Lem. 20.7 for T',

$$\kappa_g = \frac{\mathrm{d}\psi}{\mathrm{d}s} + \omega_{12}\left(\beta'\right).$$

Note: κ_g is different from Frenet-Serret κ, and not non-negative! Nevertheless, the derivation of Lem. 7.2 proceeds analogously for covariant acceleration, so for any regular curve α (check!),

$$\alpha' = vT, \quad \alpha'' = \frac{\mathrm{d}v}{\mathrm{d}t}T + \kappa_g v^2 N.$$

Now obviously $\alpha'' = 0$ if, and only if, $\frac{dv}{dt} = 0$ and $\kappa_g = 0$, whence

Lemma 21.5

A regular curve in a geometric surface is a geodesic if, and only if, it has constant speed and its geodesic curvature vanishes.

21.2 Clairaut parametrization

A particularly simple situation occurs when the surface admits

Definition 21.6: Clairaut Parametrization

A Clairaut parametrization $\boldsymbol{x} : D \to M$ is orthogonal with E, G depending on x_1 only, that is, $F = 0, E_2 = G_2 = 0$. The x_1^-/x_2^- parameter curves are called meridians/parallels.

Example Surfaces of revolution (cf. Lecture 15) are a subset of the class of surfaces admitting Clairaut parametrization.

Lemma 21.7

Suppose $\alpha = \boldsymbol{x}\left(x_1, x_2\right)$ is a Clairaut-parametrized unit speed geodesic. If ψ is the angle from $\boldsymbol{x}_1$ to α', then the function

$$c = G\left(x_1\right) x_2' = \sqrt{G} \sin\psi,$$

is a constant along α (that is, a constant of motion)

Proof. Given the metric warping functions, cf. Lecture 19, we have

$$g\left(\alpha', \boldsymbol{x}_2\right) = g\left(x_1'\boldsymbol{x}_1 + x_2'\boldsymbol{x}_2, \boldsymbol{x}_2\right) = Fx_1' + Gx_2' = Gx_2',$$
$$g\left(\alpha', \boldsymbol{x}_2\right) = \underbrace{\left|\alpha'\right|}_{1} \left|\boldsymbol{x}_2\right| \cos\left(\frac{\pi}{2} - \psi\right) = \sqrt{G}\sin\psi,$$

as required. For constancy of c along α, see (2) of Thm. 21.2,

$$0 = Gx_2'' + G_1 x_1' x_2' = Gx_2'' + G'x_2' = \left(Gx_2'\right)' = c',$$

using Clairaut parametrization Def. 21.6 and chain rule for G'. □

Note Given c, unit-speed geodesics are confined to a region of M wherein $G \geq c^2$. The case $c = 0$ corresponds to meridians.

Lemma 21.8

Any Clairaut-parametrized unit speed geodesic α with α' never orthogonal to meridians can be recast as $\beta(u) = \boldsymbol{x}(u, v(u))$ with

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \pm\frac{c\sqrt{E}}{\sqrt{G}\sqrt{G - c^2}}$$

Proof. Using unit-speed $Ex_1'^2 + Gx_2'^2 = 1$ and $x_2' = \frac{c}{G}$ from Lem. 21.7,

$$x_1' = \pm\frac{\sqrt{G - c^2}}{\sqrt{EG}}$$

Now $x_1' \neq 0$ and hence $\frac{dv}{du} = \frac{x_2'(s)}{x_1'(s)}$ yields the result. □

21.3 Hyperbolic plane revisited

Recall from last lecture the hyperbolic plane (H, g_h) but now with a polar parametrization $\boldsymbol{x}(r, \vartheta) = (r\cos\vartheta, r\sin\vartheta)$ so that we have the disk $r < 2$ with conformal factor $h = 1 - \frac{r^2}{4}$ and (check!)

$$\begin{aligned} E(r,\vartheta) &= g_h(\boldsymbol{x}_r, \boldsymbol{x}_r) = \frac{1}{h^2(r)}, \\ F(r,\vartheta) &= g_h(\boldsymbol{x}_r, \boldsymbol{x}_\vartheta) = 0, \\ G(r,\vartheta) &= g_h(\boldsymbol{x}_\vartheta, \boldsymbol{x}_\vartheta) = \frac{r^2}{h^2(r)}. \end{aligned}$$

Notice that is, in fact, a Clairaut parametrization according to Def. 21.6. Thus, r-parameter curves - straight lines through the origin of the disk - are geodesic meridians, as noted above.

