Calculus 3 for Computer Science 8/18/14
MATH 2605

Diagnostic Assessment: (at Recitation tomorrow)
1) Dot/cross products, $\angle$ between vectors
2) Determinants (2×2, 3×3)
3) Planes and lines in 3-D
4) Solving systems
5) Finding matrix for linear transformation
6) Gram-Schmidt
7) Eigenvalues, eigenvectors, diagonalization

8/20/14

$Proj_{\vec{v}}\vec{u}$ is vector projection of $\vec{u}$ onto $\vec{v}$
Since length is $\|\vec{u}\|\cos\theta$ and direction is $\frac{\vec{v}}{\|\vec{v}\|}$, $Proj_{\vec{v}}\vec{u} = (\|\vec{u}\|\cos\theta)\frac{\vec{v}}{\|\vec{v}\|}$
$\Rightarrow Proj_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

Scalar component of $\vec{u}$ in direction of $\vec{v}$: $\lambda = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$
$\Rightarrow Proj_{\vec{v}}\vec{u} = \lambda\frac{\vec{v}}{\|\vec{v}\|}$

Box product: $(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Any point in line: $\vec{r} = \vec{r}_0 + t\vec{v}$, $t \in \mathbb{R}$

8/21/14

Find equation for set of points equidistant from planes $y=3$ and $y=1$
$|y-3| = |y-1|$
$(y-3)^2 = (y-1)^2$
$(y-3+y-1)(y-3+1-y) = 0$
$y = 2$

Find equation for set of points equidistant from point $(0,0,2)$ and xy-plane ($z=0$)
$\sqrt{x^2+y^2+(z-2)^2} = |z|$
$x^2+y^2+(z-2)^2 = z^2$

Find the point on the sphere $x^2+(y-3)^2+(z+5)^2=4$ nearest xy-plane.
$z=0$: $x^2+(y-3)^2 = 4-25 < 0$ so $z=0$ is not on sphere
$x^2+(y-3)^2 = 4-(z+5)^2 \geq 0$
$-2 \leq z+5 \leq 2$
$-7 \leq z \leq -3$
$z=-3$ is closest to $z=0$
$x^2+(y-3)^2 = 0 \Rightarrow x=0, y=3$

MATH 2605-G2

8/22/14

Vector ∥ to line of intersection of:

$$\begin{cases} 3x-6y-2z=15 \\ 2x+y-2z=5 \end{cases}$$

First, $\vec{n}_1=(3,-6,-2)$ and $\vec{n}_2=(2,1,-2)$.

The vector $\vec{v}$ is ⊥ to both normals so $\vec{v}=\vec{n}_1\times\vec{n}_2$

$$\vec{v}=\det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}=(14,2,15)$$

Line of intersection is $\vec{r}=\vec{r}_0+t\vec{v}$:

$3x-6y=15 \Rightarrow 3x-6(5-2x)=15 \Rightarrow x=3$

$2x+y=5 \Rightarrow y=5-2(3)=-1$

$\therefore \vec{r}=(3,-1,0)+t(14,2,15)$

Another way:

$x=\frac{14}{15}z+3$

$y=-2\left(\frac{14}{15}z+3\right)+2z+5$

$=\frac{2}{15}z-1$

$\vec{r}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 14/15 \\ 2/15 \\ 1 \end{pmatrix}z+\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix},\ z\in\mathbb{R}$

Parametric equation of plane:

$\vec{r}=s\vec{u}+t\vec{v}+\vec{w};\ s,t\in\mathbb{R}$

Week 1 Homework

8/24/14

12.1

21) a) spherical shell of thickness 1 between $r=1$ and $r=2$ centered at origin
b) solid hemisphere of radius 1 centered at origin with only positive z half

35) $z\in[0,1],\ x\in\mathbb{R},\ y\in\mathbb{R}$

43) $\sqrt{(4-1)^2+(-2-4)^2+(7-5)^2}=\sqrt{49}=7$

47) center: $(-2,0,2)$ radius: $2\sqrt{2}$

55) $(x+2)^2+y^2+(z-2)^2=8$ same as 47

57) $\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+\left(y^2+\frac{1}{2}y+\frac{1}{16}\right)+\left(z^2+\frac{1}{2}z+\frac{1}{16}\right)=\frac{9}{2}+\frac{3}{16}$
center: $\left(-\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right)$ radius: $\sqrt{\frac{75}{16}}=\frac{5\sqrt{3}}{4}$

63) $|y-3|=|y+1| \Rightarrow (y-3)^2=(y+1)^2 \Rightarrow (y-3+y+1)(y-3-y-1)=0 \Rightarrow y=1$

