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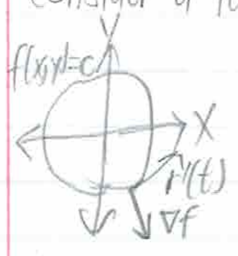
iii) Any direction orthogonal to $\nabla f\ (\neq \vec{0})$ is a direction of zero change:
$D_{\vec{u}}f = \|\nabla f\| \cos\frac{\pi}{2} = 0$

Ex.) Directions of zero change in last ex.: $\vec{u}_1 = \frac{1}{\sqrt{2}}(1,-1)$ and $\vec{u}_2 = \frac{1}{\sqrt{2}}(-1,1)$

Consider a level curve of $f(x,y)$: $f(x,y)=c$. If $\vec{r}(t) = (g(t), h(t))$ is a parametrization of the curve, then $f(\vec{r}(t)) = f(g(t), h(t)) = c$.



Taking derivative w/ respect to t: $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$

$\Leftrightarrow \frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t) = 0$

$\Leftrightarrow \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (g'(t), h'(t)) = 0$

$\Leftrightarrow \nabla f \cdot \vec{r}'(t) = 0$

Since $\vec{r}'(t)$ is tangent to the curve and orthogonal to ∇f, ∇f is <u>normal</u> to the curve.
The equation for the line tangent to the curve $f(x,y)=c$ is

$\nabla f|_{(x_0,y_0)} \cdot ((x,y)-(x_0,y_0)) = 0$

$\Leftrightarrow \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x-x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y-y_0) = 0$

Ex.) Find an equation for the line tangent to $\frac{x^2}{4}+y^2=2$ at $(-2,1)$.

Let $f(x,y) = \frac{x^2}{4}+y^2$ so that $f(x,y)=2$.
Then $\nabla f(x,y) = (\frac{1}{2}x, 2y) \Rightarrow \nabla f(-2,1) = (-1,2)$.
Tangent line: $\frac{\partial f}{\partial x}\Big|_{(-2,1)}(x-(-2)) + \frac{\partial f}{\partial y}\Big|_{(-2,1)}(y-1) = 0$

$\Leftrightarrow (-1)(x+2)+2(y-1)=0$

$\Leftrightarrow -x+2y=4$

<u>Gradient rules:</u>

1) $\nabla(f+g) = \nabla f + \nabla g$
2) $\nabla(fg) = f\nabla g + g\nabla f$
3) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

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Recitation

The purpose of directional derivatives is to render a curve in one variable.

Ex.) For $z=f(x,y)=x^2+y^2$ along the direction $y=x$, $g(x)=f(x,x)=x^2+x^2=2x^2$. $g'(x)=4x$.
$\nabla f \cdot \vec{u} = (2x, 2y)\cdot(1,1) = 2x+2y = 4x$, $y=x$.

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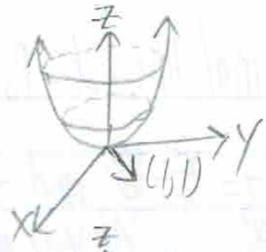
1) $f(x,y)=\sqrt{x^2+y^2}$, $(x_0,y_0)=(1,1)$, $\vec{\alpha}=(1,1)$

$\nabla f=\left(\frac{2x}{2\sqrt{x^2+y^2}}, \frac{2y}{2\sqrt{x^2+y^2}}\right)$ $\quad \vec{u}=\frac{\vec{\alpha}}{\|\vec{\alpha}\|}=\frac{1}{\sqrt{2}}(1,1)$

$\nabla f|_{(x_0,y_0)}\cdot u=\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)=1$



2) $f(x,y)=x^2+xy+y^2$, $(x_0,y_0)=(1,1)$, $\vec{\alpha}=(1,1)$

$\nabla f=(2x+y, x+2y)$ $\quad \vec{u}=\frac{\vec{\alpha}}{\|\vec{\alpha}\|}=\frac{1}{\sqrt{2}}(1,1)$

$\nabla f|_{(x_0,y_0)}\cdot \vec{u}=\left(\frac{3}{\sqrt{2}}\right)+\left(\frac{3}{\sqrt{2}}\right)=3\sqrt{2}$

$f(x,y)=(x+\frac{y}{2})^2+y^2-\frac{y^2}{4}=z$

$z=(x+\frac{y}{2})^2+\frac{3y^2}{4}$

Ellipse

3) $f(x,y)=x^2-xy+y^2-y$; Find the unit direction vector $\vec{u}$ such that $D_{\vec{u}}f(1,-1)$

a) is the largest.

$\nabla f(1,-1)=(2x-y, -x+2y-1)|_{(1,-1)}=(3,-4)$

$\nabla f(1,-1)\cdot\vec{u}=\|\nabla f(1,-1)\|\cos\theta=5$

$u=\frac{\nabla f}{\|\nabla f\|}\Big|_{(1,-1)}=\left(\frac{3}{5},-\frac{4}{5}\right)$

b) is the smallest.

Opposite direction of a: $u=\left(\frac{-3}{5},\frac{4}{5}\right)$

c) $(f_x,f_y)|_{(1,-1)}\cdot(u_1,u_2)=0$

$(u_1,u_2)=(-f_y,f_x),(f_y,-f_x)=\left(\frac{4}{5},\frac{3}{5}\right),\left(\frac{-4}{5},\frac{-3}{5}\right)$

d) $=4$ $\quad (f_x,f_y)|_{(1,-1)}\cdot(u_1,u_2)=4$

$\|\nabla f(1,-1)\|\cos\theta=4$

$\cos\theta=\frac{4}{5}$

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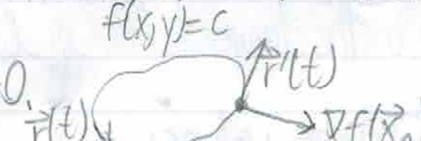
For $f(x,y)$: At a level curve $f(x,y)=c$, ∇f is normal to curve.

