CS711008Z Algorithm Design and Analysis

Lecture 10. Algorithm design technique: Network flow and its applications [1]

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[1] The slides are made based on Chapter 7 of Introduction to algorithms, Combinatorial optimization algorithm and complexity by C. H. Papadimitriou and K. Steiglitz. Some slides are excerpted from the presentation by K. Wayne with permission.

Outline

- MAXIMUMFLOW problem: FORD-FULKERSON algorithm, MAXFLOW-MINCUT theorem;
- A duality explanation of FORD-FULKERSON algorithm and MAXFLOW-MINCUT theorem;
- 加速 Scaling technique to improve FORD-FULKERSON algorithm;
- Solving the dual problem: Push-Relabel algorithm; 最常用
- 拓展. Extensions of MAXIMUMFLOW problem: lower bound of capacity, multiple sources & multiple sinks, indirect graph;

A brief history of MINIMUMCUT problem I

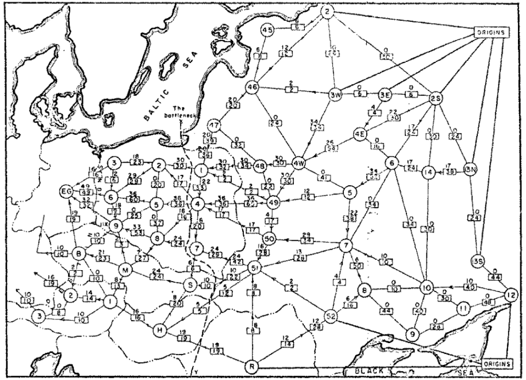


Figure: Soviet Railway network, 1955

A brief history of MINIMUMCUT problem II

- *"From Harris and Ross [1955]: Schematic diagram of the railway network of the Western Soviet Union and Eastern European countries, with a maximum flow of value 163,000 tons from Russia to Eastern Europe, and a cut of capacity 163,000 tons indicated as 'The bottleneck' ..."*
- A recently declassified U.S. Air Force report indicates that the original motivation of minimum-cut problem and Ford-Fulkerson algorithm is to disrupt rail transportation the Soviet Union [A. Shrijver, 2002]. 最小切 ← 炸铁路.

A brief history of algorithms to MinimumCut problem

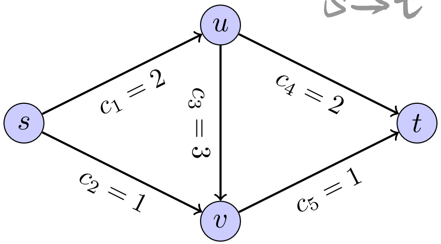
Year	Developers	Time-complexity
1956	Ford and Fulkerson	$O(mC)$ and $O(m^2 \log C)$
1970	Dinitz	$O(n^2 m)$
1972	Edmonds and Karp	$O(m^2 n)$
1974	Karzanov	$O(n^3)$
1986	Sleator and Tarjan	$O(nm \log n)$
1988	Goldberg and Tarjan	$O(n^2 m \log(\frac{n^2}{m}))$
2012	Orlin	$O(nm)$

MaximumFlow problem

MaximumFlow problem

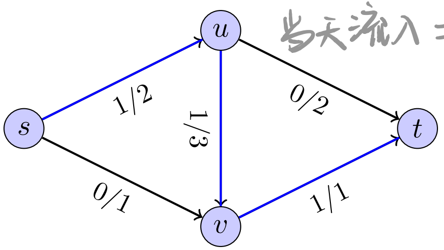
INPUT:
A directed graph $G =< V, E >$. Each edge e has a capacity C_e.
Two special points: **source** s and **sink** t;
OUTPUT:
For each edge $e = (u, v)$, to assign a flow $f(u, v)$ such that $\sum_{u,(s,u)\in E} f(s, u)$ is maximized.

$s \rightarrow t$



Intuition: to push as many commodity as possible from **source** s to **sink** t.

Flow 流



当天流入 == 当天流出

Definition (Flow)

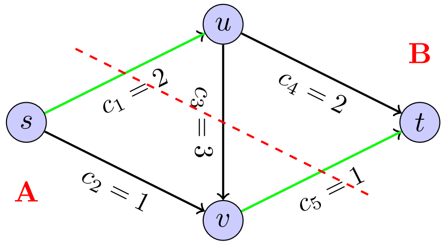
$f : E \to R^+$ is a $s-t$ **flow** if:

1. (Capacity constraints): $0 \leq f(e) \leq C_e$ for all edge e;
2. (Conservation constraints): for any intermediate vertex $v \in V - \{s,t\}$, $f^{in}(v) = f^{out}(v)$, where $f^{in}(v) = \sum_{e \text{ into } v} f(e)$ and $f^{out}(v) = \sum_{e \text{ out of } v} f(e)$.
   (Intuition: input = output for any intermediate vertex.)

The **value of flow** f is defined as $V(f) = f^{out}(s)$.

Definition ($s-t$ cut)

An **$s-t$ cut** is a partition (A, B) of V such that $s \in A$ and $t \in B$.
The **capacity of a cut** (A, B) is defined as
$C(A, B) = \sum_{e \text{ from } A \text{ to } B} C(e)$.



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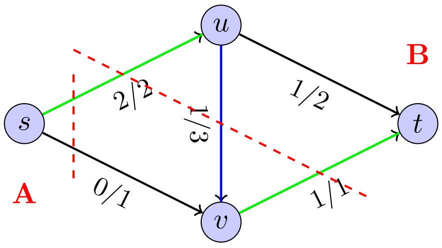
$$C(A, B) = 3$$

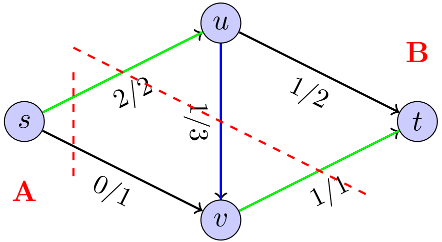
Flow value lemma

Lemma

(Flow value lemma) Give a flow f. For **any** *$s-t$ cut (A,B), the flow across the cut is a constant $V(f)$. Formally,*
$V(f) = f^{out}(A) - f^{in}(A)$.



$$V(f) = 2 + 0 = 2$$
$$f^{out}(A) - f^{in}(A) = 2 + 1 - 1 = V(f)$$



Proof.

- We have: $0 = f^{out}(v) - f^{in}(v)$ for any node $v \neq s$ and $v \neq t$.
- Thus, we have:

$$
\begin{aligned}
V(f) &= f^{out}(s) - f^{in}(s) \qquad\qquad //\text{Hint: } f^{in}(s) = 0; \\
&= \sum\nolimits_{v \in A} (f^{out}(v) - f^{in}(v)) \\
&= (\sum\nolimits_{\text{e from A to B}}^{\text{目的地B}} f(e) + \sum\nolimits_{\text{e from A to A}}^{\text{边向内部}} f(e)) \\
&\quad -(\sum\nolimits_{\text{e from B to A}} f(e) + \sum\nolimits_{\text{e from A to A}} f(e)) \\
&= f^{out}(A) - f^{in}(A) \quad \Rightarrow \text{和割没关系.}
\end{aligned}
$$

Ford-Fulkerson algorithm [1956]


Figure: Lester Randolph Ford Jr. and Delbert Ray Fulkerson

Trial 1: Dynamic programming technique

- Dynamic programming doesn't seem to work.
- In fact, there is no algorithm known for MAXIMUM FLOW problem that can really be viewed as belonging to the dynamic programming paradigm.
- We know that the MAXIMUMFLOW problem is in $\mathbf{P}$ since it can be formulated as a linear program (See Lecture 8).
- However, the network structure has its own property to enable a more efficient algorithm, informally called **network simplex**.

网络单纯形.

Trial 2: IMPROVEMENT strategy

①分
②不分.
③聪明的枚举

- Let's return to the general IMPROVEMENT strategy:

IMPROVEMENT(f)

```
x = x_0; //starting from an initial solution;
while TRUE do
    x = IMPROVE(x); //move one step towards optimum;
    if STOPPING(x, f) then
        break;
    end if
end while
return x;
```

Three key questions of iteration framework

- Three key questions:
  1. How to construct an initial solution? 最简初始解
     - For MaximumFlow problem, an initial solution can be easily obtained by setting $f(e) = 0$ for any e (called 0-flow).
     - It is easy to verify that both CONSERVATION and CAPACITY constraints hold for the 0-flow.
  2. How to improve a solution?
  3. When shall we stop?