65) b) $0^2+4^2+0\neq4$ so pt. not on sphere

$x^2+(y-7)^2+(z+5)^2=d \Rightarrow$ minimize d

$-\left(x^2+(y-3)^2+(z+5)^2=4\right)$

$-14y+49+6y-9=d-4$

$d=-8y+44 \Rightarrow$ maximize y

$x^2+(z+5)^2=4-(y-3)^2\geq0 \Rightarrow y\leq5$

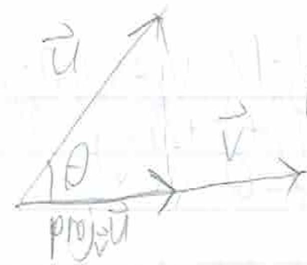
$y=5,\ x^2+(z+5)^2=0$ so $x=0$ and $z=-5$

8/25/14

$\|\overrightarrow{PS}\|\sin\theta = d$

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$

Distance from pt. to line: $d = \frac{\|\overrightarrow{PS}\times\vec{v}\|}{\|\vec{v}\|}$



$\text{proj}_{\vec{v}}\vec{u} = \vec{u}\cos\theta$

$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

Distance from pt. to plane: $d = \frac{\overrightarrow{PS}\cdot\vec{n}}{\|\vec{n}\|}$

12.2

7) $\langle \frac{9}{5}-\frac{8}{5}, -\frac{6}{5}+4\rangle = \langle \frac{1}{5}, \frac{14}{5}\rangle$ $\sqrt{(\frac{1}{5})^2+(\frac{14}{5})^2} = \frac{\sqrt{197}}{5}$

17) $\langle -3, 2, -1\rangle$

21) $\langle 3, 5, -8\rangle$

31) a) $2\hat{i}$ b) $\sqrt{3}\hat{k}$ c) $\frac{3}{10}\hat{j}+\frac{4}{5}\hat{k}$ d) $\frac{6}{7}\hat{i}-\frac{2}{7}\hat{j}+\frac{3}{7}\hat{k}$

35) a) $(3, 4, -5)$ b) $(\frac{1}{2}, 3, \frac{5}{2})$

41) $2\hat{i}+\hat{j} = a\hat{i}+a\hat{j}+b\hat{i}-b\hat{j}$

$\begin{cases} a+b=2 \\ a-b=1 \end{cases} \Rightarrow \begin{matrix} a=1.5 \\ b=0.5 \end{matrix}$

45) $\begin{cases} F_1\cos 30° = F_2\cos 45° \\ F_1\sin 30° + F_2\sin 45° = 100(10) \end{cases}$

$F_1 = \frac{\sqrt{2}}{\sqrt{3}}F_2$

$\frac{\sqrt{2}}{2\sqrt{3}}F_2 + \frac{\sqrt{2}}{2}F_2 = 1000$

$F_2\left(\frac{\sqrt{2}+\sqrt{6}}{2\sqrt{3}}\right) = 1000$

$F_2 \approx 897\text{ N}$ $F_1 \approx 732\text{ N}$

12.3

1) a) $2(-2)-16-5 = -25$ b) $-25 = \sqrt{(2^2+4^2+5)(2^2+4^2+5)}\cos\theta \Rightarrow \cos\theta = -1$

c) $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} = \frac{-25}{5} = -5$ d) $\frac{-5}{5}\langle 2, -4, \sqrt{5}\rangle = \langle -2, 4, -\sqrt{5}\rangle$

3) a) $\vec{v}\cdot\vec{u} = 11\cdot 3 - 2\cdot 4 = 25$ $\|\vec{v}\| = 15$ $\|\vec{u}\| = 5$ b) $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{25}{15(5)} = \frac{1}{3}$

c) $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} = \frac{25}{15} = \frac{5}{3}$ d) $\frac{5}{3}\langle 10, 11, -2\rangle$

7) a) $10+\sqrt{17} = \vec{v}\cdot\vec{u}$ b) $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{10+\sqrt{17}}{\sqrt{26}(\sqrt{21})}$ c) $\frac{10+\sqrt{17}}{\sqrt{26}(\sqrt{21})}\sqrt{21}$ d) $\frac{10+\sqrt{17}}{\sqrt{26}}\langle 5, 1\rangle$

9) $\cos\theta = \frac{2+2}{\sqrt{5}(\sqrt{6})}$ 33) $x+2y = c$ 47) $v_1 = \langle\sqrt{3}, -1\rangle$ $\cos\theta = \frac{\sqrt{3}+\sqrt{3}}{2(2)} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$

27) a $c = 2+2 = 4$ $v_2 = \langle 1, -\sqrt{3}\rangle$

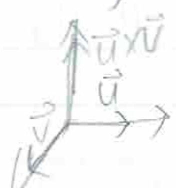
12.4

1) $\vec{u}\times\vec{v} = -2(-\hat{j})-2(-\hat{k})+2\hat{i}-\hat{j} = 2\hat{i}+\hat{j}+2\hat{k}$ $\|\vec{u}\times\vec{v}\| = 3$

19) 8

29) a) $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$ b) $u\times v$

4)



