Finite Difference Time Domain Modeling of 1D Transmission Lines

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I. PROJECT DESCRIPTION

The objective of this project was to develop software that performs a FDTD solution of the 1D transmission line equations that are either lossy or lossless for various transmission lines of finite length excited by a Thévenin voltage source and additionally terminated by restive and reactive loads. This FDTD software is capable of simulating and plotting transmission line voltages and currents as a function of time and produce simulation videos. The software is also validated for a number of examples.

The software developed in this project is be capable of simulating the time-dependent line voltages and currents on a lossy transmission-line by utilizing the second-order accurate FDTD approximation of the lossy transmission line equations. The transmission line in the FDTD simulation is characterized by per unit length capacitance, inductance, conductance and resistance and length. In this software,the transmission line is excited by a Thévenin voltage source with an internal resistance. The Thevenin voltage signal generator in this software can be set to a trapezoidal or gaussian pulse of desired amplitude. The transmission line is terminated by a load network. In this FDTD simulation software, the load network can be set to be a purely resistive load or a reactive load network which can either be a series RL load or a parallel RC load. Additionally, the load can be set to be an open or short circuit. The theory of the FDTD simulation of the 1D lossy transmission line will be discussed in the next section. In the following section the top-level code design of the FDTD software will be briefly discussed, and finally the FDTD software will be validated by running a series of basic general simulations.

II. THEORY

To numerically solve the lossy transmission line equations, a Finite Difference Time domain (FDTD) method is employed. A brief over view of theory of the FDTD update equations and the source and load voltage update equations used in the code are described in this section. The discrete time and spatial samples of voltage and current along the transmission line are described by the expressions:

$$V_i^{n+\frac{1}{2}} = V\big((n+\frac{1}{2})\Delta t, i\Delta x\big) \tag{1}$$

$$I_{i+\frac{1}{2}}^{n} = I\big(n\Delta t, (i+\frac{1}{2})\Delta x\big) \tag{2}$$

The line voltage update for the interior voltage nodes of the lossless transmission line is given by the expression:

$$V_i^{n+1} = V_i^{n-1} - \big(\frac{\Delta t}{C\Delta z}\big)(I_{i+1}^n - I_{i-1}^n) \tag{3}$$

The line current update for the interior current nodes of the lossless transmission line is given by the expression:

$$I_i^{n+1} = I_i^{n-1} - \big(\frac{\Delta t}{L\Delta z}\big)(V_{i+1}^n - V_{i-1}^n) \tag{4}$$

The line voltage update for the interior voltage nodes of the lossy transmission line is given by the expression:

$$V_i^{n+\frac{1}{2}} = \Big[\frac{\frac{C}{\Delta t} - \frac{G}{2}}{\frac{C}{\Delta t} + \frac{G}{2}}\Big] V_i^{n-\frac{1}{2}} - \Big[\frac{1}{\frac{C}{\Delta t} + \frac{G}{2}}\Big] \frac{I_{i+\frac{1}{2}}^n - I_{i-\frac{1}{2}}^n}{\Delta x} \tag{5}$$

The line current update for the interior current nodes of the lossy transmission line is given by the expression

$$I_{i+\frac{1}{2}}^{n+1} = \Big[\frac{\frac{L}{\Delta t} - \frac{R}{2}}{\frac{L}{\Delta t} + \frac{R}{2}}\Big] I_{i+\frac{1}{2}}^{n} - \Big[\frac{1}{\frac{L}{\Delta t} + \frac{R}{2}}\Big] \frac{V_{i+1}^{n+\frac{1}{2}} - V_i^{n+\frac{1}{2}}}{\Delta x} \tag{6}$$

The update equations (5) and (6) for the lossy transmission line reduce to equations (3) and (4) when the per unit length resistance and conductance go to zero. Next the source and load update equations used in the FDTD code are presented. The load update for a open circuit termination is given by the condition:

$$I_{N-1}^{n+1} = 0 \tag{7}$$

The load update for a short circuit termination is given by the condition:

$$V_{N-1}^{n+\frac{1}{2}} = 0 \tag{8}$$

The 2nd order accurate source update for the source voltage node is given by the expression:

$$V_i^{n+\frac{1}{2}} = (C\frac{\Delta x}{2\Delta t} + \frac{1}{2Rg})^{-1}\big[(C\frac{\Delta x}{2\Delta t} - \frac{1}{2Rg})\ V_0^{n-\frac{1}{2}} - I_{\frac{1}{2}}^{n} + \frac{V_g^n}{Rg}\big] \tag{9}$$

The 2nd order accurate update for the load voltage node for the purely restive load is given by the expression:

$$V_N^{n+\frac{1}{2}} = (C\frac{\Delta x}{2\Delta t} + \frac{1}{2R_L})^{-1}\big[(C\frac{\Delta x}{2\Delta t} - \frac{1}{2R_L})\ V_N^{n-\frac{1}{2}} + (C\frac{\Delta x}{2\Delta t} + \frac{1}{2R_L})^{-1} I_{N-\frac{1}{2}}^{n}\big] \tag{10}$$

The 2nd order accurate update for the load voltage node for the parallel RC load is given by the expression:

$$V_N^{n+\frac{1}{2}} = \Big[\frac{(\frac{C\Delta x+2C_L}{2\Delta t}) - \frac{1}{2}(\frac{G\Delta x}{2} + \frac{1}{R_L})}{(\frac{C\Delta x+2C_L}{2\Delta t}) + \frac{1}{2}(\frac{G\Delta x}{2} + \frac{1}{R_L})}\Big] V_N^{n-\frac{1}{2}} + \Big[\frac{1}{(\frac{C\Delta x+2C_L}{2\Delta t}) + \frac{1}{2}(\frac{G\Delta x}{2} + \frac{1}{R_L})}\Big] I_{N-1}^{n} \tag{11}$$

