number of possibilities. We introduced **optimization**—the process of selecting values of decision variables that *minimize* or *maximize* some quantity of interest—in Chapter 9. Recall that we developed a simple optimization model for setting the best price to maximize revenue:

$$\text{Maximize revenue} = -2.794 \times Price^2 + 3149 \times Price$$

We showed that we could estimate the optimal price rather easily by developing a simple data table in a spreadsheet that evaluates this function for various values of the price, or use Excel's *Solver* to find the best price exactly.

Optimization models have been used extensively in operations and supply chains, finance, marketing, and other disciplines for more than 50 years to help managers allocate resources more efficiently and make lower-cost or more-profitable decisions. Optimization is a very broad and complex topic; in this chapter, we introduce you to the most common class of optimization models—linear optimization models. In the next chapter, we will discuss more complex types of optimization models, called integer and nonlinear optimization models.

BUILDING LINEAR OPTIMIZATION MODELS

Developing any optimization model consists of four basic steps:

1. Define the decision variables.
2. Identify the objective function.
3. Identify all appropriate constraints.
4. Write the objective function and constraints as mathematical expressions.

Decision variables are the unknown values that the model seeks to determine. Depending on the application, decision variables might be the quantities of different products to produce, amount of money spent on R&D projects, the amount to ship from a warehouse to a customer, the amount of shelf space to devote to a product, and so on. The quantity we seek to minimize or maximize is called the **objective function**; for example, we might wish to maximize profit or revenue, or minimize cost or some measure of risk. Any set of decision variable values that maximizes or minimizes (in generic terms, *optimizes*) the objective function is called an **optimal solution.**

Most practical optimization problems consist of many decision variables and numerous **constraints**—limitations or requirements that decision variables must satisfy. Some examples of constraints are as follows:

- The amount of material used to produce a set of products cannot exceed the available amount of 850 square feet.
- The amount of money spent on research and development projects cannot exceed the assigned budget of $300,000.
- Contractual requirements specify that at least 500 units of product must be produced.
- A mixture of fertilizer must contain exactly 30% nitrogen.
- We cannot produce a negative amount of product (this is called a *nonnegativity constraint*).

The presence of constraints along with a large number of variables usually makes identifying an optimal solution considerably more difficult and necessitates the use of powerful software tools.

The essence of building an optimization model is to translate constraints and the objective function into mathematical expressions. Constraints are generally expressed mathematically as algebraic inequalities or equations. Note that the phrase "cannot exceed" specifies a " $\leq$ " inequality, "at least" specifies a " $\geq$ " inequality, and "must contain exactly" specifies an " $=$ " relationship. All constraints in optimization number of possibilities. We introduced **optimization**—the process of selecting values of decision variables that *minimize* or *maximize* some quantity of interest—in Chapter 9. Recall that we developed a simple optimization model for setting the best price to maximize revenue:

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- Amount of material used $\leq$ 850 square feet
- Amount spent on research and development $\leq$ $300,000
- Number of units of product produced $\geq$ 500
- Amount of nitrogen in mixture/total amount in mixture $=$ 0.30
- Amount of product produced $\geq$ 0

The left-hand side of each of these expressions is called a **constraint function**. A constraint function is a function of the decision variables in the problem. For example, suppose that in the first case, we are producing three products. Further assume that the material requirements of these three products are 3.0, 3.5, and 2.3 square feet per unit, respectively. If A, B, and C represent the number of units of each product to produce, then $3.0A$ represents the amount of material used to produce A units of product A, $3.5B$ represents the amount of material used to produce B units of product B, and $2.3C$ represents the amount of material used to produce C units of product C. Note that dimensions of these terms are (square feet/unit)(units) = square feet. Hence, "amount of material used" can be expressed mathematically as the constraint function $3.0A + 3.5B + 2.3C$. Therefore, the constraint that limits the amount of material that can be used is written as:

$$3.0A + 3.5B + 2.3C \leq 850$$

As another example, if two ingredients contain 20% and 33% nitrogen, respectively, then the fraction of nitrogen in a mixture of x pounds of the first ingredient and y pounds of the second ingredient is expressed by the constraint function:

$$(0.20x + 0.33y)/(x + y)$$

If the fraction of nitrogen in the mixture must be 0.30, then we would have:

$$(0.20x + 0.33y)/(x + y) = 0.3$$

This can be rewritten as:

$$(0.20x + 0.33y) = 0.3(x + y)$$

and simplified as:

$$-0.1x + 0.03y = 0$$

In a similar fashion, we must translate the objective function into a mathematical expression involving the decision variables. To see this entire process in action, let us examine a typical decision scenario:

> Sklenka Ski Company (SSC) is a small manufacturer of two types of popular all-terrain snow skis, the Jordanelle and Deercrest models. The manufacturing process consists of two principal departments: fabrication and finishing. The fabrication department has 12 skilled workers, each of whom works 7 hours per day. The finishing department has three workers, who also work a 7-hour shift. Each pair of Jordanelle skis requires 3.5 labor hours in the fabricating department and one labor hour in finishing. The Deercrest model requires four labor hours in fabricating and 1.5 labor hours in finishing. The company operates five days per week. SSC makes a net profit of $50 on the Jordanelle model, and $65 on the Deercrest model. In anticipation of the next ski sale season, SSC must plan its production of these two models. Because

of the popularity of its products and limited production capacity, its products are in high demand and SSC can sell all it can produce each season. The company anticipates selling at least twice as many Deercrest models as Jordanelle models. The company wants to determine how many of each model should be produced on a daily basis to maximize net profit.

Step 1: Define the decision variables. SSC wishes to determine how many of each model skis to produce. Thus, we may define

Jordanelle = number of pairs of Jordanelle skis produced/day

Deercrest = number of pairs of Deercrest skis produced/day

We usually represent decision variables by short, descriptive names, abbreviations, or subscripted letters such as X_1 and X_2. For many mathematical formulations involving many variables, subscripted letters are often more convenient; however, in spreadsheet models, we recommend using more descriptive names to make the models and solutions easier to understand. Also, it is very important to clearly specify the dimensions of the variables, for example, "pairs/day" rather than simply "Jordanelle skis."

Step 2: Identify the objective function. The problem states that SSC wishes to maximize profit. In some problems, the objective is not explicitly stated, and you must use logic and business experience to identify the appropriate objective.

Step 3: Identify the constraints. From the information provided, we see that labor hours are limited in both the fabrication department and finishing department. Therefore, we have the constraints:

Fabrication: Total labor used in fabrication cannot exceed the amount of labor available.

Finishing: Total labor used in finishing cannot exceed the amount of labor available.

