Assignment4

T1：

1 Linear-inequality feasibility

Given a set of m linear inequalities on n variables $x_1, x_2, \cdots, x_n$, the **linear-inequality feasibility problem** asks if there is a setting of the variables that simultaneously satisfies each of the inequalities.

Show that if we have an algorithm for linear programming, we can use it to solve the linear-inequality feasibility problem. The number of variables and constraints that you use in the linear-programming problem should be polynomial in n and m.

Solution:

本题是指现在有一些不等式约束，还有一个求解线性规划的算法，问能否使用这个算法去判断给定的不等式约束是否有可行解。所以只要给不等式约束加一个目标函数就可以了，比如加 Max 0，或者 Min 0，它的约束还是原来的不等式约束。然后利用线性规划的算法就能知道是否有可行解了。

T2：

With human lives at stake, an air traffic controller has to schedule the airplanes that are landing at an airport in order to avoid airplane collision. Each airplane i has a time window $[s_i, t_i]$ during which it can safely land. You must compute the exact time of landing for each airplane that respects these time windows. Furthermore, the airplane landings should be stretched out as much as possible so that the minimum time gap between successive landings is as large as possible.

For example, if the time window of landing three airplanes are [10:00-11:00], [11:20-11:40], [12:00-12:20], and they land at 10:00, 11:20, 12:20 respectively, then the smallest gap is 60 minutes, which occurs between the last two airplanes.

Given n time windows, denoted as $[s_1, t_1], [s_2, t_2], \cdots, [s_n, t_n]$ satisfying $s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n$, you are required to give the exact landing time of each airplane, in which the smallest gap between successive landings is maximized.

Please formulate this problem as an LP, construct an instance and use GLPK or Gurobi or other similar tools to solve it.

Solution：

设共有 n 架飞机，第 i 架飞机可降落的时间段为$[s_i,t_i]$，设第 i 架飞机降落时间点为 x_i,同时记 d 为任意连续两架飞机降落时间间隔中最短的一个，那么根据题意，可以列出如下线性规划：

$$\begin{aligned} & Max \quad d \\ & s.t. \quad s_1 < t_1 < s_2 < \cdots < s_n < t_n \\ & \qquad s_i \le x_i \le t_i \quad i = 1,2\cdots n \\ & \qquad x_i - x_{i-1} \ge d \quad i = 2,3\cdots n \\ & \qquad x_i, s_i, t_i, d \ge 0 \quad i = 1,2\cdots n \end{aligned}$$

举例计算：

例，有 5 架飞机，它们可降落时段分别是：[7:00,7:30], [7:50,8:30], [9:00,9:20], [9:40,10:00], [10:50,11:20],为了方便表示，不妨将 s_1 记为 0，则区间变为：[0,30], [50,90], [120,140], [160,180], [230,260].则根据上述线性规划条件，有：

maxmize z=d subject to

x1 >=0 　　x1 <=30
x2 >=50 　　x2 <=90
x3 >=120 　　x3 <=140
x4 >=160 　　x4 <=180
x5 >=230 　　x5 <=260
x2-x1-d >=0 　　x3-x2-d >=0
x4-x3-d >=0 　　x5-x4-d >=0

解之得：d = 60, x1 = 0, x2 = 60, x3 = 120, x4 = 180, x5 = 240.

即，各飞机的降落时间分别是 7:00,8:00,9:00,10:00,11:00。

T3

A teaching building has m classrooms in total, and n courses are trying to use them. Each course i $(i = 1, 2, \cdots, n)$ only uses one classroom during time interval $[S_i, F_i)$ $(F_i > S_i > 0)$. Considering any two courses can not be carried on in a same classroom at any time, you have to select as many courses as possible and arrange them without any time collision. For simplicity, suppose $2n$ elements in the set $\{S_1, F_1, \cdots, S_n, F_n\}$ are all different.

1. Please use ILP to solve this problem, then construct an instance and use GLPK or Gurobi or other similar tools to solve it.

2. If you relax the integral constraints and change ILP to an LP (e.g. change $x \in \{0, 1\}$ to $0 \leqslant x \leqslant 1$), will solution of the LP contains only integers, regardless of values of all S_i and F_i? If it's true, prove it; if it's false, give a counter example. You can use the following lemma for help.

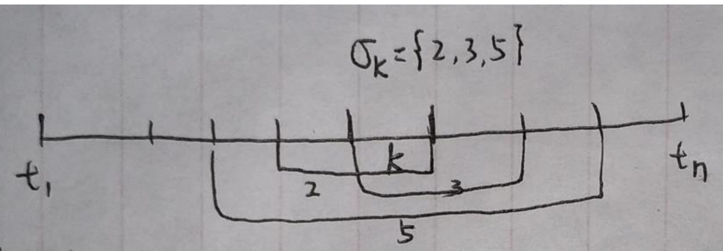
   LEMMA If matrix A has only 0, +1 or -1 entries, and each column of A has at most one +1 entry and at most one -1 entry. In addtion, the vector b has only integral entries. Then the vertex of polytope $\{x | Ax \leqslant b, x \geqslant 0\}$ contains only integral entries.