The 2nd order accurate update for the load voltage node for the series RL load is given by the system of equations in the form of a matrix equation given by the expression:

$$\begin{bmatrix} (\frac{C\Delta x}{2\Delta t} + \frac{G\Delta x}{4}) & \frac{1}{2} \\ \frac{1}{2} & -(\frac{L_L}{\Delta t} + \frac{R_L}{2}) \end{bmatrix} \begin{bmatrix} V_N^{n+\frac{1}{2}} \\ I_L^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} (\frac{C\Delta x}{2\Delta t} - \frac{G\Delta x}{4}) & -\frac{1}{2} \\ -\frac{1}{2} & -(\frac{L_L}{\Delta t} - \frac{R_L}{2}) \end{bmatrix} \begin{bmatrix} V_N^{n-\frac{1}{2}} \\ I_L^{n-\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} I_{N-1}^{n} \\ 0 \end{bmatrix}$$

All the nodes representing the voltage and current in space are stored in 1 dimensional arrays which are updated by these update equations for each time step.

III. TOP-LEVEL DESIGN AND CODE DESCRIPTION

In this section a brief overview of the top level design of the 1dimensional FDTD transmission line simulation code is presented. The code first reads in all the physical data and simulation data top perform the FDTD simulation by reading in a user edited text file. Next, in the setup FDTD function, all transmission line parameters and update equation coefficients are precalculated and the voltage, and current node arrays are initialize to zero. After all parameters and data arrays are initialized, the FDTD algorithm is initiated. This algorithm consist of a for loop that runs from the first time step to the last time step, where at each time step, the following array updating functions are executed in the order as now presented: the line voltage update, source voltage update, load voltage update, and line current update are update. At the end of every iteration of the update, a function is called to save all necessary data. Once the FDTD algorithm is complete, the post processing function is called. This function writes and save a video of the simulation and produces plots of selected quantities.

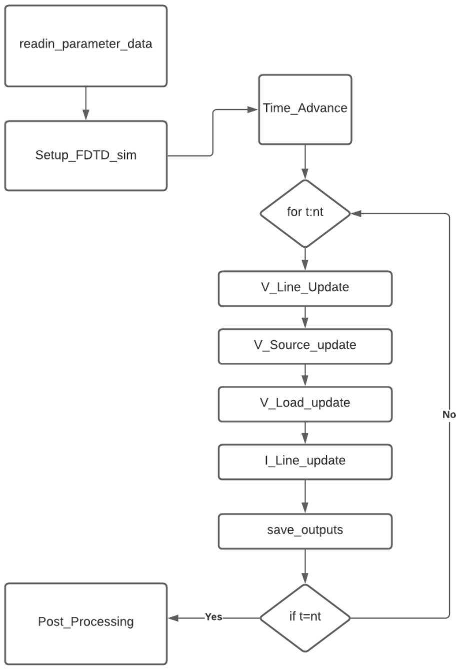

FIG. 1: Block Diagram of the top level functional design of the FDTD python script

IV. VALIDATING CASE STUDIES

In this section the case studies that were performed with the FDTD software are documented to serve as proof of validation of the FDTD model implemented in the software.

A. Case Study i)

The first validation case study performed was a simulation of a uniform, lossless transmission-line with per unit length capacitance and inductance C = 100 pF/m and L = 250 nH/m and a length of 0.5 m. The line was discretized with dx=0.01 m. The line was excited by a voltage source with a trapezoidal pulse of 2 V peak value. The trapezoidal pulse had a rise and fall time of 200 ps and a 500 ps duration. The Thevenin source impedance was set at 50 ohms and the load impedance was set 50 ohms. The time step size was set to the magic time step by setting the CNL number to 1.The FDTD simulation time was set to 5 ns, which is time for the pulse to make 1 full round trip on the line.

