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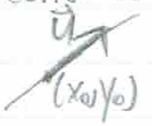
<u>Directional derivatives and gradient:</u>

Consider $f(x,y)$ and (x_0,y_0) in its domain. Let $\vec{u}$ be a unit vector.



The rate of change of f at (x_0,y_0) in the direction of $\vec{u}$ is

$$\left(\frac{df}{ds}\right)\Big|_{\vec{u}}^{(x_0,y_0)} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2)-f(x_0,y_0)}{s}$$

The line that goes through (x_0,y_0) and has $\vec{u}$ as direction vector is

$$\vec{r}=(x(s),y(s))=(x_0,y_0)+s\vec{u}=(x_0+su_1, y_0+su_2)$$

Then, from the chain rule:

$$\left(\frac{df}{ds}\right)_{\vec{u}}^{(x_0,y_0)} = \frac{\partial f}{\partial x}\frac{dx}{ds}+\frac{\partial f}{\partial y}\frac{dy}{ds} = \frac{\partial f}{\partial x}u_1+\frac{\partial f}{\partial y}u_2 = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\cdot(u_1,u_2)$$

Gradient of f: $\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$

Directional derivative in direction of $\vec{u}$: $\left(\frac{\partial f}{\partial s}\right)\Big|_{(x_0,y_0)}^{\vec{u}}=\nabla f\cdot\vec{u}$

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Recitation

Find $\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$:

1) $f(x,y)=xe^{x^2y}+\frac{1}{y}e^x$

$\nabla f=\left(x(e^{x^2y})(2xy)+e^{x^2y}(1)+\frac{1}{y}e^x,\ x(e^{x^2y})(x^2)-\frac{1}{y^2}e^x\right)$

$=\left(2x^2ye^{x^2y}+e^{x^2y}+\frac{1}{y}e^x,\ x^3e^{x^2y}-\frac{1}{y^2}e^x\right)$

2) $f(x,y)=x^2+2xy+3y^2$

$\nabla f=(2x+2y,\ 2x+6y)$

3) $f(x,y)=x^y$

$\nabla f=(yx^{y-1},\ x^y\ln x)$

Given ∇f, how do you find f? Use partial integration.

$\nabla f(x,y)=(-e^{-2y}\sin x,\ -2e^{-2y}\cos x+y)$

$\left(\int -e^{-2y}\sin x\,\partial x,\ \int(-2e^{-2y}\cos x+y)\,\partial y\right)$

$=\left(\cos x\,e^{-2y}+f(y),\ e^{-2y}\cos x+\frac{1}{2}y^2+f(x)\right)$

$f(x,y)=e^{-2y}\cos x+\frac{1}{2}y^2+C$

For $\nabla f=(x,x)$, does f exist?

Theorem: if $(f_x)_y=(f_y)_x$, then f exists. Otherwise f does not exist.

$(f_x)_y=0\neq(f_y)_x=1$ so f does not exist for $\nabla f=(x,x)$.

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<u>Notes about project:</u>

Part 1) Gauss-Newton method

Iterative equation:

$$\beta^{(s+1)} = \beta^{(s)} - (J^T J)^{-1} J^T \vec{r}$$

If $J = QR$:

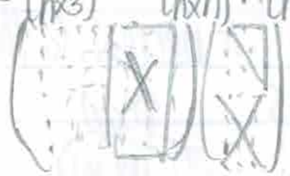
$$\beta^{(s+1)} = \beta^{(s)} - R^{-1} Q^T \vec{r} = \beta^{(s)} - \vec{x}$$

with $\vec{x} = R^{-1} Q^T \vec{r}$

$\Leftrightarrow R\vec{x} = Q^T \vec{r}$

$\begin{pmatrix} \cdot & \cdot & \cdot \\ & \cdot & \cdot \\ & & \cdot \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$ Solve for x using back-substitution.