Then by Lem. 21.8, non-meridian geodesics satisfy the equation

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}r} = \pm \frac{ch}{r^2\sqrt{1 - \left(\frac{ch}{r}\right)^2}}$$

Now in order to solve this differential equation, we substitute

$$w = \frac{c}{r\sqrt{1+c^2}}\left(1 + \frac{r^2}{4}\right)$$

since this turns out to have the properties (check!)

$$\left(1+c^2\right)\left(1-w^2\right) = 1 - \frac{c^2h^2}{r^2}, \quad \frac{\mathrm{d}w}{\mathrm{d}r} = -\frac{ch}{r^2\sqrt{1+c^2}}$$

whence our differential equation becomes

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}w} = \mp \frac{1}{\sqrt{1-w^2}} = \pm \frac{\mathrm{d}\arccos w}{\mathrm{d}w}$$

yielding simply $w = \cos(\vartheta - \vartheta_0)$, say. Hence we obtain the solution

$$r^2 - \frac{4\sqrt{1+c^2}}{c} r\cos(\vartheta - \vartheta_0) + 4 = 0.$$

Now a Euclidean circle of radius R centered at (r_0, ϑ_0) obeys

$$r^2 + r_0^2 - 2rr_0\cos(\vartheta - \vartheta_0) = R^2,$$

hence so do points (r, ϑ) of geodesics, whereby $r_0^2 = 4 + R^2$.

Since $r_0 > 2$, the circle centre is outside the Poincaré disk, and hence geodesics of H are circle segments within, orthogonally approaching the rim (why?).

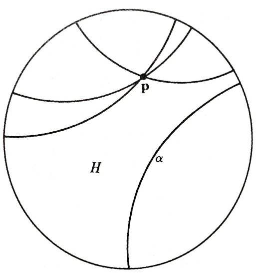


Figure 11: O'Neill Fig. 7.11

Note Hyperbolic space thus provides a famous counterexample to Euclid's classical parallel postulate (cf. figure above). Historically, this opened up the field of non-Euclidean (Riemannian) geometries!

22 Lecture 22 - Feb 28

22.1 Gauss-Bonnet formula

Finally, we are ready to uncover a deep connection between surface properties that are intrinsically geometrical or local, and those that are topological or global, in various statements of Gauss-Bonnet.

We start by integrating the geodesic curvature met last time:

Definition 22.1

The total geodesic curvature of a regular curve segment $\alpha : [a,b] \to M$ in an oriented geometric surface (M, g) is

$$\int_\alpha \kappa_g \, \mathrm{d}s = \int_a^b \kappa_g(s(t)) \frac{\mathrm{d}s}{\mathrm{d}t} \, \mathrm{d}t.$$

Example For a circle $\int_\alpha \kappa_g \, \mathrm{d}s = \pm 2\pi$ depending on orientation.

We can immediately relate this to the connection form as follows,

Lemma 22.2

Let $\alpha : [a,b] \to M$ be a regular curve in (M, g) oriented by $\{E_1, E_2\}$, with angle ψ from E_1 to α', then

$$\int_\alpha \kappa_g \, \mathrm{d}s = \psi(b) - \psi(a) + \int_\alpha \omega_{12}.$$

Proof. This follows directly from Liouville's formula (Lecture 21).

While Liouville's formula was for unit speed curves, note that

$$\begin{aligned} \int_{s(a)}^{s(b)} \omega_{12}\left(\alpha'(s)\right) \mathrm{d}s &= \int_{s(a)}^{s(b)} \omega_{12}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right) \frac{\mathrm{d}t}{\mathrm{d}s} \, \mathrm{d}s \\ &= \int_a^b \omega_{12}\left(\alpha'(t)\right) \mathrm{d}t = \int_\alpha \omega_{12} \end{aligned}$$

□

Recall from Lecture 11 (e.g. Stokes' Thm. 11.9) that we carry out integration over closed 2-segments $\mathrm{x} : R = [a,b] \times [c,d] \to M$ and their boundary curve $\partial \boldsymbol{x}$. By contrast, our surfaces M are covered by open sets or without boundary (e.g. sphere, torus).