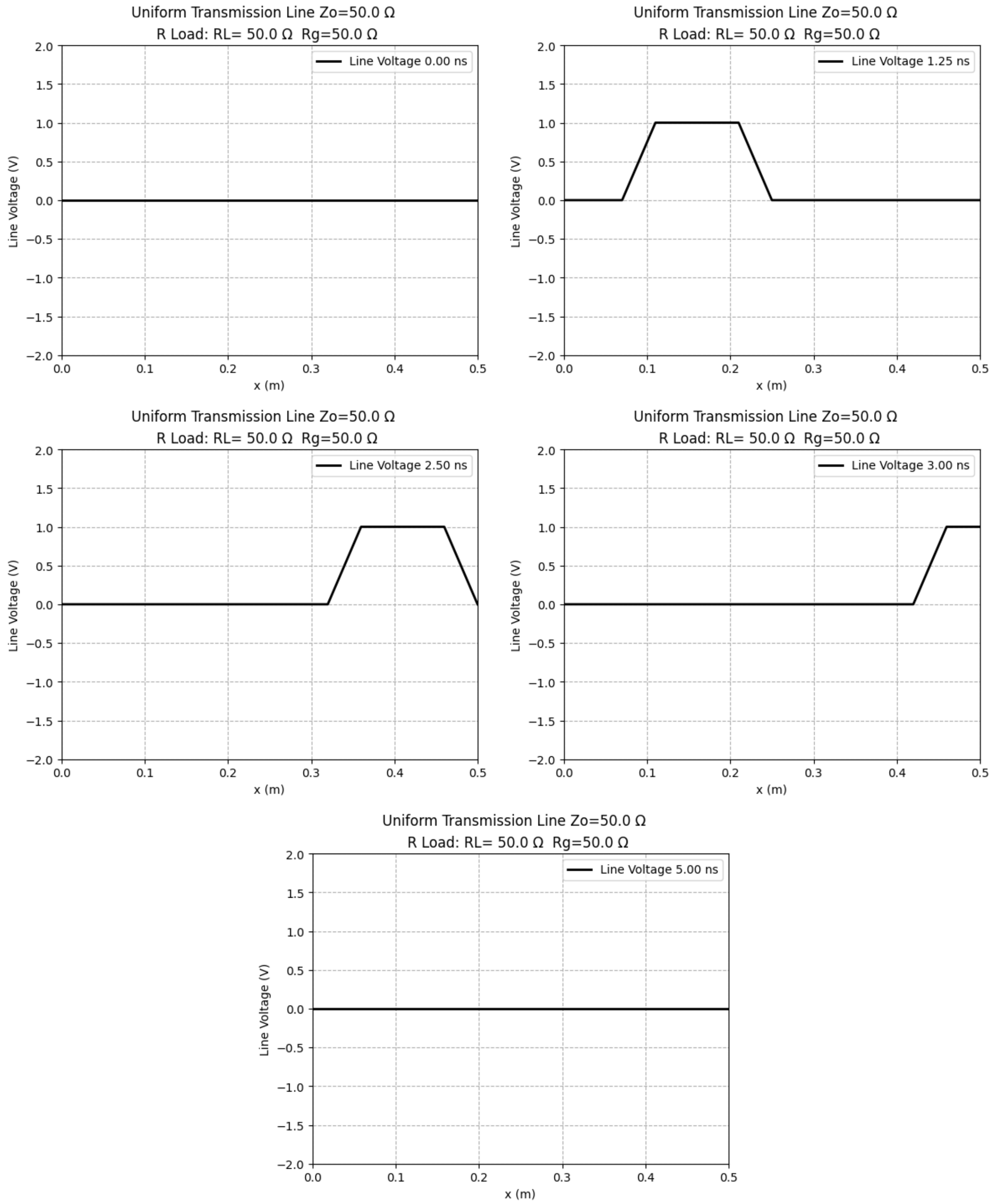

FIG. 2: Plots of the line voltage along the transmission line at times, 0.00ns, 1.25ns, 2.25ns, 3.00ns, 5.00ns, for the case with 2 volts amplitude trapezoidal pulse and Rl=50 ohms and Rs=50 ohms

The results of this case simulation show what would be expected from theory. The results are shown in FIG. 2. A trapezoidal pulse is observed propagating down the transmission line and it its completely absorbed by the load, There is no reflection of the pulse from the load because the reflection coefficient is zero due to the matched line characteristic impedance and load impedance.

B. Case Study ii)

This case study is a repetition of case study i) but this time the simulation is performed using a short circuit load, and subsequently open circuit load.

