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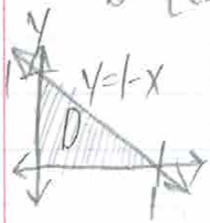
$J(u,v) = \det\begin{pmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{pmatrix} = \frac{1}{3}$

To find the boundary, transform the original D:

$D = \{(x,y): 0 \le x \le 1,\ 0 \le y \le 1-x\} \Rightarrow \underset{①}{x=0},\ \underset{②}{y=0},\ \underset{③}{x+y=0}$ (origin), $\underset{④}{y=1-x}$



① $\frac{u-v}{3} = 0 \Rightarrow u = v$ ③ $x+y = u = 0$

② $\frac{2u+v}{3} = 0 \Rightarrow u = \frac{-v}{2}$ ④ $\frac{2u+v}{3} = 1 - \left(\frac{u-v}{3}\right) \Rightarrow u = 1$

(cont. below)

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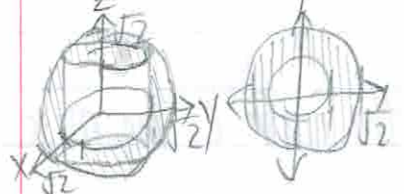
Recitation

1) Find the volume of the region that lies inside $x^2+y^2+z^2=2$ and outside $x^2+y^2=1$.



$R: \{(z,r,\theta): -\sqrt{2-r^2} \le z \le \sqrt{2-r^2},\ 1 \le r \le \sqrt{2},\ 0 \le \theta \le 2\pi\}$

$V = \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{2}} 2\sqrt{2-r^2}\,dr\,d\theta = \ldots$

2) Find the volume of the solid cut from $x^2+y^2 \le 1$ by sphere $x^2+y^2+z^2=4$.

$R: \{(z,r,\theta): -\sqrt{4-r^2} \le z \le \sqrt{4-r^2},\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$V = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2\sqrt{4-r^2}\,dr\,d\theta = \ldots$

3) Find the volume of the solid bounded above by $x^2+y^2+z^2=2$ and below by $z=x^2+y^2$.

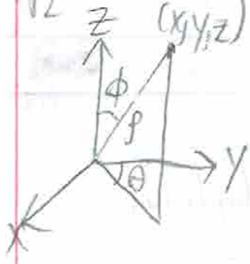
Cylindrical:

Boundary: $z+z^2=2$

$(z-1)(z+2)=0$

$z=1$

$V = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (\sqrt{2-r^2}-r^2)\,dr\,d\theta = \ldots$



Spherical:

$V = \int_0^{2\pi}\left(\int_0^{\pi/4}\int_0^{\cos\phi/\sin^2\phi} \rho\sin\phi\,d\rho\,d\phi + \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \rho\sin\phi\,d\rho\,d\phi\right)d\theta$

comes from $x^2+y^2 = \rho\cos\phi$ (z)

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Ex.) (cont.) $\int_0^1\int_{-2u}^{u} \sqrt{u}\,v^2\left(\frac{1}{3}\right)dv\,du = \frac{1}{3}\int_0^1 \sqrt{u}\,\frac{v^3}{3}\Big|_{v=-2u}^{u} du = \frac{1}{9}\int_0^1 \sqrt{u}(u^3+8u^3)\,du = \int_0^1 u^{7/2}\,du = \frac{2}{9}u^{9/2}\Big|_0^1 = \frac{2}{9}$

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Ex.) Jacobian for spherical coordinates:

$x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$

$$J(\rho,\phi,\theta)=\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}=\begin{vmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & \rho\sin\phi & 0\end{vmatrix}$$

$$=\rho^2\sin\phi$$

<u>Line Integrals</u>:

Let $f(x,y,z)$ be defined over a region containing the curve C, where

$$C=\vec{r}(t)=g(t)\hat{i}+h(t)\hat{j}+k(t)\hat{k}=(g(t),h(t),k(t)),\ a\le t\le b.$$

Problem: Integrate f over the curve:

$$f(\vec{r}(t))=f(g(t),h(t),k(t))$$

with respect to <u>arc length</u>.

For instance, f can be the density of a wire occupying the curve C. The mass is then represented by

$$\int_C f(x,y,z)\,ds\equiv\int_C f(\vec{r}(t))s'(t)\,dt$$

Remember that the arc length function is

$$s(t)=\int_a^t\|\vec{r}'(\tau)\|\,d\tau$$

and $s'(t)=\|\vec{r}'(t)\|$.

Then $\int_C f(x,y,z)\,ds=\int_a^b f(\vec{r}(t))\|\vec{r}'(t)\|\,dt$.

Ex.) $f(x,y,z)=\frac{\sqrt{3}}{x^2+y^2+z^2}$ over $\vec{r}(t)=t\hat{i}+t\hat{j}+t\hat{k},\ 1\le t<\infty$ $\quad (x,y,z)=(t,t,t)$

$$\int_C\frac{\sqrt{3}}{x^2+y^2+z^2}\,ds=\int_1^\infty\frac{\sqrt{3}}{3t^2}\|(1,1,1)\|\,dt$$

$$=\int_1^\infty\frac{1}{\sqrt{3}t^2}\sqrt{3}\,dt$$

$$=-\frac{1}{t}\Big|_1^\infty$$

$$=1$$

Ex.) Integrate $f(x,y,z)=x-3y^2+z$ over the segment joining $(0,0,0)$ and $(1,1,1)$.

$\vec{r}(t)=(0,0,0)+t(1,1,1)=(t,t,t),\ 0\le t\le 1$

$\Rightarrow\|\vec{r}'(t)\|=\sqrt{3}$

$$\int_C(x-3y^2+z)\,ds=\int_0^1(t-3t^2+t)\sqrt{3}\,dt=\sqrt{3}(t^2-t^3)\Big|_0^1=0$$

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What if $C_1 \cup C_2$ was considered instead?

Ex.) 

$$\int_{C_1\cup C_2} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

$C_1: \vec{r}(t) = (t,t,0),\ 0 \le t \le 1 \Rightarrow \vec{r}\,'(t) = (1,1,0) \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{2}$

$C_2: \vec{r}(t) = (1,1,t),\ 0 \le t \le 1 \Rightarrow \vec{r}\,'(t) = (0,0,1) \Rightarrow \|\vec{r}\,'(t)\| = 1$

$$\int_{C_1\cup C_2} f(x,y,z)\,ds = \int_0^1 (t-3t^2+0)\sqrt{2}\,dt + \int_0^1 (1-3(1)^2+t)(1)\,dt$$

$$= \sqrt{2}\left(\frac{t^2}{2} - t^3\right)\Big|_0^1 + \left(-2t + \frac{t^2}{2}\right)\Big|_0^1$$

$$= \sqrt{2}\left(-\frac{1}{2}\right) - \frac{3}{2}$$

$$= \frac{-\sqrt{2}-3}{2}$$