Notes on a generative model of hypergraphs

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1 Problem

We want to find a way of generating hypergraphs with prescribed expected vertex degrees. We proceed to find a generalization of the, so-called, Chung-Lu model for hypergraphs. This will be done using an exponential random hypergraph formulation. We begin by considering 3-regular hypergraphs. The result is then generalized to g-regular hypergraphs.

A new part that was added in August, extends the results to g-regular hypergraphs with α-dimensional joint prescribed expected vertex degrees.

On September 12th a small correction was made that added a factor of $\frac{1}{2}$ to Eqs. 83 through 84. and Eqs. 92 through 95. This change did not affect any other part. This is version 3 of these notes.

2 Prescribed Expected Vertex Degrees

2.1 3-regular Hypergraphs

We wish to randomly generate hypergraphs in with prescribed expected degrees

$$\bar{\mathbf{k}} = \left\{\bar{k}_1,\ \bar{k}_2,\ \ldots,\ \bar{k}_N\right\} \tag{1}$$

where there are assumed to be N vertices. Calling the ensemble of all possible simple 3-regular hypergraphs $\mathcal{G}_3$, the expected degree of vertex i is

$$\bar{k}_i = \sum_{G\in\mathcal{G}_3} k_i(G)\ P(G) \tag{2}$$

where $k_i(G)$ is the degree of vertex i in a graph G and $P(G)$ is the probablity of generating that graph.

Maximizing entropy while attempting to enforce the prescribed degree constraints, the probability of generating a hypergraph is

$$P(G;\vec{\beta}) = \frac{1}{\mathcal{Z}(\vec{\beta})} e^{-\vec{\beta}\cdot\mathbf{k}} \tag{3}$$

where $\vec{\beta}$ is a vector of Laplace coefficients, and $\mathbf{k}$ is the vector of vertex degrees. In what follows we will choose the elements of $\vec{\beta}$ so that the prescribed degree constraints are enforced.

Now, a 3-regular hypergraph can be represented by a 3-dimensional adjacency tensor $\mathbf{a}$, with elements of

$$a_{ijk} = \begin{cases} 1 & \text{If a hyperedge exists connecting vertices } i,\ j,\ \text{and } k. \\ 0 & \text{Otherwise.} \end{cases} \tag{4}$$

Require, of course, that

$$a_{ijk} = a_{ikj} = a_{jik} = a_{jki} = a_{kij} = a_{kji}, \qquad \forall\ i,\ j,\ k, \tag{5}$$

and that

$$a_{ijj} = a_{jij} = a_{jji} = 0, \qquad \forall\ i,\ j. \tag{6}$$

Note that

$$k_i = \sum_{j=1}^{N}\sum_{k=1}^{N} a_{ijk} \tag{7}$$

Recognizing Eqn. 3 as a Gibbs distribution, identify the "Hamiltonian"

$$\begin{aligned}
\mathcal{H}(G;\vec{\beta}) &= \vec{\beta}\cdot\mathbf{k} && (8)\\
&= \sum_{i=1}^{N} \beta_i k_i && (9)\\
&= \sum_{i=1}^{N} \beta_i \sum_{j=1}^{N}\sum_{k=1}^{N} a_{ijk} && (10)\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \beta_i a_{ijk} && (11)\\
&= \sum_{i<j<k} \beta_i a_{ijk} + \sum_{i<k<j} \beta_i a_{ijk} + \sum_{j<i<k} \beta_i a_{ijk} \\
&\quad + \sum_{j<k<i} \beta_i a_{ijk} + \sum_{k<i<j} \beta_i a_{ijk} + \sum_{k<j<i} \beta_i a_{ijk} && (12)\\
&= 2\left(\sum_{i<j<k} \beta_i a_{ijk} + \sum_{i<j<k} \beta_j a_{ijk} + \sum_{i<j<k} \beta_k a_{ijk}\right) && (13)\\
&= 2\sum_{i<j<k} \left(\beta_i + \beta_j + \beta_k\right) a_{ijk} && (14)
\end{aligned}$$

