MATH 2605-G2 10/1/14

<u>Numerical computations of eigenvalues:</u>

Want to solve

$$A\vec{x}=\lambda\vec{x}$$

where $p(\lambda)=\det(A-\lambda I)=0$, $(A-\lambda I)\vec{x}=0$.

Ex.) $A=\begin{pmatrix} 0 & & & \varepsilon \\ 1 & 0 & & \\ & 1 & 0 & \\ 0 & & \ddots & \\ & & 1 & 0 \end{pmatrix}_{n\times n}$

Here $p(\lambda)=\lambda^n-\varepsilon$ so $p(\lambda)\Rightarrow\lambda=\sqrt[n]{\varepsilon}$.

If $\varepsilon=0$, then all $\lambda=0$.

If $\varepsilon\neq0$, there are $n-1$ complex eigenvalues.

Choose $\varepsilon=10^{-40}$ so that $n=40$.

Then $\lambda=(10^{-40})^{1/40}=0.1$.

In other words, a perturbation of 10^{-40} produces an error of 0.1 in λ.

Power Method:

If A has a full set of eigenvectors, then a vector $\vec{u}_0$ is:

$\vec{u}_0=c_1\vec{v}_1+c_2\vec{v}_2+\dots+c_n\vec{v}_n$.

Then $A^k\vec{u}_0=c_1A^k\vec{v}_1+c_2A^k\vec{v}_2+\dots+c_nA^k\vec{v}_n$

$=c_1\lambda_1^k\vec{v}_1+c_2\lambda_2^k\vec{v}_2+\dots+c_n\lambda_n^k\vec{v}_n$.

Suppose λ_1 is the largest eigenvalue. If $c_1\neq0$:

$$\frac{A^k\vec{u}_0}{\lambda_1^k}=c_1\vec{v}_1+c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\vec{v}_2+\dots+c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\vec{v}_n.$$

As k increases, $A^k\vec{u}_0$ will keep the same direction as $\vec{v}_1$ and converge to the eigenvector corresponding to λ_1.

The algorithm:

Make an initial guess $\vec{u}_0\neq0$. Then:

$\vec{u}_1=A\vec{u}_0$

$\vec{u}_2=A\vec{u}_1=A^2\vec{u}_0$

$\vdots$

$\vec{u}_k=A^k\vec{u}_0$

Define $\vec{u}_{k+1}=\frac{1}{\alpha_k}A\vec{u}_k$, where α_k is the 1st component in $\vec{u}_k$. Then:

$\lim_{k\to\infty} \vec{u}_k = \vec{v}_1$ $\lim_{k\to\infty} \alpha_k = \lambda_1$

Convergence factor: $r = \left|\frac{\lambda_2}{\lambda_1}\right|$

The closer r is to 1, the "slower" the convergence.

Variants of power method:

* Block power method: instead of working with just $\vec{u}_k$, use p orthonormal vectors multiplied by A, then apply Gram-Schmidt to regain orthormality.
* Inverse power method: use A^{-1} instead of A; then the convergence will be to the eigenvector corresponding with the smallest eigenvalue λ_n.
  $\vec{u}_{k+1} = A^{-1}\vec{u}_k \Leftrightarrow A\vec{u}_{k+1} = \vec{u}_k$ (Solve using $A=LU$ or $A=QR$)
* Shifted inverse power method: use $(A-\alpha I)^{-1}$ instead of A^{-1}; each eigenvalue is shifted by α.
  $r = \left|\frac{\lambda_n - \alpha}{\lambda_1 - \alpha}\right|$ ← useful when λ_n is close to α.
  $\vec{u}_{k+1} = (A-\alpha I)^{-1}\vec{u}_k \Leftrightarrow (A-\alpha I)\vec{u}_{k+1} = \vec{u}_k$

Note: Power methods are advantageous only for large and sparse matrices (lots of zeros).

Otherwise:

Use $A_0 = Q^{-1}AQ$, where Q is orthogonal (and sparse because it is obtained using Householder or Givens)

A_0 and A follow the format for similar matrices. Therefore, they share the same eigenvectors.

Use power method to find eigenvalues, and use shifted inverse power method to find eigenvectors.

This concludes the numerical linear algebra portion of the course.

MATH 2605 — G2

10/1/14

Vector calculus: (Chp. 13, Thomas)

A curve in space is defined as a vector function

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, t \in I$

so the position at time t is the point (x, y, z) with

$x = f(t), y = g(t), z = h(t)$.

The path is the set of points $\vec{r}(t)$ with $t \in I$.

Ex.) $\vec{r}(t) = (\cos t, \sin t, t), t \in [0, 4\pi]$.