Why Chung-Lu model fails for scale-free network with $\gamma < 3$

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1 Chung-Lu model

For a given degree sequence $D = \{d_1, d_2, ..., d_N\}$ where $d_{max}^2 < \sum_{i=1}^{N} d_i$, each potential edge between vertex i and j is chosen with probability $p_{ij} = \frac{d_i d_j}{\sum_{k=1}^{N} d_k}$ and is independent of other edges. [1]

2 scale-free network

For scale-free network, the probability of a node having degree k is $P(k) = \frac{k^{-\gamma}}{H_{N-1,\gamma}}$, where $H_{a,b}$ is the a^{th} generalized harmonic number of exponent b, $H_{a,b} = \sum_{t=1}^{a} t^{-b}$.

When $N >> 1$ and $\gamma > 1$

$$H_{N-1,\gamma} = \sum_{t=1}^{N-1} t^{-\gamma} \approx \int_1^N t^{-\gamma} dt = \frac{N^{1-\gamma} - 1^{1-\gamma}}{1-\gamma} \approx \frac{1}{\gamma - 1} \tag{1}$$

is a constant.

3 expected maximum degree

The expected maximum degree of a scale free sequence is [2]

$$\hat{d} = max\{x : N \sum_{k=x}^{N-1} \frac{k^{-\gamma}}{H_{N-1,\gamma}} \geq 1\} \tag{2}$$

When $N >> 1$ and $\gamma > 1$

$$N \int_{k=x}^{N-1} \frac{k^{-\gamma}}{H_{N-1,\gamma}} dk = 1 \tag{3}$$

$$\frac{N}{(1-\gamma)H_{N-1,\gamma}}[(N-1)^{1-\gamma} - x^{1-\gamma}] = 1 \tag{4}$$

$$x = [\frac{N}{(1-\gamma)H_{N-1,\gamma}}]^{\frac{1}{\gamma-1}} \sim N^{\frac{1}{\gamma-1}} \tag{5}$$

4 expected average degree

The expected average degree is

$$\bar{d} = \sum_{k=1}^{N-1} kP(k) = \sum_{k=1}^{N-1} k\frac{k^{-\gamma}}{H_{N-1,\gamma}} = \frac{\sum_{k=1}^{N-1} k^{-(\gamma-1)}}{H_{N-1,\gamma}} = \frac{H_{N-1,\gamma-1}}{H_{N-1,\gamma}} \tag{6}$$

When $N >> 1$ and $\gamma > 2$,

$$\bar{d} \approx \frac{\gamma-1}{\gamma-2} \tag{7}$$

is a constant.

5 expected summation of degree

When $N >> 1$ and $\gamma > 2$,

$$\sum_{i=1} d_i = N\bar{d} \approx N\frac{\gamma-1}{\gamma-2} \sim N \tag{8}$$

6 Chung-Lu condition for scale-free sequence

Chung-Lu model requires $d_{max}^2 < \sum_{i=1}^{N} d_i$.

Since when $N >> 1$ and $\gamma > 2$, we have $\hat{d} \sim N^{\frac{1}{\gamma-1}}$, $\sum_{i=1}^{N} d_i \sim N$, Chung-Lu model requires

$$N^{\frac{2}{\gamma-1}} < N \tag{9}$$

which is true only when $\gamma > 3$.

This is why Chung-Lu model cannot work when $\gamma < 3$.

References

[1] Fan Chung and Linyuan Lu, *Complex Graphs and Networks*. Page 97.

[2] Charo I. Del Genio, Thilo Gross, and Kevin E. Bassler *All scale-free networks are sparse*