A failure start: augmenting flow along a path in the original graph

- Let p be a simple $s - t$ path in the network G.

```
Initialize f(e) = 0 for all e.
while there is an s − t path in graph G do
    arbitrarily choose an s − t path p in graph G;
    f = AUGMENT(p, f);
end while
return f;
```

Augmenting flow along a path

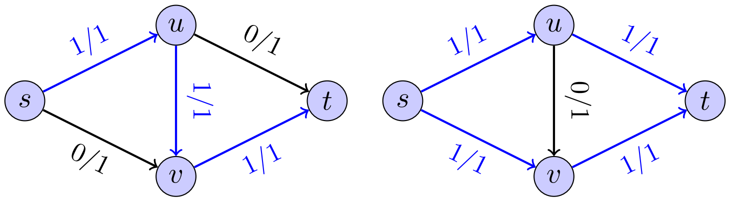
- We define $bottleneck(p, f)$ as the minimum residual capacity of edges in path p.

  $\textsc{Augment}(p, f)$

  Let $b = bottleneck(p, f)$;
  for each edge $e = (u, v) \in p$ **do**
      increase $f(u, v)$ by b;
  end for

Why we fail?

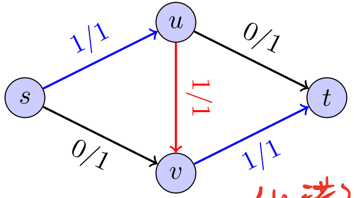
- We start from 0-flow. In order to increase the value of f, we find a $s-t$ path, say $p = s \rightarrow u \rightarrow v \rightarrow t$, to transmit one unit of commodity.
- However we cannot find a $s-t$ path in G to increase f further (left panel) although the maximum flow value is 2 (right panel).



$V(f) = 1$ $V(f) = 2$

Ford-Fulkerson algorithm: “undo” functionality

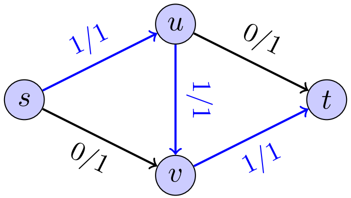
- Key observation:
  - When constructing a flow f, one might commit errors on some edges, i.e. the edges should not be used to transmit commodity. For example, the edge $u \rightarrow v$ should not be used.



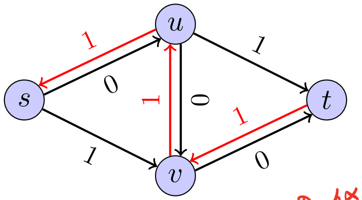
$V(f) = 1$ 做错了,退货回去.

  - To improve the flow f, we should work out ways to **correct these errors**, i.e. “undo” the transmission assigned on the edges.

- But how to implement the "undo" functionality?
- **Adding backward edges!**
- Suppose we add a **backward** edge $v \to u$ into the original graph. Then we can correct the transmission via pushing back commodity from v to u.



Flow f



Backward edges 退货.

Residual graph with "backward" edges to correct errors

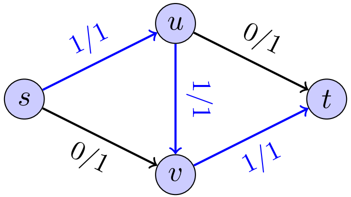
剩余图

Definition (Residual Graph)

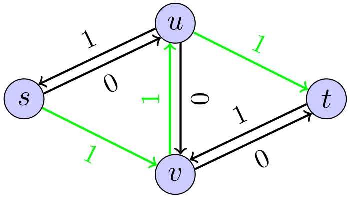
Given a directed graph $G =< V, E >$ with a flow f, we define **residual graph** $G_f =< V, E' >$. For any edge $e = (u, v) \in E$, two edges are added into E' as follows:

1. (**Forward edge** (u, v) with leftover capacity):
   If $f(e) < C(e)$, add edge $e = (u, v)$ with capacity $C(e) = C(e) - f(e)$. 剩余.
2. (**Backward edge** (v, u) with undo capacity):
   If $f(e) > 0$, add edge $e' = (v, u)$ with capacity $C(e') = f(e)$.

Finding an $s-t$ path in G_f rather than G

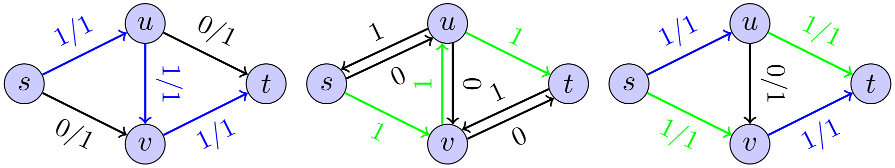


Flow f



Residual graph G_f

Note: the path contains a backward edge (v, u)



Flow f + An $s-t$ path in G_f = New flow f'

对偶DRP

Note:

- By using the backward edge $v \to u$, the initial transmission from u to v is pushed back.
- More specifically, the first commodity transferred through flow f changes its path (from $s \to u \to v \to t$ to $s \to u \to t$), while the second one uses the path $s \to v \to t$.

Ford-Fulkerson algorithm

- Let p be a simple $s - t$ path in residual graph G_f, called **augmentation path**. We define $bottleneck(p, f)$ as the minimum capacity of edges in path p.

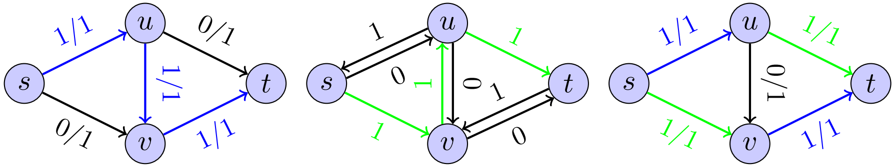
  Ford-Fulkerson algorithm:

```
Initialize f(e) = 0 for all e.
while there is an s − t path in residual graph G_f do
    arbitrarily choose an s − t path p in G_f;
    f = AUGMENT(p, f);
end while
return f;
```

Correctness and time-complexity analysis

Theorem

The operation $f' = \text{AUGMENT}(p, f)$ *generates a new flow* f' *in* G.



Flow f + An $s-t$ path in G_f = New flow f'

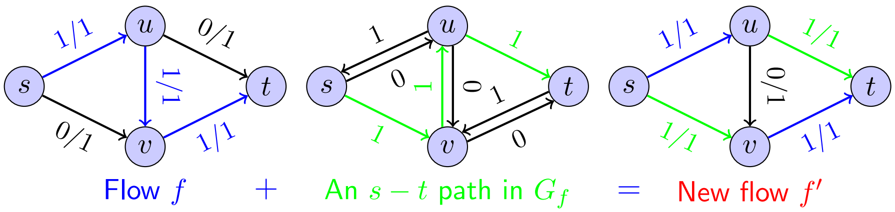
Proof.

- Checking **capacity constraints**: Consider two cases of edge $e = (u, v)$ in path p:
  1. (u, v) is a forward edge arising from $(u, v) \in E$:
     $0 \leq f(e) \leq f'(e) = f(e) + bottleneck(p, f) \leq$
     $f(e) + (C(e) - f(e)) \leq C(e)$
  2. (u, v) is a backward edge arising from $(v, u) \in E$:
     $C(e) \geq f(e) \geq f'(e) = f(e) - bottleneck(p, f) \geq$
     $f(e) - f(e) = 0$
- Checking **conservation constraints**:
  On each node v, the change of the amount of flow entering v is the same as the change in the amount of flow exiting v.

$\square$

Theorem

(Monotonically increasing) $V(f') > V(f)$



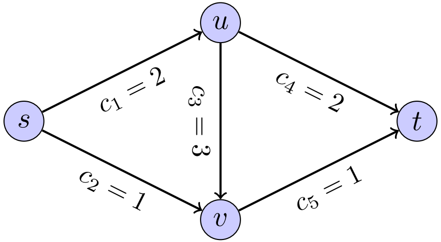
- Hint: $V(f') = V(f) + bottleneck(p, f) > V(f)$ since $bottleneck(p, f) > 0$.

Property 3: a trivial upper bound of flow

Theorem

$V(f)$ has an upper bound $C = \sum_{e \text{ out of } s} C(e)$.

(Intuition: the edges out of s are completely saturated with flow.)



Property 4: augmentation step

Theorem

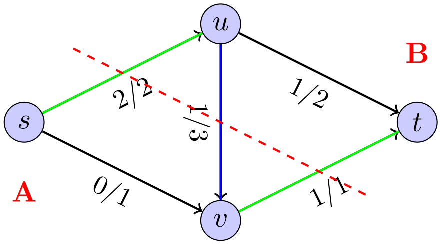
Assume all edges have integer capacities. At every intermediate stage of the Ford-Fulkerson algorithm, both flow value $V(f)$ and residual capacities are integers. Thus, $bottleneck(p, f) \geq 1$, and there is at most C iterations of the `while` loop.

