

CS711008Z Algorithm Design and Analysis

Lecture 10. Algorithm design technique: Network flow and its applications¹

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¹The slides are made based on Chapter 7 of *Introduction to algorithms, Combinatorial optimization algorithm and complexity* by C. H. Papadimitriou and K. Steiglitz, the classical papers by Kuhn, Edmonds, etc. in the book *50 Years of Integer Programming 1958-2008: From the Early Years to the State-of-the-Art*.

- Extensions of MAXIMUMFLOW problem: undirected network;
CIRCULATION with multiple sources & multiple sinks;
CIRCULATION with lower bound of capacity; MINIMUM
COST FLOW;
- Solving practical problems using network flow and primal-dual techniques:
 - 1 Partitioning a set: IMAGESEGMENTATION,
PROJECTSELECTION, PROTEINDOMAINPARSING;
 - 2 Finding paths: FLIGHTSCHEDULING, DISJOINT PATHS,
BASEBALL ELIMINATION;
 - 3 Decomposing numbers: BASEBALL ELIMINATION;
 - 4 Constructing matches: BIPARTITE MATCHING,
SURVEY DESIGN;
- Extensions of matching: BIPARTITE MATCHING,
WEIGHTED BIPARTITE MATCHING, GENERAL GRAPH MATCHING,
WEIGHTED GENERAL GRAPH MATCHING;
- A brief history of network flow.

Extensions of MAXIMUMFLOW problem

Four extensions of MAXIMUMFLOW problem:

- 1 MAXIMUMFLOW for undirected network;
- 2 CIRCULATION with multiple sources and multiple sinks;
- 3 CIRCULATION with lower bound for capacity;
- 4 MINIMUM COST FLOW;

Extension 1: MAXIMUM FLOW for undirected network

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INPUT:

an **undirected** network $G = \langle V, E \rangle$, each edge e has a capacity $C(e) > 0$. Two special nodes: source s and sink t ;

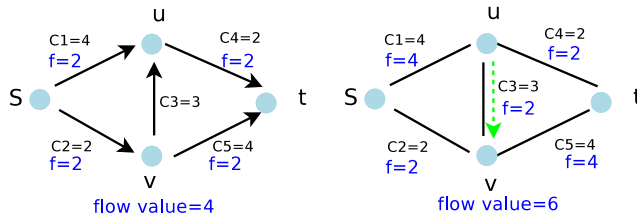
OUTPUT:

for each edge e , to assign a flow $f(e)$ to maximize the flow value $\sum_{e=(s,v)} f(e)$.

Flow properties:

- 1 (Capacity restriction): $0 \leq f(u, v) + f(v, u) \leq C(u, v)$ for any $(u, v) \in E$;
- 2 (Conservation restriction): $f^{in}(v) = f^{out}(v)$ for any node $v \in V$ except for s and t .

Example



Note: On the directed network, the maximum flow value is 4; in contrast, on the undirected network, the maximum flow value is 6.

Maximum-flow algorithm for undirected network G

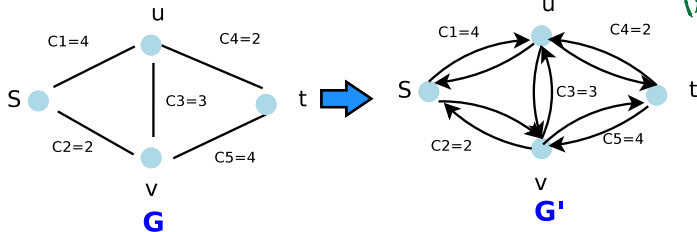
- ★ 1: Transforming the undirected network G to a directed network G' ; 将无向图化为有向图.
- 2: Calculating the maximum flow for G' by using Ford-Fulkerson algorithm;
- 3: Revising the flow to meet the capacity restrictions;

Step 1: changing undirected network to directed network

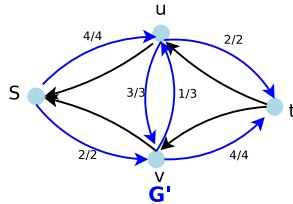
- Transformation: an undirected network G is transformed into a directed network G' through:

- 1 adding edges: for each edge (u, v) of G , introducing two edges $e = (u, v)$ and $e' = (v, u)$ to G' ;
- 2 setting capacities: setting $C(e') = C(e)$.

将一条边化为 $(u, v), (v, u)$ 两条边。

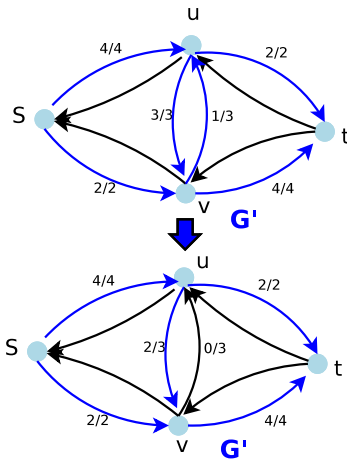


Step 2: calculating the maximum flow for G'



Note: the only trouble is the violation of capacity restriction: for edge $e = (u, v)$, $f(e) + f(e') = 4 > C(e) = 3$.

Step 3: revising flow to meet capacity restriction

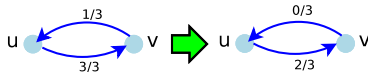


Note: for an edge violating capacity restriction, say $e = (u, v)$, the flow $f(e)$ and $f(e')$ were revised.

