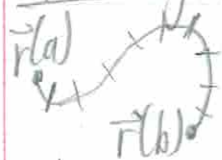
MATH 2605 - G2

10/06/14

Arc length:



Intuition: measure each segment as a straight line segment. Make the segments infinitely small.

$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

$L_k \approx \|\vec{r}(t_k) - \vec{r}(t_{k-1})\|$

$\approx \sqrt{[f(t_k)-f(t_{k-1})]^2 + [g(t_k)-g(t_{k-1})]^2 + [h(t_k)-h(t_{k-1})]^2}$

The curve is continuous, satisfying the mean value theorem. There exist $t_k^*, t_k^{**}, t_k^{***}$ such that

$L_k \approx \sqrt{[f'(t_k^*)(t_k - t_{k-1})]^2 + [g'(t_k^{**})(t_k - t_{k-1})]^2 + [h'(t_k^{***})(t_k - t_{k-1})]^2}$

$\approx \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2 + [h'(t_k^{***})]^2}\,(t_k - t_{k-1})$

Taking the limit of the sum of all segments $L = \sum L_k$ as $\Delta t \to 0$:

$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$

$= \int_a^b \|\vec{r}'(t)\|\, dt$

Ex.) $\vec{r}(t) = (2+t)\hat{i} - (t+1)\hat{j} + t\hat{k},\ t \in [0,3]$

$L = \int_0^3 \|(1,-1,1)\|\, dt = \int_0^3 \sqrt{3}\, dt = 3\sqrt{3}$

Ex.) $\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + t^2\hat{k},\ t \in [0, \frac{\pi}{2}]$

$\Rightarrow \vec{r}'(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 2t\hat{k}$

$\Rightarrow \|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 4t^2} = 2\sqrt{1+t^2}$

$\Rightarrow L = \int_0^{\pi/2} 2\sqrt{1+t^2}\, dt$

$= \left[\sqrt{1+t^2}\,t + \ln\left|\sqrt{1+t^2} + t\right|\right]_0^{\pi/2}$

$= \sqrt{1+\frac{\pi^2}{4}}\left(\frac{\pi}{2}\right) + \ln\left|\sqrt{1+\frac{\pi^2}{4}} + \frac{\pi}{2}\right|$

Recitation

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1) $\vec{r}(t) = \cos 2t\,\hat{i} + 3\sin 2t\,\hat{j}$

$\vec{v}(t) = -2\sin 2t\,\hat{i} + 6\cos 2t\,\hat{j}$

*Use $v(t)$ to find direction of parametrization.

2) $\vec{f}(t) = te^t\hat{i} + e^t\hat{j} + \hat{k}$

$\int_0^1 \vec{f}(t) = \left(\int_0^1 te^t\,dt, \int_0^1 e^t\,dt, \int_0^1 dt\right)$

$u(t) = t,\ dv(t) = e^t\,dt$

$du(t) = dt,\ v(t) = \int dv(t) = e^t + C$

$\int u(t)\,dv(t) = te^t - \int e^t\,dt = te^t - e^t + C$

$\int_0^1 \vec{f}(t) = \left(te^t - e^t\big|_0^1, e^t\big|_0^1, t\big|_0^1\right) = (1, e-1, 1)$

3) $\vec{v}(t) = \frac{te^{t^2}\sin^2 t}{\sqrt{1+t^{2014}}}\hat{i} + \cos t\, e^t\hat{j}$

$\int_{-1}^1 \vec{v}(t)\,dt = \left(0, \int_{-1}^1 \cos t\, e^t\,dt\right)$

$\boxed{\text{why? Odd function}}$

u	dv
$\cos t$	e^t
$-\sin t$	e^t
$-\cos t$	e^t

$\int \cos t\, e^t\,dt = \cos t\, e^t - \left[(-\sin t\, e^t) - \int(-\cos t)e^t\,dt\right]$

$2\int \cos t\, e^t\,dt = e^t\cos t + e^t\sin t$

$\int \cos t\, e^t\,dt = \frac{e^t(\cos t + \sin t)}{2}$

$\int_{-1}^1 \vec{v}(t)\,dt = \left(0, \frac{e(\cos 1 + \sin 1)}{2} - \frac{e^{-1}(\cos(-1) + \sin(-1))}{2}\right)$

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Arc length parameter:

Choose a pt. $\vec{r}(t_0)$ in a smooth curve.

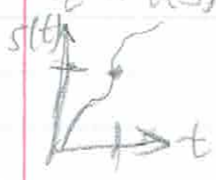
Define the arc length distance as the function:

$s(t) = \int_{t_0}^{t} \|r'(\tau)\|\,d\tau$

Note: $\frac{ds}{dt} = \|\vec{r}'(t)\| = \|v(t)\| \geq 0$

Now t can be solved for as a function of s:

$t = t(s)$



If $\vec{v}(t) \neq 0$ then $\|\vec{v}\| > 0 \Rightarrow s'(t) > 0$.

Then the parametrization by arc length is $\vec{R}(s) = \vec{r}(t(s))$.

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$\vec{r}(t)=\cos t\,\hat{\imath}+\sin t\,\hat{\jmath}+t\hat{k}$, $t_0=0$.