- Intuition: Under a reasonable assumption that all capacities are integers, we have $bottleneck(p, f) \geq 1$ at every stage; thus, $V(f') \geq V(f) + 1$.
- Time complexity: $O(mC)$.
  - $O(C)$ iterations
  - At each iteration, it takes $O(m + n)$ time to find an $s - t$ path in G_f using DFS or BFS technique.

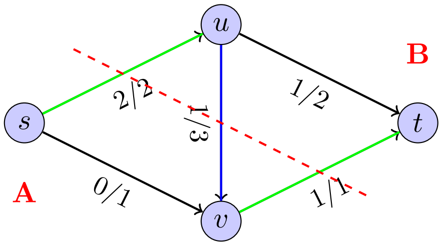
Property 5: A tighter upper bound

Theorem

(Tight upper bound) Given a flow f. For any $s-t$ cut (A, B), we have $V(f) \leq C(A, B)$.



$$V(f) = 2 \leq C(A, B) = 3$$



Proof.

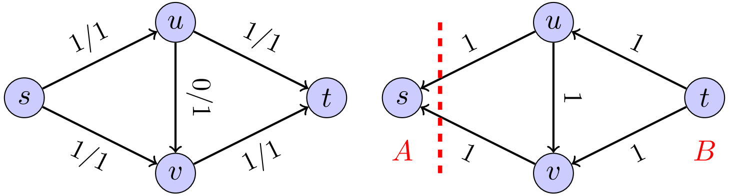
$$
\begin{aligned}
V(f) &= f^{out}(A) - f^{in}(A) && \text{(by flow value lemma)} \\
&\leq f^{out}(A) && \text{(by } f^{in}(A) \geq 0\text{)} \\
&= \sum_{\mathrm{e} \in A \to B} f(e) \\
&\leq \sum_{\mathrm{e} \in A \to B} C(e) && \text{(by } f(e) \leq C(e)\text{)} \\
&= C(A, B)
\end{aligned}
$$

$\square$

Correctness

Theorem

FORD-FULKERSON *ends up with a maximum flow* f *and a minimum cut* (A, B).



Flow f Residual graph G_f

Proof.

- Ford-Fulkerson algorithms ends when there is no $s-t$ path in the residual graph G_f.
- Let A be the set of nodes reachable from s in G_f, and $B = V - A$. (A, B) forms a $s-t$ cut. $(A \neq \phi, B \neq \phi)$.
- Consider two types of edges $e = (u, v) \in E$ across cut (A, B):

  1. $u \in A, v \in B$: we have $f(e) = C(e)$ (Otherwise, A should be extended to include v since (u, v) is in G_f.)
  2. $u \in B, v \in A$: we have $f(e) = 0$ (Otherwise, A should be extended to include u since (v, u) is in G_f.)

- Thus we have

$$
\begin{aligned}
V(f) &= f^{out}(A) - f^{in}(A) \\
&= f^{out}(A) \qquad \text{(by } f^{in}(A) = 0\text{)} \\
&= \sum\nolimits_{e \in A \to B} f(e) \\
&= \sum\nolimits_{e \in A \to B} C(e) \qquad \text{(by } f(e) = C(e)\text{)} \\
&= C(A, B)
\end{aligned}
$$

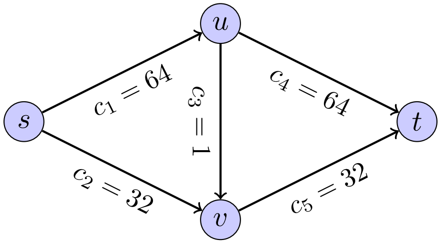
FORD-FULKERSON algorithm: bad example 1

The integer restriction is important

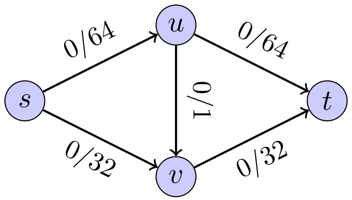
- In the analysis of FORD-FULKERSON algorithm, the integer restriction of capacities is important: the bottleneck edge leads to an increase of at least 1.
- The analysis doesn't hold if the capacities can be irrational.
- In fact, the flow might be increased by a smaller and smaller number and the iteration will be endless.
- Worse yet, this endless iteration might not converge to the maximum flow.

(See an example by Uri Zwick)

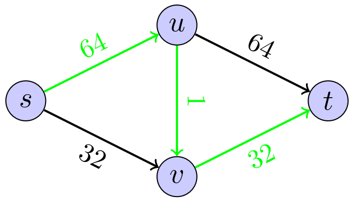
FORD-FULKERSON algorithm: bad example 2



A bad example of FORD-FULKERSON algorithm: Step 1

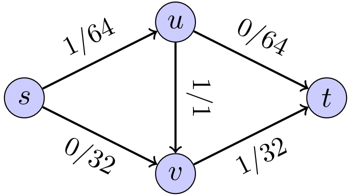


Flow $f : V(f) = 0$

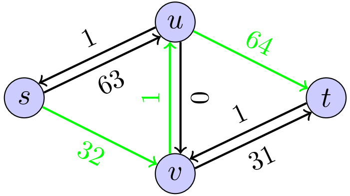


An $s - t$ path in G_f

A bad example of Ford-Fulkerson algorithm: Step 2

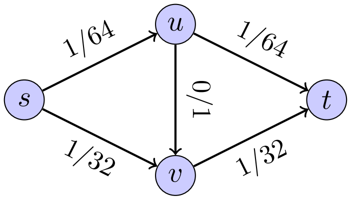


Flow $f : V(f) = 1$



An $s - t$ path in G_f

A bad example of Ford-Fulkerson algorithm: Step 3



Flow $f : V(f) = 2$

Note:

1. After two iterations, the problem is similar to the original problem except for the capacities on $(s, u), (s, v), (u, t), (v, t)$ decrease by 1.
2. Thus, Ford-Fulkerson algorithm will end after 64+32 iterations. (Why? $bottleneck = 1$ at all stages.)

Ford-Fulkerson algorithm: weakness

- Ford-Fulkerson algorithm doesn't specify how to choose an augmentation path, leading to some weaknesses:
  - A path with small bottleneck capacity is chosen as augmentation path;
  - We put flow on too many edges than necessary.
- It should be pointed out that the original max-flow paper lists several heuristics for improvement.

Improvements of FORD-FULKERSON algorithm

- Various strategies to select augmentation path in G_f:
  1. Fat pipes:
     - To select the augmentation path with **the largest bottleneck capacity**;
     - Scaling technique: an efficient way to find an augmentation path with **large** improvement;
  2. Short pipes:
     - Edmonds-Karp: to find **the shortest $s-t$ path** in *BFS tree*.
     - Dinitz' algorithm: to extend *BFS tree* to **layered network**, find augmentation path in the layered network, and perform **amortized** analysis;
     - Dinic's algorithm: running **DFS** to find a collection of augmentation paths (called **blocking flow**) in the **layered network**.
- It should be pointed out that the complexity of EDMONDS-KARP and DINIC'S algorithms do not depend on edge capacities, and thus avoid the limitation of Ford-Fulkerson algorithm for the networks with irrational edge capacities.

Improvement 1: Scaling technique for speed-up

Scaling technique

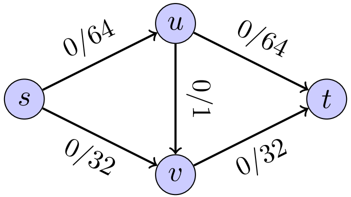
- Question: can we choose a **large** augmentation path? The larger $bottleneck(p, f)$, the less iterations.
- An $s - t$ path p in G_f with the **largest** $bottleneck(p, f)$ can be found using binary search, or a slight change of Dijkstra's algorithm in $O(m + n \log n)$ time; however, it is still somewhat inefficient.
- Basic idea: we can relax the **"largest"** requirement to **"sufficiently large"**.
- Specifically, we can set up a lower bound Δ for $bottleneck(P, f)$: **simply removing the "small" edges**, i.e. the edges with capacities less than Δ from $G(f)$. This residual graph is called $G_f(\Delta)$.
- Δ will be scaled as iteration proceeds.

- Scaling FORD-FULKERSON algorithm:

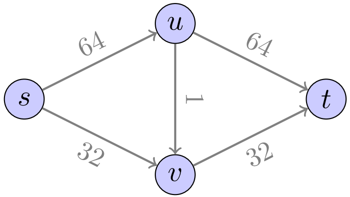
  Initialize $f(e) = 0$ for all e.
  Let $\Delta = C$;
  while $\Delta \geq 1$ **do**
      while there is an $s - t$ path in $G_f(\Delta)$ **do**
          choose an $s - t$ path p;
          $f = \text{AUGMENT}(p, f)$;
      end while
      $\Delta = \Delta/2$;
  end while
  return f;

- Intuition: flow is augmented in a large step size whenever possible; otherwise, the step size is reduced. Step size is controlled via removing the "small" edges out of residual graph.
- Note: Δ turns to be 1 finally; thus no edge in residual graph will be neglected.

