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Therefore, $\vec{F}\cdot\vec{n} = M\frac{dy}{ds} - N\frac{dx}{ds}$.

Hence, Flux of $\vec{F}$ across $C = \int_C (M\frac{dy}{ds} - N\frac{dx}{ds})ds$

$= \int M dy - N dx$

$= \oint_C M dy - N dx$

Recitation:

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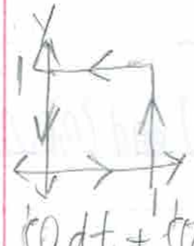
1) $\int \vec{F}\cdot d\vec{r}(t)$ for $\vec{F} = (y+z, z+x, x+y)$ over $C_1: r(t) = (t,t,t), 0 \le t \le 1$ and $C_2: r(t) = (t, t^2, t^4), 0 \le t \le 1$

$\int_0^1 (2t(1) + 2t(1) + 2t(1))dt + \int_0^1 [(t^2+t^4)(1) + (t^4+t)(2t) + (t+t^2)(4t^3)]dt$

$= 3t^2|_0^1 + \int_0^1 (t^2 + t^4 + 2t^5 + 2t^2 + 4t^4 + 4t^5)dt$

$= 3 + (t^3 + t^5 + t^6)|_0^1$

$= 6$

2) 

$\int \vec{F}\cdot dr(t)$ for $\vec{F} = (y, 2x+y)$ over the path shown on the left

$C_1: (t, 0), 0 \le t \le 1$ $\quad C_2: (1, t), 0 \le t \le 1$ $\quad C_3: (1-t, 1), 0 \le t \le 1$ $\quad C_4: (0, 1-t)$

$r'(t) = (1, 0)$ $\quad r'(t) = (0, 1)$ $\quad r'(t) = (-1, 0)$ $\quad r'(t) = (0, -1)$

$\int_0^1 0\,dt + \int_0^1 (2+t)dt + \int_0^1 (-1+0)dt + \int_0^1 (t-1)dt = (2t + \frac{t^2}{2})|_0^1 - t|_0^1 + (\frac{t^2}{2} - t)|_0^1 = 2 + \frac{1}{2} - 1 - \frac{1}{2} = 1$

Fundamental Theorem:

$F = (y, x) = \nabla J$

$J = xy \Rightarrow \nabla J = (y, x)$

$\int_C \nabla J \cdot d\vec{r}(t) = J(\vec{r}(t))|_{t=start}^{t=end}$

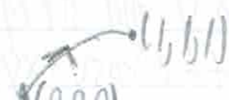
3) $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = (y+z, z+x, x+y)$ over $C: r(t) = (t, \frac{t^3}{3}, \frac{e^t}{e}), 0 \le t \le 1$

J is the "anti-gradient" of F:

$\int (y+z)dx = xy + xz$ $\quad \int (z+x)dy = zy + xy$ $\quad \int (x+y)dz = xz + yz$

$J = xy + xz + yz + C$

$\int_C \vec{F}\cdot d\vec{r} = J(1, \frac{1}{3}, 1) - J(0, 0, \frac{1}{e}) = (\frac{1}{3} + 1 + \frac{1}{3}) - 0 = \frac{5}{3}$

4) $\vec{F} = e^{y+2z}(\hat{i} + x\hat{j} + 2x\hat{k})$ over 

$\int \vec{F}\cdot d\vec{r} = xe^{y+2z}|_{(1,1,1)} - xe^{y+2z}|_{(0,0,0)}$

$= e^3$

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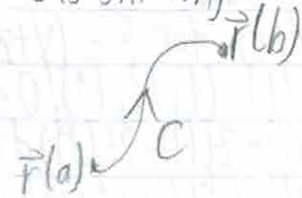
Conservative fields:

Consider $\int_C \vec{F}\cdot d\vec{r}$. If this line integral retains the same quantity regardless of the path taken, then it is path independent and $\vec{F}$ is conservative. For the gradient field $\vec{F}=\nabla f$, f is called the potential function of $\vec{F}$.

