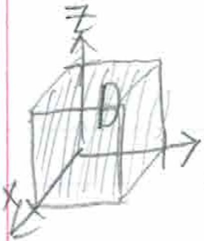
MATH 2605-G2

11/12/14



$= \int_0^2\int_0^2 xy\frac{z^2}{2}\Big|_{z=0}^{z=2} dy\,dx$ $= (\int_0^2 x\,dx)(\int_0^2 y\,dx)(\int_0^2 z\,dz)$

$= \int_0^2\int_0^2 2xy\,dy\,dx$

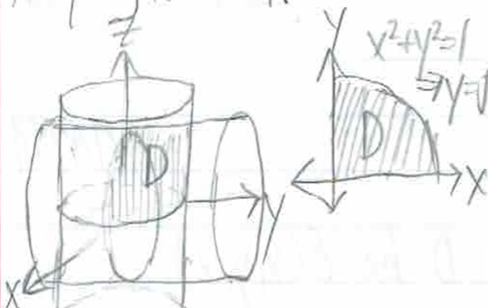
$= \int_0^2 2x(2)\,dx$ following the same pattern.

$= 2(2)(2) = 8$

Then the average is $\frac{8}{8} = 1$.

Ex) Find the volume of the region common to the interior of the cylinders $x^2+y^2=1$, $x^2+z^2=1$.

First find volume of first octant.



$D: \{(x,y,z): 0\le x\le 1, 0\le y\le\sqrt{1-x^2}, 0\le z\le\sqrt{1-x^2}\}$

$\text{Volume}(D) = \iiint 1\,dV$

$= \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx$

$= \int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx$

$= \int_0^1 (1-x^2)\,dx$

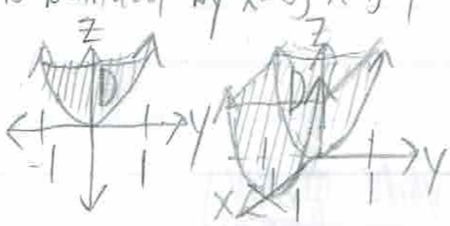
$= x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$

Since $\frac{2}{3}$ is the volume of the first octant, the total volume is $8(\frac{2}{3}) = \frac{16}{3}$.

11/13/14

<u>Recitation</u>

1) Find the volume of the region between the cylinder $z=y^2$ and the xy-plane that is bounded by $x=0$, $x=1$, $y=-1$, and $y=1$.



$D: \{(x,y,z): 0\le x\le 1, -1\le y\le 1, 0\le z\le y^2\}$

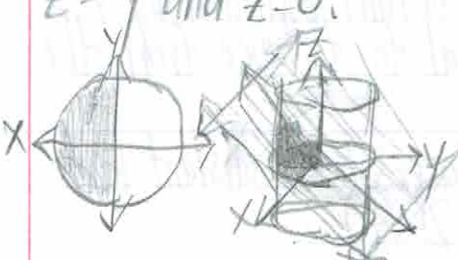
$V = \int_0^1\int_{-1}^1\int_0^{y^2} dz\,dy\,dx$

$= \int_0^1\int_{-1}^1 y^2\,dy\,dx$

$= \int_0^1 (\frac{y^3}{3})\Big|_{-1}^1 dx$

$= \frac{2}{3}x\Big|_0^1 = \frac{2}{3}$

2) Find the volume of the wedge cut from the cylinder $x^2+y^2=1$ by the planes $z=-y$ and $z=0$.



$D: \{(x,y,z): -1\le x\le 0, -\sqrt{1-x^2}\le y\le\sqrt{1-x^2}, 0\le z\le -y\}$

$V = \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{-y} dz\,dy\,dx = \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -y\,dy\,dx = 0$

$z^2 = c^2(1-x^2-\frac{y^2}{4}) \Rightarrow z = \pm c\sqrt{1-x^2-\frac{y^2}{4}}$

3) For what value of c does the volume of $x^2+(\frac{y}{2})^2+(\frac{z}{c})^2=1$ equal 8π?

$D: \{(x,y,z): -1\le x\le 1, -2\sqrt{1-x^2}\le y\le 2\sqrt{1-x^2}, -c\sqrt{1-x^2-\frac{y^2}{4}}\le z\le c\sqrt{1-x^2-\frac{y^2}{4}}\}$

MATH 2605-G2

11/13/14

$$V=\int_{-1}^{1}\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}}\int_{-c\sqrt{1-x^2-\frac{y^2}{4}}}^{c\sqrt{1-x^2-\frac{y^2}{4}}} dz\,dy\,dx=8\pi$$

Ways to Solve:

<u>Change of coordinates:</u>

$u=x$, $v=\frac{y}{2}$, $w=\frac{z}{c}$ $\Rightarrow u^2+v^2+w^2=1$, call this new domain S.

Now $V=\iiint_S k\,du\,dv\,dw$, where k turns out to be the <u>Jacobi determinant</u> $\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} & \frac{dx}{dw} \\ \frac{dy}{du} & \frac{dy}{dv} & \frac{dy}{dw} \\ \frac{dz}{du} & \frac{dz}{dv} & \frac{dz}{dw} \end{vmatrix}$.