Note Adding a boundary to M significantly affects applicability of earlier theorems using compactness (e.g.

in Seminar 17)! However, we do not pursue surfaces with boundary in this course.

From now on we shall assume that 2-segments are injective and regular in all R (not just int R, cf. Def. 18.5 of pavings), with piecewise injective and regular boundary (cf. Lecture 12)

$$\partial \boldsymbol{x} = \alpha \cup \beta \cup (-\gamma) \cup (-\delta)$$

where, as before in Def. 11.8 of a 2-segment, α, γ are u-parameter curves and β, δ are v-parameter curves intersecting at the vertices

$$p_1 = x(a,c), \quad p_2 = x(b,c), \quad p_3 = x(b,d), \quad p_4 = x(a,d)$$

Of particular interest now are the jumping angles that velocities of the regular boundary segments undergo at the vertices in turn:

Definition 22.3: Exterior and Interior Angles

Consider $\boldsymbol{x} : R \to M$ oriented by $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ with $\psi \in (0,\pi)$ between $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$. The exterior angles ε at the vertices p_1, p_2, p_3, p_4 are

$$\begin{aligned} \varepsilon_1 &= \varepsilon(-\delta', \alpha') = \pi - \psi_1, \quad & \varepsilon_2 &= \varepsilon(\alpha', \beta') = \psi_2 \\ \varepsilon_3 &= \varepsilon(\beta', -\gamma') = \pi - \psi_3, \quad & \varepsilon_4 &= \varepsilon(-\gamma', -\delta') = \psi_4 \end{aligned}$$

respectively, and the interior angles are complementary, $\iota = \pi - \varepsilon$.

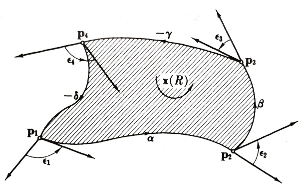


Figure 12: O'Neill Fig. 7.14

Theorem 22.4: Gauss-Bonnet Formula

Let $\boldsymbol{x} : R \to M$ be an injective regular 2-segment with Gaussian curvature K, geodesic curvature κ_g on the piecewise regular boundary $\partial \boldsymbol{x}$ with exterior angle ε_i at vertex $p_i (1 \le i \le 4)$, then

$$\iint_{\boldsymbol{x}} K \, \mathrm{d}M + \int_{\partial \boldsymbol{x}} \kappa_g \, \mathrm{d}s + \sum_i \varepsilon_i = 2\pi.$$

Proof. We work in the associated frame of Def. 19.4 and its coframe and connection form ω_{12}, and orientation so that $\mathrm{d}M(E_1, E_2) = +1$.

Then by the second structural equation, Gaussian curvature obeys

$$\mathrm{d}\omega_{12} = -K \theta_1 \wedge \theta_2$$

Now applying Stokes' Thm. 11.9 and Def. 11.7, we obtain

$$\begin{aligned} \int_{\partial \boldsymbol{x}} \omega_{12} = \iint_{\boldsymbol{x}} \mathrm{d}\omega_{12} &= -\iint_{\boldsymbol{x}} K \theta_1 \wedge \theta_2 \\ &= -\iint_R K (\theta_1 \wedge \theta_2)(\boldsymbol{x}_u, \boldsymbol{x}_v) \, \mathrm{d}u \, \mathrm{d}v \\ &= -\iint_R K \sqrt{EG} \, \mathrm{d}u \, \mathrm{d}v \\ &= -\iint_R K \, \mathrm{d}M(\boldsymbol{x}_u, \boldsymbol{x}_v) \, \mathrm{d}u \, \mathrm{d}v \\ &= -\iint_{\boldsymbol{x}} K \, \mathrm{d}M \quad (*) \end{aligned}$$

On the other hand, we use Lem. 22.2 to compute

$$\int_{\partial \boldsymbol{x}} \omega_{12} = \int_\alpha \omega_{12} + \int_\beta \omega_{12} + \int_{-\gamma} \omega_{12} + \int_{-\delta} \omega_{12}.$$

Now α is a u-parameter curve with $\psi = 0$ so $\int_\alpha \kappa_g \, \mathrm{d}s = \int_\alpha \omega_{12}$. For β, we have $\int_\beta \kappa_g \mathrm{d}s = \psi_3 - \psi_2 + \int_\beta \omega_{12}$, so with Def. 22.3,

$$\int_\beta \omega_{12} = \varepsilon_2 + \varepsilon_3 - \pi + \int_\beta \kappa_g \, \mathrm{d}s.$$