In addition, the problem notes that the company anticipates selling at least twice as many Deercrest models as Jordanelle models. Thus, we need a constraint that states:

The number of pairs of Deercrest skis must be at least twice the number of Jordanelle skis.

Finally, we must ensure that negative values of the decision variables cannot occur. Nonnegativity constraints are assumed in nearly all optimization models.

Step 4: Write the objective function and constraints as mathematical expressions Because SSC makes a net profit of $50 on the Jordanelle model, and $65 on the Deercrest model, the objective function is:

$$\text{Maximize total profit} = 50\,\textit{Jordanelle} + 65\,\textit{Deercrest}$$

For the constraints, we will use the approach described earlier in this chapter. First, consider the fabrication and finishing constraints. Write these as:

Fabrication: Total labor used in fabrication $\leq$ the amount of labor available

Finishing: Total labor used in finishing $\leq$ the amount of labor available

Now translate both the constraint functions on the left and the limitations on the right into mathematical or numerical terms. Note that the amount of labor available in fabrication is (12 workers) (7 hours/day) = 84 hours/day, while in finishing we have (3 workers) (7 hours/day) = 21 hours/day. Because each pair of Jordanelle skis requires 3.5 labor hours and Deercrest skis require 4 labor hours in the fabricating department, the total labor used in fabrication is 3.5 *Jordanelle* + 4 *Deercrest*. Note that the dimensions of these terms are (hours/pair of skis) (number of pairs of skis produced) = hours. Similarly, for the finishing department, the total labor used is 1 *Jordanelle* + 1.5 *Deercrest*. Therefore, the appropriate constraints are:

$$\text{Fabrication: } 3.5\,\textit{Jordanelle} + 4\,\textit{Deercrest} \leq 84$$
$$\text{Finishing: } 1\,\textit{Jordanelle} + 1.5\,\textit{Deercrest} \leq 21$$

For the market mixture constraint, "Number of pairs of Deercrest skis must be at least twice the number of pairs of Jordanelle skis," we have:

$$\textit{Deercrest} \geq 2\,\textit{Jordanelle}$$

It is customary to write all the variables on the left-hand side of the constraint. Thus, an alternative expression for this constraint is:

$$\textit{Deercrest} - 2\,\textit{Jordanelle} \geq 0$$

The difference between the number of pairs of Deercrest skis and twice the number of pairs of Jordanelle skis can be thought of as the excess number of pairs of Deercrest skis produced over the minimum market mixture requirement. Finally, nonnegativity constraints are written as:

$$\textit{Deercrest} \geq 0$$
$$\textit{Jordanelle} \geq 0$$

The complete optimization model is:

$$\begin{aligned} \text{Maximize total profit} &= 50\,\textit{Jordanelle} + 65\,\textit{Deercrest} \\ 3.5\,\textit{Jordanelle} + 4\,\textit{Deercrest} &\leq 84 \\ 1\,\textit{Jordanelle} + 1.5\,\textit{Deercrest} &\leq 21 \\ \textit{Deercrest} - 2\,\textit{Jordanelle} &\geq 0 \\ \textit{Deercrest} &\geq 0 \\ \textit{Jordanelle} &\geq 0 \end{aligned}$$

Characteristics of Linear Optimization Models

A *linear* optimization model (often called a **linear program**, or **LP**) has two basic properties. First, the objective function and all constraints are *linear functions* of the decision variables. This means that each function is simply a sum of terms, each of which is some constant multiplied by a decision variable. The SSC model has this property. Recall the constraint example that we developed earlier for the nitrogen requirement. Notice that the constraint function on the left-hand side of the constraint:

$$(0.20x + 0.33y)/(x + y) = 0.3$$

as originally written is not linear. However, we were able to convert it to a linear form using simple algebra. This is advantageous, as special, highly efficient solution algorithms are used for linear optimization problems.

The second property of a linear optimization problem is that all variables are *continuous*, meaning that they may assume any real value (typically, nonnegative). Of course, this assumption may not be realistic for a practical business problem (you cannot produce half a refrigerator!). However, because this assumption simplifies the solution method and analysis, we often apply it in many situations where the solution would not be seriously affected. In the next chapter, we will discuss situations where it is necessary to force variables to be whole numbers (integers). For all examples and problems in this chapter, we will assume continuity of the variables.

IMPLEMENTING LINEAR OPTIMIZATION MODELS ON SPREADSHEETS

We will learn how to solve optimization models using an Excel tool called *Solver.* To facilitate the use of *Solver,* we suggest the following spreadsheet engineering guidelines for designing spreadsheet models for optimization problems:

- **Put the objective function coefficients, constraint coefficients, and right-hand values in a logical format in the spreadsheet.** For example, you might assign the decision variables to columns and the constraints to rows, much like the mathematical formulation of the model, and input the model parameters in a matrix. If you have many more variables than constraints, it might make sense to use rows for the variables and columns for the constraints.
- **Define a set of cells (either rows or columns) for the values of the decision variables.** In some models, it may be necessary to define a matrix to represent the decision variables. The names of the decision variables should be listed directly above the decision variable cells. Use shading or other formatting to distinguish these cells.
- **Define separate cells for the objective function and each constraint function (the left-hand side of a constraint).** Use descriptive labels directly above these cells.

We will illustrate these principles for the Sklenka Ski example. Figure 13.1 shows a spreadsheet model for the product mix example (Excel file *Sklenka Skis*). The *Data* portion of the spreadsheet provides the objective function coefficients, constraint coefficients, and right-hand sides of the model. Such data should be kept separate from the actual model so that if any data are changed, the model will automatically be updated. In the *Model* section, the number of each product to make is given in cells B14 and C14. Also in the *Model* section are calculations for the constraint functions:

$$3.5\,\textit{Jordanelle} + 4\,\textit{Deercrest} \quad \text{(hours used in fabrication, cell D15)}$$
$$1\,\textit{Jordanelle} + 1.5\,\textit{Deercrest} \quad \text{(hours used in finishing, cell D16)}$$
$$\textit{Deercrest} - 2\,\textit{Jordanelle} \quad \text{(market mixture, cell D19)}$$

and the objective function, 50 *Jordanelle* + 65 *Deercrest* (cell D22).