Correctness of revising flow

Theorem

There exists a maximum flow f for network G , where $f(u, v) = 0$ or $f(v, u) = 0$.



Proof.

- Suppose f' is a maximum flow for undirected network G' , where $f'(u, v) > 0$ and $f'(v, u) > 0$. We change f' to f as follows:
- Let $\delta = \min\{f'(u, v), f'(v, u)\}$.
- Define $f(u, v) = f'(u, v) - \delta$, and $f(v, u) = f'(v, u) - \delta$. We have $f(u, v) = 0$ or $f(v, u) = 0$.
- It is obvious that both capacity restrictions and conservation restrictions hold.
- f has the same value to f' and thus optimal.

Extension 2: CIRCULATION problem with multiple sources and multiple sinks

Extension 2: CIRCULATION problem with multiple sources and multiple sinks

INPUT:

a network $G = \langle V, E \rangle$, where each edge e has a capacity $C(e) > 0$; multi sources s_i and sinks t_j . A sink t_j has demand $d_j > 0$, while a source s_i has supply d_i (described as a negative demand $d_i < 0$).

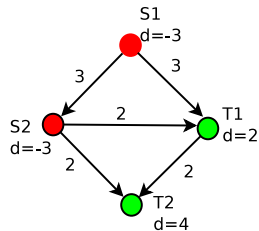
OUTPUT:

a **feasible circulation** f to satisfy all demand requirements using the available supply, i.e.,

- ① Capacity restriction: $0 \leq f(e) \leq C(e)$;
- ② Demand restriction: $f^{in}(v) - f^{out}(v) = d_v$;

Note: For the sake of simplicity, we define $d_v = 0$ for any node v except for s_i and t_j . Thus we have $\sum_i d_i = 0$, and denote $D = \sum_{d_v > 0} d_v$ as the **total demands**.

An example



Note: The differences between CIRCULATION and MULTICOMMODITIES problem:

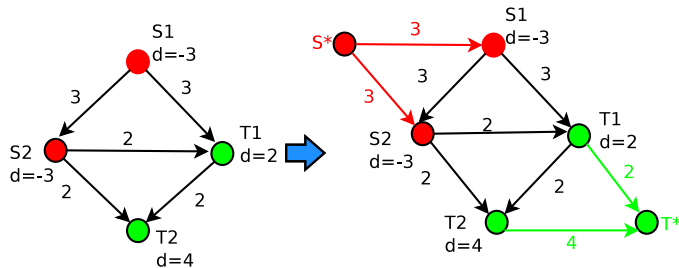
- 1 CIRCULATION problem: There is ONLY one type of commodity: a sink t_i can accept commodity from any source. In other words, the combination of commodities from all sources constitutes the demand of t_i .
- 2 MULTICOMMODITIES problem: There are multiple commodities, say transferring *food* and *oil* in the same network. Here t_i (say demands *food*) accepts commodity k_i from s_i (say sending *food*) only. Linear programming is the only known polynomial-time algorithm for the MULTICOMMODITIES problem.

Algorithm for circulation:

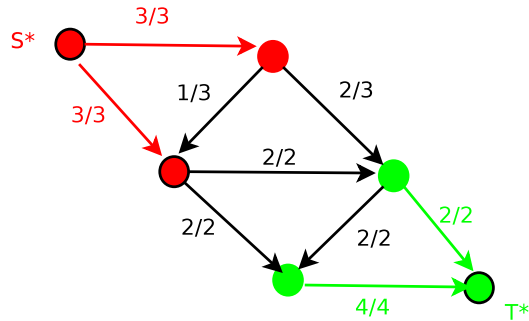
- 1: Constructing an expanded network G' via adding super source S^* and super sink T^* ;
- 2: Calculating the maximum flow f for G' by using Ford-Fulkerson algorithm;
- 3: Return flow f if the maximum flow value is equal to $D = \sum_{v:d_v>0} d_v$.

Step 1: constructing an expanded network G'

Transformation: constructing a network G' through adding a super source s^* to connect each s_i with capacity $C(s^*, s_i) = -d_i$. Similarly, adding a super sink t^* to connect to each t_j with capacity $C(t_j, t^*) = d_j$.

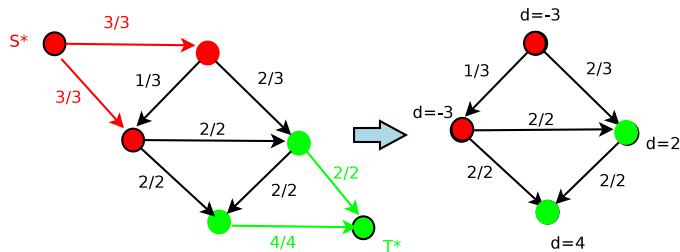


Step 2: calculating the maximum flow for G'



Note: a/b means $f(e) = a$, and capacity $C(e)=b$.

Step 3: checking the maximum flow for G'



Note: a/b means $f(e) = a$, and capacity $C(e)=b$.

The maximum flow value is $6 = \sum_{v, d_v > 0} d_v$. Thus, we obtained a feasible solution to the original circulation problem.

Theorem

There is a feasible solution to CIRCULATION problem iff the maximum $s^ - t^*$ flow in G' is D .*

Proof.

• \Leftarrow

Simply removing all (s^*, s_i) and (t_j, t^*) edges. It is obvious that both capacity constraint and conservation constraint still hold for all s_i and t_j .

