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Sect 4.2

4. Since $\begin{cases} y_0 = f(x_0) \\ y_1 = f(x_1) \end{cases}$ $\begin{cases} \pi_1(Y, y_0) = f_\pi(\pi_1(X, x_0)) \\ \pi_1(Y, y_1) = f_\pi(\pi_1(X, x_1)) \end{cases}$

path $\begin{cases} w: x_0 \rightsquigarrow x_1 \\ f\circ w: y_0 \rightsquigarrow y_1 \end{cases}$ $\begin{cases} w_\#: \pi_1(X, x_0) \rightsquigarrow \pi_1(X, x_1) \\ (f\circ w)_\#: \pi_1(Y, y_0) \rightsquigarrow \pi_1(Y, y_1) \end{cases}$

Then we only need to prove: $f_\pi \circ w_\# = (f\circ w)_\# \circ f_\pi$

$\forall x \in \pi_1(X, x_0)$, then $f_\pi \circ w_\#(x) = f\circ(w^{-1} x w)$

$(f\circ w)_\# \circ f_\pi(x) = (f\circ w)^{-1}(f\circ x)(f\circ w)$

Since f is a homeomorphism, then

$(f\circ w)(f\circ w^{-1}) = (f\circ w w^{-1}) = Id$, $(f\circ w)^{-1} = f\circ w^{-1}$

$$\begin{aligned} (f\circ w)_\# \circ f_\pi(x) &= (f\circ w^{-1})(f\circ x)(f\circ w) \\ &= (f\circ w^{-1} x)(f\circ w) \\ &= f\circ(w^{-1} x w) \\ &= f_\pi \circ w_\#(x) \end{aligned}$$

Then the homeomorphic graph is exchangable.

5. Since $r: X \to A$, $i: A \hookrightarrow X$, where $r|_A = i = id|_A$, then $r\circ i = id|_A$

Then $r_\pi \circ i_\pi = (r\circ i)_\pi = id$,

Since $r_\pi \circ i_\pi: \pi_1(A, x_0) \to \pi_1(A, x_0)$

Then r_π is an injective, i_π is a surjective.

6. Suppose b^{-1} is the inverse path of b, then ab^{-1} is a loop path.

Since X is simple connected, then X has constant basic group.

Then path ab^{-1} is homotopic to constant point path e, i.e. $ab^{-1} \simeq e$.

Then $b \simeq eb \simeq ab^{-1}b \simeq ae \simeq a$

Thus. $a \simeq b$.

7. $w_\# = w'_\#$

$\Leftrightarrow \forall \alpha \in \pi_1(X, x_1)$, $w^{-1}\alpha w = w'^{-1}\alpha w'$

(left times w, right times w'^{-1})

$\Leftrightarrow \forall \alpha \in \pi_1(X, x_1)$, $\alpha w w'^{-1} = w w'^{-1} \alpha$

$\Leftrightarrow w w'^{-1}$ is in the center of $\pi_1(X, x_0)$

[M] Sect 52.

7. (a) Since $f, g \in \Omega(G, x_0)$, then $f(s) \cdot g(s)$ is also a loop based on x_0.

Then $f \otimes g \in \Omega(G, x_0)$

Since the point path e_{x_0} at x_0 is unit element in $\Omega(G, x_0)$

the inverse path $\bar{\alpha}$ is the inverse element of α in $\Omega(G, x_0)$, $e = \alpha \otimes \bar{\alpha}$

Then $(\Omega(X, x_0), \otimes)$ is a group.

(b) $\forall w_1, w_2 \in \pi_1(G, x_0)$, $\alpha \in \Omega(G, x_0)$

$$\begin{aligned} \text{Then } w_1(\alpha) \otimes w_2(\alpha) &= w_1^{-1}\alpha w_1 \otimes w_2^{-1}\alpha w_2 \\ &= w_1^{-1}\alpha w_1 w_2^{-1}\alpha w_2 \\ &\in \Omega(G, x_0) \end{aligned}$$

Then $\otimes$ induces the group operation on $\pi_1(G, x_0)$

(c) $\forall \alpha \in (X, x_0)$, $(f * e_{x_0}) \otimes (e_{x_0} * g)(\alpha)$

$$\begin{aligned} &= (f \otimes g)(\alpha) \\ &= f(\alpha) \cdot g(\alpha) \\ &= (f * g)(\alpha) \\ &= (f * e_{x_0}) * (e_{x_0} * g)(\alpha) \end{aligned}$$

Then $(f * e_{x_0}) \circ (e_{x_0} * g) = (f * e_{x_0}) * (e_{x_0} * g)$

Thus, $\circ$ and $*$ are the same.

(d) $\forall w_1, w_2 \in \pi_1(G, x_0)$,

$$\begin{aligned} w_1 \otimes w_2 &= w_1 * w_2 \\ &= w_2 * w_1 \\ &= w_2 \otimes w_1 \end{aligned}$$

Then $\pi_1(G, x_0)$ is abelian.