To show the correspondence between the mathematical model and the spreadsheet model more clearly, we will write the model in terms of the spreadsheet cells:

$$\text{Maximize profit} = \text{D22} = \text{B9*B14} + \text{C9*C14}$$

subject to the constraints:

$$\text{D15} = \text{B6*B14} + \text{C6*C14} \le \text{D6} \quad \text{(fabrication)}$$
$$\text{D16} = \text{B7*B14} + \text{C7*C14} \le \text{D7} \quad \text{(finishing)}$$

	A	B	C	D
1	**Sklenka Skis**			
2				
3	**Data**			
4		**Product**		
5	**Department**	Jordanelle	Deercrest	Limitation (min.)
6	Fabrication	3.5	4	84
7	Finishing	1	1.5	21
8				
9	Profit/unit	$ 50.00	$ 65.00	
10				
11				
12	**Model**			
13		Jordanelle	Deercrest	
14	Quantity Produced	0	0	Hours Used
15	Fabrication	0	0	0
16	Finishing	0	0	0
17				
18				Excess Deercrest
19	Market mixture			0
20				
21				Total Profit
22	Profit Contribution	$ -	$ -	$ -

	A	B	C	D
1	**Sklenka Skis**			
2				
3	**Data**			
4		**Product**		
5	**Department**	Jordanelle	Deercrest	Limitation (min.)
6	Fabrication	3.5	4	84
7	Finishing	1	1.5	21
8				
9	Profit/unit	50	65	
10				
11				
12	**Model**			
13		Jordanelle	Deercrest	
14	Quantity Produced	0	0	Hours Used
15	Fabrication	=B6*$B$14	=C6*$C$14	=B15+C15
16	Finishing	=B7*$B$14	=C7*$C$14	=B16+C16
17				
18				Excess Deercrest
19	Market mixture			=C14-2*B14
20				
21				Total Profit
22	Profit Contribution	=B9*$B$14	=C9*$C$14	=B22+C22

FIGURE 13.1 Sklenka Skis Model Spreadsheet Implementation

$$\text{D19} = \text{C14} - 2\text{*B14} \ge 0 \quad \text{(market mixture)}$$
$$\text{B14} \ge 0,\ \text{C14} \ge 0 \quad \text{(nonnegativity)}$$

Observe how the constraint functions and right-hand-side values are stored in separate cells within the spreadsheet.

In Excel, the pair-wise sum of products of terms can easily be computed using the SUMPRODUCT function. This often simplifies the model-building process, particularly when many variables are involved. For example, the objective function formula could have been written as:

$$\text{B9*B14} + \text{C9*C14} = \text{SUMPRODUCT(B9:C9, B14:C14)}$$

Similarly, the labor limitation constraints could have been expressed as:

$$\text{B6*B14} + \text{C6*C14} = \text{SUMPRODUCT(B6:C6, B14:C14)}$$
$$\text{B7*B14} + \text{C7*C14} = \text{SUMPRODUCT(B7:C7, B14:C14)}$$

Excel Functions to Avoid in Modeling Linear Programs

Several common functions in Excel can cause difficulties when attempting to solve linear programs using *Solver* because they are discontinuous (or "nonsmooth") and no longer satisfy the conditions of a linear model. For instance, in the formula IF (A12 < 45, 0, 1), the cell value jumps from 0 to 1 when the value of cell A12 crosses 45. In such situations, the correct solution may not be identified. Common Excel functions to avoid are ABS, MIN, MAX, INT, ROUND, IF, and COUNT. While these are useful in general modeling tasks with spreadsheets, you should avoid them in linear optimization models.

SOLVING LINEAR OPTIMIZATION MODELS

To solve an optimization problem, we seek values of the decision variables that maximize or minimize the objective function and also satisfy all constraints. Any solution that satisfies all constraints of a problem is called a **feasible solution**. Finding an optimal solution among the infinite number of possible feasible solutions to a given problem is not an easy task. A simple approach is to try to manipulate the decision variables in the spreadsheet models to find the best solution possible; however, for many problems, it might be very difficult to find a feasible solution, let alone an optimal solution. You might try to find the best solution possible for the Sklenka Ski problem by using the spreadsheet model. With a little experimentation and perhaps a bit of luck, you might be able to zero in on the optimal solution or something close to it. However, to guarantee finding an optimal solution, some type of systematic mathematical solution procedure is necessary. Fortunately, such a procedure is provided by the Excel *Solver* tool, which we discuss next.

Solver is an add-in packaged with Excel that was developed by Frontline Systems Inc. (*www.solver.com*) and can be used to solve many different types of optimization problems. *Premium Solver*, which is part of *Risk Solver Platform* that accompanies this book, is an improved alternative to the standard Excel-supplied *Solver*. *Premium Solver* has better functionality, numerical accuracy, reporting, and user interface. We will show how to solve the SSC model using both the standard and premium versions; however, we highly recommend using the premium version, and we will use it in the remainder of this chapter.

Solving the SSC Model Using Standard Solver

The standard *Solver* can be found in the *Analysis* group under the *Data* tab in Excel 2010 (see Chapter 9 for installation directions). Figure 13.2 shows the completed *Solver Parameters* dialog for this example. The objective function cell in the spreadsheet (D22) is defined in the *Set Objective* field. Either enter the cell reference or click within the field and then in the cell in the spreadsheet. Click the appropriate radio button for *Max* or *Min*. Decision variables (cells B14 and C14) are entered in the field called *By Changing Variable Cells;* click within this field and highlight the range corresponding to the decision variables in your spreadsheet.

To enter a constraint, click the *Add* button. A new dialog, *Add Constraint*, appears (see Figure 13.3). In the left field, *Cell Reference*, enter the cell that contains the constraint function (left-hand side of the constraint). For example, the constraint function for the fabrication constraint is in cell D15. Make sure that you select the correct type of constraint (≤, ≥, or =) in the drop down box in the middle of the dialog. The other options are discussed in the next chapter. In the right field, called *Constraint*, enter the numerical value of the right-hand side of the constraint or the cell reference corresponding to it. For the fabrication constraint, this is cell D6. Figure 13.3 shows the completed dialog for the fabrication constraint. To add other constraints, click the *Add* button.