• \Rightarrow

We construct a $s^* - t^*$ flow and prove that it is a maximum flow:

- 1 Define a flow f as follows: $f(s^*, s_i) = -d_i$ and $f(t_j, t^*) = d_j$.
- 2 Consider a special cut (A, B) , where $A = \{s^*\}$, $B = V - A$.
- 3 We have $C(A, B) = D$. Thus f is a maximum flow since it reaches the maximum value.



Extension 3: CIRCULATION with lower bound for capacity

Extension 3: CIRCULATION with lower bound of capacity

INPUT:

a network $G = \langle V, E \rangle$, where each edge e has a capacity **upper bound** $C(e)$ and a **lower bound** $L(e)$; multi sources s_i and sinks t_j . A sink t_j has demand $d_j > 0$, while a source s_i has supply d_i (described as a negative demand $d_i < 0$).

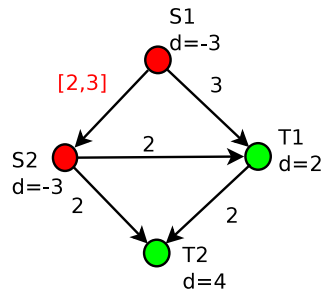
OUTPUT:

a feasible circulation f to satisfy all demand requirements using the available supply, i.e.,

- 1 Capacity restriction: $L(e) \leq f(e) \leq C(e)$;
- 2 Conservation restriction: $f^{in}(v) - f^{out}(v) = d_v$;

Note: For the sake of simplicity, we define $d_v = 0$ for any node v except for s_i and t_j . Thus we have $\sum_i d_i = 0$, and define $D = \sum_{d_v > 0} d_v$ be the *total demands*.

An example



$[a,b]$ denotes $L(e)=a$, and $C(e) = b$.

Advantages of lower bound: By setting lower bound $L(e) > 0$, we can force edge e to be used by flow, e.g. edge (s_1, s_2) should be used in the flow.

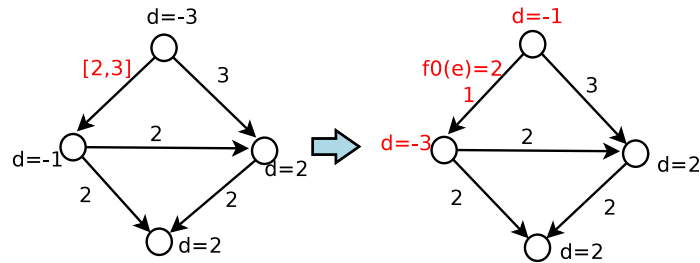
Algorithm for circulation with lower-bound for capacity

- 1: Building **an initial flow** f_0 by setting $f_0(e) = L(e)$ for $e = (u, v)$;
- 2: Solving a new circulation problem for G' without capacity lower bound. Specifically, G' was made by revising an edge $e = (u, v)$ with lower bound capacity:
 - ① nodes: $d'_u = d_u + L(e)$, $d'_v = d'_v - L(e)$,
 - ② edge: $L(e) = 0$, $C(e) = C(e) - L(e)$.

Denote f' as a feasible circulation to G' .

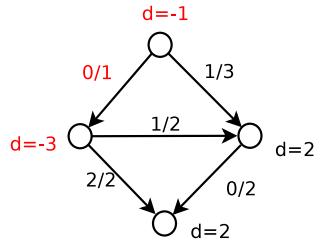
- 3: Return $f = f' + f_0$.

Step 1: Building an initial flow f_0



Note: $a/[l,b]$ means $f(e) = a$, and capacity $L(e)=l$, and $C(e)=b$.

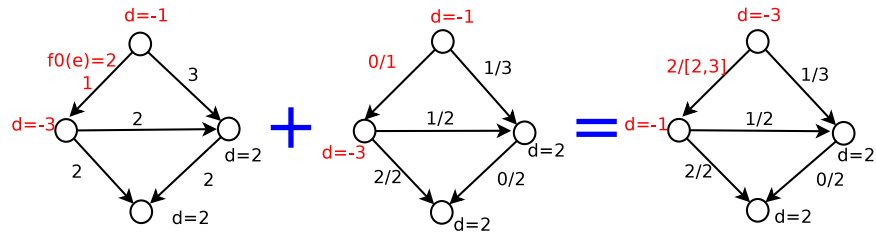
Step 2: Solving the new circulation problem



Note: $a/[l,b]$ means $f(e) = a$, and capacity $L(e)=l$, and $C(e)=b$.

We found a feasible circulation f' for the network G' .

Step 3: Adding f_0 and f'



We get f to the original problem as: $f = f_0 + f'$.

Theorem

There is a circulation f to G (with lower bounds) iff there is a circulation f' to G' (without lower bounds).

Proof.

- Define $f'(e) = f(e) + L_e$.
- It is easy to verify both capacity constraints and conservation constraints hold.



Extension 4: MINIMUM COST FLOW problem

Extension 4: MINIMUM COST FLOW

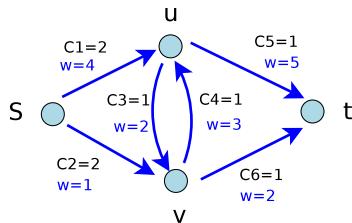
INPUT:

a network $G = \langle V, E \rangle$, where each edge e has a capacity $C(e) > 0$, and a cost $w(e)$ for transferring a unit through edge e .
Two special node: source s and sink t . A flow value v_0 .

