MATH 2605-G2

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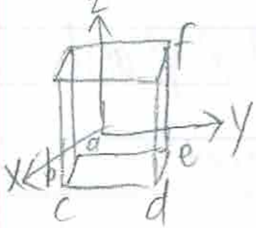
Triple Integrals:

Let $f(x,y,z)$ be defined over a closed, bounded region D.

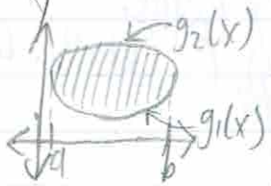
We take a partition element ΔV_k and a point (x_k, y_k, z_k) in ΔV_k. Then we calculate the sum $\sum_k (x_k, y_k, z_k)\Delta V_k$ with the limit as the size of the partition elements approaches zero to get

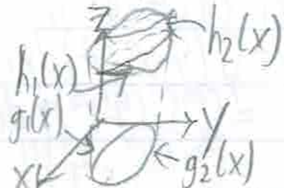
$$\iiint_D f(x,y,z)\,dV = \iiint_D f(x,y,z)\,dx\,dy\,dz.$$

If $D = \{(x,y,z) : (a \le x \le b,\ c \le y \le d,\ e \le z \le f)\}$, (shaped like a box)



$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$





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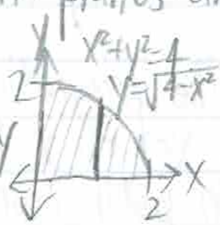
More general regions:

$D = \{(x,y,z) : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ h_1(x,y) \le z \le h_2(x,y)\}$

Here, $\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$

If $f(x,y,z) = 1$, then $\iiint_D dV = \iiint_D dx\,dy\,dz$ is the volume of D.

Ex.) Evaluate $\iiint_D xy\,dx\,dy\,dz$ where D is the solid in the first octant bounded by the coordinate planes and the upper half of the sphere $x^2+y^2+z^2=4$.



$D: \{(x,y,z) : 0 \le x \le 2,\ 0 \le y \le \sqrt{4-x^2},\ 0 \le z \le \sqrt{4-x^2-y^2}\}$

$$\iiint_D xy\,dx\,dy\,dz = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xy\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} xyz\Big|_{z=0}^{\sqrt{4-x^2-y^2}} dy\,dx$$

$u = 4-x^2$

$du = -2x\,dx$

$\int x u^{3/2} \frac{du}{-2x}$

$= -\frac{1}{2} u^{5/2}\left(\frac{2}{5}\right)$

$$= \int_0^2 \left(-\frac{1}{2}\right) x \frac{(4-x^2-y^2)^{3/2}}{3/2}\Big|_{y=0}^{\sqrt{4-x^2}} dx$$

$$= -\frac{1}{3}\int_0^2 x\left[(4-x^2-(4-x^2))^{3/2} - (4-x^2)^{3/2}\right]dx$$

$$= +\frac{1}{3}\int_0^2 x(4-x^2)^{3/2}\,dx$$

$$= \frac{1}{3}\left(-\frac{1}{2}\right)\left(\frac{2}{5}\right)(4-x^2)^{5/2}\Big|_0^2$$

$$= -\frac{1}{15}(0-32) = \frac{32}{15}$$

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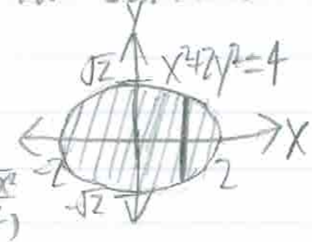
Ex1) Find the volume of the region D enclosed by the surfaces $z=x^2+3y^2$ and $z=8-x^2-y^2$.

Intersection between the two surfaces:

$z=x^2+3y^2=8-x^2-y^2$

$\Leftrightarrow x^2+2y^2=4$

$\Leftrightarrow y=\pm\sqrt{\frac{4-x^2}{2}}$



$D=\{(x,y,z): -2\le x\le 2,\ -\sqrt{\frac{4-x^2}{2}}\le y\le\sqrt{\frac{4-x^2}{2}},\ x^2+3y^2\le z\le 8-x^2-y^2\}$

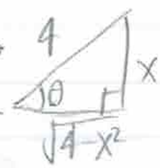
$$V=\iiint_D dV=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2}dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}(8-2x^2-4y^2)\,dy\,dx$$

$$=\int_{-2}^{2}\left(8y-2x^2y-\frac{4y^3}{3}\right)\Bigg|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}dx$$

$$=\int_{-2}^{2}\left(4\sqrt{4-x^2}-x^2\sqrt{4-x^2}-\frac{2}{3}\sqrt{4-x^2}^3+4\sqrt{4-x^2}-x^2\sqrt{4-x^2}-\frac{2}{3}\sqrt{4-x^2}^3\right)dx$$

$$=\frac{2}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx \longrightarrow$$



$\sin\theta=\frac{x}{4}\Rightarrow\theta=\sin^{-1}\left(\frac{x}{4}\right)$

$dx=4\cos\theta\,d\theta$

$\sqrt{4-x^2}=4\cos\theta$

$$=\frac{2}{3}\int_{-\pi/6}^{\pi/6}(4\cos\theta)^3\,4\cos\theta\,d\theta$$

$$=\frac{512}{3}\int_{-\pi/6}^{\pi/6}\left[\left(\frac{1+\cos 2\theta}{2}\right)^2\right]d\theta$$

$$=\frac{512}{3}\int_{-\pi/6}^{\pi/6}\frac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$=\frac{128}{3}\int_{-\pi/6}^{\pi/6}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$=\frac{128}{3}\left(\frac{3}{2}\theta+\sin 2\theta+\frac{\sin 4\theta}{8}\right)\Bigg|_{-\pi/6}^{\pi/6}$$

$$=\frac{128}{3}\left(\frac{\pi}{4}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{16}+\frac{\pi}{4}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{16}\right)$$

$$=\frac{64}{3}\left(\pi+\frac{9\sqrt{3}}{4}\right)$$