You may also define a group of constraints that all have the same algebraic form (either all ≤, all ≥, or all=) and enter them together. For example, the department resource limitation constraints are expressed within the spreadsheet model as:

$$D15 \le D6$$
$$D16 \le D7$$

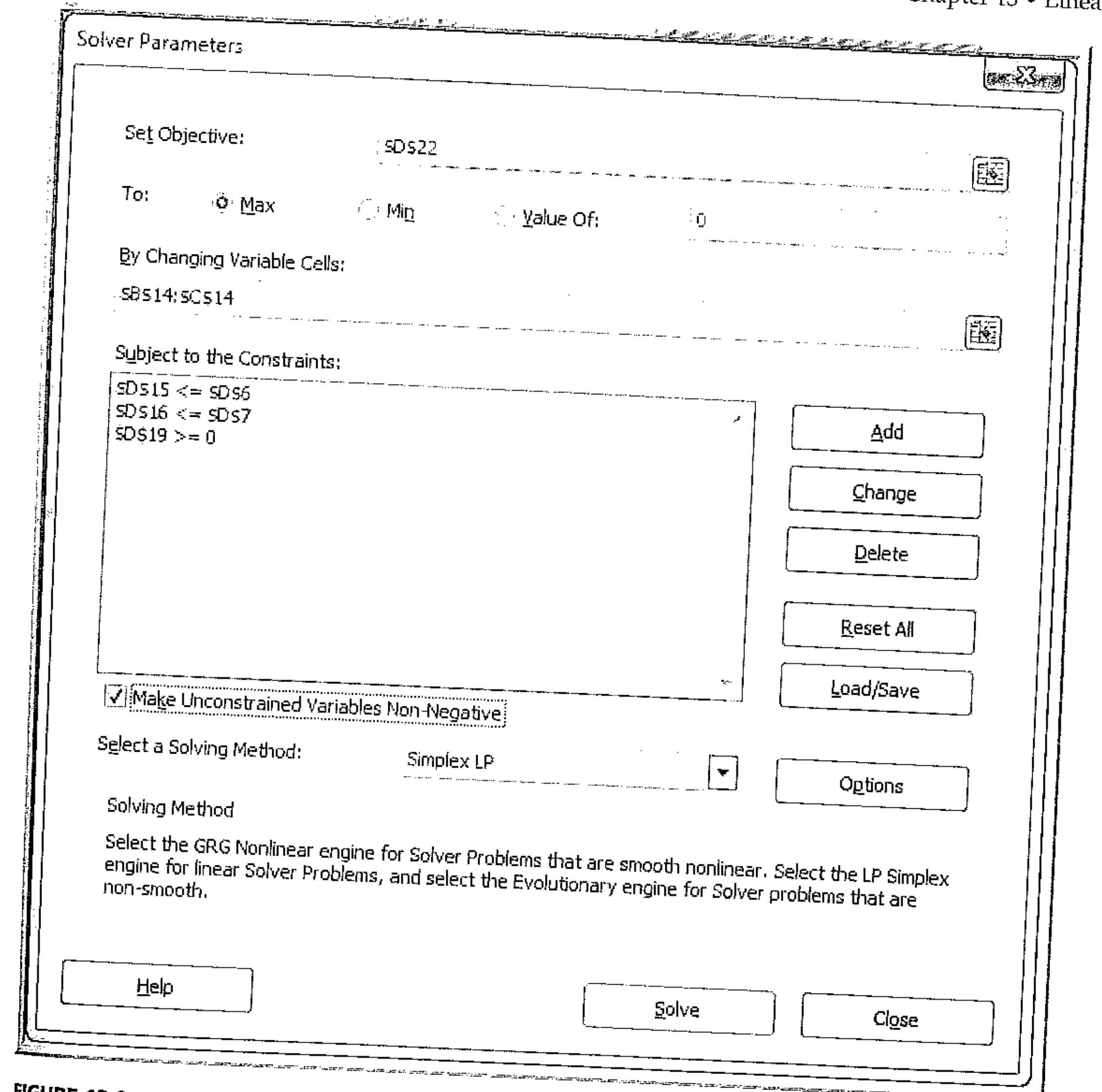

FIGURE 13.2 *Solver Parameters* Dialog

Because both constraints are ≤ types, we could define them as a group by entering the range D15:D16 in the *Cell Reference* field and D6:D7 in the *Constraint* field to simplify the input process. When all constraints are added, click *OK* to return to the *Solver Parameters* dialog box. You may add, change, or delete these as necessary by clicking the

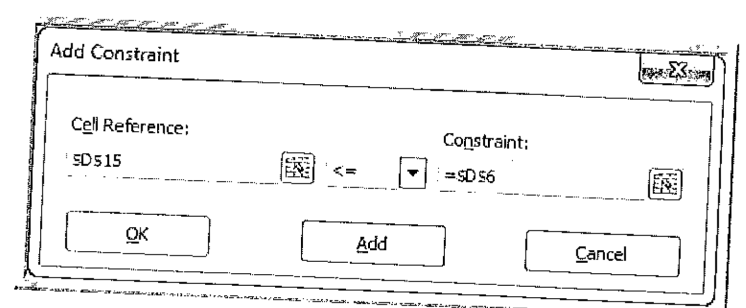

FIGURE 13.3 *Add Constraint* Dialog

appropriate buttons. You need not enter nonnegativity constraints explicitly. Just check the box in the dialog *Make Unconstrained Variables Non-Negative*.

For linear optimization problems it is very important to select the correct solving method. The standard Excel *Solver* provides three options for the solving method:

1. *Standard GRG Nonlinear*—used for solving nonlinear optimization problems
2. *Standard LP Simplex*—used for solving linear and linear integer optimization problems
3. *Standard Evolutionary*—used for solving complex nonlinear and nonlinear integer problems

In the field labeled *Select a Solving Method*, choose *Simplex LP*. Then click the *Solve* button to solve the problem. The *Solver Results* dialog will appear, as shown in Figure 13.4, with the message "Solver found a solution." If a solution could not be found, *Solver* will notify you with a message to this effect. This generally means that you have an error in your model or you have included conflicting constraints that no solution can satisfy. In such cases, you will need to reexamine your model.

Solver generates three reports as listed in Figure 13.4: Answer, Sensitivity, and Limits. To add them to your Excel workbook, click on the ones you want, and then click *OK*. The optimal solution will be shown in the spreadsheet as in Figure 13.5. The maximum profit is $945, obtained by producing 5.25 pairs of Jordanelle skis and 10.5 pairs of Deercrest skis per day (remember that linear models allow fractional values for the decision variables!). If you save your spreadsheet after setting up a *Solver* model, the *Solver* model will also be saved.