An example: Step 1

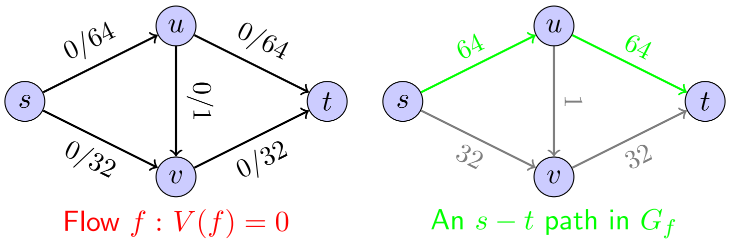


Flow $f: V(f) = 0$



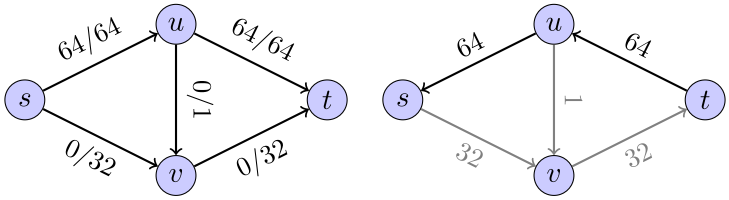
No $s-t$ path in G_f

- Flow: 0 flow;
- Δ: $\Delta = 96$;
- $G_f(\Delta)$: the edges in light blue were removed since capcities are less than 96.
- $s-t$ path: cannot find. Thus Δ is scaled: $\Delta = \Delta/2 = 48$.



- Flow: 0 flow;
- Δ: $\Delta = 48$;
- $G_f(\Delta)$: the edges in light blue were removed since capcities are less than 48.
- $s-t$ path: a path $s-u-t$ appears. Perform augmentation operation.

An example: Step 3

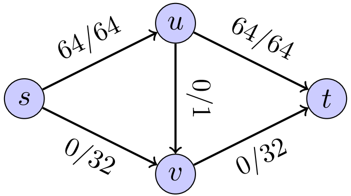


Flow $f : V(f) = 64$

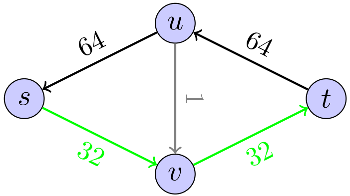
No $s - t$ path in G_f

- Flow: 64;
- Δ: $\Delta = 48$;
- $G_f(\Delta)$: the edges in light blue were removed since capcities are less than 48.
- $s - t$ path: no path found. Perform scaling: $\Delta = \Delta/2 = 24$.

An example: Step 4



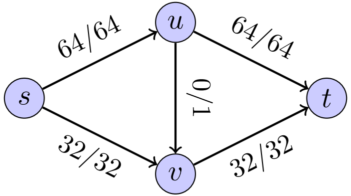
Flow $f : V(f) = 64$

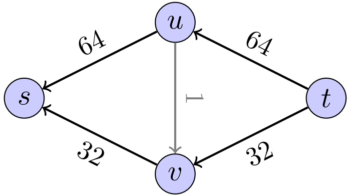


An $s - t$ path in G_f

- Flow: 64;
- Δ: $\Delta = 24$;
- $G_f(\Delta)$: the edges in light blue were removed since capcities are less than 24.
- $s - t$ path: find a path: $s - v - t$. Perform augmentation.

An example: Step 5





- Flow: 96. Maximum flow obtained.
- Δ: $\Delta = 24$;
- $G_f(\Delta)$: the edges in light blue were removed since capcities are less than 24.
- $s - t$ path: cannot find a $s - t$ path.

Theorem

(Outer `while` *loop number) The* `while` *iteration number is at most* $1 + \log_2 C$.

Scaling Ford-Fulkerson algorithm:

Initialize $f(e) = 0$ for all e.
Let $\Delta = C$;
while $\Delta \geq 1$ **do**
  while there is an $s - t$ path in $G_f(\Delta)$ **do**
    choose an $s - t$ path p;
    $f = \text{AUGMENT}(p, f)$;
  end while
  $\Delta = \Delta/2$;
end while
return f;

Analysis: inner `while` loop

Theorem

(Inner `while` loop number) In a scaling phase, the number of augmentations is at most $2m$.

Scaling Ford-Fulkerson algorithm:

Initialize $f(e) = 0$ for all e.
Let $\Delta = C$;
while $\Delta \geq 1$ **do**
    while there is an $s - t$ path in $G_f(\Delta)$ **do**
        choose an $s - t$ path p;
        $f = \text{AUGMENT}(p, f)$;
    end while
    $\Delta = \Delta/2$;
end while
return f;

Analysis: inner `while` loop cont'd

Proof.

1. Let f be the flow that a Δ-scaling phase ends up with, and f^* be the maximum flow. We have $V(f) \geq V(f^*) - m\Delta$. (Intuition: $V(f)$ is not too bad; the distance to maximum flow is small.)
2. In the subsequent $\frac{\Delta}{2}$-scaling phase, each augmentation will increase $V(f)$ at least $\frac{\Delta}{2}$.

Thus, there are at most $2m$ augmentations in the $\frac{\Delta}{2}$-scaling phase. □

- Time-complexity: $O(m^2 \log_2 C)$.
  - $O(\log_2 C)$ outer `while` loop;
  - $O(m)$ inner loops;
  - Each augmentation takes $O(m)$ time.

But why $V(f) \geq V(f^*) - m\Delta$?

Proof.

- Let A be the set of nodes reachable from s in the residual graph $G_f(\Delta)$, and $B = V - A$. Thus (A, B) forms a cut $(A \neq \phi, B \neq \phi)$.
- Consider two types of edges $e = (u, v) \in E$.
  1. $u \in A, v \in B$: we have $f(e) \geq C(e) - \Delta$ (Otherwise, A should be extended to include v since (u, v) in $G_f(\Delta)$.)
  2. $v \in A, u \in B$: we have $f(e) \leq \Delta$ (Otherwise, A should be extended to include v since (u, v) in $G_f(\Delta)$.)
- Thus we have:

$$
\begin{aligned}
V(f) &= \sum\nolimits_{\mathrm{e} \in A \rightarrow B} f(e) - \sum\nolimits_{\mathrm{e} \in B \rightarrow A} f(e) \\
&\geq \sum\nolimits_{\mathrm{e} \in A \rightarrow B} (C(e) - \Delta) - \sum\nolimits_{\mathrm{e} \in B \rightarrow A} \Delta \\
&\geq \sum\nolimits_{\mathrm{e} \in A \rightarrow B} C(e) - m\Delta \\
&= C(A, B) - m\Delta \\
&\geq V(f^*) - m\Delta
\end{aligned}
$$

Implementation 2: Edmonds-Karp algorithm using $O(m^2 n)$ time

Edmonds-Karp algorithm [1972]

Figure: Jack Edmonds, and Richard Karp

Note: The algorithm was first published by Yefim Dinic in 1970 and independently published by Jack Edmonds and Richard Karp in 1972.

EDMONDS-KARP algorithm

EDMONDS-KARP algorithm:

```
Initialize f(e) = 0 for all e.
while there is a s − t path in G_f do
    choose the shortest s − t path p in G_f using BFS;
    f = AUGMENT(p, f);
end while
return f;
```

(a demo)

Analysis

Theorem

Edmonds-Karp algorithm runs in $O(m^2 n)$ time.

Proof.

- During the execution of Edmonds-Karp algorithm, an edge $e = (u, v)$ serves as **bottleneck** edge at most $\frac{n}{2}$ times
- Thus, the `while` loop will be executed at most $\frac{n}{2}m$ times since there are m edges in total
- It takes $O(m)$ time to find the shortest path using BFS, and augment flow using the path.

□

Lemma

An edge $e = (u, v)$ serves as a bottleneck edge at most $\frac{n}{2}$ times.

Proof.