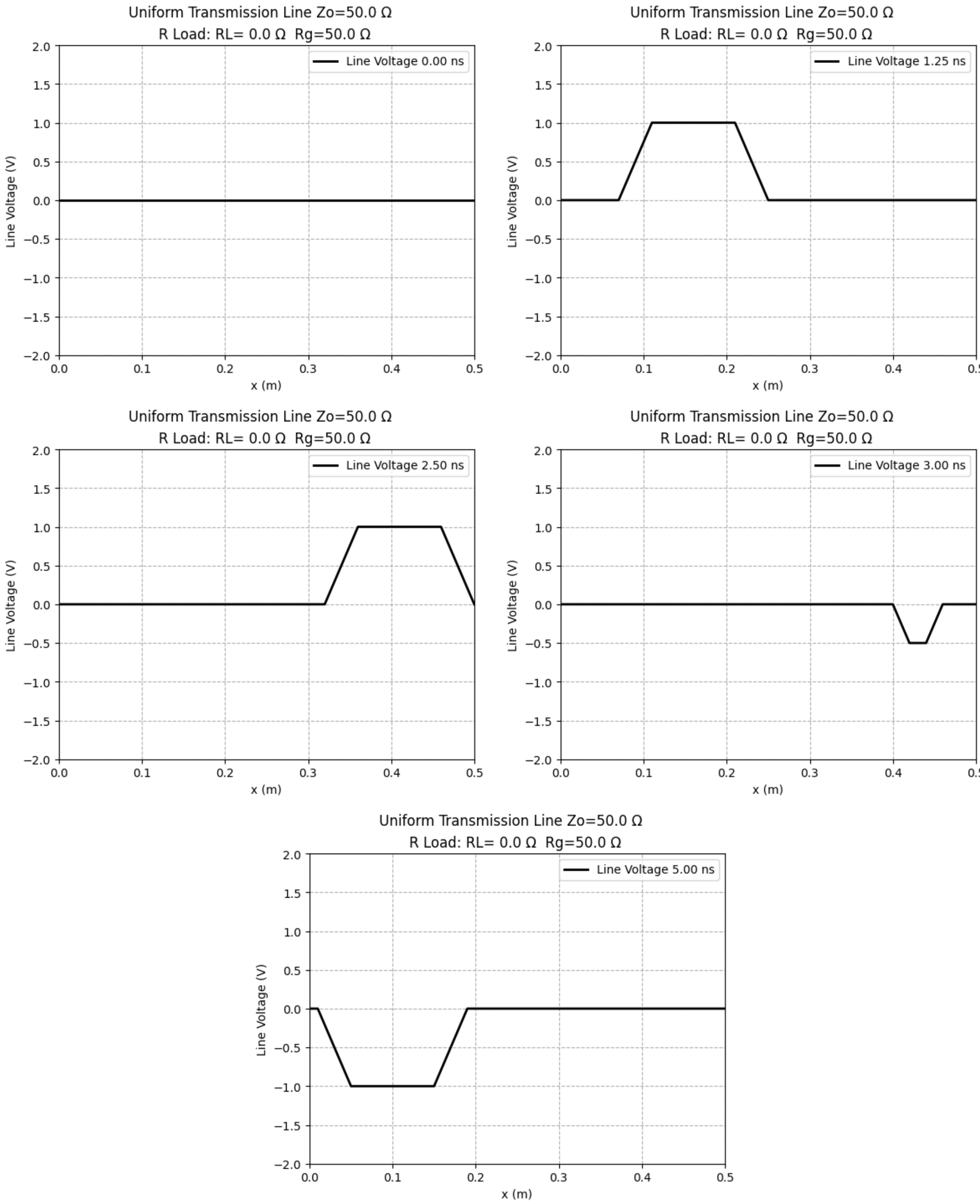

FIG. 3: Plots of the line voltage along the transmission line at times, 0.00ns, 1.25ns, 2.25ns, 3.00ns, 5.00ns, for the case with 2V amplitude trapezoidal pulse and Rl=0 ohms and Rs=50 ohms

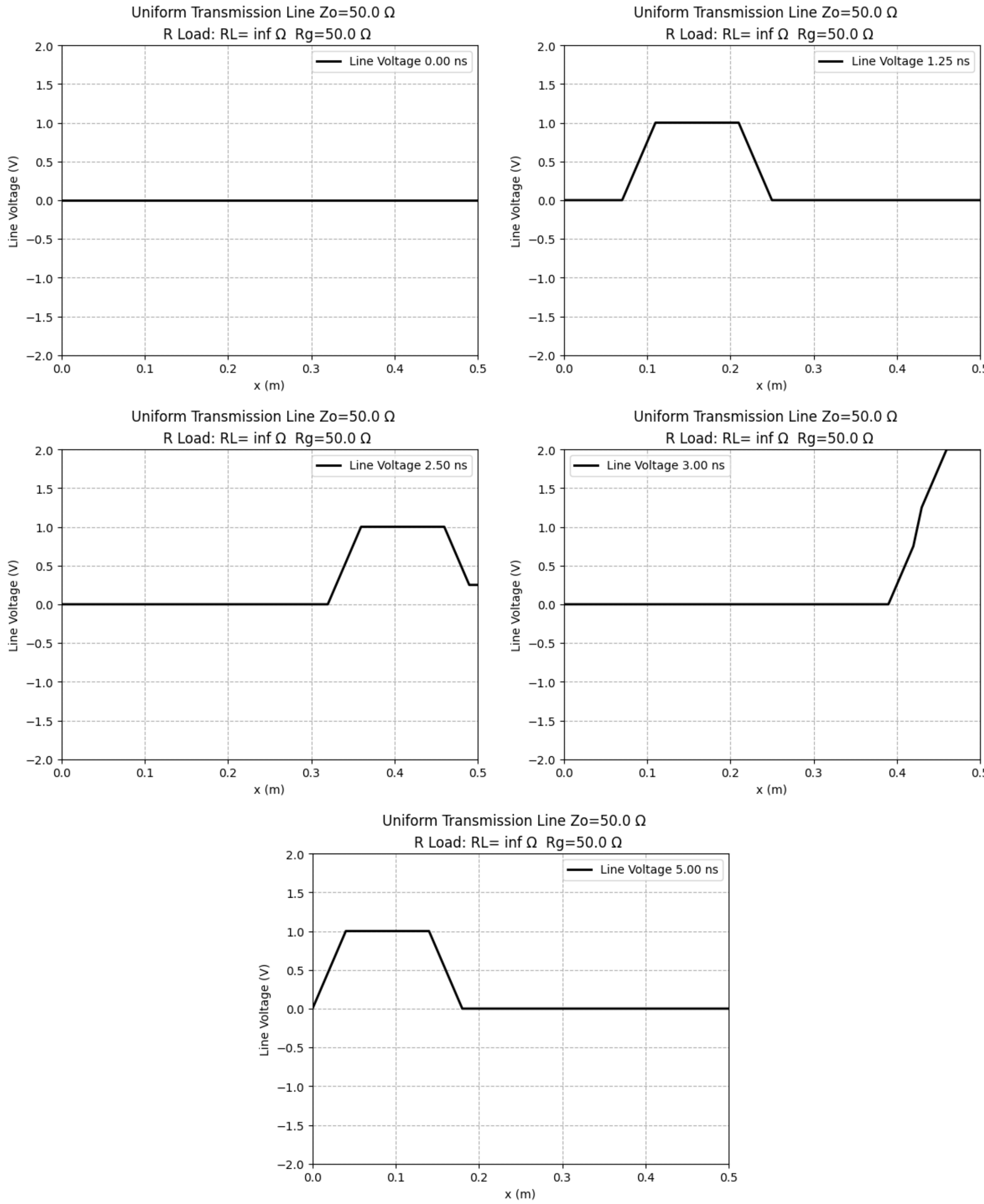

FIG. 4: Plots of the line voltage along the transmission line at times, 0.00ns, 1.25ns, 2.25ns, 3.00ns, 5.00ns, for the case with 2V amplitude trapezoidal pulse and Rl=infinity ohms and Rs=50 ohms

