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Ex.) Velocity vectors in a wind tunnel:

Ex.) Gradient vector field: assign $\nabla f(x,y,z)$ to each point (x,y,z) in the domain of f.
Recall that ∇f points in the direction of greatest increase in $f(x,y,z)$.

<u>Line integrals of vector fields:</u>
Assume that $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ has continuous components, and that the curve C has a smooth parametrization
$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}, \quad a \le t \le b$$
If the unit tangent vector is $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$, then $\vec{F}\cdot\vec{T}$ is the tangential component of $\vec{F}$ along the curve C.
Note: $\vec{T}(t) = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds}$ where $s(t) = \int_a^t \|\vec{r}'(\xi)\|\,d\xi$

Then the line integral is defined as
$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right)ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b [\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)]dt,$$
a special case of the general line integral.

Ex.) $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = z\hat{i} + xy\hat{j} + y^2\hat{k}$ along $C: \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\hat{k}$, $0 \le t \le 1$.
$\Rightarrow \vec{F}(\vec{r}(t)) = \sqrt{t}\hat{i} + (t^2)t\hat{j} + (t)^2\hat{k} = (\sqrt{t}, t^3, t^2)$
$\vec{r}'(t) = (2t, 1, \frac{1}{2\sqrt{t}})$
$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = 2t^{3/2} + t^3 + \frac{1}{2}t^{3/2} = t^3 + \frac{5}{2}t^{3/2}$
$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t^3 + \frac{5}{2}t^{3/2})\,dt = \frac{t^4}{4} + t^{5/2}\Big|_0^1 = \frac{5}{4}$

Ex.) $\int_C \vec{h}\cdot d\vec{r}$ for $\vec{h} = x\hat{i} + x^2y\hat{j}$ along $C: \vec{r}(u) = \cos u\,\hat{i} + \sin u\,\hat{j}$, $u \in [0, \frac{\pi}{2}]$

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$\Rightarrow \vec{h}(\vec{r}(u)) = \cos u\,\hat{i} + \cos^2 u \sin u\,\hat{j}$
$\vec{r}'(u) = -\sin u\,\hat{i} + \cos u\,\hat{j}$
$\vec{h}(\vec{r}(u))\cdot\vec{r}'(u) = -\cos u \sin u + \cos^3 u \sin u$
$\int_C \vec{h}\cdot d\vec{r} = \int_0^{\pi/2} (-\cos u \sin u + \cos^3 u \sin u)\,du$
$= \left(\frac{\cos^2 u}{2} - \frac{\cos^4 u}{4}\right)\Big|_0^{\pi/2} = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

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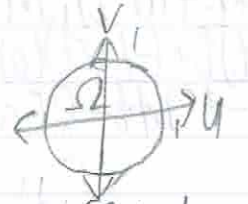
Recitation

1) Find the area enclosed by $x^2+xy+y^2=1$.

$(x^2+xy+(\frac{y}{2})^2)+\frac{3}{4}y^2=1$

$(x+\frac{y}{2})^2+(\frac{\sqrt{3}y}{2})^2=1$

$u=x+\frac{y}{2}$ $\quad v=\frac{\sqrt{3}y}{2}$ $\quad \Rightarrow u^2+v^2=1$



$x=u-\frac{v}{\sqrt{3}} \leftarrow y=\frac{2v}{\sqrt{3}}$

area of circle

$A=\iint_{\Omega}\left|\begin{matrix}\frac{dx}{du} & \frac{dx}{dv}\\ \frac{dy}{du} & \frac{dy}{dv}\end{matrix}\right|du\,dv=\iint_{\Omega}\left|\begin{matrix}1 & -\frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{3}}\end{matrix}\right|du\,dv=\frac{2}{\sqrt{3}}\iint_{\Omega}du\,dv=\frac{2\pi}{\sqrt{3}}$

2) Find the volume of $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{6}=1$.

$u=\frac{x}{2}, \quad v=\frac{y}{\sqrt{5}}, \quad w=\frac{z}{\sqrt{6}} \Rightarrow u^2+v^2+w^2=1$

volume of sphere

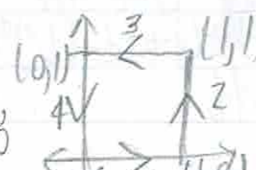
$V=\iiint_{\Omega}\left|\begin{matrix}2 & & \\ & \sqrt{5} & \\ & & \sqrt{6}\end{matrix}\right|du\,dv\,dw=2\sqrt{30}\iiint_{\Omega}du\,dv\,dw=2\sqrt{30}\left(\frac{4\pi}{3}\right)=\frac{8\sqrt{30}\pi}{3}$

3) Find the line integral with respect to arc length of $\int_C(x+y)\,ds$, where C is the line segment from $(0,1)$ to $(1,0)$.