The "partition function" then is

$$\mathcal{Z}(\vec{\beta}) = \sum_{G\in\mathcal{G}_3} e^{-\mathcal{H}(G;\vec{\beta})} \tag{15}$$

$$= \sum_{a_{123}=0}^{1}\sum_{a_{124}=0}^{1}\cdots\sum_{a_{N-2,N-1,N}=0}^{1} e^{-2\sum_{i<j<k}(\beta_i+\beta_j+\beta_k)a_{ijk}} \tag{16}$$

$$= \prod_{i<j<k}\sum_{a_{ijk}=0}^{1} e^{-2(\beta_i+\beta_j+\beta_k)a_{ijk}} \tag{17}$$

$$= \prod_{i<j<k}\left(1+e^{-2(\beta_i+\beta_j+\beta_k)}\right) \tag{18}$$

The probability that a hyperedge exists between vertices r, s, and t is (assume that $r<s<t$)

$$p_{rst} = \langle a_{rst}\rangle \tag{19}$$

$$= \sum_{G\in\mathcal{G}_3} a_{rst}(G)\ P(G;\vec{\beta}) \tag{20}$$

$$= \frac{1}{\mathcal{Z}(\vec{\beta})}\sum_{a_{123}=0}^{1}\sum_{a_{124}=0}^{1}\cdots\sum_{a_{N-2,N-1,N}=0}^{1} a_{rst}\ e^{-2\sum_{i<j<k}(\beta_i+\beta_j+\beta_k)a_{ijk}} \tag{21}$$

$$= \frac{\left[\prod'_{i<j<k}\left(1+e^{-2(\beta_i+\beta_j+\beta_k)}\right)\right]\sum_{a_{rst}=0}^{1} a_{rst}\ e^{-2(\beta_r+\beta_s+\beta_t)a_{rst}}}{\prod_{i<j<k}\left(1+e^{-2(\beta_i+\beta_j+\beta_k)}\right)} \tag{22}$$

where the prime on the product in the numerator indicates a product over all vertex combinations *except* rst. Continuing,

$$p_{rst} = \frac{\sum_{a_{rst}=0}^{1} a_{rst}\ e^{-2(\beta_r+\beta_s+\beta_t)a_{rst}}}{1+e^{-2(\beta_r+\beta_s+\beta_t)}} \tag{23}$$

$$= \frac{e^{-2(\beta_r+\beta_s+\beta_t)}}{1+e^{-2(\beta_r+\beta_s+\beta_t)}} \tag{24}$$

$$= \frac{1}{1+e^{2(\beta_r+\beta_s+\beta_t)}} \tag{25}$$

In the sparse hypergraph limit require $p_{rst}\ll 1$, which implies $e^{2(\beta_r+\beta_s+\beta_t)}\gg 1$. In this case,

$$p_{rst} \approx e^{-2(\beta_r+\beta_s+\beta_t)} \tag{26}$$

$$= e^{-2\beta_r}e^{-2\beta_s}e^{-2\beta_t} \tag{27}$$

Now, we still need to determine the elements of $\vec{\beta}$. To do this, note that

$$\begin{aligned}
\langle k_i \rangle &= \sum_s{}' \sum_t{}' p_{ist} && (28) \\
&= \sum_s{}' \sum_t{}' e^{-2\beta_i} e^{-2\beta_s} e^{-2\beta_t} && (29) \\
&= e^{-2\beta_i} \sum_s{}' \sum_t{}' e^{-2\beta_s} e^{-2\beta_t} && (30) \\
&= e^{-2\beta_i} \sum_s{}' e^{-2\beta_s} \sum_t{}' e^{-2\beta_t} && (31) \\
&\approx e^{-2\beta_i} \left(\sum_s{}' e^{-2\beta_s} \right)^2 && (32)
\end{aligned}$$

where the primes on the sums indicate that repeated indices are excluded. Now, assume that

$$\sum_s{}' e^{-2\beta_s} \approx \sum_s e^{-2\beta_s} = C \tag{33}$$

where C is a constant. Then,

$$e^{-2\beta_i} = \frac{\langle k_i \rangle}{C^2} \tag{34}$$

To determine C note that the average number of hyperedges is

$$\begin{aligned}
\langle M \rangle &= \sum_{i<j<k} p_{ijk} && (35) \\
&= \frac{1}{3!} \sum_{i,j,k} p_{ijk} && (36) \\
&\approx \frac{1}{3!} \left(\sum_i e^{-2\beta_i} \right)^3 && (37) \\
&= \frac{1}{3!} C^3 && (38)
\end{aligned}$$

or

$$C = (3! \ \langle M \rangle)^{1/3} \tag{39}$$

Thus, finally, for sparce 3-regular simple hypergraphs

$$p_{rst} \approx \frac{\langle k_r \rangle \langle k_s \rangle \langle k_t \rangle}{(3! \ \langle M \rangle)^2} \tag{40}$$