The results of this case simulation show what would be expected from theory. The results of the short termination are shown in FIG. 3. In the first case of the trial with the end of the transmission line terminated with a short, a trapezoidal pulse with the expected characteristics is observed propagating down the transmission line. Once it reaches the short it is observed that the voltage at the short is zero throughout the whole scattering of the incident voltage pulse. The trapezoidal pulse is reflected with a 180 degree phase shift, and the incident and reflected pulse destructively interfere with each other at the short load. The results of the open termination are shown in FIG. 4. In the second case of the trial with the end of the transmission line terminated with a open circuit load, a trapezoidal pulse with the expected characteristics is observed propagating down the transmission line. Once it reaches the open termination it is observed that the voltage at the short is non zero throughout the whole scattering of the incident voltage pulse. The trapezoidal pulse is reflected with no phase shift, and the incident and reflected pulse constructively interfere with each other at the open circuit load.

C. Case Study iii)

In this case study, the same transmission line setup from the first validating example was used, however with a source resistance 25Ω and a load resistance of 150Ω. The line voltage at the source end (x = 0) and the load end (x = 0.5 m) was computed and plotted over the time interval 0-20 ns with the Courant number set to 1 and dx=0.01 m.

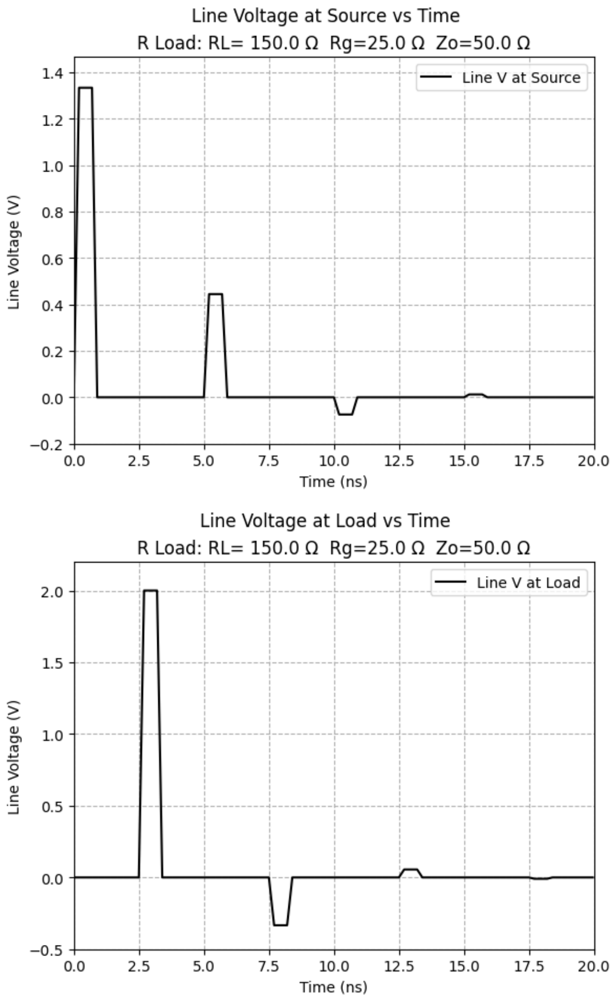

FIG. 5: Plots of the line voltage a the source and load versus time for the transmission line excited with a trapezoidal pule of amplitude 2v with a source resistance of 25 ohms and a load Resistance of 150 ohms

The results of this case simulation show what would be expected from theory. The results are shown in FIG. 5. In this case we see reflections of the trapezoidal pulse from both the load and the source due to the unmatched impedances. The load and source reflection coefficients are calculated below.

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_o} = \frac{150 - 50}{150 + 50} = 0.5 \tag{12}$$

$$\Gamma_L = \frac{Z_g - Z_0}{Z_g + Z_o} = \frac{25 - 50}{25 + 50} = -\frac{1}{3} \tag{13}$$

the plot of the voltage at the source show the initial pulse departed the thevenin source. The voltage pulse entering the transmission line at the source end has a voltage of 1.33. theory predicts (-1/3)x2 + 2 which equals 1.33 V at the source end which agrees with the observed initial voltage pulse of 1.333 V entering the line at the source end. The voltage at the load node shows the constructive interference of the incident and reflect trapezoidal pulse, leading to a total voltage of 2V at reflection from the load. Theory predicts (0.5x1.33 + 1.33) which equals 2 which agrees with the observed voltage at the load end when the incident and reflected voltage pulses interfere at the load end. The trapezoidal pulse continues to be reflected by both the load and the source impedances until the voltage of the pulse converges to zero.

D. Case Study iv)

This case study is a repetition of case study i), but this time using CN = 0.99.

