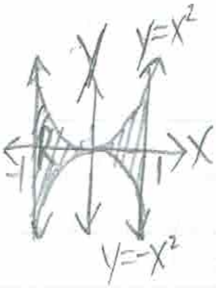
MATH 2605-G2 11/3/14

Ex.)



$\iint_R (xy+5)\,dx\,dy \quad R=\{(x,y): -1\le x\le 1, -x^2\le y\le x^2\}$

$=\int_{-1}^{1}\int_{-x^2}^{x^2}(xy+5)\,dy\,dx$

$=\int_{-1}^{1}\left(\frac{xy^2}{2}+5y\right)\Big|_{y=-x^2}^{y=x^2}dx$

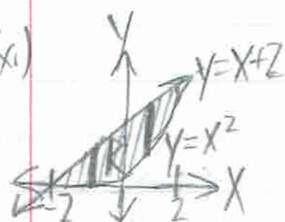
$=\int_{-1}^{1}\left[\frac{x(x^2)^2}{2}+5(x^2)-\left(x\frac{(-x^2)^2}{2}+5(-x^2)\right)\right]dx$

$=\int_{-1}^{1}10x^2\,dx$

$=\frac{10x^3}{3}\Big|_{-1}^{1}$

$=\frac{20}{3}$

Ex.)



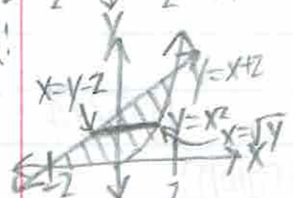
$\iint_R (xy-y^3)\,dy\,dx$

$=\int_{-2}^{0}\int_{0}^{x+2}(xy-y^3)\,dy\,dx$
$+\int_{0}^{2}\int_{x^2}^{x+2}(xy-y^3)\,dy\,dx$

$x^2=x+2 \Leftrightarrow x^2-x-2=0 \Leftrightarrow (x-2)(x+1)=0$

$R=\{(x,y): -2\le x\le 0, 0\le y\le x+2\}\cup\{(x,y): 0\le x\le 2, x^2\le y\le x+2\}$

OR:



$\iint_R (xy-y^3)\,dx\,dy \quad R=\{(x,y): 0\le y\le 4, y-2\le x\le \sqrt{y}\}$

$=\int_{0}^{4}\int_{y-2}^{\sqrt{y}}(xy-y^3)\,dx\,dy$

$=\int_{0}^{4}\left(\frac{x^2}{2}y-xy^3\right)\Big|_{x=y-2}^{x=\sqrt{y}}dy$

$=\int_{0}^{4}\left[\frac{y}{2}y-\sqrt{y}\,y^3-\frac{(y-2)^2}{2}y-(y-2)y^3\right]dy$

Recitation 11/4/14

1) Min./Max. of $x+y$ such that $x^2+y^2=1$

$\nabla f=(1,1)=\lambda\nabla g=\lambda(2x,2y)$

$\Rightarrow x=y=\frac{1}{2\lambda}$

$y^2=1-x^2$

$\Rightarrow x^2=1-x^2$

$2x^2=1$

$x=\sqrt{\frac{1}{2}}\Rightarrow y=\sqrt{\frac{1}{2}}$

$\lambda=\frac{1}{2}(\sqrt{2})=\frac{\sqrt{2}}{2}$

$x+y=\sqrt{2}$

2) Min./Max. of x^2+y^2 such that $x+y\le 1$

i) $\nabla f(x,y)=0$

$\Rightarrow (2x,2y)=0$

$\Rightarrow (x,y)=(0,0)$

ii) Check the boundary:

Min./Max. $f(x,y)$ such that $x+y=1\Rightarrow x=1-y$

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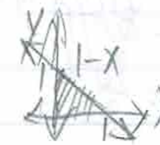
one way: Min/Max. $g(y)=x^2+y^2=(1-y)^2+y^2$

$g'(y)=0 \Rightarrow -2(1-y)+2y=0 \Rightarrow y=\frac{1}{2} \Rightarrow (\frac{1}{2},\frac{1}{2})$ so $x^2+y^2=\frac{1}{2}$ is max

$f(0,0)=0$ is min, no min. along boundary

or: $\nabla f=\lambda \nabla g \Rightarrow (2x,2y)=\lambda(1,1) \Rightarrow x=y=\frac{\lambda}{2} \Rightarrow x=1-x \Rightarrow x=\frac{1}{2},\ y=\frac{1}{2}$

3) $\iint (x+y)\,dx\,dy$ in bounded region $x=0,\ y=0,\ y=1-x$



$R: \{(x,y): 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$\int_0^1\int_0^{1-x}(x+y)\,dy\,dx = \int_0^1 (xy+\frac{y^2}{2})\Big|_0^{1-x}dx$

$= \int_0^1 \left(x(1-x)+\frac{(1-x)^2}{2}\right)dx$

$= \frac{x^2}{2}-\frac{x^3}{3}-\frac{(1-x)^3}{6}\Big|_0^1$

$= \frac{1}{6}$

Other parametrization: $\int_0^1\int_0^{1-y}(x+y)\,dx\,dy$

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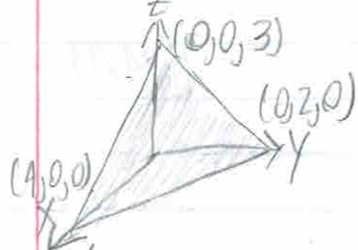
Ex) Find the volume of the solid in the first octant of the region:



