091M4041H - Assignment Four
Linear programming

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1 Linear-inequality feasibility

1.1 Problem Description

Given a set of m linear inequalities on n variables $x_1, x_2, , x_n$, the linearinequality feasibility problem asks if there is a setting of the variables that simultaneously satisfies each of the inequalities.

Show that if we have an algorithm for linear programming, we can use it to solve the linear-inequality feasibility problem. The number of variables and constraints that you use in the linear-programming problem should be polynomial in n and m.

1.2 Solution

为了确定是否有可行解，可以设计一个辅助线性规划。设L是一个标准型的线性规划，引入一个新的变量x_0来构造如下面所示带有$(n+1)$个变量的辅助线性规划L_{aux}。

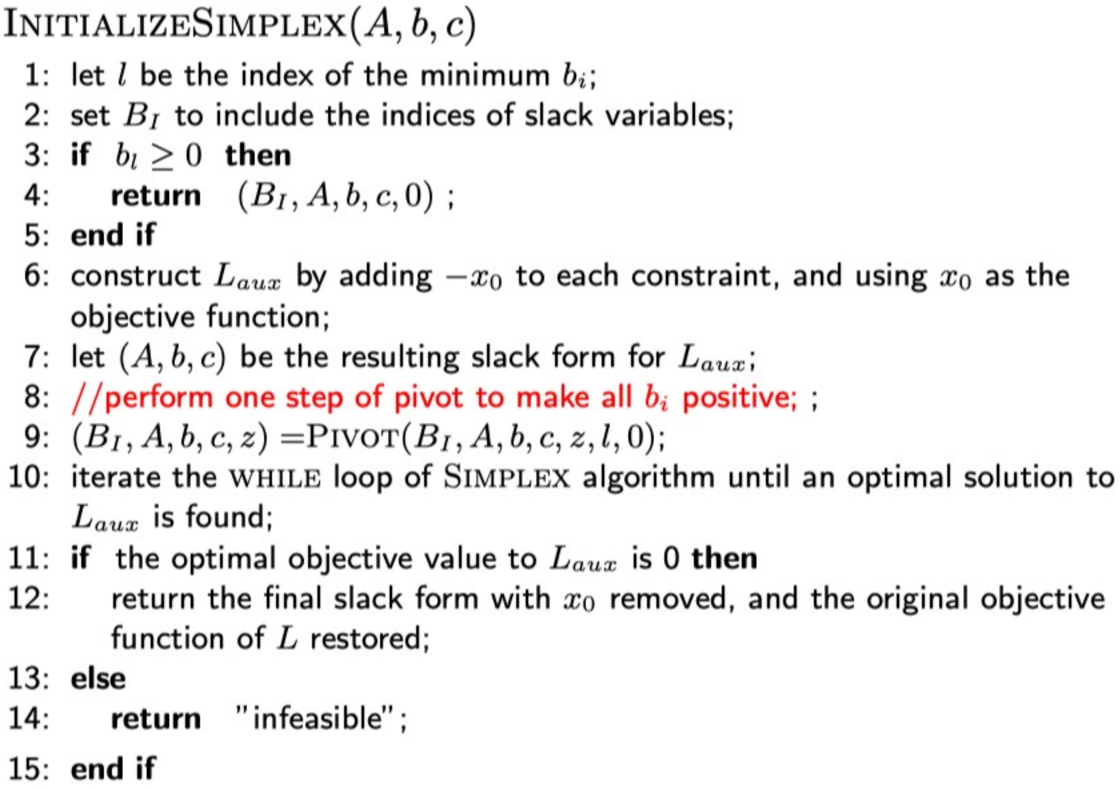
$$\begin{array}{llll} & min & -x_0 & \\ s.t & \displaystyle\sum_{j=1}^{n} a_{ij}x_j - x_0 \le b_i & & for\ all\ i = 1, 2, \dots, m \\ & & s_i \ge 0 & for\ all\ i = 0, 1, 2, \dots, n \end{array}$$

当且仅当L_{aux}的最优解为0，即x_0为0时，原线性规划L有可行解。

证明：假设L有一个可行解$\hat{x} = (\hat{x_1}, \hat{x_2}, \dots, \hat{x_n})$,设解$\hat{x_0} = 0$，则$\hat{x_0}$并上$\hat{x}$是$L_{aux}$的一个可行解，目标值为0；因为$\hat{x} \ge 0$是$L_{aux}$的一个约束，又因为目标函数是最大化$-x_0$，所以这个解对于$L_{aux}$肯定是最优的。