Then, setting

$$\langle k_r \rangle = \bar{k}_r \tag{41}$$

the prescribed average degree, and

$$\langle M \rangle = \frac{1}{3} \sum_{i=1}^{N} \bar{k}_i \tag{42}$$

get

$$p_{rst} \approx \frac{\bar{k}_r \bar{k}_s \bar{k}_t}{\left(2\sum_{i=1}^{N}\bar{k}_i\right)^2} \tag{43}$$

Hypergraphs generated with this hyperedge probability are the generalization of the Chung-Lu model for graphs.

2.2 g-regular Hypergraphs

It is easy to generalize the above result to g-regular hypergraphs. The results are that

$$\mathcal{H}(G;\vec{\beta}) = (g-1)! \sum_{i_1<i_2<\cdots<i_g} \left(\sum_{q=1}^{g} \beta_{i_q}\right) a_{i_1 i_2 \ldots i_g} \tag{44}$$

$$\mathcal{Z}(\vec{\beta}) = \prod_{i_1<i_2<\cdots<i_g} \left(1 + e^{-(g-1)!\sum_{q=1}^{g}\beta_{i_q}}\right) \tag{45}$$

$$p_{i_1 i_2 \ldots i_g} = \frac{1}{1 + e^{(g-1)!\sum_{q=1}^{g}\beta_{i_q}}} \tag{46}$$

In the sparse hypergraph limit

$$p_{i_1 i_2 \ldots i_g} \approx e^{-(g-1)!\sum_{q=1}^{g}\beta_{i_q}} = \prod_{q=1}^{g} e^{-(g-1)!\beta_{i_q}} \tag{47}$$

$$e^{-(g-1)!\beta_i} = \frac{\langle k_i \rangle}{C^{g-1}} \tag{48}$$

$$\langle M \rangle \approx \frac{1}{g!} C^g \tag{49}$$

or

$$C \approx (g!\ \langle M \rangle)^{1/g} \tag{50}$$

Thus,

$$p_{i_1 i_2 \ldots i_g} \approx \frac{\prod_{q=1}^{g} \langle k_{i_q} \rangle}{(g!\ \langle M \rangle)^{g-1}} \tag{51}$$

or

$$p_{i_1 i_2 \ldots i_g} \approx \frac{\prod_{q=1}^{g} \bar{k}_{i_q}}{\left((g-1)!\ \sum_{j=1}^{N} \bar{k}_j\right)^{g-1}} \tag{52}$$

3 Prescribed Expected Joint Vertex Degrees

We wish to extend the earlier formulaton to randomly generate g-regular hypergraphs with α-dimensional prescribed expected degrees

$$\bar{\mathbf{K}} \tag{53}$$

which is an $\alpha < g$ dimensional tensor, with elements

$$\bar{k}_{i_1,i_2,\ldots,i_\alpha} \tag{54}$$

specifying the number of expected hyperedges incident on vertices $i_1,\ i_2,\ \ldots,\ i_\alpha$. Assuming there are N vertices, there are N^α elements of $\bar{\mathbf{K}}$.

Calling the ensemble of all possible simple g-regular hypergraphs $\mathcal{G}_g$, the expected joint degrees are

$$\bar{k}_{i_1,i_2,\ldots,i_\alpha} = \sum_{G\in\mathcal{G}_g} k_{i_1,i_2,\ldots,i_\alpha}(G)\ P(G) \tag{55}$$

where $k_{i_1,i_2,\ldots,i_\alpha}(G)$ is the joint degree of vertices $i_1,\ i_2,\ \ldots,\ i_\alpha$ in a graph G and $P(G)$ is the probablity of generating that graph.

Maximizing entropy while attempting to enforce the prescribed degree constraints, the probability of generating a hypergraph is

$$P(G;\mathbf{B}) = \frac{1}{\mathcal{Z}(\mathbf{B})}e^{-\mathbf{B}\cdot\mathbf{K}} = \frac{1}{\mathcal{Z}(\mathbf{B})}e^{-\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_\alpha=1}^{N}\beta_{i_1,i_2,\ldots,i_\alpha}k_{i_1,i_2,\ldots,i_\alpha}} \tag{56}$$

where $\mathbf{B}$ is an α-dimensional tensor of Laplace coefficients, and $\mathbf{K}$ is the α-dimensional tensor of joint vertex degrees. Only elements of $\mathbf{B}$ with nonrepeating elements are relevent, and all elements with permuted elements are equivalent.