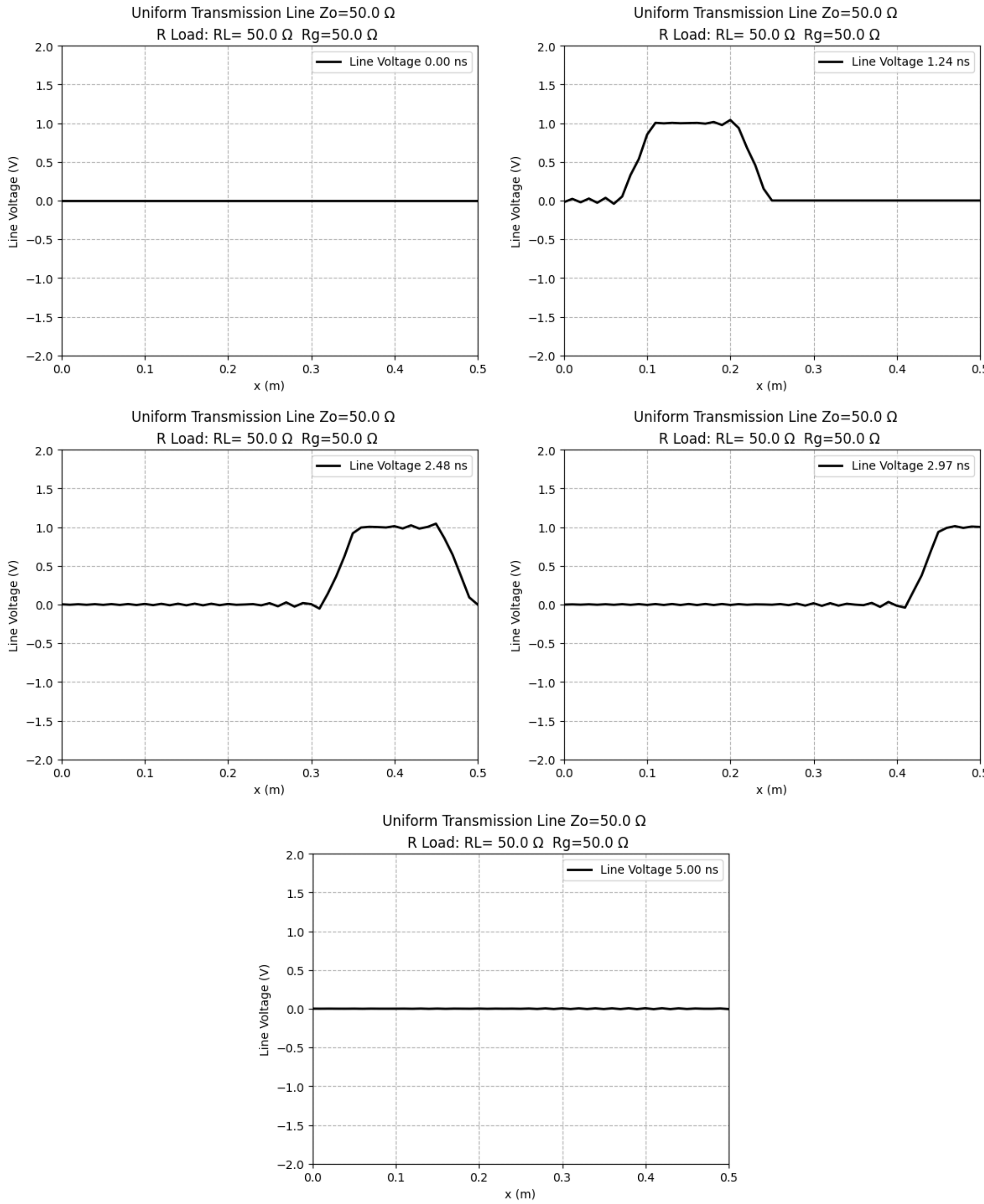

FIG. 6: Plots of the line voltage along the transmission line at times, 0.00ns, 1.25ns, 2.25ns, 3.00ns, 5.00ns, for the case with 2V amplitude trapezoidal pulse and Rl=50 ohms and Rs=50 ohms with the Courant number set to 0.99 in the FDTD simulation

The results of this case simulation show what would be expected from theory. The results are shown in FIG. 6. The trapezoidal pulse with the expected characteristics is observed propagating down the transmission line, however, with the Courant number set to 0.99, the FDTD simulation is not a the magic time step and some errors are observed in the voltage profile of the. These errors can be attributed to the numerical dispersion errors due to the inherent error of the central difference approximations.

E. Case Study v)

This case study is a repetition of case study i), but this time using CN = 1.01, simulating for 2.5 ns.

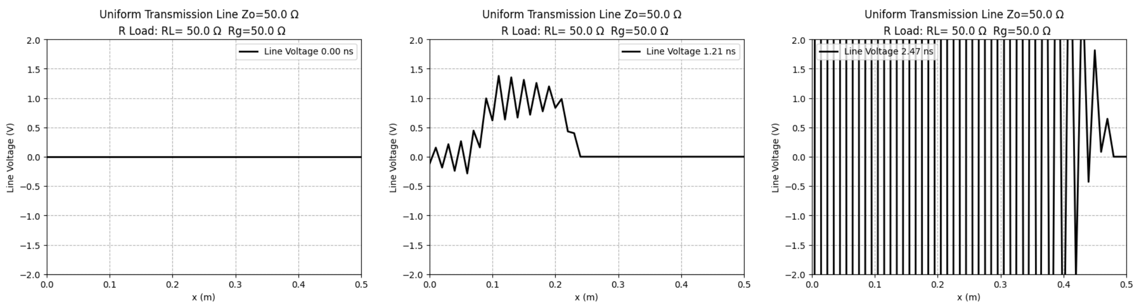

FIG. 7: Plots of the line voltage along the transmission line at times, 0.00ns, 1.25ns, and 2.25ns for the case with 2V amplitude trapezoidal pulse and Rl=50 ohms and Rs=50 ohms with the Courant number set to 1.01 in the FDTD simulation

The results of this case simulation show what would be expected from theory. The results are shown in FIG. 7., with the Courant number set to 1.01. With the Courant number no longer less then or equal to the one, the discretized signal is no longer restricted from travelling more than a signal spatial step per single time step. This leads to a violation of casualty, as information of the signal is traveling faster then the propagation velocity of the signal and hence total energy bounded in the signal grows exponentially as the trapezoidal pulse propagates. It is observed inf FIG 7. that the voltage amplitude grows exponentially.

