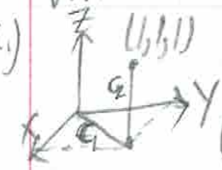
MATH 2605-G2 11/21/14

What if $C_1 \cup C_2$ was considered instead?

Ex.)



$$\int_{C_1 \cup C_2} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

$C_1: \vec{r}(t) = (t,t,0),\ 0 \le t \le 1 \Rightarrow \vec{r}\,'(t) = (1,1,0) \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{2}$

$C_2: \vec{r}(t) = (1,1,t),\ 0 \le t \le 1 \Rightarrow \vec{r}\,'(t) = (0,0,1) \Rightarrow \|\vec{r}\,'(t)\| = 1$

$$\int_{C_1 \cup C_2} f(x,y,z)\,ds = \int_0^1 (t - 3t^2 + 0)\sqrt{2}\,dt + \int_0^1 (1 - 3(1)^2 + t)(1)\,dt$$
$$= \sqrt{2}\left(\frac{t^2}{2} - t^3\right)\Big|_0^1 + \left(-2t + \frac{t^2}{2}\right)\Big|_0^1$$
$$= \sqrt{2}\left(-\frac{1}{2}\right) - \frac{3}{2}$$
$$= \frac{-\sqrt{2} - 3}{2}$$

Changing the path changes the integral.

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Moments and mass:

Let $\delta(x,y,z)$ be the density. Then the mass is

$$M = \int_C \delta\,ds$$

and the moments are

$$M_{yz} = \int_C x\delta\,ds \quad M_{xz} = \int_C y\delta\,ds \quad M_{xy} = \int_C z\delta\,ds.$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$$

and the moment of inertia of a distance r from a point in the curve to the line L is

$$I_L = \int_C r^2 \delta\,ds.$$

With respect to coordinate axes:

$$I_x = \int_C (y^2 + z^2)\delta\,ds \quad I_y = \int_C (x^2 + z^2)\delta\,ds \quad I_z = \int_C (x^2 + y^2)\delta\,ds$$

Vector fields:

A vector field assigns a vector to each point in its domain:

$$\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$$

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Ex.) Velocity vectors in a wind tunnel:

Ex.) Gradient vector field: assign $\nabla f(x,y,z)$ to each point (x,y,z) in the domain of f.
Recall that ∇f points in the direction of greatest increase in $f(x,y,z)$.

Line integrals of vector fields:
Assume that $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ has continuous components, and that the curve C has a smooth parametrization

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}, \quad a \le t \le b$$

If the unit tangent vector is $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$, then $\vec{F}\cdot\vec{T}$ is the tangential component of $\vec{F}$ along the curve C.

Note: $\vec{T}(t) = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds}$ where $s(t) = \int_a^t \|\vec{r}'(\xi)\|\, d\xi$

Then the line integral is defined as

$$\int_C \vec{F}\cdot\vec{T}\, ds = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b [\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)]\,dt,$$

a special case of the general line integral.

Ex.) $\int \vec{F}\cdot d\vec{r}$ for $\vec{F} = z\hat{i} + xy\hat{j} + y^2\hat{k}$ along $C: \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\hat{k}$, $0 \le t \le 1$.

$\Rightarrow \vec{F}(\vec{r}(t)) = \sqrt{t}\hat{i} + (t^2)t\hat{j} + (t)^2\hat{k} = (\sqrt{t}, t^3, t^2)$

$\vec{r}'(t) = (2t, 1, \frac{1}{2\sqrt{t}})$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = 2t^{3/2} + t^3 + \frac{1}{2}t^{3/2} = t^3 + \frac{5}{2}t^{3/2}$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t^3 + \frac{5}{2}t^{3/2})\,dt = \frac{t^4}{4} + t^{5/2}\Big|_0^1 = \frac{5}{4}$

Ex.) $\int_C \vec{h}\cdot d\vec{r}$ for $\vec{h} = x\hat{i} + x^2y\hat{j}$ along $C: \vec{r}(u) = \cos u\,\hat{i} + \sin u\,\hat{j}$, $u \in [0, \frac{\pi}{2}]$

$\Rightarrow \vec{F}(\vec{r}(t)) = \cos u\,\hat{i} + \cos^2 u \sin u\,\hat{j}$

$\vec{r}'(t) = -\sin u\,\hat{i} + \cos u\,\hat{j}$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = -\cos u \sin u + \cos^3 u \sin u$

$\int_C \vec{F}\cdot d\vec{r}$