MATH 2605-G2 10/27/14

Extreme Values (cont.)

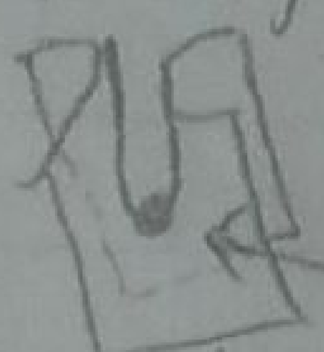
Since the normal vector is vertical, the tangent plane is horizontal:

$\nabla F(a,b,f(a,b)) \cdot ((x,y,z)-(a,b,f(a,b)))=0$

$\Leftrightarrow (0,0,-1)\cdot((x,y,z)-(a,b,f(a,b)))=0$

$\Leftrightarrow z=f(a,b)$, the equation for the tangent plane

(x_0,y_0) is a <u>saddle point</u> if it is a critical point, and in an open set around (a,b) there are both points (a,b) with $f(a,b)<f(x_0,y_0)$ and with $f(a,b)>f(x_0,y_0)$. (x_0,y_0) is neither a maximum nor a minimum.



saddle point

Ex.) Find the critical points of $f(x,y)=2x^2+4xy+5y^2+2x-y$.
No boundary points since the domain is $\mathbb{R}^2$.

$\begin{cases} \frac{\partial f}{\partial x}=4x+4y+2=0 \\ \frac{\partial f}{\partial y}=4x+10y-1=0 \end{cases} \Rightarrow \quad 6y=3 \Rightarrow y=\frac{1}{2}, \quad x=\frac{-1-2y}{2}=-1$

$(-1,\frac{1}{2})$ is the only critical point.

<u>Theorem</u>: Suppose $f(x,y)$ and all of its first and second derivatives are continuous and disk centered at (a,b), and that

$$\frac{\partial f}{\partial x}(a,b)=\frac{\partial f}{\partial y}(a,b)=0.$$

Then let

$$A=\frac{\partial^2 f}{\partial x^2}(a,b) \quad B=\frac{\partial^2 f}{\partial x \partial y}(a,b) \quad C=\frac{\partial^2 f}{\partial y^2}(a,b)$$

$$D=\det\begin{pmatrix} A & B \\ B & C\end{pmatrix}=AC-B^2.$$

i) If $A>0$ and $D>0$, then $f(a,b)$ is a local minimum.
ii) If $A<0$ and $D>0$, then $f(a,b)$ is a local maximum.
iii) If $D<0$, then $f(a,b)$ is a saddle point.
iv) If $D=0$ then the test is inconclusive.

In the example, $\frac{\partial^2 f}{\partial x^2}=4=A$, $\frac{\partial^2 f}{\partial x \partial y}=4=B$, and $\frac{\partial^2 f}{\partial y^2}=10=C$.
Then $D=AC-B^2=4\cdot10-4^2=24$. Since $A>0$ and $D>0$, (a,b) is a local min.
Since $f(x,y)$ is unbounded, it has no global extrema.

Ex.) Let $f(x,y)=e^{xy}$. Locate and classify all critical points.

$\frac{\partial f}{\partial x}=ye^{xy}=0 \quad \frac{\partial f}{\partial y}=xe^{xy}=0 \quad \Rightarrow x=y=0$