同理，假设于L_{aux}的最优解为0，也就是$\hat{x_0} = 0$，同时，其他变量$\hat{x}$的值满足L的约束。

伪代码如下所示：



```
InitializeSimplex($(A, b, c)$)
let $l$ be the index of the minimum $b_i$;
set $B_I$ to include the indices of slack variables;
if  $b_l \ge 0$  then
    return  $(B_I, A, b, c, 0)$ ;
end if
construct $L_{aux}$ by adding $-x_0$ to each constraint, and using $x_0$ as the
objective function;
let $(A, b, c)$ be the resulting slack form for $L_{aux}$;
//perform one step of pivot to make all $b_i$ positive; ;
$(B_I, A, b, c, z)$ =Pivot($(B_I, A, b, c, z, l, 0)$);
iterate the while loop of Simplex algorithm until an optimal solution to
$L_{aux}$ is found;
if  the optimal objective value to $L_{aux}$ is 0 then
    return the final slack form with $x_0$ removed, and the original objective
    function of $L$ restored;
else
    return  "infeasible";
end if
```

Figure 1: Linear-inequality feasibility

2 Interval Scheduling Problem

2.1 Problem Description

A teaching building has m classrooms in total, and n courses are trying to use them. Each course $i(i = 1, 2, , n)$ only uses one classroom during time interval $[S_i, F_i)(F_i > S_i > 0)$. Considering any two courses can not be carried on in a same classroom at any time, you have to select as many courses as possible and arrange them without any time collision. For simplicity, suppose $2n$ elements in the set $S_1, F_1, , S_n, F_n$ are all different.

2.2 Solution

由题意可知，所给的$2n$个时间都是不同的，所以我们可以有$2n$个时间点，又因为题目要求每个时间段都最多只能有m门课同时上课，我们可以转化为每个时间点最多只有m门课在上课。比如在时间区间$[S_k, F_k]$中，假设有如下的时间点$S_k, S_i, S_j, F_i, F_j, F_k$，若第$k$门课在上课，又因为第$k$门课的时间区间为$[S_k, F_k)$，那么我们可以将第$k$门课的这些时间点设为$(1, 1, 1, 1, 1, 0)$，其中1表示第$k$门课目前正在上课，0表示不在上课。

首先，我们定义一些参数:

m, the maximum number of courses

$c_i, i \in [1, n]$

$x_{ij}, i \in [1, n], j \in [1, 2n]$

$$c_i = \begin{cases} 1, & if\ course\ i\ is\ choosed \\ 0, & otherwise \end{cases}$$

$$x_{ij} = \begin{cases} 1, & course\ i\ occupy\ time\ point\ j \\ 0, & course\ i\ don't\ occupy\ time\ point\ j \end{cases}$$

因此问题可表达为：

$$\begin{aligned} max \quad & \textstyle\sum_{i=1}^{n} c_i \\ s.t \quad & \textstyle\sum_{i=1}^{n} x_{ij} && \leq m, for\ all\ j = 1, 2, \ldots, 2n \\ & c_i, x_{ij} && \in \{0, 1\}, i \in [1, n], j \in [1, 2n] \end{aligned}$$

上面的公式也是标准型，不需要转换

2.3 Demo

设共有10门课,2个教室，每门课的时间如下所示：时间范围是[8,20]，总计20个时间节点。

各课程的时间如下：((14.5 18.5) (8.5 11.) (15.5 16.5) (11.5 18.) (10.5 15.) (12.5 16.) (10. 14.) (12. 19.) (13. 13.5) (9. 19.5))