OUTPUT:

to find a circulation f with flow value v_0 and the cost is minimized.

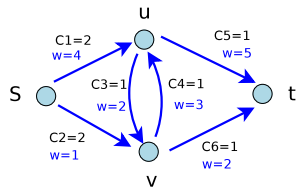
An example



指定运货量.

- Objective: how to transfer $v_0 = \underline{2}$ units commodity from s to t with the minimal cost?
- Basic idea: the cost w_e makes it difficult to find the minimal cost flow by simply expanding G to G' as we did for the CIRCULATION problem. Then we return to the primal_dual idea.

Primal_Dual technique: LP formulation



$$\begin{array}{ll}
 \min & 4y_1 + y_2 + 2y_3 + 3y_4 + 5y_5 + 2y_6 \\
 \text{s.t.} & \underbrace{y_1 + y_2}_{=2} \\
 & -y_1 + y_3 - y_4 + y_5 = -2 \\
 & -y_2 - y_3 + y_4 + y_6 = 0 \\
 & y_i \leq C_i \\
 & y_i \geq 0
 \end{array}$$

→ 发2吨货

=2 node s
= -2 node t → 收2吨货.

=0 node u
=0 node v

Intuition: y_i denotes the flow on edge i .

$$\begin{array}{llllllll}
 \text{max} & -4y_1 & -y_2 & -2y_3 & -3y_4 & -5y_5 & -2y_6 & \\
 \text{s.t.} & y_1 & +y_2 & & & & & \leq 2 \quad \text{node } s \\
 & & & & & -y_5 & -y_6 & \leq -2 \quad \text{node } t \\
 & -y_1 & & +y_3 & -y_4 & +y_5 & & \leq 0 \quad \text{node } u \\
 & & -y_2 & -y_3 & +y_4 & & +y_6 & \leq 0 \quad \text{node } v \\
 & & & & & y_i & \leq C_i \\
 & & & & & y_i & \geq 0
 \end{array}$$

Rewrite the LP into standard DUAL form via:

- Objective function: using max instead of min.
- Constraints: Simply replacing “=” with “ \leq ”. (Why? Notice that if all inequalities were satisfied, they should be equalities. For example, inequalities (2), (3) and (4) force $y_1 + y_2 \geq 2$, thus change \leq into $=$ for inequality (1). So do other inequalities.

Finding a valid circulation with value v_0 first.

- We need to find a valid circulation with value $v_0 = 2$ first.
- This is easy: CIRCULATION problem.
- Thus we have a feasible solution to D .

得到一个可行解

$$\begin{array}{llllllll}
 \max & -4y_1 & -y_2 & -2y_3 & -3y_4 & -5y_5 & -2y_6 & \\
 s.t. & y_1 & +y_2 & & & & & \leq 0 \quad \text{node } s \\
 & & & & & y_5 & +y_6 & \leq 0 \quad \text{node } t \\
 & y_1 & & -y_3 & +y_4 & -y_5 & & \leq 0 \quad \text{node } u \\
 & & y_2 & +y_3 & -y_4 & & -y_6 & \leq 0 \quad \text{node } v \\
 & & & & & y_i & \leq 0 & \text{for full arc} \\
 & & & & & -y_i & \leq 0 & \text{for empty arc} \\
 & & & & & y_i & \leq 1 & \text{for any arc}
 \end{array}$$

Recall the rules to construct DRP from D:

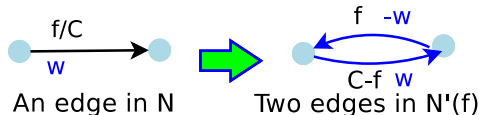
- Replacing the right hand items with 0.
- Removing the constraints not in J (J contains the constraints in D where $=$ holds).
- Adding constraints $y_i \geq -1$ for any arcs.

Solving DRP: combinatorial technique rather than simplex

Definition (Cycle flow)

A flow f is called **cycle flow** if input equal output for any node (including s and t).

- Suppose we have already obtained a flow for network N .
- Solving the corresponding DRP is essentially finding a cycle in a new network $N'(f)$, which is constructed as follows:
 - 1 For each edge $e = (u, v)$ in N , two edges $e = (u, v)$ and $e' = (v, u)$ were introduced to $N'(f)$;
 - 2 The capacities for e and e' in $N'(f)$ are set as $C(e) - f(e)$ and $\cancel{f}(e)$, respectively;
 - 3 The costs are set as $w(e') = -w(e)$;



Theorem

f is the minimum cost flow in network $N \Leftrightarrow$ network $N'(f)$ contains no cycle with negative cost.

Proof.

f is the minimum cost flow in network N

\Leftrightarrow The optimal solution to DRP is 0.

$\Leftrightarrow N'(f)$ has no cycle flow with negative cost.

$\Leftrightarrow N'(f)$ has no cycle with negative cost. □

Intuition: Suppose that we have obtained a cycle in $N'(f)$.

Pushing a unit flow along the cycle leads to a cycle flow (denoted as \bar{f}). Then $f + \bar{f}$ is also a flow for the original network N .