- For a residual graph G_f, we first category all nodes into levels $L_0, L_1, ...$, where $L_0 = \{s\}$, and L_i contains all nodes v such that the shortest path from s to v has i edges. We use $L(u)$ to denote the level number of node u.
- Consider the two consecutive occurrences of edge $e = (u, v)$ in G_f as bottleneck, say at step k and step k'''.
- We have $L(v) = L(u) + 1$ at step k, and after flow augmentation, the bottleneck edge $e = (u, v)$ will be reversed in G_f.
- At step k''', $e = (u, v)$ becomes a bottleneck edge again.
- This means that $e' = (v, u)$ should be reversed first before step k''', say at step k''. We have $L''(u) = L''(v) + 1$. Thus $L''(u) = L''(v) + 1 \geq L'(v) + 1 \geq L(u) + 2$.
- For any node, its maximal level is at most n and its level never decreases (why?). Thus the lemma holds.

Analysis of the Edmonds-Karp algorithm

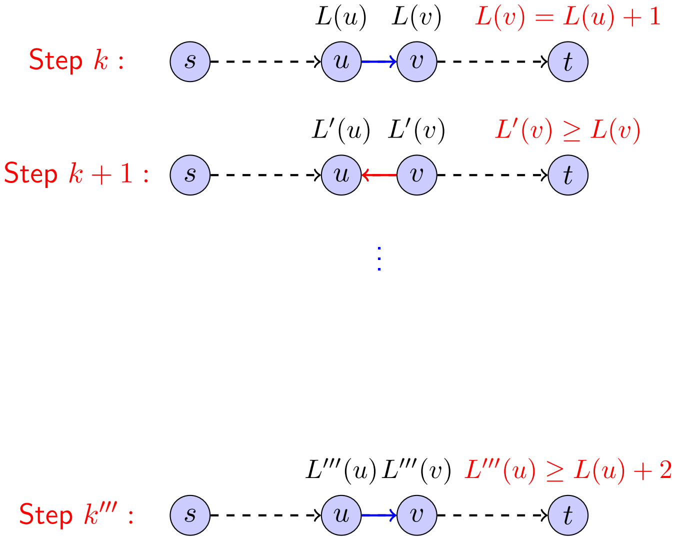
$L(u)$ $L(v)$ $L(v) = L(u) + 1$

Step k : $s \dashrightarrow u \rightarrow v \dashrightarrow t$

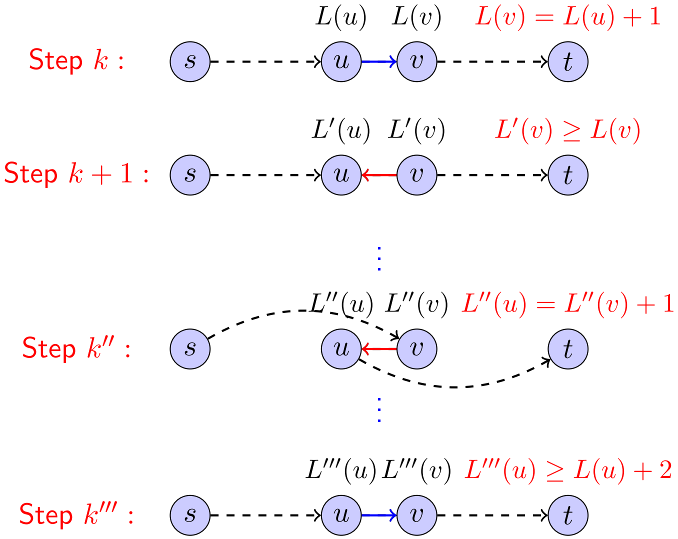
$L'''(u)$ $L'''(v)$ $L'''(u) \geq L(u) + 2$

Step k''' : $s \dashrightarrow u \rightarrow v \dashrightarrow t$

Analysis of the Edmonds-Karp algorithm



Analysis of the Edmonds-Karp algorithm



Node's level never decreases I

Lemma

For any node v, its shortest-path distance $L_f(v)$ in residual graph G_f never decreases during the execution of Edmonds-Karp algorithm.

Node's level never decreases II

Proof.

- Consider a flow f, and the Edmonds-Karp algorithm selects the shortest-path p from s to t in the residual graph G_f for augmentation, forming a new flow f'.
- Assume for contradiction that there is a node v such that $L_{f'}(v) < L_f(v)$
- Without loss of generality, let v be such a node with the minimum $L_{f'}(v)$ in $G_{f'}$. Let $p' = s \rightsquigarrow u \to v$ be the shortest-path to v in G'_f. Thus we have:
  - $L_{f'}(v) = L_{f'}(u) + 1$
  - $L_{f'}(u) \geq L_f(u)$ due to the selection of v.
- Let's investigate whether $u \to v$ appears in G_f. There are two cases:
  - $u \to v \in G_f$
  - $u \to v \notin G_f$ but $\in G_{f'}$

  and in both cases, a contradiction occurs. Thus the lemma follows.

In both cases, a contradiction occurs

- $u \to v \in G_f$:
  - We have $L_f(u) + 1 \geq L_f(v)$.
  - Thus $L_{f'}(v) = L_{f'}(u) + 1 \geq L_f(u) + 1 \geq L_f(v)$. A contradiction with the assumption.
- $u \to v \notin G_f$ but $\in G_{f'}$:
  - This can happen only if $v \to u$ exists in p, one of the shortest paths from s to t in G_f.
  - Thus, we have $L_f(u) = L_f(v) + 1$.
  - Thus another contradiction occurs: $L_{f'}(v) = L_{f'}(u) + 1 \geq L_f(u) + 1 = L_f(v) + 2$.

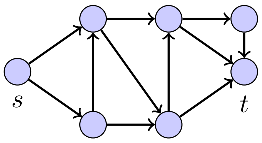
Implementation 3: Dinitz' algorithm and its variant Dinic's algorithm

Figure: Yefim Dinitz

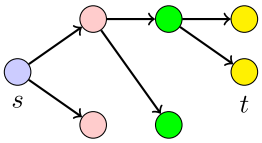
The original Dinitz' algorithm

Motivations:

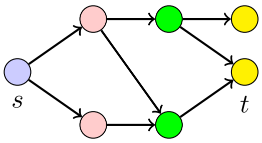
- The initial intention was just to accelerate FORD-FULKERSON algorithm by means of a smart data structure;
- Notice that finding an augmentation path takes $O(m)$ time and becomes a bottleneck of FORD-FULKERSON algorithm;
- It is valuable to save **all information** achieved at a BFS search for subsequent iterations;
- Specifically, the **BFS tree** is enriched to **layered network**:
  - BFS tree: includes **only the first edge found to a node v;**
  - Layered network: keeping **all the edges residing on all the shortest $s-v$ path**. Note that layered network has an advantage to record all shortest $s-t$ paths.



Residual graph G_f



BFS tree



Layered network G_L

Dinic's algorithm: layered network + blocking flow

- Shimon Even and Alon Itai understood the paper by Y. Dinitz and that by A. Karzanov except for the layered network maintenance (removing the "dead-end" nodes). The gaps were spanned by using:
  1. **blocking flow** (first proposed by A. Karzanov) to prove that the levels of layered network increases from phase to phase;
  2. **DFS** to search an augmentation path guided by the layered network.
- Note: when running on bi-partite graph, the Dinic's algorithm turns into the Hopcroft-Karp algorithm.

DINIC'S algorithm I

```
Initialize f(e) = 0 for all e.
while TRUE do
    Construct layered network G_L from residual graph G_f
    using BFS;
    if dist(s, t) = ∞ then
        break;
    end if
    find a blocking flow f' in G_L using DFS technique guided
    by the layered network;
    augment flow f by f';
end while
return f;
```

- The execution of the algorithm can be divided into **phases**, each phase consisting of construction of layered network, and finding blocking flow in it.

DINIC's algorithm II

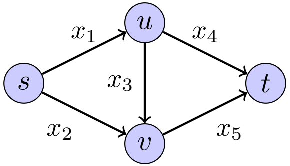
- Here, a **blocking flow** refers to a flow such that in the corresponding residual graph, there is no $s-t$ path.
- Intuition: after acquiring a layered network using $O(m)$ time, a blocking flow (containing **a collection of** $s-t$ paths) is found for further augmentation. In contrast, EDMONDS-KARP algorithm augment **only one** $s-t$ path.