15) a) $(2\hat{i}-2\hat{k})\times(2\hat{j}+\hat{k})/2$

$= (4\hat{k}-2\hat{j}+4\hat{i})/2$

$= 2\hat{i}-\hat{j}+2\hat{k}$

b) $\|\text{above}\| = 3$ $\vec{u} = \frac{2\hat{i}-\hat{j}+2\hat{k}}{3}$

c) $(u\times v)\times w$ d) $(u\times v)\cdot w$

e) $(u\times v)\times(u\times w)$ f) $\|\vec{u}\|\frac{\vec{v}}{\|\vec{v}\|}$

47) $v_1 = \langle -1, 2, 0\rangle$ $v_2 = \langle 0, 1, -2\rangle$

$A_\Delta = \frac{\|v_1\times v_2\|}{2} = \frac{\|1-\hat{k}-2\hat{j}-4\hat{i}\|}{2} = \frac{\sqrt{21}}{2}$

MATH 2605-G2

If $x = c_1\bar{v}_1 + \cdots + c_n\bar{v}_n$, then $\psi_{\bar{V}}\bar{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. 8/26/14

MATH 2605-G2

1) a) $\vec{v}_1 = \langle 2,-1,-1\rangle$ $\vec{v}_2 = \langle 1,0,-2\rangle$
$\vec{v}_1 \times \vec{v}_2 = 4\hat{j} + \hat{k} + 2\hat{i} - \hat{j} = 2\hat{i} + 3\hat{j} + \hat{k}$
$2x+3y+z=0$

b) $\dfrac{|2\hat{i}+3\hat{j}+\hat{k}|}{2} = \dfrac{\sqrt{4+9+1}}{2} = \dfrac{\sqrt{14}}{2}$

2) a) $y = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} t,\ t \in \mathbb{R}$ b) $y = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} t,\ t \in \mathbb{R}$ $d = \dfrac{2\cdot 3 - 3\cdot 4 + 0\cdot 5}{\sqrt{3^2+4^2+5^2}} = \dfrac{-6}{5\sqrt{2}}$

Linear Algebra Review Homework 8/27/14

1) $v_2 = \langle 1,0,0,0\rangle$, $v_3 = \langle 0,2,-1,0\rangle$, $v_4 = \langle 0,1,0,-1\rangle$

2) $\langle 1,0,0,1\rangle$, $\langle 1,2,1,0\rangle$

3) $E_2E_1A = E_2\begin{bmatrix} 1 & & \\ -1 & 1 & \\ -2 & & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix}$ rank=2 nullity=2

4) 2

5) $c_1\beta_1 + c_2\beta_2 + c_3\beta_3 = \langle 2,4,5\rangle$
$c_1+c_3=2$ $c_1+c_2+c_3=4$ $c_1+c_2=5$
$c_1 = 2+1 = 3 \longleftarrow c_3 = -1$
$c_2 = 5-c_1 = 2$
$3\beta_1 + 2\beta_2 - \beta_3 = \alpha$

6) $\bar{V}_1 = \bar{W}_1 + \bar{W}_2 + \bar{W}_3 = \langle 1,1,1\rangle^t_{\bar{W}}$
$\bar{V}_2$: $c_1+c_3=7$ $c_1+c_2+c_3=10$ $c_1+c_2=6$
$c_2=3$ $c_1=3$ $c_3=4$
$\bar{V}_2 = 3\bar{W}_1 + 3\bar{W}_2 + 4\bar{W}_3 = \langle 3,3,4\rangle^t_{\bar{W}}$
$M = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 3 & 3 \\ 1 & 4 & 3 \end{bmatrix}$

7) $\dfrac{d\vec{v}}{dt} = \langle 1, 2x-1, 3x^2-1\rangle^t = A_1\vec{v} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -3/2 & 2 & 0 & 0 \\ -3/2 & 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$
$= A_2\vec{w} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & 2 & 0 \\ -3/2 & -3 & 3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

13) $\beta_1 = \dfrac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\beta_2 = \dfrac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ $\alpha_1 \cdot \alpha_2 = 0$
$\beta_3 = \beta_1 \times \beta_2 = \dfrac{1}{\sqrt{30}}(-\hat{j} + 2\hat{k} + \hat{i} - 4\hat{j}) = \dfrac{1}{\sqrt{30}}\begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$

8) x-axis: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$ y-axis: $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ z-axis: $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9) Stretched by factor s_x and s_y in x and y dimensions
Yes, because $A(c\bar{x}) = cA\bar{x}$, and $A(c_1\bar{x} + c_2\bar{x}) = A(c_1+c_2)\bar{x}$

10) No, because $T(cx,y,z) \neq cT(x,y,z)$

14) $\bar{u}_1 = \begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \end{pmatrix}\dfrac{1}{\sqrt{30}}$ $\bar{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}\dfrac{1}{\sqrt{5}}$ $\bar{u}_1 \cdot \bar{u}_2 = 0$