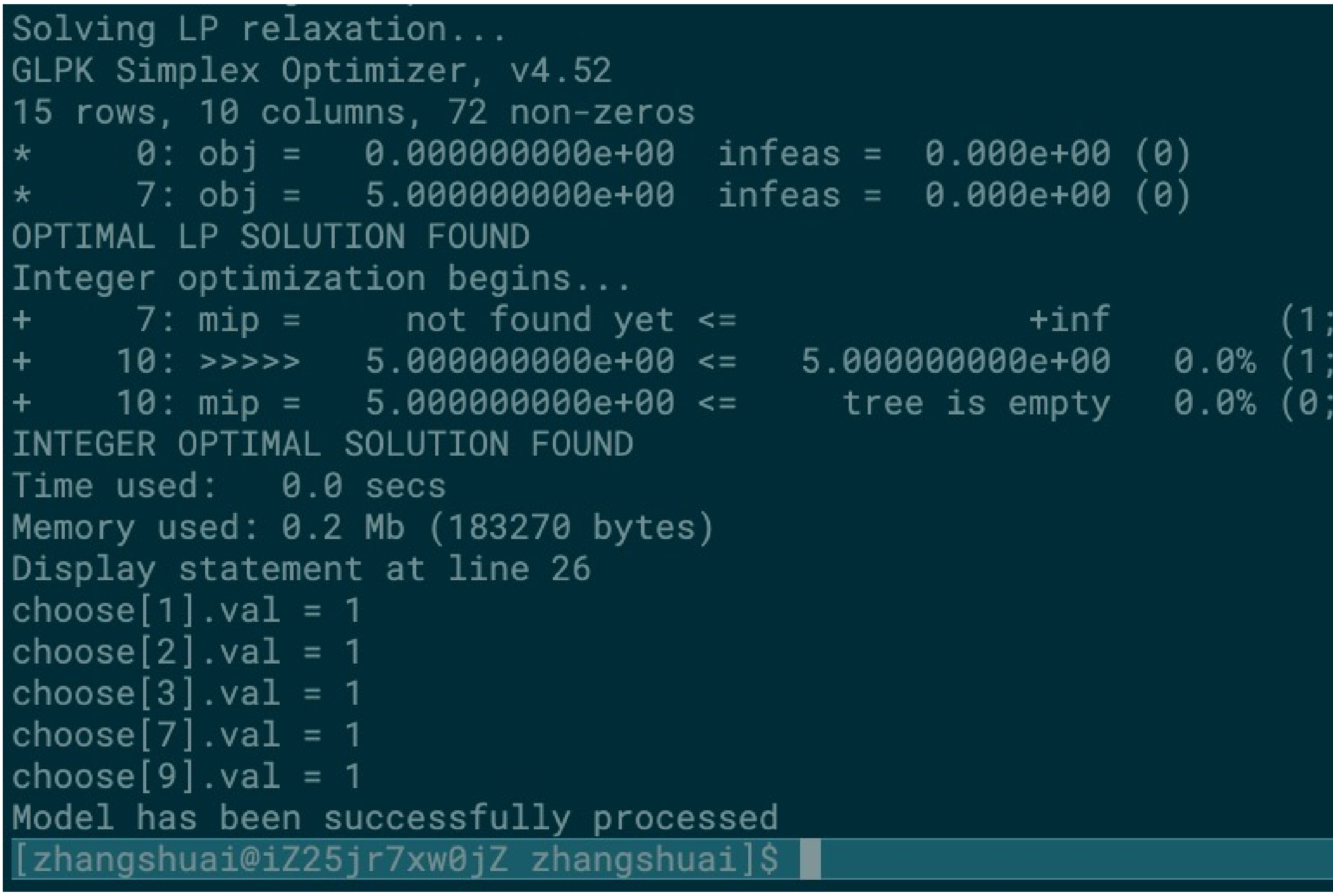
时间节点排序如下：

8.5 9. 10. 10.5 11. 11.5 12. 12.5 13. 13.5 14. 14.5 15. 15.5 16. 16.5 18. 18.5 19. 19.5

计算出每个课程占用的时间点如下所示：维度大小为$2n * n$:

```
0 1 2 3 4 5 6 7 8 9 10
1 0 1 0 0 0 0 0 0 0 0
2 0 1 0 0 0 0 0 0 0 1
3 0 1 0 0 0 0 1 0 0 1
4 0 1 0 0 1 0 1 0 0 1
5 0 0 0 0 1 0 1 0 0 1
6 0 0 0 1 1 0 1 0 0 1
7 0 0 0 1 1 0 1 1 0 1
```