(a demo here)

Analysis

- Total time: $O(mn^2)$
  - $\#\texttt{WHILE} = O(n)$. (why? Each phase increases the level of t by at least 1. Similar argument to the analysis of Edmonds-Karp algorithm.)
  - At each step, it takes $O(m)$ time to construct layered network using extended BFS;
  - and it takes $O(mn)$ time to find blocking flow. The reason is:
    1. it takes $O(n)$ time to find a $s - t$ path using DFS in a layered network;
    2. at least one bottleneck edge in the path will be saturated;
    3. thus it needs at most m iterations to find a blocking flow;

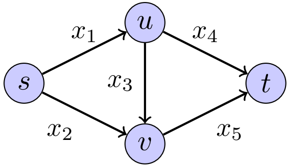
Understanding network-flow from the dual point of view



DUAL: set variables for edges (Intuition: x_i denotes $flow$ via edge i)

$$
\begin{array}{rrrrrrrl}
\max & & & & & & f & \\
s.t. & x_1 & +x_2 & & & & -f & = 0 \text{ vertex } s \\
 & & & & -x_4 & -x_5 & +f & = 0 \text{ vertex } t \\
 & -x_1 & & +x_3 & +x_4 & & & = 0 \text{ vertex } u \\
 & & -x_2 & -x_3 & & +x_5 & & = 0 \text{ vertex } v \\
 & x_1 & & & & & & \leq C_1 \\
 & & x_2 & & & & & \leq C_2 \\
 & & & x_3 & & & & \leq C_3 \\
 & & & & x_4 & & & \leq C_4 \\
 & & & & & x_5 & & \leq C_5 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & & \geq 0
\end{array}
$$

An equivalent version



$$
\begin{array}{llllllll}
\max & & & & & & f & \\
s.t. & x_1 & +x_2 & & & & -f & \leq 0 \text{ vertex } s \\
 & & & & -x_4 & -x_5 & +f & \leq 0 \text{ vertex } t \\
 & -x_1 & & +x_3 & +x_4 & & & \leq 0 \text{ vertex } u \\
 & & -x_2 & -x_3 & & +x_5 & & \leq 0 \text{ vertex } v \\
 & x_1 & & & & & & \leq C_1 \\
 & & x_2 & & & & & \leq C_2 \\
 & & & x_3 & & & & \leq C_3 \\
 & & & & x_4 & & & \leq C_4 \\
 & & & & & x_5 & & \leq C_5 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & & \geq 0
\end{array}
$$

Note: the constraints (1), (2), (3), and (4) force $-x_2 - x_3 + x_5 = 0$. So do other constraints.

Duality explanation of MaxFlow-MinCut: Primal problem

PRIMAL: set variables for nodes.

$$
\begin{array}{lccccccccccl}
\min & & & & & C_1z_1 & +C_2z_2 & +C_3z_3 & +C_4z_4 & +C_5z_5 & \\
s.t. & y_s & & -y_u & & +z_1 & & & & & \geq 0 \\
 & y_s & & & -y_v & & +z_2 & & & & \geq 0 \\
 & & & y_u & -y_v & & & +z_3 & & & \geq 0 \\
 & & -y_t & +y_u & & & & & +z_4 & & \geq 0 \\
 & & -y_t & & +y_v & & & & & +z_5 & \geq 0 \\
 & -y_s & +y_t & & & & & & & & \geq 1 \\
 & y_s, & y_t, & y_u, & y_v, & z_1, & z_2, & z_3, & z_4, & z_5 & \geq 0
\end{array}
$$

Note:

1. Since the constraints involves the difference among y_s, y_u, y_v and y_t, one of them can be fixed without effects. Here, we fix $y_s = 0$. Thus we have $y_t \geq 1$ (by the constraint $-y_s + y_t \geq 1$).
2. Constraint (4) requires $z_4 \geq y_t - y_u$, and the objective is to minimize a function containing C_4z_4, forcing $y_t = 1$.
3. Constraint (1) requires $z_1 \geq y_u$, and the objective is to minimize a function containing C_1z_1, forcing $z_1 = y_u$. So does constraint (2).

An equivalent version

PRIMAL: set variables for nodes.

$$
\begin{array}{lccccccccccl}
\min & & & & & C_1 z_1 & +C_2 z_2 & +C_3 z_3 & +C_4 z_4 & +C_5 z_5 & \\
s.t. & & & -y_u & & +z_1 & & & & & = 0 \\
 & & & & -y_v & & +z_2 & & & & = 0 \\
 & & & y_u & -y_v & & & +z_3 & & & \geq 0 \\
 & & & y_u & & & & & +z_4 & & \geq 1 \\
 & & & & y_v & & & & & +z_5 & \geq 1 \\
 & y_s & & & & & & & & & = 0 \\
 & & y_t & & & & & & & & = 1 \\
 & & & y_u, & y_v, & z_1, & z_2, & z_3, & z_4, & z_5 & \geq 0
\end{array}
$$

Note: the coefficient matrix of constraints (3), (4) and (5) is totally uni-modular, implying the optimal solution is an integer solution.

An equivalent version

PRIMAL: set variables for nodes.

$$
\begin{array}{lccccccccccr}
\min & & & & & C_1 z_1 & +C_2 z_2 & +C_3 z_3 & +C_4 z_4 & +C_5 z_5 & \\
s.t. & & & -y_u & & +z_1 & & & & & = 0 \\
 & & & & -y_v & & +z_2 & & & & = 0 \\
 & & & y_u & -y_v & & & +z_3 & & & \geq 0 \\
 & & & y_u & & & & & +z_4 & & \geq 1 \\
 & & & & y_v & & & & & +z_5 & \geq 1 \\
 & y_s & & & & & & & & & = 0 \\
 & & y_t & & & & & & & & = 1 \\
 & & & y_u, & y_v, & z_1, & z_2, & z_3, & z_4, & z_5 & = 0/1
\end{array}
$$

MaxFlow-MinCut: strong duality

$$
\begin{array}{lllllllllll}
\min & & & & & C_1 z_1 & +C_2 z_2 & +C_3 z_3 & +C_4 z_4 & +C_5 z_5 & \\
s.t. & & & -y_u & & +z_1 & & & & & = 0 \\
 & & & & -y_v & & +z_2 & & & & = 0 \\
 & & & y_u & -y_v & & & +z_3 & & & \geq 0 \\
 & & & y_u & & & & & +z_4 & & \geq 1 \\
 & & & & y_v & & & & & +z_5 & \geq 1 \\
 & y_s & & & & & & & & & = 0 \\
 & & y_t & & & & & & & & = 1 \\
 & & & y_u, & y_v, & z_1, & z_2, & z_3, & z_4, & z_5 & = 0/1
\end{array}
$$

Observations:

- Intuition of primal variables: if node i is in A, $y_i = 0$; and $y_i = 1$ otherwise.
- The primal problem is essentially to find a cut. (Note: $z_1 = 1$ iff $y_s = 0$ and $y_u = 1$, i.e., edge (s, u) is a cut edge.)
- By weak duality, we have $f \leq c$. This is exactly the MaximumFlow-MinimumCut theorem.

FORD-FULKERSON algorithm is essentially a primal-dual algorithm

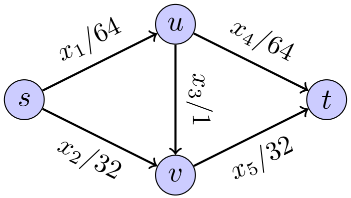
Primal-dual algorithm

- Recall that the generic primal-dual algorithm can be described as follows.

```
Initialize x as a dual feasible solution;
while TRUE do
    Construct DRP corresponding to x;
    Let ω_opt be the optimal solution to DRP;
    if ω_opt = 0 then
        return x;
    else
        Improve x according to the optimal solution to DRP;
    end if
end while
```

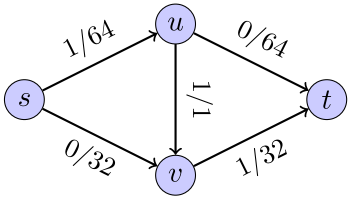
- We will show that solving the DRP is equivalent to finding an augmentation path in residual graph.

Dual problem and DRP I



- DUAL D: set variables for edges;

$$
\begin{array}{lcccccccl}
\max & & & & & & f & & \\
s.t. & x_1 & +x_2 & & & & -f & \leq 0 & \text{vertex } s \\
 & & & & -x_4 & -x_5 & +f & \leq 0 & \text{vertex } t \\
 & -x_1 & & +x_3 & +x_4 & & & \leq 0 & \text{vertex } u \\
 & & -x_2 & -x_3 & & +x_5 & & \leq 0 & \text{vertex } v \\
 & x_1 & & & & & & \leq 64 & \\
 & & x_2 & & & & & \leq 32 & \\
 & & & x_3 & & & & \leq 1 & \\
 & & & & x_4 & & & \leq 64 & \\
 & & & & & x_5 & & \leq 32 & \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & & \geq 0 &
\end{array}
$$



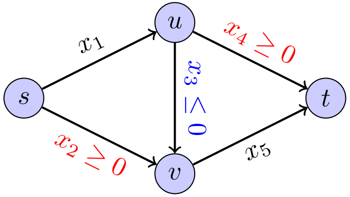
- Let's consider a dual feasible solution $\mathbf{x} = (1, 0, 1, 0, 1)$. Recall how to write DRP from D:
  - Replacing the right-hand side C_i with 0;
  - Adding constraints: $x_i \leq 1$, $f \leq 1$;
  - Keep only the tight constraints J. Here we category J into two sets, i.e. $J = J^S \cup J^E$, where the saturated arcs $J^S = \{i | x_i = C_i\}$, and the empty arcs $J^E = \{i | x_i = 0\}$. Here, $J_S = \{3\}$, and $J_E = \{2, 4\}$.