$V=\iint_R z\,dx\,dy,\ z=f(x,y)$

The equation of the plane that contains $(4,0,0)$, $(0,2,0)$ and $(0,0,3)$:

$n=((0,2,0)-(4,0,0))\times((0,0,3)-(4,0,0))$

$=(-4,2,0)\times(-4,0,3)$

$=(6,12,8)$

$\Rightarrow (6,12,8)\cdot((x,y,z)-(4,0,0))=0$

$\Leftrightarrow 6(x-4)+12y+8z=0$

$\Leftrightarrow z=\frac{12-3x-6y}{4}$

$R=\{(x,y): 0 \le x \le 4,\ 0 \le y \le -\frac{1}{2}x+2\}$

$\Rightarrow V=\int_0^4\int_0^{-\frac{1}{2}x+2}\frac{12-3x-6y}{4}\,dy\,dx$

$=\int_0^4 \left(3y-\frac{3xy}{4}-\frac{3y^2}{4}\right)\Big|_0^{-\frac{1}{2}x+2}dx$

$=\int_0^4\left[3(-\frac{x}{2}+2)-\frac{3}{4}x(-\frac{x}{2}+2)-\frac{3}{4}(-\frac{x}{2}+2)^2\right]dx$

$=\int_0^4\left(-\frac{3}{2}x+6+\frac{3}{8}x^2-\frac{3}{2}x-\frac{3}{16}x^2+\frac{3}{2}x-3\right)dx$

$=-\frac{3}{4}x^2+3x+\frac{3}{16}\frac{x^3}{3}\Big|_0^4$

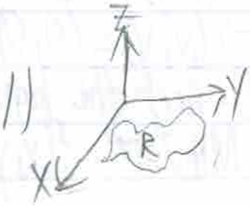
$=4$

Check:

Area of tetrahedron $=\frac{1}{3}(\frac{1}{2}bwh)=\frac{1}{6}(3)(4)(2)$

$=4$

Area by double integration: $\iint_R 1\,dx\,dy=\iint_R dx\,dy$ $\quad (f(x,y)=1)$



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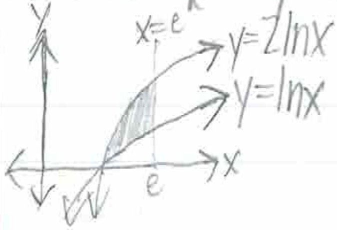
Find the area of the region bounded by $y = \ln x$, $y = 2\ln x$, and $x = e$.

Ex.) $\text{Area}(R) = \iint_R dx\,dy = \int_1^e \int_{\ln x}^{2\ln x} dy\,dx = \int_1^e (2\ln x - \ln x)\,dx = \int_1^e \ln x\,dx$



$u = \ln x \quad dv = dx$

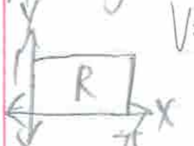
$du = \frac{1}{x} \quad v = x$

$\Rightarrow x\ln x\Big|_1^e - \int_1^e 1\,dx$

$= e\ln e - 0 - (e-1) = 1$

Average of $f(x,y)$ over a region R is $\frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dx\,dy$. Intuitively, this can be thought of as the total value of all the points in the region divided by the number of points in the region.

Ex.) Average of $f(x,y) = x\cos(x,y)$ over $0 \le x \le \pi$, $0 \le y \le 1$:



$V = \iint_R [x\cos(x,y)]\,dx\,dy$

$= \int_0^\pi \int_0^1 x\cos(x,y)\,dx\,dy$

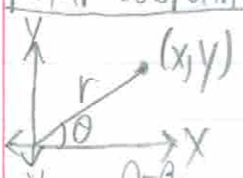
$= \int_0^\pi \sin(xy)\Big|_{y=0}^{y=1} dx$

$= \int_0^\pi \sin x\,dx$

$= -\cos x\Big|_0^\pi$

$= 2$ Average: $\frac{2}{(\pi-0)(1-0)} = \frac{2}{\pi}$

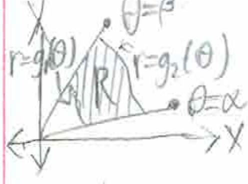
Polar coordinates:



$x = r\cos\theta \Rightarrow f(x,y) = f(r\cos\theta, r\sin\theta) = f(r,\theta)$

$y = r\sin\theta$

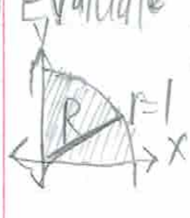
$R = \{(r,\theta) : \alpha \le \theta \le \beta,\ g_1(\theta) \le r \le g_2(\theta)\}$



$\iint_R f(x,y)\,dx\,dy = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,\underline{r}\,dr\,d\theta$

extra r: read book for more

Ex.) Evaluate $\iint_R xy\,dx\,dy$.



$= \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$

$= \int_0^{\pi/2} \int_0^1 \frac{r^3 \sin 2\theta}{2}\,dr\,d\theta$

$= \int_0^{\pi} \frac{r^4}{8} \sin 2\theta \Big|_{r=0}^{r=1} d\theta$

$= \frac{1}{8}\left(\frac{-\cos 2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{8}\left(-\frac{1}{2} + \frac{1}{2}\right)$