11) $\cancel{\begin{bmatrix} 7 & -8 & -8 \\ 4 & -16 & 18 \\ -5 & 11 & 13 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix}}$ $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

12) $J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $J^3 = \begin{bmatrix} 9 & 6 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 27 & 27 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 125 \end{bmatrix}$

$J^2 = \begin{bmatrix} 9 & 6 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$ $J^3 + 7J + 1 = \begin{bmatrix} 49 & 35 & 1 \\ 1 & 49 & 1 \\ 1 & 1 & 161 \end{bmatrix}$

$c\bar{u}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}\cdot\dfrac{1}{\sqrt{30}}\begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \end{pmatrix}\right)\bar{u}_1 - \left(\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}\cdot\bar{u}_2\right)\bar{u}_2$
$= \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \dfrac{9}{30}\begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \dfrac{2}{5}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -7/5 \\ 3/10 \\ 2/5 \\ 1/5 \end{pmatrix}$

$\bar{u}_3 = \dfrac{(\ \vdots\ )}{\|c\bar{u}_3\|}$

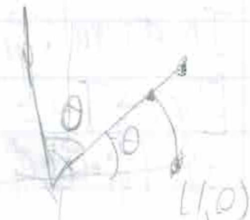
MATH 2605 - G2 8/28/14

1) $\begin{vmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0$ so V_1, V_2, V_3 are indep.
$V_4 = 4V_3 - V_2 - V_1$ so $\vec{c} = \begin{bmatrix} -1 \\ -1 \\ 4 \\ -1 \end{bmatrix}$

2) 3, dimension

3) If $a=0$ or $f=0$, $\min(i,j)$ is at most 2, so the nullity is at least 1.
If $d=0$, then $\det(U) = adf = 0$. A nullspace or zero determinant signifies dependency.

4) $\det(U) = adf \neq 0$



5) a) Independent, because similar to identity matrix
b) Dependent, because $V_1 + V_2 + V_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

8/29/14

$R_\theta = \begin{pmatrix} \cos 2\theta & \cos(\frac{\pi}{2} - 2\theta) \\ \sin 2\theta & -\sin(\frac{\pi}{2} - 2\theta) \end{pmatrix} = \begin{pmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{pmatrix}$

Rotation around line θ Projection onto line θ

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \sin\theta$ $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cos\theta$

$P_\theta = T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$

Change of basis:
Identity transformation is $T = I_d : V \to V$
${}_{\vec{w}}Id_{\vec{v}} = (\psi_w(\vec{v}_1) \mid \psi_w(\vec{v}_2) \mid \cdots \mid \psi_w(\vec{v}_n))$

9/2/14

1) True: if $m=n$, row space = column space
2) Column space: 1 Row space: 1 Nullspace: 3 Left nullspace: 1
1) $A = PR = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos 90° & -\sin 90° \\ +\sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} \cos 90° & -\sin 90° \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$
2) The zero matrix
3) Rotation and reflection
6) Gives it a y-component of $y = 3x$

MATH 2605-G2

9/3/14

$T \to$ transformation matrix
$\psi \to$ transformation vector

$_{\vec w}T_{\vec v} = (\psi_{\vec w}(\vec v_1) \mid \psi_{\vec w}(\vec v_2) \mid \cdots \mid \psi_{\vec w}(\vec v_n))$

If $\vec u = a_1\vec v_1 + a_2\vec v_2 + \cdots + a_n\vec v_n$

$\psi_{\vec v}\vec u = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

Then $\psi_{\vec w}\vec u = {}_{\vec w}T_{\vec v}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

If <u>axis</u> is rotated, use negative direction of θ

If W is transformation from S.B. to $\vec w$, then $_{\vec w}T_{\vec v} = W^{-1}$

In general: $_{\vec w}T_{\vec v} = W^{-1}V$

Ex: $_{\vec q}T_{\vec e} = Q^{-1}I = Q^T$

$\vec u = (\vec q_1 \cdot \vec u)\vec q_1 + \cdots + (\vec q_n \cdot \vec u)\vec q_n$

$\text{proj}_{\vec q_1}\vec v_1 = |\vec v_2|\cos\theta\,\frac{\vec q_1}{|\vec q_1|}$

$\vec q_1 \cdot \vec v_2 = |\vec q_1||\vec v_2|\cos\theta$

$= \frac{\cancel{|\vec v_2|}(\vec q_1 \cdot \vec v_2)}{|\vec q_1|\cancel{|\vec v_2|}}\vec q_1$