Now, a g-regular hypergraph can be represented by a g-dimensional adjacency tensor $\mathbf{a}$, with elements of

$$a_{i_1 i_2 \ldots i_g} = \begin{cases} 1 & \text{If a hyperedge exists connecting vertices } i_1,\ i_2,\ \ldots,\ i_{g-1} \text{ and } i_g. \\ 0 & \text{Otherwise.} \end{cases} \tag{57}$$

Require that all elements that have permuted indices are equal, and all with any repeated elements are zero.

Note that

$$k_{i_1,i_2,\ldots,i_\alpha} = \sum_{i_{\alpha+1}=1}^{N}\sum_{i_{\alpha+2}=1}^{N}\cdots\sum_{i_g=1}^{N} a_{i_1 i_2 \ldots i_g} \tag{58}$$

Recognizing Eqn. 56 as a Gibbs distribution, identify the "Hamiltonian"

$$\begin{aligned}
\mathcal{H}(G;\mathbf{B}) &= \mathbf{B}\cdot\mathbf{K} && (59)\\
&= \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_\alpha=1}^{N}\beta_{i_1,i_2,\ldots,i_\alpha}k_{i_1,i_2,\ldots,i_\alpha} && (60)\\
&= \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_\alpha=1}^{N}\beta_{i_1,i_2,\ldots,i_\alpha}\sum_{i_{\alpha+1}=1}^{N}\sum_{i_{\alpha+2}=1}^{N}\cdots\sum_{i_g=1}^{N} a_{i_1 i_2 \ldots i_g} && (61)\\
&= \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_g=1}^{N}\beta_{i_1,i_2,\ldots,i_\alpha}a_{i_1 i_2 \ldots i_g} && (62)
\end{aligned}$$

To learn how to proceed, consider the case with $g = 3$ and $\alpha = 2$, then

$$\begin{aligned}
\mathcal{H}(G;\mathbf{B}) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\beta_{ij}a_{ijk} && (63)\\
&= \sum_{i<j<k}\beta_{ij}a_{ijk} + \sum_{i<k<j}\beta_{ij}a_{ijk} + \sum_{j<i<k}\beta_{ij}a_{ijk}\\
&\quad + \sum_{j<k<i}\beta_{ij}a_{ijk} + \sum_{k<i<j}\beta_{ij}a_{ijk} + \sum_{k<j<i}\beta_{ij}a_{ijk} && (64)\\
&= \sum_{i<j<k}\beta_{ij}a_{ijk} + \sum_{i<j<k}\beta_{ik}a_{ijk} + \sum_{i<j<k}\beta_{ji}a_{ijk}\\
&\quad + \sum_{i<j<k}\beta_{jk}a_{ijk} + \sum_{i<j<k}\beta_{ki}a_{ijk} + \sum_{i<j<k}\beta_{kj}a_{ijk} && (65)\\
&= 2\left(\sum_{i<j<k}\beta_{ij}a_{ijk} + \sum_{i<j<k}\beta_{ik}a_{ijk} + \sum_{i<j<k}\beta_{jk}a_{ijk}\right) && (66)\\
&= 2\sum_{i<j<k}\left(\beta_{ij} + \beta_{ik} + \beta_{jk}\right)a_{ijk} && (67)
\end{aligned}$$

So then, for general g and α get

$$\begin{aligned}
\mathcal{H}(G;\mathbf{B}) &= \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_g=1}^{N}\beta_{i_1,i_2,\ldots,i_\alpha}a_{i_1 i_2\ldots i_g} && (68)\\
&= (g-\alpha)!\,\alpha!\sum_{i_1<i_2<\cdots<i_g}\left(\sum_{q_1=1}^{g}\sum_{q_2>q_1}^{g}\cdots\sum_{q_\alpha>q_{\alpha-1}}^{g}\beta_{i_{q_1},i_{q_2},\ldots,i_{q_\alpha}}\right)a_{i_1 i_2\ldots i_g} && (69)
\end{aligned}$$