$(x,y)=(t,1-t)$ for $t\in[0,1]$

$\int_0^1(t+(1-t))\cdot\|\vec{r}'(t)\|\,dt=\int_0^1\sqrt{2}\,dt=\sqrt{2}$

4) $\int_C\frac{ds}{x^2+y^2+1}$, C:



Line 1: $r(t)=(t,0)$, $0\le t\le1 \Rightarrow x=t,\ y=0$

$\int_0^1\frac{1}{t^2+1}\cdot\|r'(t)\|\,dt$

$=\tan^{-1}t\Big|_0^1$

$=\frac{\pi}{4}$

Add up line integrals from lines 2, 3, and 4 as well.

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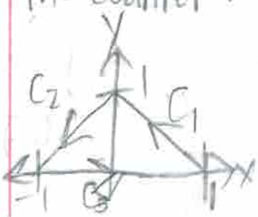
Ex.) $\int_C\vec{F}\cdot d\vec{r}$ for $\vec{F}=\cos x\,\hat{i}+xy\,\hat{j}$ along C, the triangle joining $(1,0)$, $(0,1)$, $(-1,0)$, in the counterclockwise direction



$\int_C\vec{F}\cdot d\vec{r}=\int_{C_1}\vec{F}\cdot d\vec{r}+\int_{C_2}\vec{F}\cdot d\vec{r}+\int_{C_3}\vec{F}\cdot d\vec{r}$

$C_1: \vec{r}(t)=(1,0)+t(-1,1)$, $0\le t\le1$

$C_2: \vec{r}(t)=(0,1)+t(-1,-1)$, $0\le t\le1$

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$$C_3: \vec{r}(t) = (-1,0) + t(2,0),\ 0 \le t \le 1$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^1 [\cos(1-t)\hat{i} + (1-t)(t)\hat{j}]\cdot(-\hat{i}+\hat{j})\,dt$$

$$+ \int_0^1 [\cos(-t)\hat{i} + (-t)(1-t)\hat{j}]\cdot(-\hat{i}-\hat{j})\,dt$$

$$+ \int_0^1 [\cos(-1+2t)\hat{i} + (-1+2t)(0)\hat{j}]\cdot(2\hat{i}+0\hat{j})\,dt$$

$$= \int_0^1 [-\cos(1-t) + t - t^2]\,dt + \int_0^1 [-\cos(-t) + t - t^2]\,dt + \int_0^1 [2\cos(-1+2t)]\,dt$$

$$= \frac{9}{3}$$

<u>Work</u>

The work done by a force $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ over an object moving along a curve

$$\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k},\ a \le t \le b$$

is

$$W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}.$$

<u>Other notations</u>: $\int_C \vec{F}\cdot d\vec{r} = \int_a^b (M, N, P)\cdot(g'(t), h'(t), k'(t))\,dt$

$$= \int_a^b [Mg'(t) + Nh'(t) + Pk'(t)]\,dt$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

If C is a closed curve, then $\int_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r}$.

Ex.) Find the work done by a force field $\vec{F} = (y-x^2)\hat{i} + (z-y^2)\hat{j} + (x-z^2)\hat{k}$ along $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ from $(0,0,0)$ to $(1,1,1)$.

$$W = \int_C (y-x^2)\,dx + (z-y^2)\,dy + (x-z^2)\,dz$$

$$= \int_0^1 [(t^2-t^2)(1) + (t^3-(t^2)^2)(2t) + (t-(t^3)^2)(3t^2)]\,dt$$

$$= \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt$$

$$= \frac{29}{60}$$

<u>Flow and Flux</u>: If

F represents the velocity field of a fluid and $C: \vec{r}(t),\ a \le t \le b$ is smooth, the <u>flow</u> along the curve from $\vec{r}(a) = A$ to $\vec{r}(b) = B$ is

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}.$$

For $\vec{F} = M\hat{i} + N\hat{j}$ and C a closed, simple (does not cross itself) curve, then the flux across C is

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_C \vec{F}\cdot(\vec{T}\times\hat{k})\,ds$$

where $\underline{\vec{n} = \vec{T}\times\hat{k}}$ is the outward-pointing unit normal vector of C.

Since $\vec{T} = \frac{d\vec{r}}{ds}$, $\vec{T}\times\hat{k} = \left(\frac{dx}{ds}, \frac{dy}{ds}, 0\right)\times(0,0,1) = \left(\frac{dy}{ds}, -\frac{dx}{ds}, 0\right)$.

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Therefore, $\vec{F}\cdot\vec{n} = M\frac{dy}{ds} - N\frac{dx}{ds}$,
Hence, Flux of $\vec{F}$ across $C = \int_C (M\frac{dy}{ds} - N\frac{dx}{ds})ds$
$= \int M dy - N dx$
$= \oint_C M dy - N dx$