9/4/14

1) Basis for nullspace of

$D = \begin{bmatrix} 1 & 0 & 5 & -2 & -1 \\ 0 & 1 & -2 & 4 & 2 \end{bmatrix}$

is $\begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Rank = 2

2) $\frac{d}{dx}\{1, x, x^2\} = \{0, 1, 2x\} = \{0, \frac{w_1}{2}, 2w_2 - w_1\}$

$_{\vec w}M_{\vec v} = \begin{bmatrix} 0 & \frac{1}{2} & -1 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & \frac{1}{2} & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x - x^2 \\ 2x^2 \end{bmatrix}$

$M\vec v = \vec w$

3) $T(x, y, z) = (2x, 2y+1, z+3)$

$T(u+v) \neq T(u) + T(v)$

Not linear

MATH 2605-G2

10a) (cont.)

$$= 3\begin{pmatrix} -(9-i^2) & -2(3-i) \\ 5 & 3-i \end{pmatrix}$$

$$= 3\begin{pmatrix} -10 & -6+2i \\ 5 & 3-i \end{pmatrix}$$

$$\begin{vmatrix} -5 & -3+i \\ 0 & 0 \end{vmatrix} = 0$$

$$x_1 = \begin{pmatrix} -3+i \\ 5 \end{pmatrix}$$

$$|A-\lambda_2| = \begin{vmatrix} -3+i & -2 \\ 5 & 3+i \end{vmatrix}$$

$$= \begin{vmatrix} -9+i^2 & -2(3+i) \\ 5 & 3+i \end{vmatrix}$$

$$= \begin{vmatrix} -9+i^2 & -6-2i \\ 5 & 3+i \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -5 & -3-i \\ 0 & 0 \end{vmatrix}$$

$$x_2 = \begin{pmatrix} -3-i \\ 5 \end{pmatrix}$$

b) $\Rightarrow \begin{vmatrix} 5 & 2+i \\ 0 & 0 \end{vmatrix}$

$$x_1 = \begin{pmatrix} 2+i \\ -5 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 2-i \\ -5 \end{pmatrix}$$

Recitation Problems

9/11/14

1) Is $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 2 & 0 & 3 \end{pmatrix}$ diagonalizable?

Columns are linearly independent, so yes.

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 2 & 0 & 3-\lambda \end{vmatrix}$$

$$= -\lambda(-\lambda(3-\lambda) - 2(0)) - 1(1(3-\lambda) - 2(2))$$

$$= -\lambda(\lambda^2 - 3\lambda) - (-1-\lambda)$$

$$= -\lambda^3 + 3\lambda^2 + \lambda + 1$$

$$R = \begin{bmatrix} e_1 \cdot v_1 & e_1 \cdot v_2 \\ 0 & e_2 \cdot v_2 \end{bmatrix}$$

2) $\vec{q}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{5}}$

$$\vec{q}_2 = \vec{q}_2 - (\vec{q}_1 \cdot \vec{v}_2)\vec{q}_1$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{5}} \left(\frac{1}{\sqrt{5}} \cdot 1 + \frac{2}{\sqrt{5}}\right)$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{3}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3/5 \\ 6/5 \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 \\ -1/5 \end{pmatrix}$$

$$\vec{q}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{\sqrt{5}}{5}$$

$$R = \begin{bmatrix} \sqrt{5} & 3/\sqrt{5} \\ 0 & 1/\sqrt{5} \end{bmatrix}$$

$$Q = \frac{\sqrt{5}}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$Q^{-1} = \sqrt{5}\begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}$$

$$A = QR = \frac{\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix}}{5} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

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9/15/14

U, V^T both reflect/rotate the original image.
Σ stretches/shrinks the image.
After finding V, you can use $\vec{u}_k = \frac{1}{\sigma_k} A\vec{v}_k$ to find V^T.
Transform in order of V^T, Σ, U.

If A is square, $A = U\Sigma V^T$ and $A^{-1} = V\Sigma^{+}U^T$, where Σ^{+} has $\frac{1}{\sigma_i}$ instead of σ_i.
If A is not square, $A^{+} = V\Sigma^{+}U^T$ and is generalized inverse.
$\vec{x} = A^{+}\vec{b}$ is another way of finding the least-squares solution to $A\vec{x} = \vec{b}$.
Also, $AA^{+}\vec{b}$ is the projection of $\vec{b}$ onto column space (range) of A.

9/16/14

1) $|A-\lambda I| = (1-\lambda)^3 + 1 + 1 - (1-\lambda) - (1-\lambda) - (1-\lambda) = 0$
$= 1 - 3\lambda + 3\lambda^2 - \lambda^3 + 3\lambda - 1 = 0$
$= \lambda^2(3-\lambda) = 0$

$\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = 3$

$x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Eigenspaces are orthogonal because A is symmetric.