8 0 0 0 1 1 1 1 1 0 1
9 0 0 0 1 1 1 1 1 1 1
10 0 0 0 1 1 1 1 1 0 1
11 0 0 0 1 1 1 0 1 0 1
12 1 0 0 1 1 1 0 1 0 1
13 1 0 0 1 0 1 0 1 0 1
14 1 0 1 1 0 1 0 1 0 1
15 1 0 1 1 0 0 0 1 0 1
16 1 0 0 1 0 0 0 1 0 1
17 1 0 0 0 0 0 0 1 0 1
18 0 0 0 0 0 0 0 1 0 1
19 0 0 0 0 0 0 0 0 0 1
20 0 0 0 0 0 0 0 0 0 0
代码间附录A，运行结果如下，可见课程1、2、3、7、9被选中：



```
Solving LP relaxation...
GLPK Simplex Optimizer, v4.52
15 rows, 10 columns, 72 non-zeros
*     0: obj =   0.000000000e+00  infeas =  0.000e+00 (0)
*     7: obj =   5.000000000e+00  infeas =  0.000e+00 (0)
OPTIMAL LP SOLUTION FOUND
Integer optimization begins...
+     7: mip =     not found yet <=              +inf        (1;
+    10: >>>>>   5.000000000e+00 <=   5.000000000e+00   0.0% (1;
+    10: mip =   5.000000000e+00 <=     tree is empty   0.0% (0;
INTEGER OPTIMAL SOLUTION FOUND
Time used:   0.0 secs
Memory used: 0.2 Mb (183270 bytes)
Display statement at line 26
choose[1].val = 1
choose[2].val = 1
choose[3].val = 1
choose[7].val = 1
choose[9].val = 1
Model has been successfully processed
[zhangshuai@iZ25jr7xw0jZ zhangshuai]$
```

Figure 2: Interval Scheduling Problem

3 Gas Station Placement

3.1 Problem Description

Let ’ s consider a long, quiet country road with towns scattered very sparsely along it. Sinopec, largest oil refiner in China, wants to place gas stations along the road. Each gas station is assigned to a nearby town, and the distance between any two gas stations being as small as possible. Suppose there are n towns with distances from one endpoint of the road being $d_1, d_2, , d_n$. n gas stations are to be placed along the road, one station for one town. Besides, each station is at most r far away from its correspond town. $d_1, d_2, , d_n$ and r have been given and satisfied $d_1, d_2, , d_n$, $0 < r < d_1$ and $d_i + r < d_{i+1}$`r for all i. The objective is to find the optimal placement such that the maximal distance between two successive gas stations is minimized.

Please formulate this problem as an LP.

3.2 Solution

由题意知，每个村镇都需要放置一个气站才能满足条件。

首先定义参数：m,要求的最大最小距离

s_i,表示第i个村镇放置的气站的位置

d_i,表示第i个村镇的位置

r，表示气站距离村镇的最大位置

然后可以将问题表述成如下形式:

$$
\begin{array}{lrl}
min & m & \\
s.t & s_{i+1} - s_i \leq m & for\ all\ i = 1, 2, \ldots, n-1 \\
 & d_i - r \leq s_i \leq d_i + r & for\ all\ i = 1, 2, \ldots, n \\
 & s_i, m \geq 0 & for\ all\ i = 1, 2, \ldots, n
\end{array}
$$