By the same token, on $-\gamma$ (where $\psi = \pi$) and $-\delta$ we obtain,

$$\int_{-\gamma} \omega_{12} = \int_{-\gamma} \kappa_g \, \mathrm{d}s, \qquad \int_{-\delta} \omega_{12} = \varepsilon_4 + \varepsilon_1 - \pi + \int_{-\delta} \kappa_g \, \mathrm{d}s.$$

so together with $(*)$, the result follows. $\square$

Note The jumping angles correspond to the geodesic curvature integration across the vertices (e.g., by smoothing the boundary).

We wish to extend the Gauss-Bonnet formula from the 2-segment image $\boldsymbol{x}(R)$ to the surface M. To this end, we introduce a refinement of the notion of pavings, Def. 18.5, as follows:

Definition 22.5: Rectangular Decomposition

A rectangular decomposition of a geometric surface M is a finite collection of injective regular 2-segments $\{(R_i, \boldsymbol{x}_i)\}$ such that $M \subset \bigcup_i \boldsymbol{x}_i(R_i)$ and any intersections of $\boldsymbol{x}_i(R_i)$ occur either in a single common vertex or a single common edge.

Exercise Visualize an example and contrast this to pavings.

Note Every compact surface M has a rectangular decomposition.

22.2 Euler characteristic

Theorem 22.6

Let $\{(R_i, \boldsymbol{x}_i)\}$ be a rectangular decomposition of a compact surface M, and v, e, f be its numbers of vertices, edges and faces, respectively. Then the integer called Euler characteristic,

$$\chi = v - e + f$$

is the same for every rectangular decomposition of M, and it is invariant under homeomorphisms of M.

Proof. Beyond our scope, cf. topology classes, e.g. MATH409. □

Note Thus, χ is truly a property of M, not of its decomposition, so we write $\chi(M)$, and it is a topological invariant.

Example A tetrahedron in $\mathbb{R}^3$ has $v = 4, e = 6, f = 4$ so $\chi = 2$.

Example A cube in $\mathbb{R}^3$ has $v = 8, e = 12, f = 6$ so again $\chi = 2$.

Note These are homeomorphic to the sphere (why?), so any rectangular decomposition of the sphere also yields $\chi = 2$.

Hence, any compact surface without boundary and $\chi = 2$ is called topological sphere.

Exercise Find $\chi(T)$ for the torus T. (Answer: $\chi(T) = 0$.)

22.3 Gauss-Bonnet theorem

Theorem 22.7: (Gauss-Bonnet)

Let (M, g) be a compact orientable geometric surface with Gaussian curvature K and Euler characteristic $\chi(M)$. Then

$$\iint_M K \ \mathrm{d}M = 2\pi\chi(M).$$

Note This reveals that total curvature is a topological invariant.

Example The sphere has $\chi = 2$, agreeing with the total curvature 4π from the standard round (or induced) geometry (Lecture 18).

Example The torus has $\chi = 0$, agreeing immediately with $K = 0$ of the flat geometry (Sem. 20), but also (check!) with the induced geometry yielding regions of $K < 0, K = 0, K > 0$ (Lecture 15).

Proof. Given an oriented rectangular decomposition of M with f faces,

$$\iint_M K \ \mathrm{d}M = \sum_{i=1}^{f} \iint_{\boldsymbol{x}_i} K \ \mathrm{d}M.$$

Then for each summand on the right-hand side, the Gauss-Bonnet formula Thm. 22.4 for x_i in terms of interior angles yields

$$\iint_{\boldsymbol{x}_i} K \ \mathrm{d}M = -\int_{\partial \boldsymbol{x}_i} \kappa_g \ \mathrm{d}s + \sum_i \iota_i - 2\pi.$$

But each edge is traversed twice in opposite direction (why?) so

$$\sum_{i=1}^{f} \int_{x_i} \kappa_g \ \mathrm{d}s = 0.$$

Moreover, at each vertex the ι add up to 2π (why?) so

$$\iint_M K \ \mathrm{d}M = 2\pi(v - f).$$

Now each face has two four edges each of which is shared by two faces whence $4f = 2e$ or $-f = f - e$ so, by Thm. 22.6, we obtain

$$\iint_M K \ \mathrm{d}M = 2\pi(v - e + f) = 2\pi\chi(M).$$

□

Note In fact, Thm. 22.4 and Thm. 22.7 can be extended to a surface M with boundary ∂M such that (cf. Homework 5),

$$\iint_M K \ \mathrm{d}M + \int_{\partial M} \kappa_g \ \mathrm{d}s + \sum_i \varepsilon_i = 2\pi\chi(M).$$