F. Case Study vi)

In this case study the same transmission line parameters are used from case i, however this time the transmission line is terminated with a with a parallel RC load with a load resistance of 150 Ω and a load capacitance of 5 pF. The source resistance is 50Ω and the Courant number was set equal to 1. The second-order accurate update for the parallel RC load voltage was utilized in this case. Below, a plot of the source voltage over the time interval interval 0-10 ns is shown.

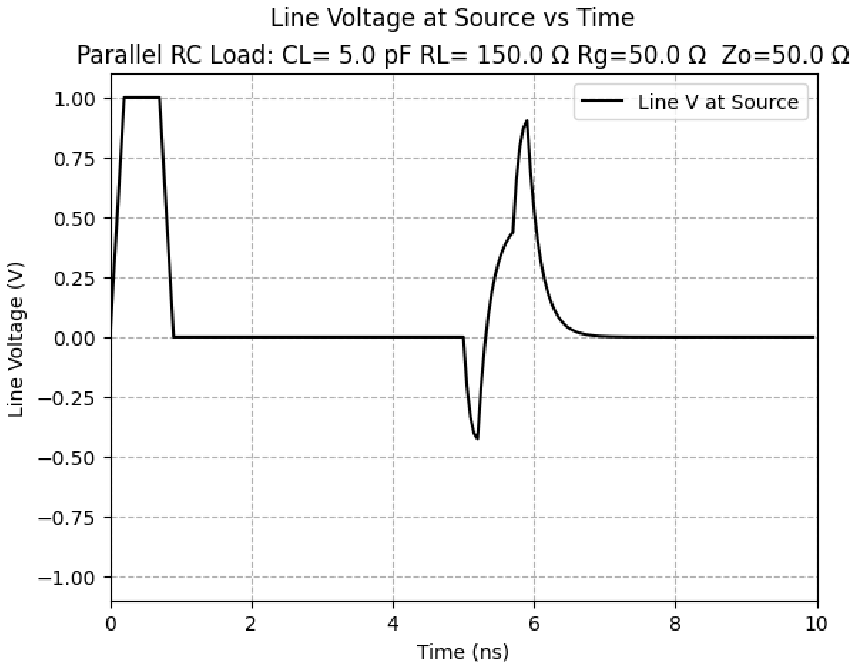

FIG. 8: Plot of the line voltage at the source vs time for a lossless transmission line terminated with a parallel RC load with with a load resistance of 150 Ω and a load capacitance of 5 pF.

The results of this case simulation show what would be expected from theory. The results are shown in FIG. 8. The graph shows the initial trapezoidal pulse propagating out of the source. At about 5 ns, the voltage due the returned reflected pulse from the parallel RC load is observed.

G. Case Study vii)

This case study is a repetition of the previous case study case vi, however, this time making the transmission line lossy with the per unit length resistance of R = 50 m/mΩ, and per unit length conductance of G = 25 mS/m.

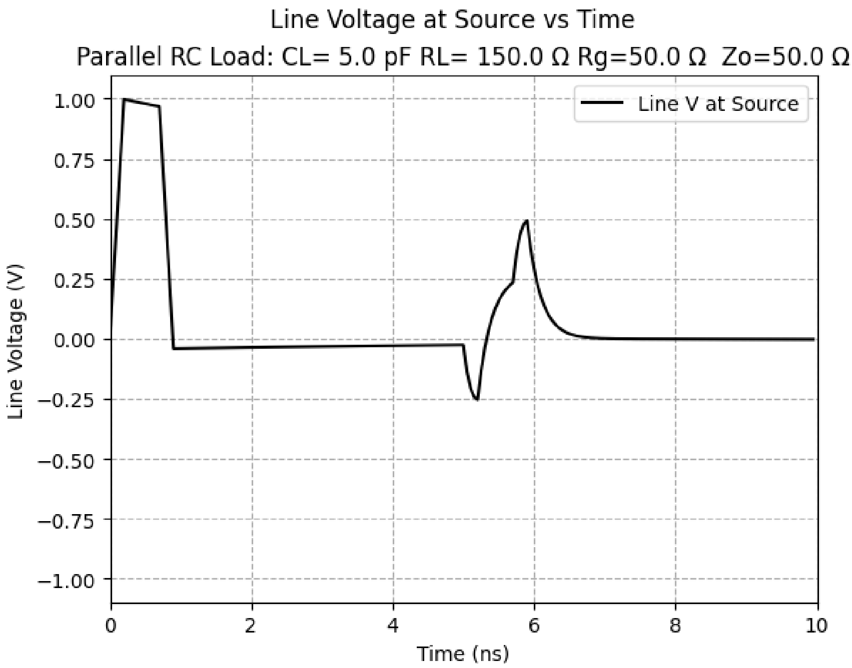

FIG. 9: Plot of the line voltage at the source vs time for a lossy line with G=25 R=250, terminated with a parallel RC load with with a load resistance of 150 Ω and a load capacitance of 5 pF.

The results of this case simulation show what would be expected from theory. The results are shown in FIG. 9. The graph shows the initial trapezoidal pulse propagating out of the source. At about 5 ns, the voltage due the returned reflected pulse from the parallel RC load is observed. The same voltage transients are observed as in case vi, except this time the voltage transient observed at the source node are of smaller magnitude do to the attenuation of the line.

