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Vector calculus: (Chp. 13, Thomas)

A curve in space is defined as a vector function

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle,\ t \in I$

so the position at time t is the point (x, y, z) with

$x = f(t),\ y = g(t),\ z = h(t)$.

The path is the set of points $\vec{r}(t)$ with $t \in I$.

Ex.) $\vec{r}(t) = (\cos t, \sin t, t),\ t \in [0, 4\pi]$

Recitation 10/2/14

1) Find the reflection matrix H for $\vec{x} = \begin{pmatrix}5\\2\end{pmatrix}$.

$\vec{V} = \vec{x} + \|\vec{x}\|\vec{e}_1 = \begin{pmatrix}5\\2\end{pmatrix} + \sqrt{29}\begin{pmatrix}1\\0\end{pmatrix} \qquad \|\vec{V}\| = \sqrt{(5+\sqrt{29})^2+4}$

$\vec{u} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{1}{\sqrt{(5+\sqrt{29})^2+4}}\begin{pmatrix}5+\sqrt{29}\\2\end{pmatrix}$

$H = I - 2uu^T = \begin{pmatrix}1 & \\ & 1\end{pmatrix} - \frac{2}{(5+\sqrt{29})^2+4}\begin{pmatrix}5+\sqrt{29}\\2\end{pmatrix}\begin{pmatrix}5+\sqrt{29} & 2\end{pmatrix}$

2) Express the solution to $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ using Jacobi's Iterations.

$S = \begin{pmatrix}1 & \\ & 4\end{pmatrix} \quad T = -\begin{pmatrix} & 2\\3 & \end{pmatrix}$

$S\vec{x}_{k+1} = T\vec{x}_k + \vec{b}$

$\begin{pmatrix}1 & \\ & 4\end{pmatrix}\begin{pmatrix}u_{k+1}\\v_{k+1}\end{pmatrix} = \begin{pmatrix} & -2\\-3 & \end{pmatrix}\begin{pmatrix}u_k\\v_k\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$

$\begin{cases} u_{k+1} = -2v_k \\ 4v_{k+1} = -3u_k + 1 \end{cases}$

3) Use the power method to find the largest eigenvalue of $A = \begin{pmatrix}2 & 2\\2 & 1\end{pmatrix}$ using $u_0 = \begin{pmatrix}1\\1\end{pmatrix}$

$Au_0 = \begin{pmatrix}2 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix} \qquad u_1 = \frac{Au_0}{\alpha_0} = \frac{1}{4}\begin{pmatrix}4\\3\end{pmatrix}$

$Au_1 = \begin{pmatrix}2 & 2\\2 & 1\end{pmatrix}\frac{1}{4}\begin{pmatrix}4\\3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}14\\11\end{pmatrix} \qquad u_2 = \frac{Au_1}{\alpha_1} = \frac{14}{4}\ )\begin{pmatrix}14\\11\end{pmatrix}$

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Vector function: $\vec{r}: \mathbb{R} \to \mathbb{R}^3$ in t

$\lim\limits_{t \to t_0} \vec{r}(t) = \vec{L}$

if for every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for all t in the domain D:

$\|\vec{r}(t) - \vec{L}\| < \varepsilon$ whenever $0 < |t - t_0| < \delta$.

If $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

and $\vec{L} = L_1\hat{i} + L_2\hat{j} + L_3\hat{k}$,

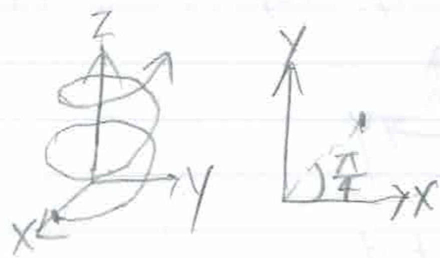
then the limit exists when

$\lim\limits_{t \to t_0} f(t) = L_1$, $\lim\limits_{t \to t_0} g(t) = L_2$, and $\lim\limits_{t \to t_0} h(t) = L_3$.

Ex.) $\lim\limits_{t \to \frac{\pi}{4}} (\cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k})$

$= (\lim\limits_{t \to \frac{\pi}{4}} \cos t)\hat{i} + (\lim\limits_{t \to \frac{\pi}{4}} \sin t)\hat{j} + (\lim\limits_{t \to \frac{\pi}{4}} t)\hat{k}$

$= \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j} + \frac{\pi}{4}\hat{k}$



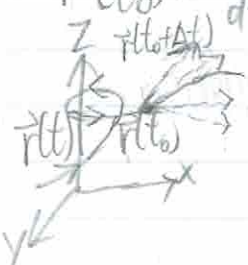
Continuity: A function $\vec{r}(t)$ is continuous at $t = t_0$ if

$\lim\limits_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$.