22.4 Poincaré-Hopf theorem

We can use Gauss-Bonnet and a vector field V to learn about the topology of M. A point where $V = 0$ (or undefined) but $V \neq 0$ in its neighbourhood is called isolated singularity (cf. MATH307).

Definition 22.8: Index

Let $p \in S \subset M$ be an isolated singularity of a vector field V, curve $\alpha([a, b]) = \partial S$ be a loop enclosing it, and ψ be the angle from a non-zero X to V on α. Then the index of V at p is

$$\mathrm{ind}(V, p) = \frac{1}{2\pi}(\psi(b) - \psi(a)) \in \mathbb{Z}$$

Thus, $\mathrm{ind}(V, p)$ is used to classify types of vector field singularites:

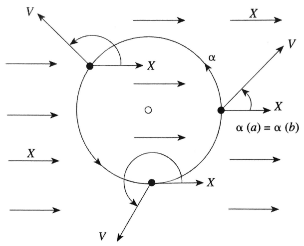


Figure 13: O'Neill Fig. 7.24

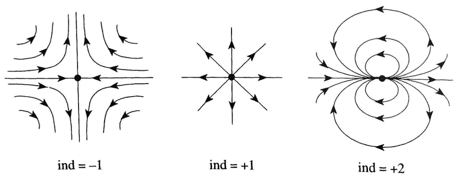


Figure 14: O'Neill Fig. 7.25

Theorem 22.9: (Poincaré-Hopf)

On a compact orientable surface M, let V be a differentiable vector field with isolated singularities at $p_i, 1 \leq i \leq k$, then

$$\sum_{i=1}^{k} \operatorname{ind}(V, p_i) = \chi(M).$$

Proof. By the Hausdorff axiom, we may cut out non-intersecting $S_i \ni p_i$ from M, yielding $\hat{M} = M \setminus \bigcup_i S_i$ such that the total curvature is

$$\iint_M K \, \mathrm{d}M = \iint_{\hat{M}} K \, \mathrm{d}M + \sum_{i=1}^{k} \iint_{S_i} K \, \mathrm{d}M.$$

Since V on $\hat{M}$ has no singularities there is an associated frame with $E_1 = V/|V|, E_2 = J(E_1)$ and $\hat{\omega}_{12}$. Then using $(*)$ of Thm. 22.4 and noting that $\partial\hat{M}$ and ∂S_i have opposite orientation (why?),

$$\iint_{\hat{M}} K \, \mathrm{d}M = -\int_{\partial\hat{M}} \hat{\omega}_{12} = \sum_{i=1}^{k} \int_{\partial S_i} \hat{\omega}_{12}.$$

On the other hand, introduce a unit X_i on each S_i so that there is a frame $\{X_i, J(X_i)\}$ with ω_{12} and $\iint_{S_i} K \, \mathrm{d}M = -\int_{\partial S_i} \omega_{12}$, whence

$$\iint_M K \, \mathrm{d}M = \sum_{i=1}^{k} \int_{\partial S_i} (\hat{\omega}_{12} - \omega_{12}) \qquad (\dagger)$$

Now introduce fiducial parallel unit vector fields P_i such that $\psi(X_i, P_i) = \psi(X_i, V) + \psi(V, P_i)$, then by Def. 22.8 and the parallel transport condition from Lem. 20.7, we obtain (check!)

$$\begin{aligned} 2\pi \operatorname{ind}(V, p_i) &= \int_a^b \psi'(X_i, V) \, \mathrm{d}t = \int_a^b (\psi'(X_i, P_i) - \psi'(V, P_i)) \, \mathrm{d}t \\ &= \int_{\partial S_i} (-\omega_{12} + \hat{\omega}_{12}) \end{aligned}$$

so the result follows from (†) and the Gauss-Bonnet Thm. 22.4. □

$\boxed{\textbf{Example}}$ Wind on earth can never be everywhere non-zero!