MATH 2605-G2

12/3/14

Green's Theorem:

If C is a smooth, simple, closed curve enclosing a region R and $\vec{F}=M\hat{i}+N\hat{j}$, then

Flux across $C=\int_C \vec{F}\cdot\vec{n}\,ds=\oint_C M\,dy-N\,dx$
$=\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dx\,dy$

Also, Circulation/flow along $C=\oint_C \vec{F}\cdot\vec{T}\,ds=\oint_C M\,dx+N\,dy$
$=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

The two statements are equivalent. This theorem is one of the main results of vector calculus.

Ex) $\vec{F}=(x-y)\hat{i}+x\hat{j}$, $C:\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}$, $0\le t\le 2\pi$

Verify that Green's theorem works for the above.

① $\oint_C M\,dy-N\,dx=\int_0^{2\pi}[(\cos t-\sin t)\cos t-\cos t(-\sin t)]\,dt$
$=\int_0^{2\pi}\frac{1-\cos 2t}{2}\,dt$
$=\frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{2\pi}$
$=\pi$

② $\iint_R(1-0)\,dx\,dy=\text{Area}(R)=\pi.$ ✓

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Recitation:

1)

$(1,1)$ Find $\oint_C \vec{F}\cdot d\vec{r}$ given $\vec{F}=(xy+y^2, x-y)$ and the path C shown on the left. $C_1:\vec{r}_1=(t,t^2)$, $0\le t\le 1$ $C_2:\vec{r}_2=(t^2,t)$, $0\le t\le 1$

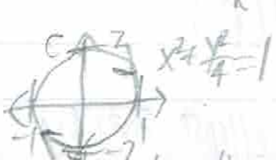
Green's Thm: $\frac{\partial N}{\partial x}=1$ $\frac{\partial M}{\partial y}=x+2y$

$\int_0^1\int_{x^2}^{\sqrt{x}}(1-x-2y)\,dy\,dx=\int_0^1(y-xy-y^2)\Big|_{x^2}^{\sqrt{x}}dx=\int_0^1(\sqrt{x}-x^{3/2}-x-x^2+x^3+x^4)\,dx=\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}-\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}\Big|_0^1$
$=\frac{2}{3}-\frac{2}{5}-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$

Green's Thm. can be used to calculate area: $\iint_R 1\,dx\,dy=\oint(0,x)\cdot d\vec{r}=\oint x\,dy$

2) $\vec{F}=(x+3y)\hat{i}+(2x-y)\hat{j}$
$\frac{\partial N}{\partial x}=2$ $\frac{\partial M}{\partial y}=3$



$\oint_C\vec{F}\cdot d\vec{r}=-\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$ because the direction of this parametrization is clockwise.
$=\iint_R(3-2)\,dx\,dy$
$=(1)(2)\pi$ Area of ellipse.
$=2\pi$

Final exam at 2:30 PM on Friday!

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Review:

Three ways to find a double integral: rectangular coordinates, polar coordinates, and Jacobian

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a) SVD: $A = U\Sigma V^T$

$U^TU = I$, $V^TV = I$, Σ is diagonal matrix of singular values

i) Diagonalize A^TA to find eigenvectors, which are the columns $\vec{v}_i$ of V

eigenvalues: $\sqrt{\lambda_i} = \sigma_i$

ii) Use $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$ to get columns of U. Alternatively, diagonalize AA^T.

b) Properties of a curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \le t \le b$:

$r'(t)$ is tangent vector

Unit tangent: $\vec{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$

Curvature: $\kappa = \|\frac{d\vec{T}}{ds}\| = \|\frac{d\vec{T}}{dt}\| / \|\vec{r}'(t)\|$

Unit normal: $\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{ds} / \|\frac{d\vec{T}}{ds}\| = \frac{d\vec{T}}{dt} / \|\frac{d\vec{T}}{dt}\|$

c) Level curve: $f(x,y) = c$

$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})|_{\vec{x}_0}$ is perpendicular to the level curve

d) Change of basis: $\vec{x}$ in standard basis to basis $W = \{\vec{w}_1, \vec{w}_2, \dots \vec{w}_n\}$

$\psi_{\vec{w}}(\vec{x}) = W^{-1}\vec{x}$

Linear transformation matrix:

$T: V \to V$

$\vec{x} \mapsto T\vec{x}$

New basis W:

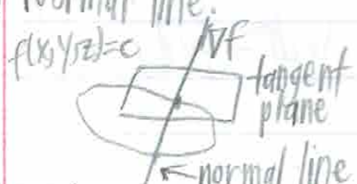
${}_{\vec{w}}T_{\vec{v}} = (\psi_{\vec{w}}(T\vec{v}_1) | \cdots | \psi_{\vec{w}}(T\vec{v}_n))$
$= W^{-1}TV$

Practice final solutions:

15) Rate of change of $f(x,y,z) = \frac{x}{z} + \frac{y}{z}$ w/ resp. to t along $\vec{r}(t) = \sin^2 t\,\hat{i} + \cos^2 t\,\hat{j} + \frac{1}{2t}\hat{k}$

Use the chain rule: $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

14) Normal line:



$\vec{r}(t) = \vec{r}_0 + t\vec{d}$

$\vec{d} = \nabla f(\vec{r}_0)$

19) Extreme values of $f(x,y) = xy$ in ellipse $3x^2 + 4y^2 = 1$

$f(x,y) =$ Max., min. of xy

$g(x,y) = 3x^2 + 4y^2 - 1$

Use Lagrange multipliers to solve $\nabla f = \lambda \nabla g$.

18) Point on $z = 6 - x^2 - y^2$ farthest from plane $x + 5y + 6z = 0$

Use $f(x,y,z) =$ distance from (x,y,z) to plane