More on Matrix Factorization

CS 111: Introduction to Computational Science

Spring 2019 Lecture #4

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Administrative

- Homework #2 due next Monday

Lecture Outline

- LU Factorization
  - Using pivots
  - Not using pivots
  - Coding the process

- Cholesky Factorization
- QR Factorization

Matrix Factorization

- Recall the $Ax = b$ example where **A** = $\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix}$ $\qquad \begin{matrix} a00 & a01 \\ a10 & a11 \end{matrix}$
- Note what happens when we try to do Gaussian elimination on A:
  - In order to make element a_{10} become zero, then I need to make:
    row2 = row2 − C.row1 where: C = −0.5
- Note that I can write **A** as:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \boxed{-0.5} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1.5 \end{bmatrix}$$

*Note: This matrix has **1** in the diagonals and the bottom part of it has the coefficient **C (-0.5)***

- Rule: I can ONLY do this if **A** is invertible!

Matrix Factorization: A = LU

- Example 1:
  Using pivoting (classical)
- Example 2:
  Without pivoting (for computational)
- Example 3:
  When no-pivoting doesn't work… (computational)

- See blackboard and our Python-ized solution!

Cholesky Factorization

- For specific cases where we have:
  - Symmetric and square matrix, **A**
  - **A** is *positive definite* (meaning: all the ***eigenvalues*** of **A** are positive)

- It means that we can factor **A** into $\mathbf{X}^T\mathbf{X}$

- Cholesky factorization is a particular form where
  X is an upper triangular with positive diagonals (called **R**)

- So, if you can find **R**, you can figure out: $\mathbf{A} = \mathbf{R}^T\mathbf{R}$

Eigenvalues?!

- Recall: When $\mathbf{Ax} = \lambda\mathbf{x}$ Where λ is a scalar

  We call λ an *eigenvalue* of **A** (there can be many of these)

  and **x** is an *eigenvector* of **A** (there can be as many of these as λs)
- This also means that the **Ax** vector is parallel to **x**
- Properties:
  - Sum of all λ is the sum of all the diagonal values in **A**
  - Product of all λ is det(A)
  - $\det(A - \lambda I) = 0$

Back to Cholesky…

- Recall: $\mathbf{A = R^T R}$
- Example:

$$\begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$$

So:

$a^2 = 4 \rightarrow a = 2$

$ab = -1 \rightarrow b = -0.5$

$b^2+c^2 = 3 \rightarrow c = \mathrm{sqrt}(2.75) = 1.6583124$

$$R = \begin{bmatrix} 2 & 0 \\ -0.5 & 1.6583 \end{bmatrix}$$

QR Factorization

- Often used to solve the **linear least squares problem**
  - An approximation of linear functions to data.
  - Re: solving statistical problems in **linear regression**

- **A = QR**, where **Q** is an orthogonal matrix based on **A**
  - Orthogonal matrix ➔ It's columns are "orthonormal"

Your To-Dos

- Sections today!
- Homework #2 – due **Monday, April 15th**

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