The "partition function" then is

$$\begin{aligned}
\mathcal{Z}(\mathbf{B}) &= \sum_{G\in\mathcal{G}_g} e^{-\mathcal{H}(G;\mathbf{B})} && (70)\\
&= \prod_{i_1<i_2<\ldots<i_g}\ \sum_{a_{i_1 i-2\ldots i_g}=0}^{1} e^{-(g-\alpha)!\,\alpha!\sum_{i_1<i_2<\cdots<i_g}\left(\sum_{q_1=1}^{g}\sum_{q_2>q_1}^{g}\cdots\sum_{q_\alpha>q_{\alpha-1}}^{g}\beta_{i_{q_1},i_{q_2},\ldots,i_{q_\alpha}}\right)a_{i_1 i_2\ldots i_g}} && (71)\\
&= \prod_{i_1<i_2<\ldots<i_g}\left(1 + e^{-(g-\alpha)!\,\alpha!\sum_{i_1<i_2<\cdots<i_g}\left(\sum_{q_1=1}^{g}\sum_{q_2>q_1}^{g}\cdots\sum_{q_\alpha>q_{\alpha-1}}^{g}\beta_{i_{q_1},i_{q_2},\ldots,i_{q_\alpha}}\right)}\right) && (72)
\end{aligned}$$

The probability that a hyperedge exists between vertices i_1, i_2, $\ldots$, and i_g is (assume that

$i_1 < i_2 < \ldots < i_g)$

$$
\begin{aligned}
p_{i_1 i_2 \ldots i_g} &= \left\langle a_{i_1 i_2 \ldots i_g} \right\rangle && (73)\\
&= \sum_{G \in \mathcal{G}_g} a_{i_1 i_2 \ldots i_g}(G)\ P(G; \mathbf{B}) && (74)\\
&= \frac{e^{-(g-\alpha)!\ \alpha! \left(\sum_{q_1=1}^{g} \sum_{q_2>q_1}^{g} \cdots \sum_{q_\alpha>q_{\alpha-1}}^{g} \beta_{i_{q_1}, i_{q_2}, \ldots, i_{q_\alpha}} \right)}}{1 + e^{-(g-\alpha)!\ \alpha! \left(\sum_{q_1=1}^{g} \sum_{q_2>q_1}^{g} \cdots \sum_{q_\alpha>q_{\alpha-1}}^{g} \beta_{i_{q_1}, i_{q_2}, \ldots, i_{q_\alpha}} \right)}} && (75)\\
&= \frac{1}{1 + e^{(g-\alpha)!\ \alpha! \left(\sum_{q_1=1}^{g} \sum_{q_2>q_1}^{g} \cdots \sum_{q_\alpha>q_{\alpha-1}}^{g} \beta_{i_{q_1}, i_{q_2}, \ldots, i_{q_\alpha}} \right)}} && (76)
\end{aligned}
$$

In the sparse hypergraph limit require $p_{i_1 i_2 \ldots i_g} \ll 1$, which implies $e^{(g-\alpha)!\ \alpha! \left(\sum_{q_1=1}^{g} \sum_{q_2>q_1}^{g} \cdots \sum_{q_\alpha>q_{\alpha-1}}^{g} \beta_{i_{q_1}, i_{q_2}, \ldots, i_{q_\alpha}} \right)} \gg 1$. In this case,

$$
\begin{aligned}
p_{i_1 i_2 \ldots i_g} &\approx e^{-(g-\alpha)!\ \alpha! \left(\sum_{q_1=1}^{g} \sum_{q_2>q_1}^{g} \cdots \sum_{q_\alpha>q_{\alpha-1}}^{g} \beta_{i_{q_1}, i_{q_2}, \ldots, i_{q_\alpha}} \right)} && (77)\\
&= \prod_{q_1=1}^{g} \prod_{q_2>q_1}^{g} \cdots \prod_{q_\alpha>q_{\alpha-1}}^{g} e^{-(g-\alpha)!\ \alpha!\ \beta_{i_{q_1}, i_{q_2}, \ldots, i_{q_\alpha}}} && (78)
\end{aligned}
$$