$J_{(n\times 3)} = Q_{(n\times n)} R_{(n\times 3)} \Rightarrow J_{(n\times 3)} = Q_{(n\times 3)} R_{(3\times 3)}$



Gradient: $\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right)$

Directional derivative in direction of $\vec{u}$, a unit vector: $[D_{\vec{u}} f](x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

Ex) $f(x,y) = xe^{xy} + \cos(xy)$, $\vec{u} = 3\hat{i} + 4\hat{j}$, $(x_0, y_0) = (2, 0)$

$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (e^{xy} + xye^{xy} - y\sin(xy), x^2 e^{xy} - x\sin(xy))$

$\nabla f(2,0) = (1, 4)$

$\text{norm}(\vec{u}) = \frac{1}{5}(3, 4)$

$[D_{\vec{u}} f](2,0) = (1,4) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{19}{5}$

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$, θ is angle between ∇f and $\vec{u}$

i) $D_{\vec{u}} f$ is most positive when $\theta = 0$, when $\vec{u}$ points in same direction as ∇f. In this case $D_{\vec{u}} f = \|\nabla f\|$. f increases the most rapidly in the direction of ∇f.

ii) Also, $D_{\vec{u}} f$ is most negative when $\theta = \pi$, when $\vec{u}$ and ∇f are antiparallel. In this case $D_{\vec{u}} f = -\|\nabla f\|$. f decreases the most rapidly in the direction of ∇f.

Ex.) Direction and magnitude of fastest increase of $f(x,y) = \frac{1}{2}(x^2 + y^2)$ at $(1,1)$:

$\nabla f(x,y) = (x, y) \Rightarrow \nabla f(1,1) = (1,1)$

Direction of fastest increase: $\vec{u} = \frac{1}{\sqrt{2}}(1,1)$

Magnitude of fastest increase: $\nabla f(1,1) \cdot \vec{u} = \frac{2}{\sqrt{2}} = \sqrt{2}$

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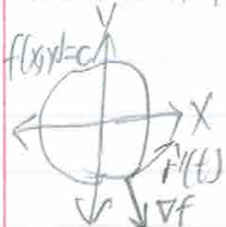
iii) Any direction orthogonal to $\nabla f\ (\neq \vec{0})$ is a direction of zero change:
$D_{\vec{u}}f = \|\nabla f\| \cos\frac{\pi}{2} = 0$

Ex.) Directions of zero change in last ex.: $\vec{u}_1 = \frac{1}{\sqrt{2}}(1,-1)$ and $\vec{u}_2 = \frac{1}{\sqrt{2}}(-1,1)$

Consider a level curve of $f(x,y)$: $f(x,y)=c$. If $\vec{r}(t)=(g(t),h(t))$ is a parametrization of the curve, then $f(\vec{r}(t)) = f(g(t),h(t)) = c$.



Taking derivative w/ respect to t: $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$

$\Leftrightarrow \frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t) = 0$

$\Leftrightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (g'(t), h'(t)) = 0$

$\Leftrightarrow \nabla f \cdot \vec{r}'(t) = 0$

Since $\vec{r}'(t)$ is tangent to the curve and orthogonal to ∇f, ∇f is <u>normal</u> to the curve.
The equation for the line tangent to the curve $f(x,y)=c$ is

$\nabla f|_{(x_0,y_0)} \cdot ((x,y)-(x_0,y_0)) = 0$

$\Leftrightarrow \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x-x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y-y_0) = 0$

Ex.) Find an equation for the line tangent to $\frac{x^2}{4}+y^2=2$ at $(-2,1)$.

Let $f(x,y) = \frac{x^2}{4}+y^2$ so that $f(x,y)=2$.
Then $\nabla f(x,y) = (\frac{1}{2}x, 2y) \Rightarrow \nabla f(-2,1) = (-1,2)$.
Tangent line: $\frac{\partial f}{\partial x}\Big|_{(-2,1)}(x-(-2)) + \frac{\partial f}{\partial y}\Big|_{(-2,1)}(y-1) = 0$

$\Leftrightarrow (-1)(x+2)+2(y-1)=0$

$\Leftrightarrow -x+2y=4$

<u>Gradient rules</u>:
1) $\nabla(f+g) = \nabla f + \nabla g$
2) $\nabla(fg) = f\nabla g + g\nabla f$
3) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$