$q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad q_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad q_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$

2) $A^TA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$|A^TA - \lambda I| = 0$
$(2-\lambda)[(1-\lambda)(2-\lambda) - 1] - 1[1(2-\lambda) - 0] = 0$
$(2-\lambda)(2 - 3\lambda + \lambda^2 - 1) - 2 + \lambda = 0$
$2 - \lambda - 6\lambda + 3\lambda^2 + 2\lambda^2 - \lambda^3 - 2 + \lambda = 0$
$-\lambda^3 + 5\lambda^2 - 6\lambda = 0$
$-\lambda(\lambda-3)(\lambda-2) = 0$

$\lambda_1 = 3 \quad \lambda_2 = 2 \quad \lambda_3 = 0$

$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$q_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad q_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad q_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$u_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$= \frac{1}{3}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$
$= \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} & & \\ & \sqrt{2} & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$

Error $\|\frac{\delta\vec{x}}{\vec{x}}\|$ is related to $\|\frac{\delta\vec{b}}{\vec{b}}\|$ by smallest eigenvalue of A. The smaller it is, the closer A is to singular and the larger the error.

Week 5 Homework

1)

2) $\sigma_{min} \leq \sigma_{max}$
$\|C^{-1}\| \leq \|C\|$
$\frac{1}{\|C^{-1}\|} > \frac{1}{\|C\|}$

3) $A^TA = \begin{bmatrix} 1/4+1/4 & 1/6+1/12 \\ 1/6+1/12 & 1/9+1/16 \end{bmatrix} = \begin{bmatrix} 13/36 & 1/4 \\ 1/4 & 25/144 \end{bmatrix}$

$\left(\frac{13}{36}-\lambda\right)\left(\frac{25}{144}-\lambda\right) - \frac{1}{16} = 0$

$\frac{13}{36}\left(\frac{25}{144}\right) - \frac{77}{144}\lambda + \lambda^2 - \frac{324}{5184} = 0$

$\lambda_1 = \frac{77}{288} + \frac{\sqrt{73}}{32} \quad \lambda_2 = \frac{77}{288} - \frac{\sqrt{73}}{32}$

$\frac{|\lambda_1|}{|\lambda_2|} \approx 1480$

$\frac{1}{2}x_1 + \frac{1}{3}x_2 = 1.4 \Rightarrow \frac{1}{8}x_1 + \frac{1}{12}x_2 = \frac{9}{25}$

$\frac{1}{3}x_1 + \frac{1}{4}x_2 = 1 \Rightarrow \frac{1}{9}x_1 + \frac{1}{12}x_2 = \frac{1}{3}$

$x_1 = \frac{48}{25} \quad x_2 = \frac{36}{25}$

$\frac{1}{2}x_1' + \frac{1}{3}x_2' = 1.5 \Rightarrow \frac{1}{8}x_1' + \frac{1}{12}x_2' = \frac{3}{8}$

$\frac{1}{3}x_1' + \frac{1}{4}x_2' = 0.9 \Rightarrow \frac{1}{9}x_1' + \frac{1}{12}x_2' = \frac{3}{10}$

$x_1' = \frac{27}{5} \quad x_2' = -\frac{18}{5}$

$\frac{\|\delta\vec{x}\|}{\|\vec{x}\|} = \varepsilon \frac{\|\delta\vec{b}\|}{\|\vec{b}\|}$

$\frac{\|\frac{1}{25}\begin{pmatrix}87\\-126\end{pmatrix}\|}{\|\frac{1}{25}\begin{pmatrix}48\\36\end{pmatrix}\|} = \varepsilon \frac{\|\begin{pmatrix}0.1\\-0.1\end{pmatrix}\|}{\|\begin{pmatrix}1.4\\1\end{pmatrix}\|}$

$\varepsilon \approx 31.046$

MATH 2605-G2 9/22/14

$R\vec{x} = Q^T\vec{b}$ does not amplify error because $\text{cond}(Q) = 1$.
Does computing QR introduce error?
1) Gram-Schmidt: $AR_1R_2R_3 \cdots R_n = Q$, where R_i are upper triangular
cond(R_i) may be large.
2) Use orthogonal transformations to make A upper-triangular: $Q_n \cdots Q_2 Q_1 A = R$
More stable
Finding Q_i: Householder Reflections and Given's Rotations
Householder Reflections:

$$H = I - 2\vec{u}\vec{u}^T, \text{ where } \|\vec{u}\| = 1.$$

$$= I - 2\frac{\vec{v}\vec{v}^T}{\|\vec{v}\|^2}, \quad \vec{v} \neq 0$$

$$\vec{u} = \frac{\vec{x} + \|\vec{x}\|\vec{e}_1}{\|\vec{x} + \|\vec{x}\|\vec{e}_1\|}$$

Note: $H = H^T$

$$A = (H_{n-1} \cdots H_1)R$$

$$H\vec{x} = (I - 2\vec{u}\vec{u}^T)\vec{x}$$
$$= \vec{x} - 2(\vec{u}^T\vec{x})\vec{u}$$
$$= \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}$$
$$= \vec{x} - 2\,\text{proj}_{\vec{u}}\vec{x}$$

