MATH 2605-G2

11/17/14

Ex.) Find the volume of the solid in the first octant bounded by $z=4-x^2-y^2$, $y=0$, and $x^2+y^2=2x$.

Using cylindrical coordinates:

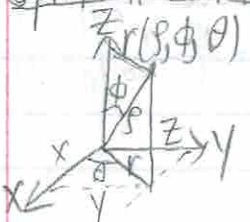
$$z=4-x^2-y^2=4-r^2$$

$$x^2+y^2=2x \Rightarrow r^2=2r\cos\theta \Rightarrow r=2\cos\theta$$

$$S=\{(r,\theta,z): 0\le\theta\le\tfrac{\pi}{2},\ 0\le r\le 2\cos\theta,\ 0\le z\le 4-r^2\}$$

$$\text{Volume}(S)=\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r^2} r\,dz\,dr\,d\theta$$
$$=\int_0^{\pi/2}\int_0^{2\cos\theta}(4r-r^3)\,dr\,d\theta$$
$$=\int_0^{\pi/2}\left(2r^2-\frac{r^4}{4}\right)\Big|_{r=0}^{r=2\cos\theta} d\theta$$
$$=\int_0^{\pi/2}(8\cos^2\theta-4\cos^4\theta)\,d\theta$$
$$=4\int_0^{\pi/2}\left[2\left(\frac{1+\cos 2\theta}{2}\right)-\left(\frac{1+\cos 2\theta}{2}\right)^2\right]d\theta$$
$$=\frac{5\pi}{4}$$

<u>Spherical coordinates:</u>



$$\rho=\sqrt{x^2+y^2+z^2}$$

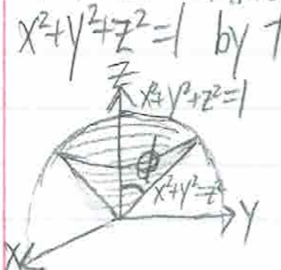
$$\tan\theta=\frac{y}{x};\quad r=\sqrt{x^2+y^2};\quad r=\rho\sin\phi$$

$$x=r\cos\theta=\rho\sin\phi\cos\theta;\quad y=r\sin\theta=\rho\sin\phi\sin\theta;\quad z=\rho\cos\phi$$

$$\sin\phi=\frac{r}{\rho}=\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}};\quad \cos\phi=\frac{z}{\rho}=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

Integral: $\iiint_D f(x,y,z)\,dV=\iiint_S f(\rho,\phi,\theta)\,\underline{\underline{\rho^2\sin\phi}}\,d\rho\,d\phi\,d\theta$

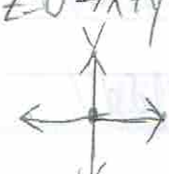
Ex.) Find the volume of the "ice cream cone" D cut from the solid sphere $x^2+y^2+z^2=1$ by the cone $x^2+y^2-z^2=0$.



$y=0\Rightarrow x=\pm z$

$x=0\Rightarrow z=\pm y$

$z=0\Rightarrow x^2+y^2=0$



$z=k\Rightarrow x^2+y^2=k^2$

The sphere $x^2+y^2+z^2=1$ in spherical coordinates is $\rho^2=1$ or $\rho=1$.

For the cone: $x^2+y^2=z^2 \Rightarrow r^2=z^2 \Rightarrow (\rho\sin\phi)^2=(\rho\cos\phi)^2$

$$\Rightarrow \sin^2\phi=\cos^2\phi$$
$$\Rightarrow \tan\phi=\pm 1 \Rightarrow \phi=\frac{\pi}{4}$$

$$S=\{(\rho,\phi,\theta): 0\le\theta\le 2\pi,\ 0\le\phi\le\tfrac{\pi}{4},\ 0\le\rho\le 1\}$$

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi/4}\frac{1}{3}\sin\phi\,d\phi\,d\theta=\frac{1}{3}\int_0^{2\pi}-\cos\phi\Big|_0^{\pi/4}d\theta=\frac{1}{3}\int_0^{2\pi}\left(1-\frac{\sqrt{2}}{2}\right)d\theta=\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$