$\vec{r}(t)$ is continuous if it is continuous at every point in its domain.

Derivatives:

$\vec{r}'(t_0) = \frac{d\vec{r}}{dt} = \lim\limits_{\Delta t \to 0} \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}$



$\vec{r}'(t_0)$ is tangent to the curve at $r(t_0)$.

So $\vec{r}'(t_0) = f'(t_0)\hat{i} + g'(t_0)\hat{j} + h'(t_0)\hat{k}$.

Motion: $\vec{r}(t)$ is the position in space of a particle at time t.

$\vec{v}(t) = \vec{r}'(t) = \frac{d\vec{r}}{dt}$ is the velocity

$\frac{\vec{v}(t)}{\|\vec{v}(t)\|}$ is the direction of motion

$\|\vec{v}(t)\|$ is the speed

$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$ is the acceleration

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$\vec{r}(t) = \frac{t}{t+1}\hat{i} + \frac{1}{t}\hat{j} - 2t\hat{k}$

Ex.) Find the velocity and acceleration vectors at $t = -\frac{1}{2}$. Then draw the path.

Take the derivative of each component separately:

$\boxed{\begin{aligned} d\left(\tfrac{u}{v}\right) &= \frac{v\,du - u\,dv}{v^2} \\ d(u^n) &= nu^{n-1}du \end{aligned}}$

$r'(t) = \frac{t+1-t}{(t+1)^2}\hat{i} + \frac{-1}{t^2}\hat{j} - 2\hat{k} = \frac{1}{(1+t)^2}\hat{i} - \frac{1}{t^2}\hat{j} - 2\hat{k}$

$\vec{v}(-\frac{1}{2}) = \vec{r}'(-\frac{1}{2}) = \frac{1}{(\frac{1}{2})^2}\hat{i} - \frac{1}{(-\frac{1}{2})^2}\hat{j} - 2\hat{k} = \langle 4, -4, -2 \rangle$

$r''(t) = \frac{-2}{(1+t)^3}\hat{i} + \frac{2}{t^3}\hat{j} + 0\hat{k}$

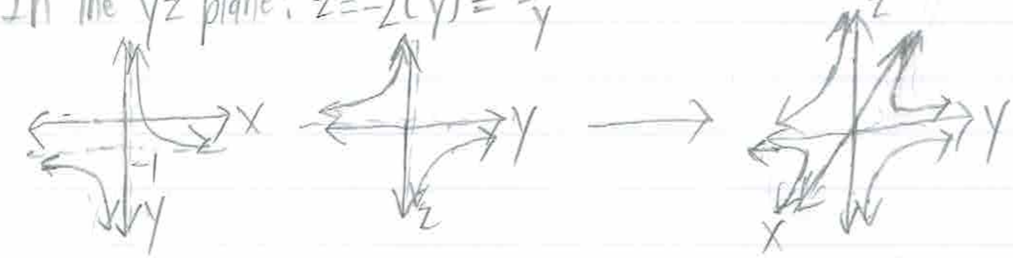
$\vec{a}(-\frac{1}{2}) = r''(-\frac{1}{2}) = \frac{-2}{(\frac{1}{2})^3}\hat{i} + \frac{2}{(-\frac{1}{2})^3}\hat{j} = \langle -16, -16, 0 \rangle$

To draw the path:

$x = \frac{t}{t+1}$, $y = \frac{1}{t}$, $z = -2t$

In the xy plane, $t = \frac{1}{y}$, so $x = \frac{\frac{1}{y}}{\frac{1}{y}+1} = \frac{1}{1+y}$. Then $y = \frac{1}{x} - 1$.

In the yz plane: $z = -2\left(\frac{1}{y}\right) = \frac{-2}{y}$



Differentiation Rules: $\vec{u}(t)$, $\vec{v}(t)$ are vector functions, $\vec{c}$ is constant vector

1) $\frac{d}{dt}(\vec{c}) = \vec{0}$
2) $\frac{d}{dt}(c\vec{u}(t)) = c\frac{d\vec{u}}{dt}$, $c \in \mathbb{R}$
3) $\frac{d}{dt}[f(t)\vec{v}(t)] = f'(t)\vec{v}(t) + f(t)\vec{v}'(t)$
4) $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt}$
5) $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
6) $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
7) $\frac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

Integrals: If $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$, then

$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\hat{i} + \left(\int_a^b g(t)\,dt\right)\hat{j} + \left(\int_a^b h(t)\,dt\right)\hat{k}$.

Ex.) $\vec{a} = \langle -3\cos t, -3\sin t, 2 \rangle$