Ex.) $\vec{r}\,'(t)=-\sin t\,\hat{\imath}+\cos t\,\hat{\jmath}+\hat{k}$

$\|\vec{r}\,'(t)\|=\sqrt{\sin^2 t+\cos^2 t+1}=\sqrt{2}$

$s(t)=\int_0^t\|\vec{r}\,'(\varphi)\|\,d\varphi=\int_0^t\sqrt{2}\,d\varphi=\sqrt{2}\varphi\big|_0^t=\sqrt{2}t$

$\Rightarrow t=\frac{1}{\sqrt{2}}s$

Parametrization by arc length: $\vec{R}(s)=\vec{r}\left(\frac{1}{\sqrt{2}}s\right)=\cos\left(\frac{1}{\sqrt{2}}s\right)\hat{\imath}+\sin\left(\frac{1}{\sqrt{2}}s\right)\hat{\jmath}+\frac{1}{\sqrt{2}}s\hat{k}$

Unit tangent vector:

$\vec{r}\,'(t)$ is tangent to the curve at $\vec{r}(t)$. Define the unit tangent vector as

$$\vec{T}=\frac{\vec{v}}{\|\vec{v}\|}.$$

Note: $\frac{dt}{ds}=\frac{1}{ds/dt}=\frac{1}{\|\vec{v}\|}=\frac{1}{\|\vec{r}\,'(t)\|}$

and $\frac{d\vec{r}}{ds}=\frac{d\vec{r}}{dt}\frac{dt}{ds}=\vec{r}\,'(t)\frac{1}{\|\vec{r}\,'(t)\|}=\frac{\vec{v}}{\|\vec{v}\|}=\vec{T}$.

Then $\vec{T}=\vec{T}(t)$ is the change of the position vector in terms of arc length.

Curvature: the rate at which the unit tangent vector changes direction with respect to arc length.

$$\kappa=\left\|\frac{d\vec{T}}{ds}\right\|=\left\|\frac{d\vec{T}}{dt}\frac{dt}{ds}\right\|=\left|\frac{dt}{ds}\right|\left\|\frac{d\vec{T}}{dt}\right\|=\frac{1}{|ds/dt|}\left\|\frac{d\vec{T}}{dt}\right\|=\frac{1}{\|\vec{r}\,'(t)\|}\left\|\frac{d\vec{T}}{dt}\right\|$$

Ex.) $\vec{r}(t)=\vec{r}_0+t\vec{d}$, $t\in\mathbb{R}$

$\vec{r}\,'(t)=\vec{d}\Rightarrow\vec{T}=\frac{\vec{r}\,'}{\|\vec{r}\,'\|}=\frac{\vec{d}}{\|\vec{d}\|}$ is constant. $\frac{d\vec{T}}{ds}=\frac{d\vec{T}}{dt}=0$.

Then the curvature $\kappa=0$.

Ex.) $\vec{r}(t)=a\cos t\,\hat{\imath}+a\sin t\,\hat{\jmath}$

$\vec{r}\,'(t)=-a\sin t\,\hat{\imath}+a\cos t\,\hat{\jmath}$

$\|\vec{r}\,'(t)\|=\sqrt{a^2\sin^2 t+a^2\sin^2 t}=a$

$\vec{T}=\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}=-\sin t\,\hat{\imath}+\cos t\,\hat{\jmath}$

Then $\frac{d\vec{T}}{dt} = -\cos t\,\hat{i} - \sin t\,\hat{j}$

and $\left\|\frac{d\vec{T}}{dt}\right\| = \sqrt{\cos^2 t + \sin^2 t} = 1.$

Finally:

$K = \frac{1}{\|\vec{r}\,'\|}\left\|\frac{d\vec{T}}{dt}\right\| = \frac{1}{a}.$

Larger circle has higher curvature.

In the special case of $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ (curve in the plane), K becomes

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}.$$

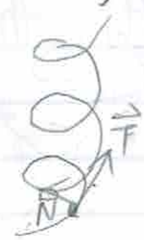
Principle unit normal:

$$\vec{N} = \frac{1}{K}\frac{d\vec{T}}{ds} = \frac{1}{\|d\vec{T}/ds\|}\frac{d\vec{T}}{ds} = \frac{d\vec{T}/ds}{\|d\vec{T}/ds\|} = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|}$$

Ex.) Helix is given by $\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\hat{k}$. Calculate K, $\vec{N}$, and $\vec{T}$.

$\vec{r}\,'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k} = \vec{v}$

$\|\vec{r}\,'(t)\| = \sqrt{a^2+b^2} = \|\vec{v}\|$



$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k})$

$K = \frac{1}{\|\vec{v}\|}\left\|\frac{d\vec{T}}{dt}\right\| = \frac{1}{\sqrt{a^2+b^2}}\left\|\frac{1}{\sqrt{a^2+b^2}}(-a\cos t\,\hat{i} - a\sin t\,\hat{j})\right\|$

$= \frac{1}{a^2+b^2}\sqrt{a^2(\cos^2 t + \sin^2 t)}$

$= \frac{a}{a^2+b^2}$

$\vec{N} = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|} = \frac{1}{a/\sqrt{a^2+b^2}}\left(\frac{1}{\sqrt{a^2+b^2}}\right)(-a\cos t\,\hat{i} - a\sin t\,\hat{j})$

$= \frac{1}{a}(-a\cos t\,\hat{i} - a\sin t\,\hat{j})$

$= -\cos t\,\hat{i} - \sin t\,\hat{j}$

Points toward z-axis.