Solving the SSC Model Using *Premium Solver*

Premium Solver has a different user interface than the standard *Solver*. After installing *Risk Solver Platform*, *Premium Solver* will be found under the *Add-Ins* tab in the Excel ribbon. Figure 13.6 shows the *Premium Solver* dialog. First click on *Objective* and then click the *Add* button. The *Add Objective* dialog appears, prompting you for the cell reference for the objective function and the type of objective (min or max) similar

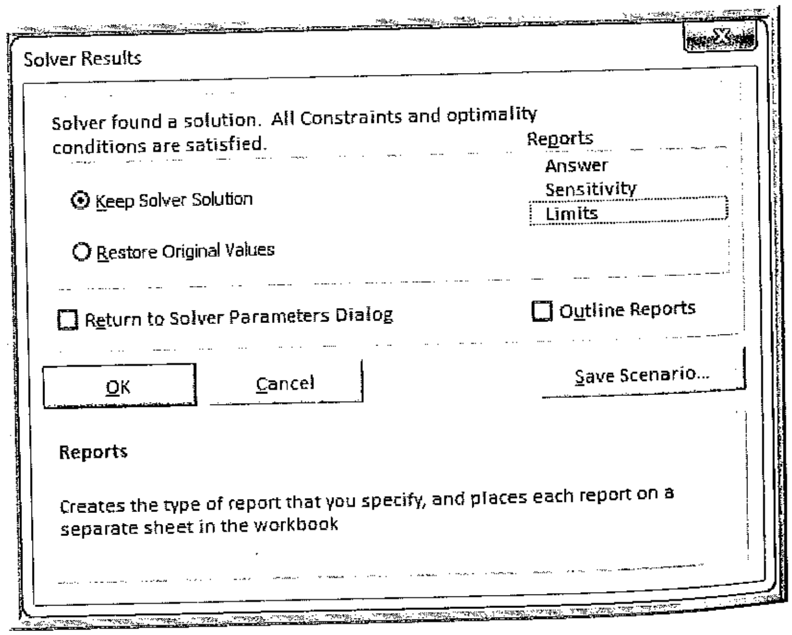

FIGURE 13.4 *Solver Results* Dialog

to the top portion of the standard *Solver Parameters* dialog. Next, highlight *Normal* under the *Variables* list and click *Add*; this will bring up an *Add Variable Cells* dialog. Enter the range of the decisions variables in the *Cell Reference* field. Next, highlight Normal under the *Constraints* list and click the *Add* button; this brings up the *Add Constraint* dialog just like in the standard version. Add the constraints in the same fashion as in the standard *Solver*. Check the box *Make Unconstrained Variables Non-Negative*. The premium version provides the same solving method options as the standard version (except that *Standard LP Simplex* is called *Standard LP/Quadratic*), so select this for linear optimization. Quadratic optimization problems have a special structure that we will not address in this book. The premium version also has three additional advanced solving methods. The *Solver Results* dialog is the same as in the standard version.

Solver Outcomes and Solution Messages

Solving a linear optimization model can result in four possible outcomes:

1. Unique optimal solution
2. Alternate optimal solutions
3. Unboundedness
4. Infeasibility

When a model has a *unique optimal solution*, it means that there is exactly one solution that will result in the maximum (or minimum) objective. The solution to the SSC model is unique. If a model has *alternate optimal solutions*, the objective is maximized (or minimized) by more than one combination of decision variables, all of which have the same objective function value. *Solver* does not tell you when alternate solutions exist and only reports one of the many possible alternate optimal solutions. However, you can use the sensitivity report information to identify the existence of alternate optimal solutions. When any of the Allowable Increase or Allowable Decrease values for changing cells are zero, then alternate optimal solutions exist, although *Solver* does not provide an easy way to find them.

A problem is *unbounded* if the objective can be increased or decreased without bound (i.e., to infinity or negative infinity) while the solution remains feasible. A model is unbounded if *Solver* reports "The Set Cell values do not converge." This generally indicates an incorrect model, usually when some constraint or set of constraints have been left out.

Finally, an *infeasible* model is one for which no feasible solution exists; that is, when there is no solution that satisfies all constraints together. When a problem is infeasible, *Solver* will report "Solver could not find a feasible solution." Infeasible problems can occur in practice, for example, when a demand requirement is higher than available capacity, or when managers in different departments have conflicting requirements or limitations. In such cases, the model must be reexamined and modified. Sometimes infeasibility or unboundedness is simply a result of a misplaced decimal or other error in the model or spreadsheet implementation, so accuracy checks should be made.

Interpreting *Solver* Reports

The Answer Report, shown in Figure 13.7 (all reports in this section were generated using *Premium Solver*), provides basic information about the solution, including the values of the optimal objective function (in the *Objective Cell* section) and decision variables (in the *Decision Variable Cells* section). In the *Constraints* section, *Cell Value* refers to the value of the constraint function using the optimal values of the decision variables. In other words, we used 60.375 minutes in the fabrication department and 21 minutes

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$22	Profit Contribution Total Profit	0	945

Decision Variable Cells

Cell	Name	Original Value	Final Value
$B$14	Quantity Produced Jordanelle	0	5.25
$C$14	Quantity Produced Deercrest	0	10.5

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$15	Fabrication Hours Used	60.375	$D$15<=$D$6	Not Binding	23.625
$D$16	Finishing Hours Used	21	$D$16<=$D$7	Binding	0
$D$19	Market mixture Excess Deercrest	0	$D$19>=0	Binding	0

FIGURE 13.7 *Solver* Answer Report

in the finishing department by producing 5.25 pairs of Jordanelle skis and 10.5 pairs of Deercrest skis. The *Status* column tells whether each constraint is binding or not binding. A **binding constraint** is one for which the *Cell Value* is equal to the right-hand side of the value of the constraint. In this example, the constraint for fabrication is not binding, while the constraints for finishing and market mixture are binding. This means that there is excess time that is not used in fabrication; this value is shown in the *Slack* column as 23.626 hours. For finishing, we used all the time available, and hence, the slack value is zero. Because we produced exactly twice the number of Deercrest skis as Jordanelle skis, the market mixture constraint is binding. It would have been not binding if we had produced more than twice the number of Deercrest skis as Jordanelle.

In general, the **slack** is the difference between the right- and left-hand sides of a constraint. Examine the fabrication constraint:

$$3.5\ Jordanelle + 4\ Deercrest \leq 84$$

We interpret this as:

$$\text{Number of fabrication hours used} \leq \text{Hours available}$$

Note that if the amount used is strictly less than the availability, we have slack, which represents the amount unused; thus,

$$\text{Number of fabrication hours used} + \text{Number of fabrication hours unused} = \text{Hours available}$$

or

$$\begin{aligned}\text{Slack} &= \text{Number of hours unused}\\ &= \text{Hours Available} - \text{Number of fabrication hours used}\\ &= 84 - (3.5 \times 5.25 + 4 \times 10.5) = 23.625\end{aligned}$$

Slack variables are always nonnegative, so for $\geq$ constraints, slack represents the difference between the left-hand side of the constraint function and the right-hand side of the requirement. The slack on a binding constraint will always be zero.