H. Case Study viii)

This case study is a repetition of the previous two case studies(case vi and vii), however, this time terminating the transmission line with a series RL load with a load resistance of 10Ω and load inductance of 10 nH.

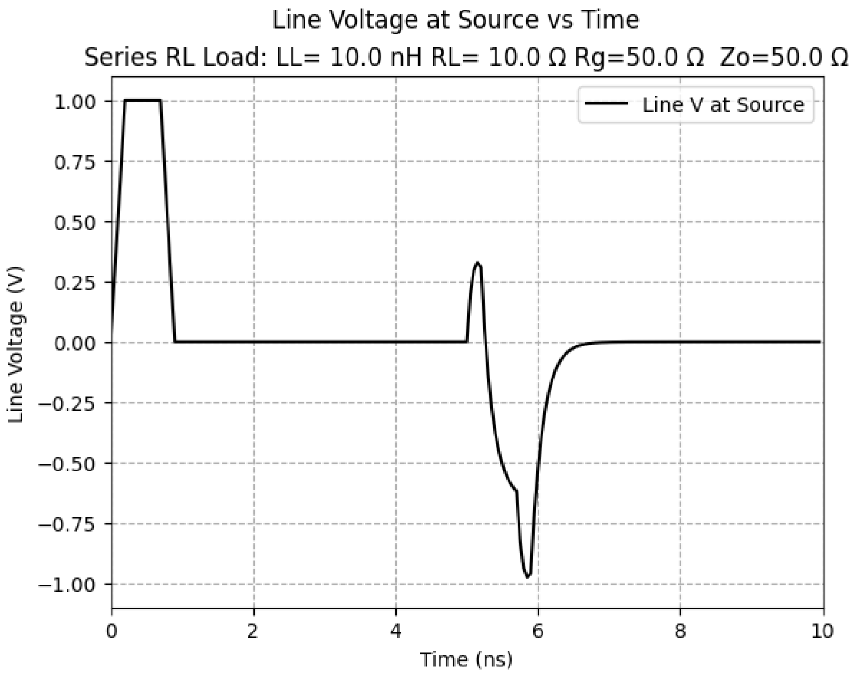

FIG. 10: Plot of the line voltage at the source vs time for a lossless transmission line terminated with a series RL load with a load resistance of 10Ω and load inductance of 10 nH.

The results of this case simulation show what would be expected from theory. The results for the lossless line are shown in FIG. 10. The graph shows the initial trapezoidal pulse propagating out of the source. At about 5 ns, the voltage due the returned reflected pulse from the series RL load is observed.

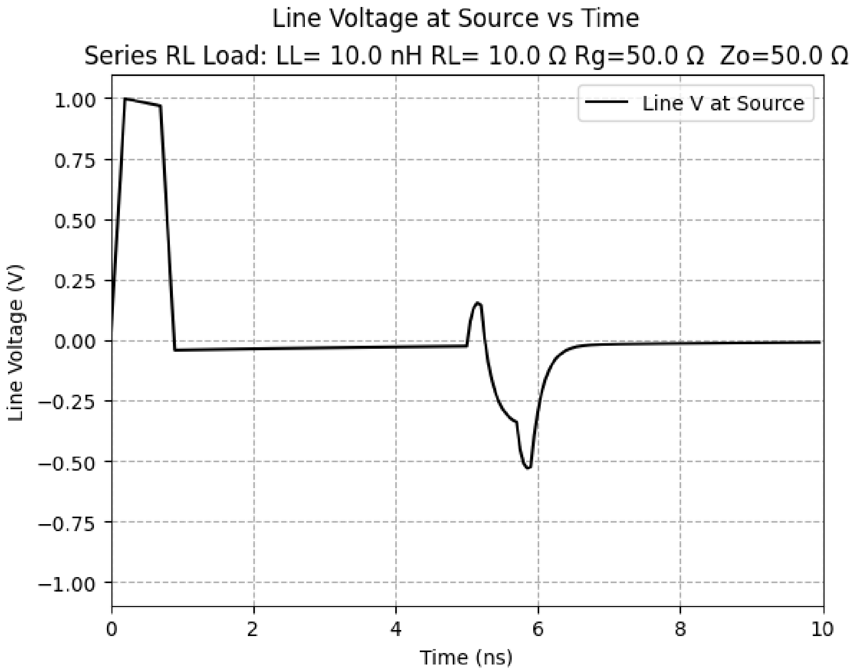

FIG. 11: Plot of the line voltage at the source vs time for a lossy line with G=25 R=250, terminated with a series RL load with a load resistance of 10Ω and load inductance of 10 nH

The results of this case simulation show what would be expected from theory. The results are shown in FIG. 11. The graph shows the initial trapezoidal pulse propagating out of the source. At about 5 ns, the voltage due the returned reflected pulse from the parallel RC load is observed. The same voltage transients are observed as in lossless case vii, except this time the voltage transient observed at the source node are of smaller magnitude do to the attenuation of the line.

I. viii) Simulation Videos

Simulation videos for case studies vi, vii, an viii can be found in the project code file.

V. SUMMARY

A code that performed a FDTD simulation of a 1 dimensional lossy transmission line was successfully developed in this project. The FDTD method used the 2nd order accurate central difference approximation and 2nd order accurate updates for load and source voltages. The code was validated against a series of test cases using a trapezoidal pulse. The results of the test cases agreed with theory, thus validating the functionality and numerical solution produced by the FDTD code.