1) a) $A^TA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$(2-\lambda)[(1-\lambda)(2-\lambda) - 1] - 1[1(2-\lambda) - 0] = 0$
$(2-\lambda)(\cancel{2} - 3\lambda + \lambda^2 - \cancel{1}) + \lambda - \cancel{2} = 0$
$-6\lambda + 2\lambda^2 + 3\lambda^2 - \lambda^3 = 0$
$-\lambda(\lambda^2 - 5\lambda + 6) = 0$
$\lambda_1 = 3 \quad \lambda_2 = 2 \quad \lambda_3 = 0$

$A^TA - \lambda_1 I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad A^TA - \lambda_2 I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{3}} \quad V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\frac{1}{\sqrt{2}} \quad V_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\frac{1}{\sqrt{6}}$

b) $u_1 = \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$u_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$

Why drop $\lambda_3 = 0$?
Because Rank$(M_{2\times3}) \leq 2$.

9/23/14

3) a) $J^2 = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 6 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

b) [illegible]

MATH 2605 - G2

9/24/14

Givens/Jacobi's Rotations

Idea: find θ such that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ 0 \end{pmatrix}$$

Solving: $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ $\sin\theta = \frac{-y}{\sqrt{x^2+y^2}}$

Use

$$R_{ij} = \begin{pmatrix} 1 & & & 0 \\ 0 & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ 0 & & & 1 \end{pmatrix}$$

Ex.) $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ Use Given's Rotations to factorize into QR.

i) Use $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\cos\theta = \frac{\sqrt{2}}{2}$ $\sin\theta = \frac{-\sqrt{2}}{2}$

$$G_1 = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G_1A = \begin{pmatrix} \sqrt{2} & & \\ 0 & & \\ 2 & & \end{pmatrix}$$

ii) Use $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 2 \end{pmatrix}$

affect 1st and 3rd elements

$\cos\theta = \frac{1}{\sqrt{3}}$ $-\sin\theta = -\sqrt{\frac{2}{3}}$

$$G_2 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 0 & \sqrt{2/3} \\ 0 & 1 & 0 \\ -\sqrt{2/3} & 0 & 1/\sqrt{3} \end{pmatrix}$$

$$G_2G_1A = \begin{pmatrix} \sqrt{6} & & \\ 0 & -1/\sqrt{2} & \\ 0 & -2/\sqrt{3} & \end{pmatrix}$$

iii) Use $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ -2/\sqrt{3} \end{pmatrix}$

$\cos\theta = \frac{-1/\sqrt{2}}{\sqrt{1/2+4/3}}$ $\sin\theta = \frac{2/\sqrt{3}}{\sqrt{1/2+4/3}}$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$G_3G_2G_1A = \begin{pmatrix} \sqrt{6} & & \\ 0 & \sqrt{\frac{1}{2}+\frac{4}{3}} & \\ 0 & 0 & \end{pmatrix}$$

G_i is rotation matrix, so $G_i^{-1} = G_i^T$.

iv) $A = G_1^{-1}G_2^{-1}G_3^{-1}R$

MATH 2605-G2

9/26/14

Iterative methods for $A\vec{x}=\vec{b}$

Motivation: find solution of $x=f(x)$, $x\in\mathbb{R}$

Iterate starting with x_0

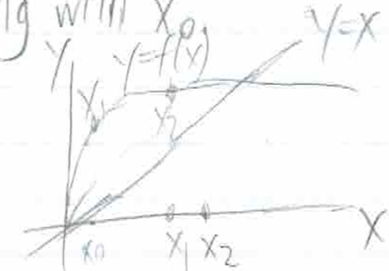
$x_1=f(x_0)$

$x_2=f(x_1)$

$\vdots$

$x_n=f(x_{n-1})$

$\vdots$



If $x^*=f(x^*)$ is solution, from Taylor expansion:

$x_n=f(x_{n-1})\cong f(x^*)+f'(x^*)(x_{n-1}-x^*)$

$=x^*+f'(x^*)(x_{n-1}-x^*)$

Then $|x_n-x^*|\cong|f'(x^*)||x_{n-1}-x^*|$

If $|f'(x^*)|<1$, the iterations will converge to the solution x^*.

9/29/14

Idea: Split $A=S-T$

$A\vec{x}=\vec{b}\Rightarrow S\vec{x}=T\vec{x}+\vec{b}$

Define:

$S\vec{x}_{k+1}=T\vec{x}_k+\vec{b}$

Find: S,T such that $\vec{x}_k$ converges to solution of $A\vec{x}=\vec{b}$

Call the solution $\vec{x}^*$, so

$S\vec{x}^*=T\vec{x}^*+\vec{b}$

Subtracting:

$S(\vec{x}_{k+1}-\vec{x}^*)=T(\vec{x}_k-\vec{x}^*)$

or $S\vec{e}_{k+1}=T\vec{e}_k$ (e for error)

If S is invertible, $\vec{e}_{k+1}=S^{-1}T\vec{e}_k$.