Objective Cell (Max)

Cell	Name	Final Value
$D$22	Profit Contribution Total Profit	945

Decision Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Quantity Produced Jordanelle	5.25	0	50	1E+30	6.6666668
$C$14	Quantity Produced Deercrest	10.5	0	65	10.0000002	90.00000013

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$15	Fabrication Hours Used	60.375	0	84	1E+30	23.625
$D$16	Finishing Hours Used	21	45	21	8.217391304	21
$D$19	Market mixture Excess Deercrest	0	-2.5	0	14	42

FIGURE 13.8 *Solver* Sensitivity Report

The Sensitivity Report (Figure 13.8) provides a variety of useful information for managerial interpretation of the solution. Specifically, it allows us to understand how the optimal objective value and optimal decision variables are affected by changes in the objective function coefficients, the impact of forced changes in certain decision variables, or the impact of changes in the constraint resource limitations or requirements. In the *Decision Variable Cells* section, the final value for each decision variable is given, along with its reduced cost, objective coefficient, and allowable increase and decrease. The **reduced cost** tells *how much the objective coefficient needs to be reduced in order for a nonnegative variable that is zero in the optimal solution to become positive.* If a variable is positive in the optimal solution, as it is for both variables in this example, its reduced cost is always zero. We will see an example later in this chapter that will help you to understand reduced costs.

The Allowable Increase and Allowable Decrease values tell how much an individual objective function coefficient can change before the optimal values of the decision variables will change (a value listed as "1E + 30" is interpreted as infinity). For example, if the unit profit for Deercrest skis either increases by more than 10 or decreases by more than 90, then the optimal values of the decision variables will change (as long as the other objective coefficient stays the same). For instance, if we increase the unit profit by 11 (to 76) and re-solve the model, the new optimal solution will be to produce 14 pairs of Deercrest skis and no Jordanelle skis. However, any increase less than 10 will keep the current solution optimal. For Jordanelle skis, we can increase the unit profit as much as we wish without affecting the current optimal solution; however, a decrease of at least 6.67 will cause the solution to change.

Note that if the objective coefficient of any one variable that has positive value in the current solution changes but stays within the range specified by the Allowable Increase and Allowable Decrease, the optimal decision variables will stay the same; however, the objective function value will change. For example, if the unit profit of Jordanelle skis were changed to $46 (a decrease of 4, within the allowable increase), then we are guaranteed that the optimal solution will still be to produce 5.25 pairs of Jordanelle and 10.5 pairs of Deercrest. However, each of the 5.25 pairs of Jordanelle skis produced and sold would realize $4 less profit—a total decrease of 5.25($4) = $21. Thus, the new value of the objective function would be $945 − $21 = $924. If an objective coeffici

changes beyond the Allowable Increase or Allowable Decrease, then we must re-solve the problem with the new value to find the new optimal solution and profit.

The range within which the objective function coefficients will not change the optimal solution provides a manager with some confidence about the stability of the solution in the face of uncertainty. If the allowable ranges are large, then reasonable errors in estimating the coefficients will have no effect on the optimal policy (although they will affect the value of the objective function). Tight ranges suggest that more effort might be spent in ensuring that accurate data or estimates are used in the model.

The *Constraints* section of the Sensitivity Report lists the final value of the constraint function (the left-hand side), the shadow price, the original constraint right-hand side, and an Allowable Increase and Allowable Decrease. The **shadow price** tells *how much the value of the objective function will change as the right-hand side of a constraint is increased by 1.* Whenever a constraint has positive slack, the shadow price is zero. For example, in the fabrication constraint, we are not using all of the available hours in the optimal solution. Thus, having one more hour available will not help us improve the solution. However, if a constraint is binding, then any change in the right-hand side will cause the optimal values of the decision variables as well as the objective function value to change.

Let us illustrate this with the finishing constraint. The shadow price of 45 states that if an additional hour of finishing time is available, then the total profit will change by $45. To see this, change the limitation of the number of finishing hours available to 22 and re-solve the problem. The new solution is to produce 5.5 pairs of Jordanelle and 11.0 pairs of Deercrest, yielding a profit of $990. We see that the total profit increases by $45 as predicted. Thus, the shadow price represents the economic value of having an additional unit of a resource.

The shadow price is a valid predictor of the change in the objective function value for each unit of increase in the constraint right-hand side up to the value of the Allowable Increase. Thus, if up to about 8.2 additional hours of finishing time were available, profit would increase by $45 for each additional hour (but we would have to resolve the problem to actually find the optimal values of the decision variables). Similarly, the negative of the shadow price predicts the change in the objective function value for each unit the constraint right-hand side is decreased, up to the value of the Allowable Decrease. For example, if one person was ill or injured, resulting in only 14 hours of finishing time available, then profit would decrease by 7($45) = $315. This can be predicted because a decrease of 7 hours is within the Allowable Decrease of 21. Beyond these ranges, the shadow price does not predict what will happen, and the problem must be re-solved.

Another way of understanding the shadow price is to break down the impact of a change in the right-hand side of the value. How was the extra hour of finishing time used? After solving the model with 22 hours of finishing time, we see that we were able to produce an additional 0.25 pairs of Jordanelle and 0.5 pairs of Deercrest skis as compared to the original solution. Therefore, the profit increased by 0.25($50) + 0.5(65) = $12.50 + 32.50 = $45. In essence, a change in a binding constraint causes a reallocation of how the resources are used.

Interpreting the shadow price associated with the market mixture constraint is a bit more difficult. If you examine the constraint, *Deercrest* − 2 *Jordanelle* ≥ 0, closely, an increase in the right-hand side from 0 to 1 results in a change of the constraint to:

$$(\mathit{Deercrest} - 1) - 2\,\mathit{Jordanelle} \geq 0$$

This means that the number of pairs of Deercrest skis produced would be one short of the requirement that it be at least twice the number of Jordanelle skis. If the problem is re-solved with this constraint, we find the new optimal solution to be 4.875 Jordanelle, 9.75 Deercrest, and profit = 942.50. The profit changed by the value of the shadow price and we see that 2 × Jordanelle = 9.75, one short of the requirement.