Fundamental theorem for line integrals: if $\vec{F}=\nabla f$ and C is smooth,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_a^b \nabla f(x(t),y(t),z(t))\cdot(x'(t),y'(t),z'(t))\,dt \\ &= \left.\begin{aligned}&\int_a^b \nabla f(\vec{r}(t))\cdot\vec{r}'(t)\,dt \\ &= \int_a^b \tfrac{d}{dt}(f(\vec{r}(t)))\,dt\end{aligned}\right\}\text{ Reverse Chain Rule} \\ &= f(\vec{r}(b))-f(\vec{r}(a))\end{aligned}$$



Only the initial and final points matter.

Theorem: $\vec{F}$ is conservative iff $\vec{F}=\nabla f$ for some f.

Theorem: $\oint_C \vec{F}\cdot d\vec{r}=0$ (C is closed and simple) iff $\vec{F}$ is conservative.

Ex.) Let $\vec{F}=\nabla f$ for $f(x,y,z)=\frac{-1}{x^2+y^2+z^2}$, and C be a curve joining $(1,0,0)$ and $(0,0,2)$.
Then $\int_C \vec{F}\cdot d\vec{r} = f(0,0,2)-f(1,0,0) = -\frac{1}{4}-(-1)=\frac{3}{4}$.

Theorem: A vector field $\vec{F}=M\hat{i}+N\hat{j}+P\hat{k}$ is conservative iff

$$\frac{\partial P}{\partial y}=\frac{\partial N}{\partial z},\quad \frac{\partial M}{\partial z}=\frac{\partial P}{\partial x},\quad \frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}.$$

In other words, the curl $\nabla\times\vec{F}$ equals zero. ($\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$.)

Ex.) Show that $\vec{F}=(e^x\cos y+yz,\ xz-e^x\sin y,\ xy+z)$ is conservative.

$$\frac{\partial P}{\partial y}=x \qquad \frac{\partial N}{\partial z}=x \quad \checkmark$$
$$\frac{\partial M}{\partial z}=y \qquad \frac{\partial P}{\partial x}=y \quad \checkmark$$
$$\frac{\partial N}{\partial x}=z-e^x\sin y \qquad \frac{\partial M}{\partial y}=-e^x\sin y+z \quad \checkmark$$

To find $f(x,y,z)$ such that $\vec{F}=\nabla f$:

① $\frac{\partial f}{\partial x}=M=e^x\cos y+yz$ ② $\frac{\partial f}{\partial y}=N=xz-e^x\sin y$ ③ $\frac{\partial f}{\partial z}=P=xy+z$

Integrate ①: $f=e^x\cos y+xyz+h(y,z)$

Take partial deriv. w/ respect to y and set equal to N: $\frac{\partial f}{\partial y}=-e^x\sin y+xz+\frac{\partial h}{\partial y}=xz-e^x\sin y$

Then $\frac{\partial h}{\partial y}=0 \Rightarrow h=\tilde{h}(z)$ and $f=e^x\cos y+xyz+\tilde{h}(z)$.

Take partial deriv. w/ respect to z and set equal to P: $\frac{\partial f}{\partial z}=xy+\frac{\partial \tilde{h}}{\partial z}=xy+z$

Then $\frac{\partial \tilde{h}}{\partial z}=z \Rightarrow \tilde{h}(z)=\frac{1}{2}z^2+k$

Finally, $f(x,y,z)=e^x\cos y+xyz+\frac{1}{2}z^2+k$.

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Green's Theorem:

If C is a smooth, simple, closed curve enclosing a region R and $\vec{F}=M\hat{i}+N\hat{j}$, then

Flux across $C=\int_C \vec{F}\cdot\vec{n}\,ds=\oint_C M\,dy-N\,dx$
$=\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dx\,dy$

Also, Circulation/flow along $C=\oint_C \vec{F}\cdot\vec{T}\,ds=\oint_C M\,dx+N\,dy$
$=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

The two statements are equivalent. This theorem is one of the main results of vector calculus.

Ex) $\vec{F}=(x-y)\hat{i}+x\hat{j}$, $C: \vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}$, $0\le t\le 2\pi$

Verify that Green's theorem works for the above.

① $\oint_C M\,dy-N\,dx=\int_0^{2\pi}[(\cos t-\sin t)\cos t-\cos t(-\sin t)]\,dt$
$=\int_0^{2\pi}\frac{1-\cos 2t}{2}\,dt$
$=\frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{2\pi}$
$=\pi$

② $\iint_R(1-0)\,dx\,dy=\text{Area}(R)=\pi.$ ✓