Minimum cost flow algorithm

Klein algorithm

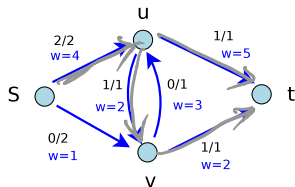
- 1: Finding a flow f with value v_0 using maximum-flow algorithm, say Ford-Fulkerson;
- 2: **while** $N'(f)$ contains a cycle C with negative cost **do**
- 3: Denote b as the bottleneck of cycle C .
- 4: Define \bar{f} as the unit flow along C .
- 5: $f = f + b\bar{f}$;
- 6: **end while**
- 7: **return** f .

Note:

- 1 The cost of flow decreases as iteration proceeds, while the flow value keeps constant.
- 2 The cycle with negative cost can be found using Bellman-Ford algorithm.

Example: Step 1

Initial flow f_0 : flow value 2, flow cost: 17.



$$4 \times 2 + 2 + 2 + 5 = 17$$

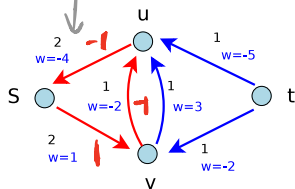
0 的边就不画了,

New network $N'(f)$:

Find a cycle.

Note: Cycle 的 cost 和为负.

bottleneck = 1, $\bar{f} = 1$?



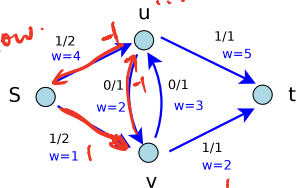
Negative cost cycle: $s \rightarrow v \rightarrow u \rightarrow s$ (in red). Cost: -5.

Example: Step 2

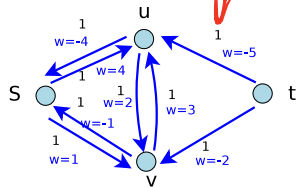
$f = f + \bar{f}$: flow value $2 - 0 = 2$, flow cost: $17 - 5 = 12$.

$$2+1+1+1$$

给每个边加 flow.



New network $N'(f)$: 怎么形成的?



Negative cost cycle: cannot find. Done!

Extension: Hitchcock TRANSPORTATION problem 1941

INPUT: n sources s_1, s_2, \dots, s_n and n sinks t_1, t_2, \dots, t_n . Source s_i has supply a_i , and a sink t_j has demand b_j . The cost from s_i to t_j is c_{ij} .

OUTPUT: arrange a schedule to minimize cost.

Note:

- 1 Frank L. Hitchcock formulated the TRANSPORTATION problem in 1941. This problem is equivalent to MINIMUM COST FLOW PROBLEM [Wagner, 1959].
- 2 In 1956, L. R. Ford Jr. and D. R. Fulkerson proposed a "labeling" technique to solve the transportation problem. This algorithm is considerably more efficient than simplex algorithm. See "Solving the Transportation Problem" by L. R. Ford Jr. and D. R. Fulkerson.
- 3 If $c_{ij} = 0/1$, then Hitchcock problem turns into assignment problem.

Applications of MAXIMUMFLOW problem

Formulating a problem into MAXIMUMFLOW problem:

- ① We should define a **network** first. Sometimes we need to construct a graph from the very scratch.
- ② Then we need to define **weight for edges**. Sometimes we need to move the weight on nodes to edges.
- ③ How to define **source s and sink t** ? Sometimes super source s^* and t^* are needed.
- ④ Finally we need to prove that **max-flow** (finding paths, matching) or **min-cut** (partition nodes) is what we wanted.

Note: most problems utilize the property that there exists a maximum integer-valued flow iff there exists a maximum flow.

Paradigm 1: Partition a set

Problem 1: IMAGESEGMENTATION problem

INPUT:

Given an image in pixel map format. The pixel $i, i \in P$ has a probability to be foreground f_i and the probability to be background b_i ; in addition, the likelihood that two neighboring pixels i and j are similar is l_{ij} ;

GOAL:

to identify foreground out of background. Mathematically, we want a partition $P = F \cup B$, such that $Q(F, B) = \sum_{i \in F} f_i + \sum_{j \in B} b_j + \sum_{i \in F} \sum_{j \in N(i) \cap F} l_{ij} + \sum_{i \in B} \sum_{j \in N(i) \cap B} l_{ij}$ is maximized.

$x_i = F/B$
极大似然.

$$\max P(x_1, x_2 \dots x_n) \\ = e^{-E(x_1, x_2 \dots x_n)}$$

→ 极小自由能.

$$\min E(x_1, x_2 \dots x_n)$$

$$E(x_1, x_2 \dots x_n) = \sum_{i=1}^n S_i(x_i)$$

$$+ \sum_{i=1}^n \sum_{j=1}^n h(x_i, x_j)$$

$$+ \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n p(x_i, x_j, x_k) + \dots$$

复杂且小.
通常忽略.



- ① 通过单个像素点, 判断是前景/背景.
- ② 通过相邻点, 判断.