Here $k=\begin{vmatrix} 1 & & \\ & 2 & \\ & & c \end{vmatrix}=2c$ and $V=8\pi=2c\cdot\frac{4}{3}\pi$

$\Rightarrow c=3$

Also, <u>spherical coordinates.</u>

11/14/14

<u>Moments and Center of Mass:</u>

Let the density (mass per unit volume) of a region D be $\delta(x,y,z)$.

Then the mass of the object occupying D is

$$M=\iiint_D \delta\,dV=\iiint_D \delta(x,y,z)\,dx\,dy\,dz$$

The "first moment" of a solid region D about the yz-plane is

$M_{yz}=\iiint_D x\,\delta\,dV$, x is the distance from a point (x,y,z) in D to the yz-plane

$$M_{xz}=\iiint_D y\,\delta\,dV \qquad M_{xy}=\iiint z\,\delta\,dV$$

Then the center of mass is

$$(\tilde{x},\tilde{y},\tilde{z})=\left(\frac{M_{yz}}{M},\frac{M_{xz}}{M},\frac{M_{xy}}{M}\right)=\left(\frac{\iiint_D x\delta\,dV}{\iiint_D \delta\,dV},\frac{\iiint_D y\delta\,dV}{\iiint_D \delta\,dV},\frac{\iiint_D z\delta\,dV}{\iiint_D \delta\,dV}\right)$$

Ex.) Calculate the mass of the solid region in the first octant bounded by $y=x$ and $z=2-\frac{x^2}{2}$, given that the density is proportional to distance from the xy-plane.

The density is $\delta(x,y,z)=kz$ for a positive constant k.

$D=\{(x,y,z): 0\le x\le 2,\ 0\le y\le x,\ 0\le z\le 2-\frac{x^2}{2}\}$

$$M=\int_0^2\int_0^x\int_0^{2-x^2/2} kz\,dz\,dy\,dx$$

$$=\int_0^2\int_0^x k\frac{z^2}{2}\Big|_0^{2-\frac{x^2}{2}}$$

$$=\frac{k}{2}\int_0^2\int_0^x \left(2-\frac{x^2}{2}\right)^2 dy\,dx$$
$$=\frac{k}{2}\int_0^2 \left(2-\frac{x^2}{2}\right)^2 x\,dx$$
$$u=2-\frac{x^2}{2}$$
$$du=-x\,dx$$
$$=\frac{k}{2}\int_0^{-2} u^2 \cancel{x}\frac{du}{-\cancel{x}}$$
$$=-\frac{k}{2}\left(\frac{u^3}{3}\right)\Big|_0^{-2}$$
$$=\frac{4k}{3}$$

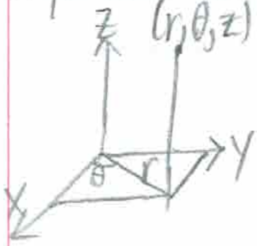
The <u>centroid</u> is the center of mass if the density is constant.

Ex.) Find the centroid of the region in the first quadrant bounded above by $y=x$ and below by $y=x^2$.

$$(\tilde{x},\tilde{y})=\left(\frac{\iint_D x\,dx\,dy}{\iint_D dx\,dy},\frac{\iint_D y\,dx\,dy}{\iint_D dx\,dy}\right)$$
$$\iint_D dx\,dy=\int_0^1\int_{x^2}^x dy\,dx=\int_0^1(x-x^2)\,dx=\frac{x^2}{2}-\frac{x^3}{3}\Big|_0^1=\frac{1}{6}$$
$$\iint_D x\,dx\,dy=\int_0^1\int_{x^2}^x x\,dy\,dx=\int_0^1 x(x-x^2)\,dx=\frac{x^3}{3}-\frac{x^4}{4}\Big|_0^1=\frac{1}{12}$$
$$\iint_D y\,dx\,dy=\int_0^1\int_{x^2}^x y\,dy\,dx=\int_0^1\left(\frac{y^2}{2}\right)\Big|_{x^2}^x dx=\int_0^1\left(\frac{x^2}{2}-\frac{x^4}{2}\right)dx=\frac{x^3}{6}-\frac{x^5}{10}\Big|_0^1=\frac{1}{15}$$
$$(\tilde{x},\tilde{y})=\left(\frac{1/12}{1/6},\frac{1/15}{1/6}\right)=\left(\frac{1}{2},\frac{2}{5}\right)$$

The moment of inertia of a rotating solid about a line L is
$$I=\iiint_D r^2\delta\,dV$$
where $r=r(x,y,z)$ is the distance from the point (x,y,z) to the line L.
If L is the x-axis, $r^2=y^2+z^2$ and
$$I_x=\iiint_D(y^2+z^2)\delta\,dV$$
Similarly,
$$I_y=\iiint_D(x^2+z^2)\delta\,dV\qquad I_z=\iiint_D(x^2+y^2)\delta\,dV$$

<u>Cylindrical coordinates</u>:



$r=\sqrt{x^2+y^2}$, $\theta=\tan^{-1}\frac{y}{x}$

or $x=r\cos\theta$, $y=r\sin\theta$, $z=z$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz=\iiint_D f(r,\theta,z)\,\underline{\underline{r}}\,dr\,d\theta\,dz$$

with $f(r,\theta,z)=f(r\cos\theta,r\sin\theta,z)$