DRP corresponds to finding an augmentation path

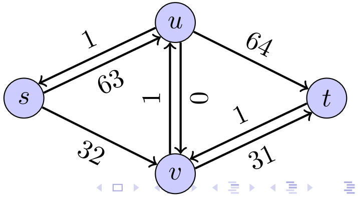
- DRP:

$$
\begin{array}{llllllll}
\max & & & & & & f & \\
s.t. & x_1 & +x_2 & & & & -f & = 0 \text{ vertex } s \\
 & & & & -x_4 & -x_5 & +f & = 0 \text{ vertex } t \\
 & -x_1 & & +x_3 & +x_4 & & & = 0 \text{ vertex } u \\
 & & -x_2 & -x_3 & & +x_5 & & = 0 \text{ vertex } v \\
 & & & x_i & & & & {\color{blue}\leq} 0 \quad i \in J^S \\
 & & & x_j & & & & {\color{red}\geq} 0 \quad j \in J^E \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & f & \leq 1
\end{array}
$$

  - $\omega_{OPT} = 0$ implies that optimal solution is found.
  - $\omega_{OPT} = 1$ implies a $s - t$ path (with unit flow) in residual graph G_f.

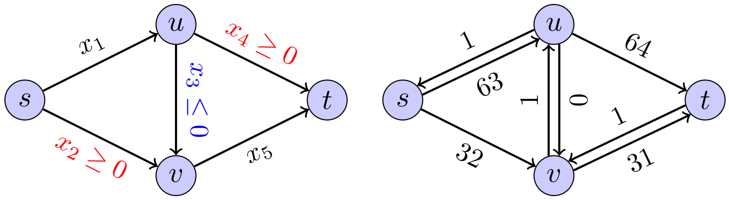


DRP



Residual graph G_f

DRP and augmentation path in residual graph



DRP Residual graph G_f

- Note that DRP corresponds to finding an augmentation path in the residual graph G_f.
  - $x_i \leq 0, i \in J^S$ denotes a backward edge,
  - $x_j \geq 0, j \in J^E$ denotes a forward edge,
  - and for other edges, there is no restriction for x_i.
- Thus FORD-FULKERSON algorithm is essentially a primal_dual algorithm.

Push-relabel algorithm [A. V. Goldberg, R. E. Tarjan, 1986]

The push-relabel algorithm is one of the most efficient algorithms to compute a maximum flow. The general algorithm has $O(n^2m)$ time complexity, while the implementation with FIFO vertex selection rule has $O(n^3)$ running time, the highest active vertex selection rule provides $O(n^2\sqrt{m})$ complexity, and the implementation with Sleator's and Tarjan's dynamic tree data structure runs in $O(nmlog(n^2/m))$ time. In most cases it is more efficient than the Edmonds-Karp algorithm, which runs in $O(nm^2)$ time.

Push-relabel algorithm

- Basic idea: the optimal solution f should meet two constraints simultaneously, namely, f is a flow, and there is no $s-t$ path in the residual graph G_f. FORD-FULKERSON algorithms maintains the first constraint while PUSH-RELABEL maintains the second constraint.
  1. Ford-Fulkerson: set variables for edges. Update flow on edges until G_f has no $s-t$ path;
  2. Push-relabel: set variables for nodes. Update a pre-flow f, maintaining the property that G_f has no $s-t$ path, until f is a flow.

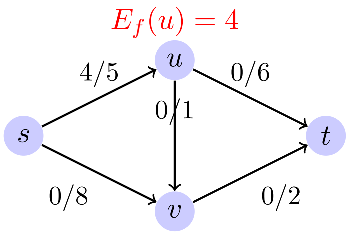
Pre-flow: a relaxation of flow

Definition (Pre-flow)

f is a pre-flow if

- (Capacity condition): $f(e) \leq C(e)$;
- (Excess condition): for any node $v \neq s$,
  $E_f(v) = \sum_{\text{e into v}} f(e) - \sum_{\text{e out of v}} f(e) \geq 0$;

- Note that a pre-flow f becomes a flow if no intermediate node has excess.



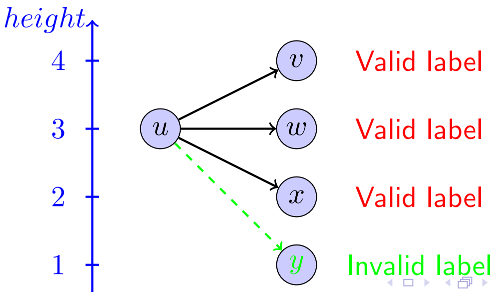
Label of nodes

Definition (Valid label)

Consider a pre-flow f. A **valid labeling** of nodes is:

- $h(s) = n$, and $h(t) = 0$;
- For each edge (u, v) in the residual graph G_f, we have $h(v) \geq h(u) - 1$;

(Intuition: $h(v)$ is height of the node v, and for an edge in G_f, its end cannot be too lower than its head.)



Valid labeling means no $s-t$ path in G_f

Theorem

There is no $s-t$ path in a residual graph G_f if there exist valid labels.

Proof.

- Suppose there is a $s-t$ path in G_f.
- Notice that $s-t$ path contains at most $n-1$ edges.
- Since $h(s)=n$ and $h(u) \leq h(v)+1$, the height of t should be great than 0. A contradiction with $h(t)=0$.

$\square$

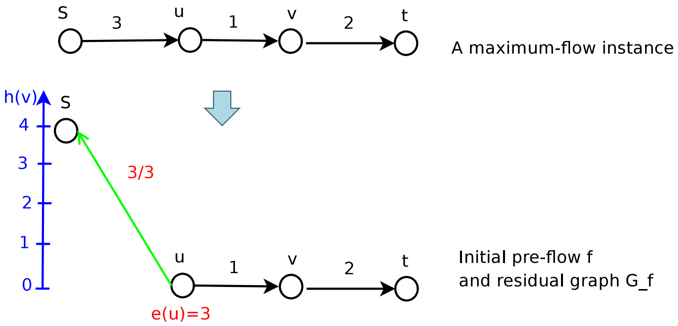
Push-relabel algorithm: basic idea

```
Set f as a pre-flow;
Set valid labels for nodes;
while TRUE do
    if no node has excess then
        return f;
    end if
    Select a node v with excess;
    if v has a neighbor w such that h(v) > h(w) then
        Push some excess from v to w;
    else
        Perform relabeling to increase h(v);
    end if
end while
```

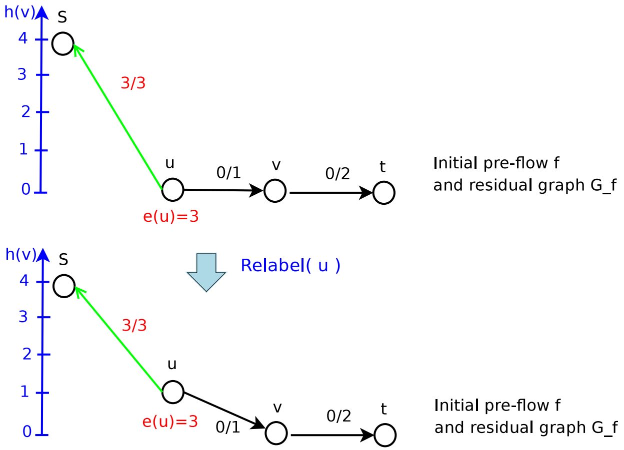
Push-relabel algorithm

$h(s) = n$; $h(v) = 0$; for any $v \neq s$;
$f(e) = C(e)$ for all $e = (s, u)$; $f(e) = 0$; for other edges;
while there exists a node v with $E_f(v) > 0$ **do**
  if there exists an edge $(v, w) \in G_f$ s.t. $h(v) > h(w)$ **then**
    //Push excess from v to w;
    if (v, w) is a forward edge **then**
      $e = (v, w)$;
      $bottleneck = \min\{E_f(v), C(e) - f(e)\}$;
      $f(e) += bottleneck$;
    else
      $e = (w, v)$;
      $bottleneck = \min\{E_f(v), f(e)\}$;
      $f(e) -= bottleneck$;
    end if
  else
    $h(v) = h(v) + 1$; //Relabel node v;
  end if
end while