An example

Naive: LP

$x_1, x_2, \dots, x_9 = 0/1$

max $a_1x_1 + 0.9(1-x_1) + \dots$
 $+ 0.2(1-y_{ij}) + \dots$

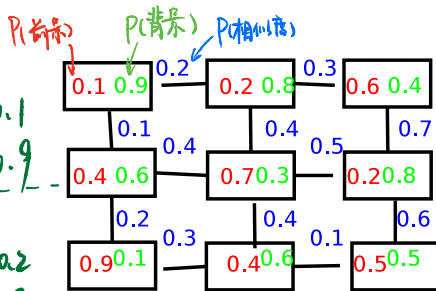
s.t. $y_{ij} \geq (x_i - x_j)$

$y_{ij} \geq (x_j - x_i)$

$x_{ij} = 0/1, y_{ij} = 0/1$

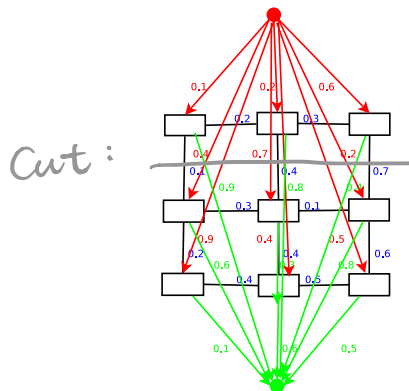
if $x=1$
 then $Q+=0.1$
 else $Q+=0.9$

if $x_1 = x_2$
 then $Q+=0.2$
 else $Q+=0$



目标: 将网络分成两部分.
 分成前景背景.

Converting to network-flow problem



*把节点的直接利边上去。

| | R | G | B | |
|---|---|---|-----|----------|
| Q | 3 | 6 | 2+7 | |
| C | 6 | 3 | 3 | |
| M | 9 | 9 | 12 | Constant |

- 1 Network: Adding two nodes source s and sink t with connections to all nodes;
- 2 Capacity: $C(s, v) = f_v$, $C(v, t) = b_v$; $C(u, v) = l_{uv}$;
- 3 Cut: a partition. Cut capacity $C(F, B) = \underline{M - Q(F, B)}$, where $\underline{M = \sum_i (b_i + f_i) + \sum_i \sum_j l_{ij}}$ is a constant.
- 4 MinCut: the optimal solution to the original problem

Problem 2: PROJECT SELECTION

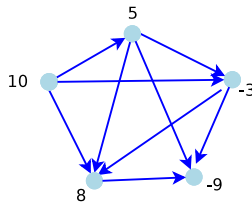
INPUT:

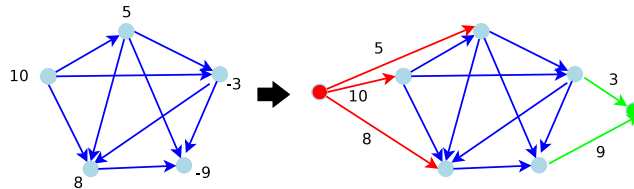
Given a directed acyclic graph (DAG). A node represents a project associated with a profit (denoted as $p_i > 0$) or a cost (denoted as $p_i < 0$), and directed edge $u \rightarrow v$ represent the prerequisite relationship, i.e. v should be finished before u .

GOAL:

to choose a subset A of projects such that:

- 1 Feasible: if a project was selected, all its prerequisites should also be selected;
- 2 Optimal: to maximize profits $\sum_{v \in A} p_v$;

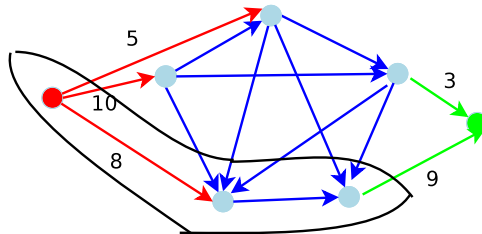




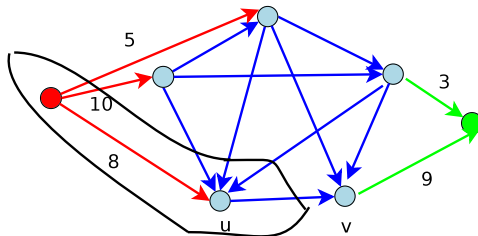
- 1 Network: introducing two nodes: s and t , s connecting the nodes with $p_i > 0$, and t connecting the nodes with $p_i < 0$;
- 2 Capacity: moving weights from nodes to edges, and set $C(u, v) = \infty$ for $\langle u, v \rangle \in E$.
- 3 Cut: a partition of nodes.

Minimum cut corresponds to maximum profit

- ① Cut capacity: $C(A, B) = C - \sum_{i \in A} p_i$, where $C = \sum_{v \in V} p_v$ ($p_v > 0$) is a constant.



- ② In the example, $C(A, B) = 5 + 10 + 9$, $\sum_{i \in A} p_i = 8 - 9$, and $C = 5 + 10 + 8$.
- ③ Min-Cut: corresponding to the maximum profit since the sum of cut capacity and profit is a constant.



- Feasible: The feasibility is implied by the infinite weights on edges, i.e. an invalid selection corresponds to a cut with infinite capacity.
- For example, if a project u was selected while its precursor v was not selected, then the edge $\langle u, v \rangle$ is a cut edge, leading to an infinite cut.

Paradigm 2: Finding paths

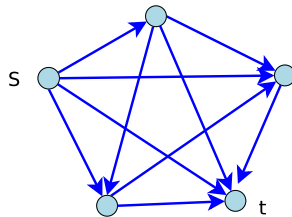
Problem 3: Disjoint paths

INPUT:

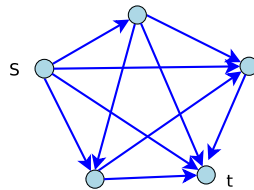
Given a graph $G = \langle V, E \rangle$, two nodes s and t , an integer k .