Cell	Objective Name	Value
$D$22	Profit Contribution Total Profit	$ 945.00

Cell	Decision Variable Name	Value	Lower Limit	Objective Result	Upper Limit	Objective Result
$B$14	Quantity Produced Jordanelle	5.25	0	$ 682.50	5.25	$ 945.00
$C$14	Quantity Produced Deercrest	10.5	10.5	$ 945.00	10.5	$ 945.00

FIGURE 13.9 *Solver* Limits Report

Why are shadow prices useful to a manager? They provide guidance on how to reallocate resources or change values over which the manager may have control. In linear optimization models, the parameters of some constraints cannot be controlled. For instance, the amount of time available for production or physical limitations on machine capacities would clearly be uncontrollable. Other constraints represent policy decisions, which, in essence, are arbitrary. Although it is correct to state that having an additional hour of finishing time will improve profit by $45, does this necessarily mean that the company should spend up to this amount for additional hours? This depends on whether the relevant costs have been included in the objective function coefficients. If the cost of labor *has not* been included in the objective function unit profit coefficients, then the company will benefit by paying less than $45 for additional hours. However, if the cost of labor *has* been included in the profit calculations, the company should be willing to pay up to an *additional* $45 over and above the labor costs that have already been included in the unit profit calculations.

The Limits Report (Figure 13.9) shows the lower limit and upper limit that each variable can assume while satisfying all constraints and holding all of the other variables constant. Generally, this report provides little useful information for decision making and can be effectively ignored.

SKILL-BUILDER EXERCISE 13.1

Make the following changes to the Sklenka Ski Company model, re-solve using *Solver*, and answer the following questions:

a. Increase the unit profit on Jordanelle skis by $10. What happens to the solution? Could you have predicted this from the Sensitivity Report (Figure 13.8)?
b. Decrease the unit profit on Jordanelle skis by $10. What happens to the solution? Could you have predicted this from the Sensitivity Report (Figure 13.8)?
c. Increase the number of finishing hours available by 10. What happens to the solution? Could you have predicted this from the Sensitivity Report (Figure 13.8)?
d. Decrease the number of finishing hours available by 10. What happens to the solution? Could you have predicted this from the Sensitivity Report (Figure 13.8)?
e. Change the unit profit for Deercrest skis to $75. What solution do you get? Do alternate optimal solutions exist? Verify that producing 0 pairs of Jordanelle and 14 pairs of Deercrest skis is an alternate optimal solution for this scenario.
f. Change the finishing and fabrication constraints to be ≥ instead of ≤ type of constraints. What happens? Why did this occur?
g. Change the finishing and fabrication constraints to = instead of ≤ type of constraints. What happens? Why did this occur?

How *Solver* Creates Names in Reports

How you design your spreadsheet model will affect on how *Solver* creates the names used in the output reports. Poor spreadsheet design can make it difficult or confusing to interpret the Answer and Sensitivity reports. Thus, it is important to understand how to do this properly.

Solver assigns names to target cells, changing cells, and constraint function cells by concatenating the text in the first cell containing text to the left of the cell with the first cell containing text above it. For example, in the SSC model, the target cell is D22. The first cell containing text to the left of D15 is "Profit Contribution" in A22, and the first cell containing text above D22 is "Total Profit" in cell D21. Concatenating these text strings yields the target cell name "Profit Contribution Total Profit," which is found in the *Solver* reports. The constraint functions are calculated in cells D15 and D16. Note that their names are "Fabrication Hours Used" and "Finishing Hours Used." Similarly, the changing cells in B14 and C14 have the names "Quantity Produced Jordanelle" and "Quantity Produced Deercrest." These names make it easy to interpret the information in the Answer and Sensitivity reports. We encourage you to examine each of the target cells, changing variable cells, and constraint function cells in your models carefully so that names are properly established.

Difficulties with *Solver*

A poorly scaled model—one in which the parameters of the objective and constraint functions differ by several orders of magnitude (as we have in the transportation example where costs are in tens and supplies/demands in thousands) may cause round-off errors in internal computations or error messages such as "The conditions for Assume Linear Model are not satisfied." This does not happen often; if it does, you should consult the Frontline Systems' Web site for additional information. Usually, all you need to do is to keep the solution that *Solver* found and run *Solver* again starting from that solution. Experts often suggest that the values of the coefficients in the objective function and constraints, as well as the right-hand sides, should not differ from each other by a factor of more than 1,000 or 10,000. *Solver* has an option called *Use Automatic Scaling*; it can be accessed by clicking the *Options* button in the *Solver Parameters* dialog, especially if *Solver* gives an error message that linearity is not satisfied.

APPLICATIONS OF LINEAR OPTIMIZATION

Linear optimization models are the most ubiquitous of optimization models used in organizations today. Applications abound in operations, finance, marketing, engineering, and many other disciplines. Table 13.1 summarizes some common types of generic linear optimization models. We already saw an example of a product mix model with the Sklenka Ski problem. This list represents but a very small sample of the many practical types of linear optimization models that are used.

Building optimization models is more of an art than a science, as there often are several ways of formulating a particular problem. Learning how to build optimization models requires logical thought but can be facilitated by studying examples of different models and observing their characteristics.

The most challenging aspect of model formulation is identifying constraints. Understanding the different types of constraints can help in proper identification and modeling. Constraints generally fall into one of the following categories:

- ***Simple Bounds.*** Simple bounds constrain the value of a single variable. You can recognize simple bounds in problem statements like "no more than $10,000 may be invested in stock XYZ" or "we must produce at least 350 units of product Y to

3.11 shows the Sensitivity Report
Reduced Cost and Shadow Price
shown in Figure 13.12 (obtained
). Thus, we *highly recommend* that
you select the reduced cost and
or three decimal places.

	G	H
ctive cient	Allowable Increase	Allowable Decrease
12.6	1E+30	3.19
14.35	1.57	1E+30
11.52	1E+30	3.75
17.58	0.34	1.57
9.75	3.19	1E+30
16.26	1E+30	1.57
8.11	3.75	1E+30
17.92	1.57	0.34

aint ide	Allowable Increase	Allowable Decrease
1200	150	0
800	1E+30	0
150	0	150
350	0	150
500	0	500
1000	0	150

ortation Model

	G	H
ctive cient	Allowable Increase	Allowable Decrease
12.6	1E+30	3.19
14.35	1.57	1E+30
11.52	1E+30	3.75
17.58	0.34	1.57
9.75	3.19	1E+30
16.26	1E+30	1.57
8.11	3.75	1E+30
17.92	1.57	0.34

aint Allowable Allowable

Interpreting Reduced Costs

The transportation model is a good example to use to discuss the interpretation of reduced costs. First, note that the reduced costs are zero for all variables that are positive in the solution. Now examine the reduced cost, 3.19, associated with shipping from Marietta to Cleveland. A question to ask is "why does the optimal solution ship nothing between these cities?" The answer is simple: It is not economical to do so! In other words, it costs too much to ship from Marietta to Cleveland; the demand can be met less expensively by shipping from Minneapolis. The next logical question to ask is "what would the unit shipping cost have to be to make it attractive to ship from Marietta instead of Minneapolis?" The answer is given by the reduced cost. If the unit cost is reduced by at least \$3.19, then the optimal solution will change and would include a positive value for the Marietta-Cleveland variable. Again, this is only true if all other data are held constant.