Now, we still need to determine the elements of $\mathbf{B}$. To see how do this, consider again the case with $g = 3$ and $\alpha = 2$. Note that

$$
p_{ijk} = e^{-2\beta_{ij}} e^{-2\beta_{ik}} e^{-2\beta_{jk}} \qquad (79)
$$

and that

$$
\begin{aligned}
\langle k_{ij} \rangle &= \sum_s{}' p_{ijs} && (80)\\
&= \sum_s{}' e^{-2\beta_{ij}} e^{-2\beta_{is}} e^{-2\beta_{js}} && (81)\\
&= e^{-2\beta_{ij}} \sum_s{}' e^{-2\beta_{is}} e^{-2\beta_{js}} && (82)
\end{aligned}
$$

$$
\begin{aligned}
\langle k_i \rangle &= \frac{1}{2} \sum_s{}' \sum_t{}' p_{ist} && (83)\\
&= \frac{1}{2} \sum_s{}' \sum_t{}' e^{-2\beta_{is}} e^{-2\beta_{it}} e^{-2\beta_{st}} && (84)
\end{aligned}
$$

where the primes on the sums indicate that repeated indices are excluded. Also note that the average number of hyperedges is

$$
\begin{aligned}
\langle M \rangle &= \sum_{i<j<k} p_{ijk} && (85)\\
&= \frac{1}{3!} \sum_{i,j,k} p_{ijk} && (86)\\
&\approx \frac{1}{3!} \sum_{i,j,k} e^{-2\beta_{ij}} e^{-2\beta_{ik}} e^{-2\beta_{jk}} && (87)
\end{aligned}
$$

From Eq. 82 get

$$
\begin{aligned}
e^{-2\beta_{ij}} &= \frac{\langle k_{ij}\rangle}{\sum_s{}' e^{-2\beta_{is}} e^{-2\beta_{js}}} && (88)\\
&\approx \frac{\langle k_{ij}\rangle}{\sum_s e^{-2\beta_{is}} e^{-2\beta_{js}}} && (89)\\
&= \frac{\langle k_{ij}\rangle}{C_{ij}} && (90)
\end{aligned}
$$

where

$$
C_{ij} \equiv \sum_s e^{-2\beta_{is}} e^{-2\beta_{js}} \qquad (91)
$$

Then also,

$$
\begin{aligned}
\langle k_i\rangle &= \frac{1}{2}\sum_s{}' \sum_t{}' e^{-2\beta_{is}} e^{-2\beta_{it}} e^{-2\beta_{st}} && (92)\\
&\approx \frac{1}{2}\sum_s \sum_t e^{-2\beta_{is}} e^{-2\beta_{it}} e^{-2\beta_{st}} && (93)\\
&= \frac{1}{2}\sum_s e^{-2\beta_{is}} \sum_t e^{-2\beta_{it}} e^{-2\beta_{st}} && (94)\\
&= \frac{1}{2}\sum_s e^{-2\beta_{is}} C_{is} && (95)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle M\rangle &\approx \frac{1}{3!}\sum_{i,j,k} e^{-2\beta_{ij}} e^{-2\beta_{ik}} e^{-2\beta_{jk}} && (96)\\
&= \frac{1}{3!}\sum_{i,j} e^{-2\beta_{ij}} \sum_k e^{-2\beta_{ik}} e^{-2\beta_{jk}} && (97)\\
&= \frac{1}{3!}\sum_{i,j} e^{-2\beta_{ij}} C_{ij} && (98)
\end{aligned}
$$

Conclusion: It seems like the best we can do is, using Eqs. 79, 89, and 91, to find the β_{ij}s by solving the simultaneous set of equations given by

$$
e^{-2\beta_{ij}} = \frac{\bar{k}_{ij}}{\sum_s e^{-2\beta_{is}} e^{-2\beta_{js}}}, \qquad \forall i,\ j \qquad (99)
$$

then use them to determine the Cs in

$$
p_{ijk} = \frac{\bar{k}_{ij}\bar{k}_{ik}\bar{k}_{jk}}{C_{ij}C_{ik}C_{jk}} \qquad (100)
$$

On the other hand, one could work directly with the Cs instead of the βs. Note that

$$
\begin{aligned}
e^{-2\beta_{ij}} &\approx \frac{\bar{k}_{ij}}{C_{ij}} && (101)\\
\sum_j e^{-2\beta_{ij}} e^{-2\beta_{jk}} &\approx \sum_j \frac{\bar{k}_{ij}\bar{k}_{jk}}{C_{ij}C_{jk}} && (102)\\
C_{ik} &\approx \sum_j \frac{\bar{k}_{ij}\bar{k}_{jk}}{C_{ij}C_{jk}} \qquad \forall\, i,\ k && (103)
\end{aligned}
$$

So that one can solve the system of simultaneus equations given by Eqns. 103 to determine the Cs, and them use them in Eq. 100 to generate the hypergraph.