GOAL:

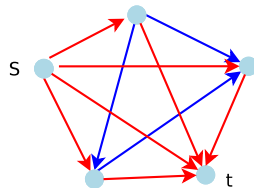
to identify k $s - t$ paths whose edges are disjoint;



Related problem: graph connectivity



- 1 Edges: the same to the original graph;
- 2 Capacity: $C(u, v) = 1$;
- 3 Flow: (See extra slides)



Theorem

k disjoint paths in $G \Leftrightarrow$ the maximum $s - t$ flow value is at least k .

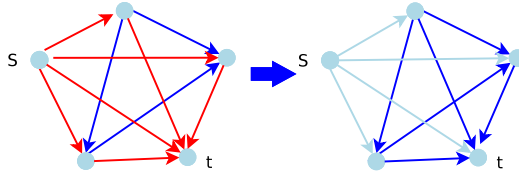
Proof.

- ① Note: maximum $s - t$ flow value is k implies an INTEGRAL flow with value k .
- ② Simply selecting the edges with $f(e) = 1$.



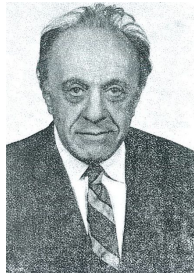
Time-complexity: $O(mn)$.

Menger theorem 1927



Theorem

The number of maximum disjoint paths is equal to the number of minimal edge removal to separate s from t .



Proof.

- ① The number of maximum disjoint paths is equal to the maximum flow;
- ② Then there is a cut (A, B) such that $C(A, B)$ is the number of disjoint paths;
- ③ The cut edges are what we wanted.



Problem 4: Survey design

$x_{ij} = 0/1$ 用产品 j 来
的印像.

max

s.t. $1 \leq x_{11} + x_{13} \leq 3$

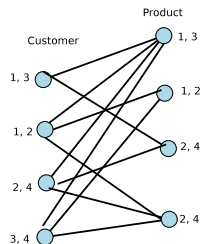
$$\sum_i \sum_j x_{ij} = 1$$

INPUT:

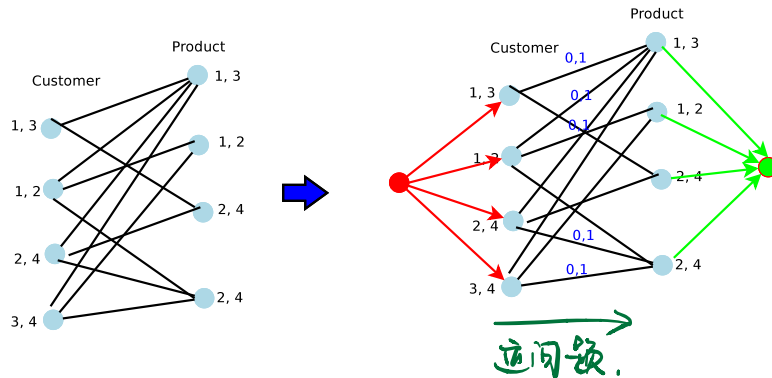
A set of customers A , and a set of products P . Let $B(i) \subseteq P$ denote the products that customer i bought. An integer k .

GOAL:

to design a survey with k questions such that for customer i , the number of questions is at least c_i but at most c'_i . On the other hand, for each product, the number of questions is at least p_i but at most p'_i .



Network construction



- 1 Edges: introducing two nodes s and t . Connecting customers with s and products with t .
- 2 Capacity: moving weights from nodes to edges; setting $C(i, j) = 1$;
- 3 Circulation: is a feasible solution to the original problem.

Paradigm 3: Matching

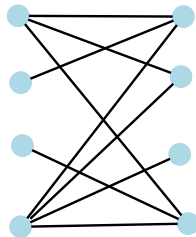
Problem 5: Matching 婚配问题

INPUT:

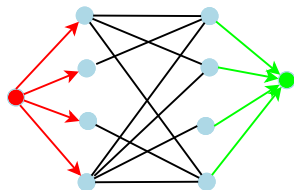
A bipartite $G = \langle V, E \rangle$;

GOAL:

to identify the maximal matching;



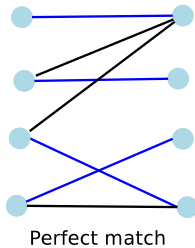
Constructing a network



- ① Edges: adding two nodes s and t ; connecting s with U and t with V ;
- ② Capacity: $C(e) = 1$ for all $e \in E$;
- ③ Flow: the maximal flow corresponds to a maximal matching;

Time-complexity: $O(mn)$

Perfect matching: Hall theorem



Definition (Perfect match)

Given a bipartite $G = \langle V, E \rangle$, where $V = X \cup Y$, $X \cap Y = \emptyset$, $|X| = |Y| = n$. A match M is a perfect match iff $|M| = n$.

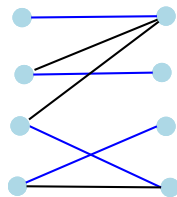
Hall theorem, Hall 1935, König 1931

Theorem

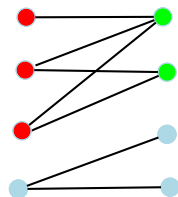
A bipartite has a perfect matching \Leftrightarrow for any $A \subseteq X$, $|\Gamma(A)| \geq |A|$, where $\Gamma(A)$ denotes the partners of nodes in A .