Multiperiod Production Planning

Many linear optimization problems involve planning over multiple time periods. Kristin's Kreations is a home-based company that makes hand-painted jewelry boxes for teenage girls. Forecasts of sales for the next year are 150 in the autumn, 400 in the winter, and 50 in the spring. Plain jewelry boxes are purchased from a supplier for \$20. The cost of capital is estimated to be 24% per year (or 6% per quarter); thus, the holding cost per item is 0.06(\$20) = \$1.20 per quarter. Kristin hires art students part-time to craft her designs during the autumn, and they earn \$5.50 per hour. Because of the high demand for part-time help during the winter holiday season, labor rates are higher in the winter, and workers earn \$7.00 per hour. In the spring, labor is more difficult to keep, and the owner must pay \$6.25 per hour to retain qualified help. Each jewelry box takes 2 hours to complete. How should production be planned over the three quarters to minimize the combined production and inventory holding costs?

The principal decision variables are the number of jewelry boxes to produce during each of the three quarters. While it might seem obvious to simply produce to the anticipated level of sales, it may be advantageous to produce more during some quarter and carry the items in inventory, thereby letting lower labor rates offset the carrying costs. Therefore, we must also define decision variables for the number of units to hold in inventory at the end of each quarter. The decision variables are:

P_A = amount to produce in autumn
P_W = amount to produce in winter
P_S = amount to produce in spring
I_A = inventory held at the end of autumn
I_W = inventory held at the end of winter
I_S = inventory held at the end of spring

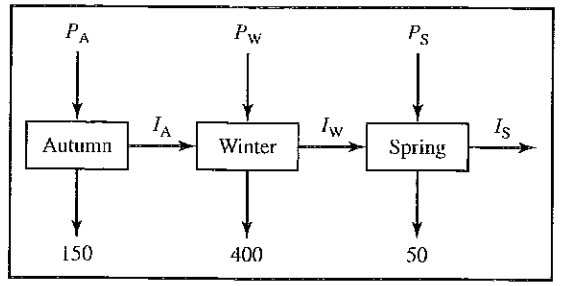

FIGURE 13.13 Material Balance Constraint Structure

The only explicit constraint is that demand must be satisfied. Note that both the production in a quarter as well as the inventory held from the *previous* time quarter can be used to satisfy demand. In addition, any amount in excess of the demand is held to the next quarter. Therefore, the constraints take the form of *inventory balance equations* that essentially say, "what is available in any time period must be accounted for somewhere." More formally,

$$\text{Production} + \text{Inventory from the previous quarter} = \text{Demand} + \text{Inventory held to the next quarter}$$

This can be represented visually using the diagram in Figure 13.13. For each quarter, the sum of the variables coming in must equal the sum of the variables going out. Drawing such a figure is very useful for any type of multiple time period planning model. This results in the constraint set:

$$P_A + 0 = 150 + I_A$$
$$P_W + I_A = 400 + I_W$$
$$P_S + I_W = 50 + I_S$$

Moving all variables to the left side results in the model:

$$\text{Minimize } 11P_A + 14P_W + 12.50P_S + 1.20I_A + 1.20I_W + 1.20I_S$$

Subject to

$$P_A - I_A = 150$$
$$P_W + I_A - I_W = 400$$
$$P_S + I_W - I_S = 50$$
$$P_i \geq 0, \quad \text{for all } i$$
$$I_j \geq 0, \quad \text{for all } j$$

As we have noted, developing models is more of an art than a science; consequently, there is often more than one way to model a particular problem. Using the ideas presented in this example, we may construct an alternative model involving only the production variables. We simply have to make sure that demand is satisfied. We can do this by ensuring that the cumulative production in each quarter is at least as great as the cumulative demand. This is expressed by the following constraints:

$$P_A \geq 150$$
$$P_A + P_W \geq 550$$

$$P_A + P_W + P_S \geq 600$$
$$P_A, P_W, P_S \geq 0$$

The differences between the left- and right-hand sides of these constraints are the ending inventories for each period (and we need to keep track of these amounts because inventory has a cost associated with it). Thus, we use the following objective function:

$$\text{Minimize } 11P_A + 14P_W + 12.50P_S + 1.20(P_A - 150) + 1.20(P_A + P_W - 550) + 1.20(P_A + P_W + P_S - 600)$$

Of course, this function can be simplified algebraically by combining like terms. Although these two models look very different, they are mathematically equivalent and will produce the same solution. The following exercise asks you to verify this.

SKILL-BUILDER EXERCISE 13.5

Implement both the original and alternative models developed for Kristin's Kreations multiperiod planning model to verify that they result in the same optimal solution. In the alternative model, how can you determine the values of the inventory variables?

Multiperiod Financial Planning

Financial planning often occurs over an extended time horizon and can be formulated as multiperiod optimization models. For example, a financial manager at D.A. Branch & Sons must ensure that funds are available to pay company expenditures in the future, but would also like to maximize investment income. Three short-term investment options are available over the next six months: *A*, a one-month CD that pays 0.25%, available each month; *B*, a three-month CD that pays 1.00%, available at the beginning of the first four months; and *C*, a six-month CD that pays 2.3%, available in the first month. The net expenditures for the next six months are forecast as $50,000, ($12,000), $23,000, ($20,000), $41,000, ($13,000). Amounts in parentheses indicate a net inflow of cash. The company must maintain a cash balance of at least $10,000 at the end of each month. The company currently has $200,000 in cash.

At the beginning of each month, the manager must decide how much to invest in each alternative that may be available. Define:

A_i = amount ($) to invest in a one-month CD at the start of month i
B_i = amount ($) to invest in a three-month CD at the start of month i
C_i = amount ($) to invest in a six-month CD at the start of month i

Because the time horizons on these alternatives vary, it is helpful to draw a picture to represent the investments and returns for each year as shown in Figure 13.14. Each circle represents the beginning of a month. Arrows represent the investments and cash flows. For example, investing in a three-month CD at the start of month 1 (B_1) matures at the beginning of month 4. It is reasonable to assume that all funds available would be invested.

From Figure 13.14, we see that investments A_6, B_4, and C_1 will mature at the end of month 6; that is, at the beginning of month 7. To maximize the amount of cash on hand at the end of the planning period, we have the objective function:

$$\text{Maximize } 1.0025A_6 + 1.00B_4 + 1.023C_1$$