Solution

没有人发这一题，简单写一下思路：

把 Si 和 Fi 全都放在一个数组里，记为 T，T 中一共有 2n 个元素，将 T 中的元素

按从小到大进行排列，每个元素记为为 ti,每个 ti 都是一个时刻，所以从 t1 到 t2n，一共有 2n 个时刻，把这段时间分割为 2n-1 个小的时间段。



σ_k表示第k个时间段与哪几节课有关，即哪些课要占用这段时间。
这里假设第2,3,5节课都占用了这个时间段。$\sigma_k=\{2,3,5\}$

$$x_i = \begin{cases} 0 & \text{第}i\text{节课不安排} \\ 1 & \cdots\cdots\text{安排} \end{cases}$$

若$m=2$，则 $x_2+x_3+x_5 \le 2$

对于每个时间段k，都要满足

$$\begin{cases} \sum\limits_{\sigma_k} x_i \le m & k=1,2\cdots 2n-1 \\ x_i=\{0,1\} \end{cases}$$

$$\max \sum_i x_i$$

第二问，不清楚。

T4：

Let's consider a long, quiet country road with towns scattered very sparsely along it. Sinopec, largest oil refiner in China, wants to place gas stations along the road. Each gas station is assigned to a nearby town, and the distance between any two gas stations being as small as possible. Suppose there are n towns with distances from one endpoint of the road being $d_1, d_2, \cdots, d_n$. n gas stations are to be placed along the road, one station for one town. Besides, each station is at most r far away from its correspond town. $d_1, \cdots, d_n$ and r have been given and satisfied $d_1 < d_2 < \cdots < d_n$, $0 < r < d_1$ and $d_i + r < d_{i+1} - r$ for all i. The objective is to find the optimal placement such that the maximal distance between two successive gas stations is minimized.

Please formulate this problem as an LP.

Solution:

一共有 n 个小镇，n 个加油站，每个小镇的位置是 d_i，设每个加油站的位置是 x_i，r 是每个加油站离对应的小镇的所允许的最大距离，设 dz 为任意连续两个

加油站的距离中最大的距离，根据题意，我们可得下面线性规划：

$$
\begin{aligned}
&Min \quad dz \\
&s.t. \quad d_1 < d_2 < \cdots < d_n \\
&\qquad d_i - r \le x_i \le d_i + r \quad i = 1, 2 \cdots n \\
&\qquad x_i - x_{i-1} \le dz \qquad i = 2, 3 \cdots n \\
&\qquad x_i, d_i, dz \ge 0 \quad i = 1, 2 \cdots n
\end{aligned}
$$

T5:

n men $(m_1, m_2, \cdots, m_n)$ and n women $(w_1, w_2, \cdots, w_n)$, where each person has ranked all members of the opposite gender, have to make pairs. You need to give a stable matching of the men and women such that there is no unstable pair. Please choose one of the two following known conditions, formulate the problem as an ILP (*hint*: Problem 1.1 in this assignment), construct an instance and use GLPK or Gurobi or other similar tools to solve it.

1. You have known that for every two possible pairs (man m_i and woman w_j, man m_k and woman w_l), whether they are stable or not. If they are stable, then $S_{i,j,k,l} = 1$; if not, $S_{i,j,k,l} = 0$. $(i, j, k, l \in \{1, 2, \cdots, n\})$
2. You have known that for every man m_i, whether m_i likes woman w_j more than w_k. If he does, then $p_{i,j,k} = 1$; if not, $p_{i,j,k} = 0$. Similarly, if woman w_i likes man m_j more than m_k, then $q_{i,j,k} = 1$, else $q_{i,j,k} = 0$. $(i, j, k \in \{1, 2, \cdots, n\})$

Solution：

Suppose $x_{ij} = \begin{cases} 1, & \text{man } i \text{ and woman } j \text{ get married} \\ 0, & \text{otherwise} \end{cases} \quad (i, j = 1, 2, \cdots, n).$

1. The ILP is as follows:

$$
\begin{array}{llcll}
\min & & & 0 & \\
\text{s.t.} & \sum_{i=1}^{n} x_{ij} & = & 1 & \text{for all } j = 1, 2, \cdots, n \\
& \sum_{j=1}^{n} x_{ij} & = & 1 & \text{for all } i = 1, 2, \cdots, n \\
& x_{ij} + x_{kl} & \leqslant & S_{i,j,k,l} + 1 & \text{for all } i, j, k, l = 1, 2, \cdots, n, i \neq k, j \neq l \\
& x_{ij} & \in & \{0, 1\} & \text{for all } i, j = 1, 2, \cdots, n
\end{array}
$$

The third constraint can be replaced by $x_{ij} + (1 - S_{i,j,k,l}) x_{kl} \leqslant 1$.