转换为标准型：

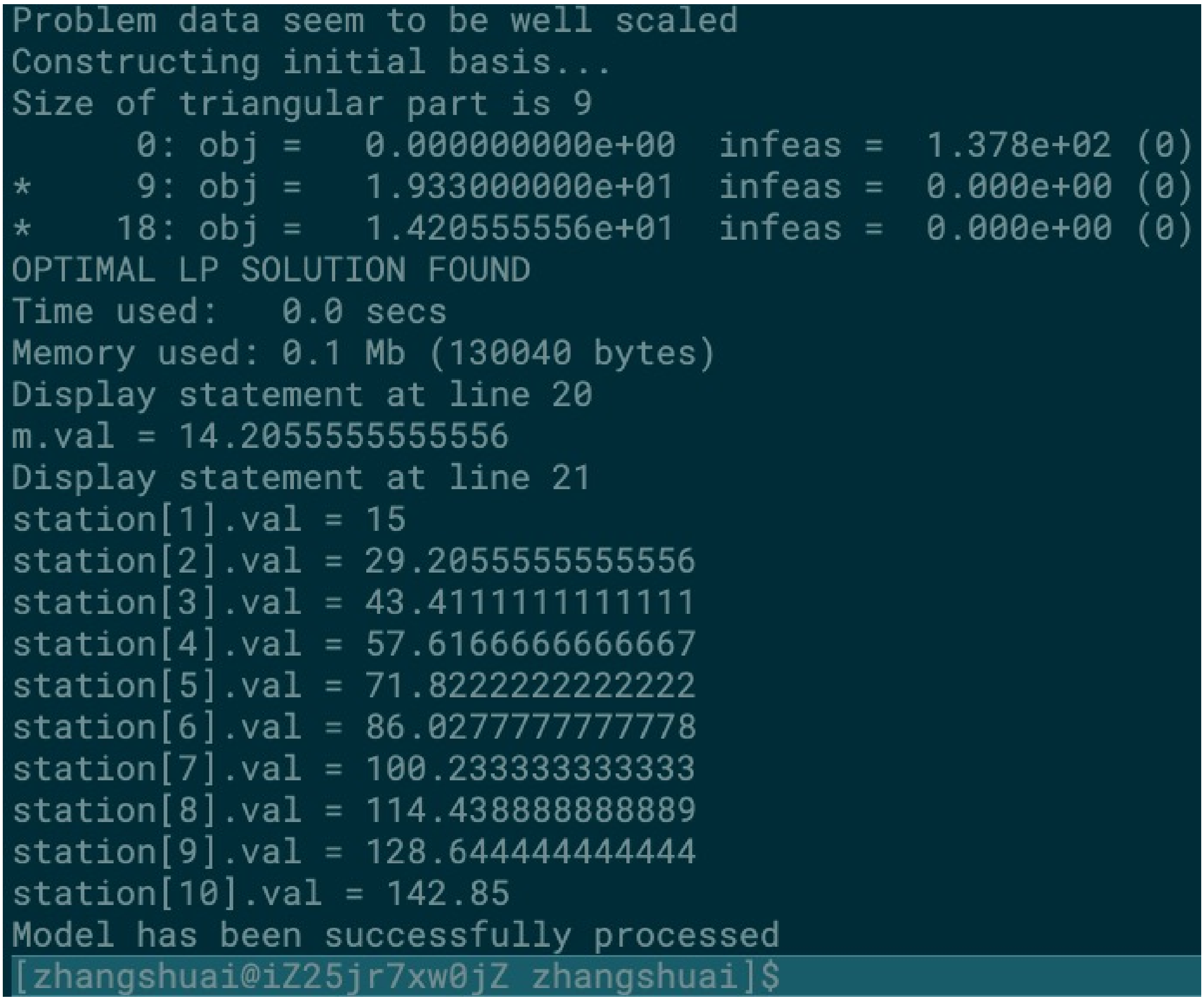
$$
\begin{array}{lrl}
min & m & \\
s.t & s_{i+1} - s_i \leq m & for\ all\ i = 1, 2, \ldots, n-1 \\
 & -s_i \leq r - d_i & for\ all\ i = 1, 2, \ldots, n \\
 & s_i \leq d_i + r & for\ all\ i = 1, 2, \ldots, n \\
 & s_i, m \geq 0 & for\ all\ i = 1, 2, \ldots, n
\end{array}
$$

3.3 Demo

设有10个村镇，它们与原点的距离为：[1] 10.0 [2] 25.89 [3] 39.48 [4] 58.42 [5] 69.72 [6] 82.82 [7] 99.6 [8] 116.77 [9] 128.52 [10] 147.85。

参数r设为5.0

代码在附录B，结果如下所示，可见，气站间的最大距离为14.21，气站的具体位置如图所示：



```
Problem data seem to be well scaled
Constructing initial basis...
Size of triangular part is 9
      0: obj =   0.000000000e+00  infeas =  1.378e+02 (0)
*     9: obj =   1.933000000e+01  infeas =  0.000e+00 (0)
*    18: obj =   1.420555556e+01  infeas =  0.000e+00 (0)
OPTIMAL LP SOLUTION FOUND
Time used:   0.0 secs
Memory used: 0.1 Mb (130040 bytes)
Display statement at line 20
m.val = 14.2055555555556
Display statement at line 21
station[1].val = 15
station[2].val = 29.2055555555556
station[3].val = 43.4111111111111
station[4].val = 57.6166666666667
station[5].val = 71.8222222222222
station[6].val = 86.0277777777778
station[7].val = 100.233333333333
station[8].val = 114.438888888889
station[9].val = 128.644444444444
station[10].val = 142.85
Model has been successfully processed
[zhangshuai@iZ25jr7xw0jZ zhangshuai]$
```

Figure 3: Gas Station Placement