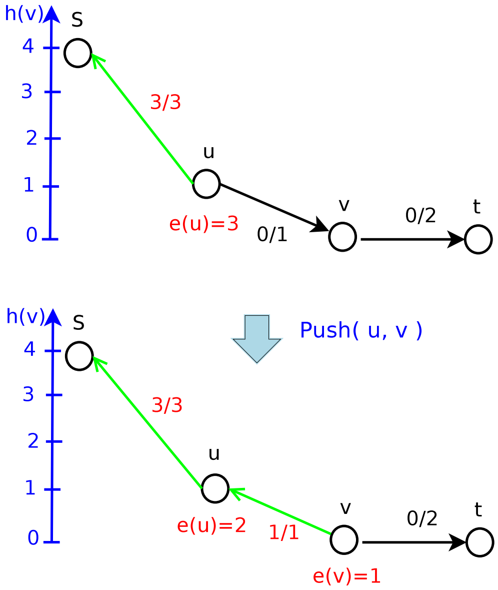
A demo of push-relabel algo: initialization



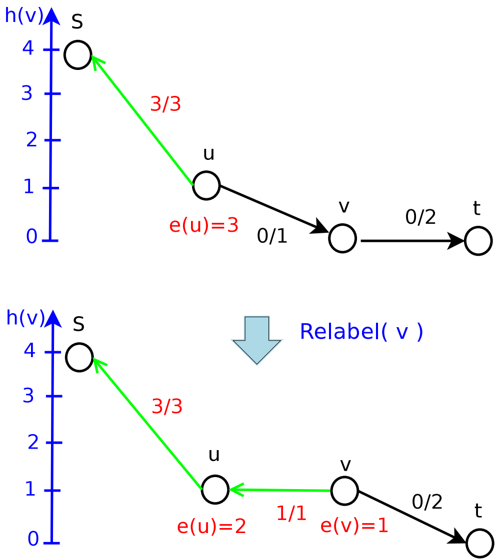
A demo of push-relabel algo: Step 1



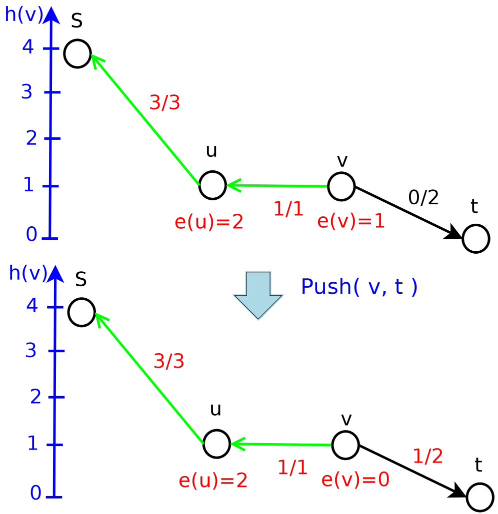
A demo of push-relabel algo: Step 2



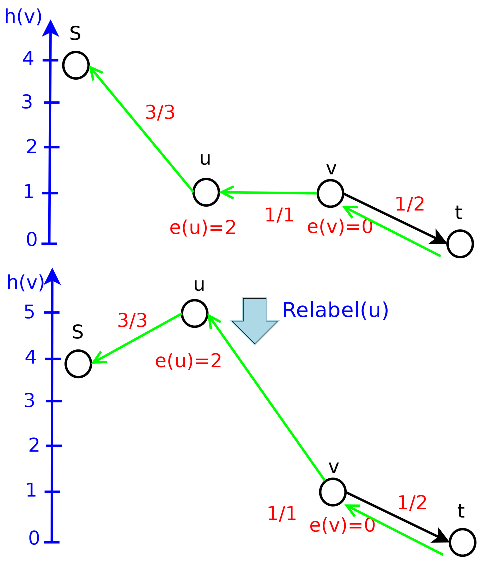
A demo of push-relabel algo: Step 3

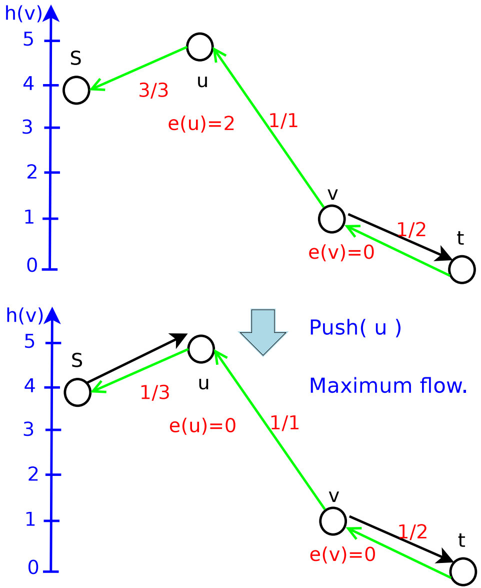


A demo of push-relabel algo: Step 4



A demo of push-relabel algo: Step 5




h(v)
5
4
3
2
1
0
s
u
3/3
e(u)=2
1/1
v
e(v)=0
1/2
t
h(v)
5
4
3
2
1
0
s
u
1/3
e(u)=0
1/1
v
e(v)=0
1/2
t
Push(u)
Maximum flow.

Correctness I

Theorem

Push-relabel algo keeps label valid, and thus outputs a maximum flow when ends.

Proof.

(Induction on the number of push and relabel operations.)

- Push operation: the new f is still a pre-flow since the capacity condition still holds.
  $Push(f, v, w)$ may add edge (w, v) into G_f. We have $h(w) < h(v)$. (pre-condition). Thus, the label is valid for the new G_f.
- Relabel operation: The pre-condition implies $h(v) \leq h(w)$ for any $(v, w) \in G_f$. $relabel(f, h, v)$ changes $h(v) = h(v) + 1$. Thus, the new $h(v) \leq h(w) + 1$.

$\square$

Time-complexity: $\#Relabel$ I

Theorem

For any node v, $\#Relabel \leq 2n - 1$. Thus, the total label operation number is less than $2n^2$.

Proof.

1. (Connectivity): For a node w with $E_f(w) > 0$, there should be a path from w to s in G_f.
   (Intuition: node w obtain a positive $E_f(w)$ through a node v by $Push(f, v, w)$. This operation also causes edge (w, v) to be added into G_f. Thus, there should be a path from w to s.
)
2. (Upper bound of $h(v)$): $h(v) < 2n - 1$ since there is a path from v to s. The length of the path is less than $n - 1$, $h(s) = n$, and $h(v) \leq h(w) + 1$ for any edge (v, w) in G_f.

□

Two types of $Push$ operations:

1. Saturated push (s-push): if $Push(f, v, w)$ causes (v, w) removed from G_f.
2. Unsaturated push (uns-push): other pushes.

$\#Push = \#\text{s-push} + \#\text{uns-push}.$

Theorem

$\#\text{s-push} \leq 2nm.$

Proof.

Consider an edge $e = (v, w)$. We will show that during the execution of algo, (v, w) appears in G_f at most $2n$ times.

- (Removing): a saturated $Push(f, v, w)$ removes (v, w) from G_f. We have $h(v) = h(w) + 1$.
- (Adding): Before applying $Push(f, v, w)$ again, (v, w) should be added to G_f first. The only way to add (v, w) to G_f is $Push(f, w, v)$. The pre-condition of $Push(f, w, v)$ requires that $h(w) \geq h(v) + 1$, i.e., $h(w)$ should be increased at least 2 since the previous $Push(f, v, w)$ operation. And we have $h(w) \leq 2n - 1$.

□

Theorem

$\#uns\text{-}push \leq 2n^2m.$

Time-complexity: $\#Push$ II

Proof.

Define a measure $\Phi(f,h) = \sum_{v:E_f(v)>0} h(v)$.

- (Increase and upper bound) $\Phi(f,h) < 4n^2m$:
  1. Relabel: a relabel operation increase $\Phi(f,h)$ by 1. The total $O(2n^2)$ relabel operations increase $\Phi(f,h)$ at most $O(2n^2)$.
  2. Saturized push: A saturated $Push(f,v,w)$ operation increases $\Phi(f,h)$ by $h(w)$ since w has excess now. $h(w) \leq 2n-1$ implies an upper bound for each operation. The total $2nm$ saturated pushes increase $\Phi(f,h)$ by at most $4n^2m$.
- (Decrease) An unsaturated $Push(f,v,w)$ will reduce $\Phi(f,h)$ at least 1.
  (Intuition: after unsaturated $Push(f,v,w)$, we have $E_f(v)=0$, which reduce $h(v)$ from $\Phi(f,h)$; on the other side, w obtains excess from v, which will increase $\Phi(f,h)$ by $h(w)$. From $h(v) \leq h(w)+1$, we have that $\Phi(f,h)$ reduces at least 1.)

$\square$