2. If m_l likes w_k more than w_j, and w_k likes m_l more than m_i, then m_i and w_j will never become the wrecker if m_l and w_k get married (but we are not sure whether m_l and w_k will get married since other strong wreckers might exist). In other words, if $p_{l,k,j} = 1$ and $q_{k,l,i} = 1$, then $x_{ik} = 0$ and $x_{lj} = 0$. Based on this, the ILP is as follows:

$$
\begin{array}{lrcll}
\min & & 0 & & \\
\text{s.t.} & \sum_{i=1}^{n} x_{ij} & = & 1 & \text{for all } j = 1, 2, \cdots, n \\
 & \sum_{j=1}^{n} x_{ij} & = & 1 & \text{for all } i = 1, 2, \cdots, n \\
 & x_{ik} + x_{lj} & \leqslant & 3 - p_{l,k,j} - q_{k,l,i} & \text{for all } i, j, k, l = 1, 2, \cdots, n, k \neq j, l \neq i \\
 & x_{ij} & \in & \{0, 1\} & \text{for all } i, j = 1, 2, \cdots, n
\end{array}
$$

The third and fourth constraints can be replaced by $x_{ij} + x_{kl} \leqslant 2 - p_{ilj} q_{jki}$.

注：这里之所以约束 3 中为 3，是 p 和 q 同时满足时，<=1，可以有(1,0)(0,0)(0,1)出现，因为不排除某个 w 或 m 与四个人之外有极强的联系，这样与之建立关系之后，相互喜欢的 w 和 m 剩下的那个可以与单恋的异性建立稳定关系。

T6.

Please write the dual problem of the MultiCommodityFlow problem in *Lec8.pdf*, and give an explanation of the dual variables.

Please also construct an instance, and try to solve both primal and dual problem using GLPK or Gurobi or other similar tools.

For simplicity, we can assume that (u, v) denotes the arc $u \to v$. Then the primal can be rewritten and corrected as:

$$
\begin{array}{lrcll}
\max / \min & 0 & & & \\
\text{s.t.} & \sum_{i=1}^{k} f_i(u, v) & \leqslant & c(u, v) & \text{for each } (u, v) \\
 & \sum_{v,(u,v)\in E} f_i(u, v) - \sum_{v,(v,u)\in E} f_i(v, u) & = & 0 & \text{for each } i \text{ and } u \in V \setminus \{s_i, t_i\} \\
 & \sum_{v,(s_i,v)\in E} f_i(s_i, v) - \sum_{v,(v,s_i)\in E} f_i(v, s_i) & = & d_i & \text{for each } i \\
 & f_i(u, v) & \geqslant & 0 & \text{for each } i, (u, v)
\end{array}
$$

If we use x_{uv} to denote the first constraints, y_{iu} the second and third constraints, then the duality is:

$$
\begin{array}{lrcll}
\min & c(u, v) x_{uv} + d_i y_{is_i} & & & \\
\text{s.t.} & x_{uv} + y_{iu} - y_{iv} & \geqslant & 0 & \text{for all } i \text{ and } u \neq t_i, v \neq t_i \\
 & x_{ut_i} + y_{iu} & \geqslant & 0 & \text{for all } i, (u, t_i) \\
 & x_{t_i v} - y_{iv} & \geqslant & 0 & \text{for all } i, (t_i, v) \\
 & x_{uv} & \geqslant & 0 & \text{for all } (u, v)
\end{array}
$$

or

$$
\begin{array}{lrcll}
\max & c(u, v) x_{uv} + d_i y_{is_i} & & & \\
\text{s.t.} & x_{uv} + y_{iu} - y_{iv} & \leqslant & 0 & \text{for all } i \text{ and } u \neq t_i, v \neq t_i \\
 & x_{ut_i} + y_{iu} & \leqslant & 0 & \text{for all } i, (u, t_i) \\
 & x_{t_i v} - y_{iv} & \leqslant & 0 & \text{for all } i, (t_i, v) \\
 & x_{uv} & \leqslant & 0 & \text{for all } (u, v)
\end{array}
$$

注：5,6 两题答案是直接从 2015 版答案搬运过来的，因为大家发过来的全都是 2,4 两题😂，没看懂的大家在群里讨论讨论。