Form an initial guess $\vec{x}_0$. Then $\vec{e}_0=\vec{x}_0-\vec{x}_k$ and

$\vec{e}_1=(S^{-1}T)\vec{e}_0$

$\vec{e}_2=(S^{-1}T)\vec{e}_1=(S^{-1}T)^2\vec{e}_0$ ...

$\vec{e}_k=(S^{-1}T)^k\vec{e}_0$

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Want $\lim_{k\to\infty} \vec{e}_k = 0$. If $(S^{-1}T)$ is diagonalizable:

$(S^{-1}T)^k = VD^kV^{-1}$

$|\lambda_1|\cdots|\lambda_n|<1$ fulfills this condition.

Spectral radius is rate of convergence.

$\rho = \max|\lambda_i|$

How to split $A=S-T$:

Take $A=L+D+U$

1) Define $S=D$. Then $T=-(L+U)$. The iteration becomes

$D\vec{x}_{k+1} = -(L+U)\vec{x}_k + b$,

the Jacobi Iteration.

2) Define $S=L+D$. Then $T=-U$. The iteration becomes

$(L+D)\vec{x}_{k+1} = -U\vec{x}_k + \vec{b}$,

the Gauss-Seidel Iteration.

3) Use a factor w to converge to solution faster. Define $S=wL+D$. Then $T=(1-w)D-wU$ and

$(D+wL)\vec{x}_{k+1} = ((1-w)D-wU)\vec{x}_k + w\vec{b}$.

Ex.) $A=\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$

a) Jacobi: $D=\begin{pmatrix}2&0\\0&2\end{pmatrix}$, $L+U=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$

$\begin{pmatrix}2&0\\0&2\end{pmatrix}\vec{x}_{k+1} = -\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\vec{x}_k + \vec{b}$

If $\vec{x}_k = \begin{pmatrix}u_k\\v_k\end{pmatrix}$:

$2u_{k+1} = v_k + b_1$
$2v_{k+1} = u_k + b_2$
$\Rightarrow$ $\boxed{\begin{aligned}u_{k+1} &= \tfrac{1}{2}v_k + \tfrac{1}{2}b_1\\ v_{k+1} &= \tfrac{1}{2}u_k + \tfrac{1}{2}b_2\end{aligned}}$

b) Gauss-Seidel: $L+D=\begin{pmatrix}2&0\\-1&2\end{pmatrix}$, $U=\begin{pmatrix}0&-1\\0&0\end{pmatrix}$

$\begin{pmatrix}2&0\\-1&2\end{pmatrix}\vec{x}_{k+1} = -\begin{pmatrix}0&-1\\0&0\end{pmatrix}\vec{x}_k + \vec{b}$

$2u_{k+1} = v_k + b_1$
$-u_{k+1} + 2v_{k+1} = b_2$
$\Rightarrow$ $\boxed{\begin{aligned}u_{k+1} &= \tfrac{1}{2}v_k + \tfrac{1}{2}b_1\\ v_{k+1} &= \tfrac{1}{2}u_{k+1} + \tfrac{1}{2}b_2\end{aligned}}$ $\leftarrow$ Twice as fast, v_{k+1} uses current approx.

MATH 2605-G2 9/29/14

Say $\vec{b} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

$\left(\begin{array}{cc|c} 2 & -1 & 6 \\ -1 & 2 & 8 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & -8 \\ 0 & 3 & 22 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & 20/3 \\ 0 & 1 & 22/3 \end{array}\right)$

so $\vec{x}^* = \begin{pmatrix} 20/3 \\ 22/3 \end{pmatrix}$.

Now use Jacobi:

Define $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\begin{cases} u_1 = \frac{1}{2} v_0 + \frac{1}{2} b_1 = 0 + \frac{1}{2}(6) = 3 \\ v_1 = \frac{1}{2} u_0 + \frac{1}{2} b_2 = \frac{1}{2}(0) + \frac{1}{2}(8) = 4 \end{cases}$

$\begin{cases} u_2 = \frac{1}{2} v_1 + \frac{1}{2} b_1 = \frac{1}{2}(4) + \frac{1}{2}(6) = 5 \\ v_2 = \frac{1}{2} u_1 + \frac{1}{2} b_2 = \frac{1}{2}(3) + \frac{1}{2}(8) = \frac{11}{2} \end{cases}$

$\vdots$

Process continues until $\|\vec{x}_{n+1} - \vec{x}_n\| < \varepsilon$, a tolerance. Ideally $\varepsilon = 0$.

Theorem: If A is dominant:

$|a_{ii}| > \sum_{i \neq j} a_{ij}$

and both iterative methods converge.

In general, eigenvalues of $S^{-1}T$ determine convergence.