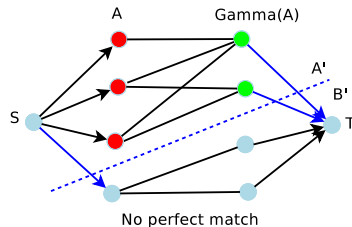
Figure: König, Egervary, and Philip Hall



Perfect match



No perfect match



Proof.

Here we only show that if there is no perfect matching, then $|\Gamma(A)| < |A|$.

- 1 Suppose there is no perfect matching, i.e., the maximal match is M , $|M| < n$;

Paradigm 4: Decomposing numbers

BASEBALL ELIMINATION problem

INPUT:

n teams T_1, T_2, \dots, T_n . A team T_i has already won w_i games, and for team T_i and T_j , there are g_{ij} games left.

GOAL:

Can we determine whether a team, say T_i , has already been eliminated from the first place? If yes, can we give an evidence?

Four teams: *New York*, *Baltimore*, *Toronto*, *Boston*

- ① w_i : NY (90), Balt (88), Tor (87), Bos (79).
- ② g_{ij} : NY:Balt 1, NY:Tor 6, Balt:Tor 1, Balt:Bos 4, Tor:Bos 4, NY:Bos 4.

It is safe to say that *Boston* has already been eliminated from the first place since:

- ① *Boston* can finish with at most $79 + 12 = 91$ wins.
- ② We can find a subset of teams, e.g. $\{NY, Tor\}$, with the total number of wins of $90 + 87 + 6 = 183$, thus at least a team finish with $\frac{183}{2} = 91.5 > 91$ wins.

Note that $\{NY, Tor, Balt\}$ cannot serve as an evidence that *Bos* has already been eliminated.

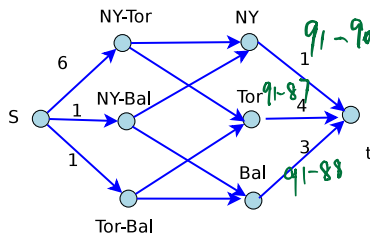
Question: For a specific team z . Can we determine whether there exists a subset of teams $S \subseteq T - \{z\}$ such that

- ① z can finish with at most m wins;
- ② $\frac{1}{|S|}(\sum_{x \in S} w_x + \sum_{x,y \in S} g_{xy}) > m$.

In other word, at least one of the teams in S will have more wins than z .

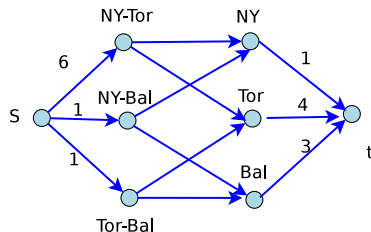
Network construction: taking $z = Boston$ as an example

- We define $m = w_z + \sum_{x \in T} g_{xz} = 91$, i.e. the total number of possible wins for team z .
- A network is constructed as follows:
 - 1 Define $S = T - \{z\}$, and $g^* = \sum_{x,y \in S} g_{xy} = 8$.
 - 2 Nodes: For each pair of teams, constructing a node $x : y$, and for each team x , constructing a node x .
 - 3 Edges:
 - For edge $s - x : y$, set capacity as $g_{x,y}$.
 - For edge $x : y - x$ and $x : y - y$, set capacity as $g_{x,y}$.
 - For edge $x - t$, set capacity as $m - w_x$.

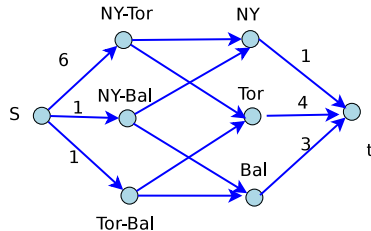


Intuition: number decomposition

Intuition: along edge $s - x : y$, we send $g_{x,y}$ wins, and at node $x : y$, this number is decomposed into two numbers, i.e. the number of wins of each team.

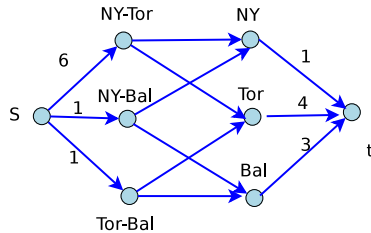


Case 1: the maximum flow value is $g^* = 8$



Theorem

There exist a flow with value $g^ = 8$ iff there is still possibility that $z = Boston$ wins the championship.*



Proof.

• \Leftarrow

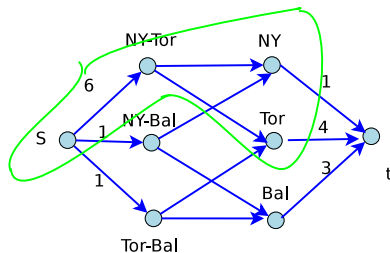
- If there is a flow with value g^* , then the capacities on edges $x - t$ guarantees that no team can finish with over m wins.
- Therefore, z still have chance to win the championship (if z wins all remaining games).

• \Leftarrow

- If there is possibility for z to win the championship
- we can define a flow with value g^* .



Case 2: the maximum flow value is less than $g^* = 8$



Theorem

If the maximum flow value is strictly smaller than g^* , the minimum cut describes a subset $S \subseteq T - \{z\}$ such that

$$\frac{1}{|S|} (\sum_{x \in S} w_x + \sum_{x, y \in S} g_{xy}) > m.$$

Proof.

(See extra slides)



Extensions of matching: ASSIGNMENT problem, Hungarian algorithm for WEIGHTED ASSIGNMENT problem, Blossom algorithm.