Equation for tangent line: $\nabla f(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0)=0$

or $\nabla f(x_0,y_0)\cdot((x,y)-(x_0,y_0))=0$.



For three variables: $f(x,y,z)=c$ is a level surface.

Take $\vec{r}(t)$ as a curve in the surface that passes through the point P_0 with $\vec{r}(t)=(g(t),h(t),k(t))$. Then $f(\vec{r}(t))=f(g(t),h(t),k(t))=c$.

Taking derivatives with respect to t: $\frac{d}{dt}(f(\vec{r}(t)))=\frac{\partial f}{\partial x}g'(t)+\frac{\partial f}{\partial y}h'(t)+\frac{\partial f}{\partial z}k'(t)=0$

$\Leftrightarrow\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\cdot(g'(t),h'(t),k'(t))=0$

$\Leftrightarrow\nabla f\cdot\vec{r}'(t)=0$

or $\frac{d}{dt}(f(\vec{r}(t)))=0$.

Then $\vec{r}'(t)$ and ∇f are orthogonal. Taking any curve $\vec{r}(t)$, ∇f is orthogonal to the tangent plane to the surface $f(x,y,z)=c$ at the point P_0. The equation for the tangent plane is $\nabla f(x_0,y_0,z_0)\cdot((x,y,z)-(x_0,y_0,z_0))=0$.

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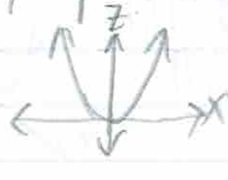
Equation for normal line: $l = P_0 + t\nabla f = (x_0, y_0, z_0) + t\nabla f(x_0, y_0, z_0)$

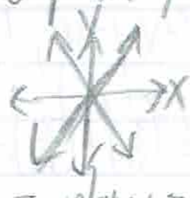
Ex.) Draw the surface $z = \frac{x^2}{4} - \frac{y^2}{9}$ and find the tangent plane at $(2,3,0)$.

At $x=0$ plane: $z = -\frac{y^2}{9}$

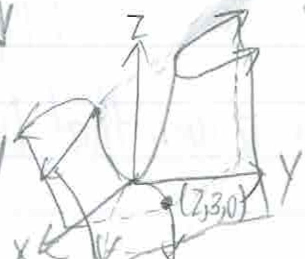
At $y=0$ plane: $z = \frac{x^2}{4}$



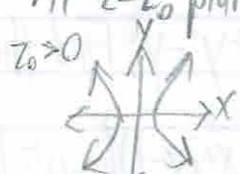
At $z=0$ plane: $y = \pm\frac{3}{2}x$

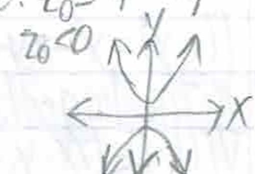


hyperbolic paraboloid



At $z=z_0$ plane: $z_0 = \frac{x^2}{4} - \frac{y^2}{9}$



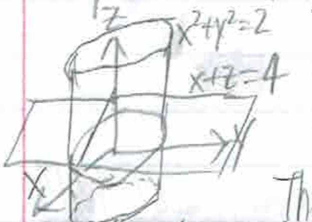


For tangent plane: define $f(x,y,z) = \frac{x^2}{4} - \frac{y^2}{9} - z$ (level surface is $f(x,y,z)=0$)

$\nabla f(2,3,0) = \left(\frac{x}{2}, \frac{-2y}{9}, -1\right)\Big|_{(2,3,0)} = \left(1, \frac{-2}{3}, -1\right)$

Tangent plane: $\nabla f(2,3,0) \cdot ((x,y,z)-(2,3,0)) = (x-2) - \frac{2}{3}(y-3) - z = x - \frac{2}{3}y - z = 0$

Ex.) Find the tangent line to the intersection curve of the surfaces $f(x,y,z) = x^2+y^2-2=0$ and $g(x,y,z) = x+z-4=0$ at $(1,1,3)$.



The tangent line is orthogonal to both ∇f and ∇g.

$\nabla f = (2x, 2y, 0) \Rightarrow \nabla f(1,1,3) = (2,2,0)$

$\nabla g = (1,0,1)$

The direction vector of the tangent line is $\vec{d} = \nabla f \times \nabla g = (2,-2,-2)$

So the tangent line is $l(t) = (1,1,3) + t(2,-2,-2)$.

Extreme values:

Definition: for $f(x,y)$ defined on R containing (a,b),

1) $f(a,b)$ is a local maximum if $f(a,b) \geq f(x,y)$ for all (x,y) in an open disk around (a,b).

2) $f(a,b)$ is a local minimum if $f(a,b) \leq f(x,y)$ for all (x,y) in an open disk around (a,b).

In either case, $g(x) = f(x,b) \Rightarrow g'(x) = 0 \Rightarrow \frac{\partial f}{\partial x}(a,b) = 0$

$h(y) = f(a,y) \Rightarrow h'(y) = 0 \Rightarrow \frac{\partial f}{\partial y}(a,b) = 0$

(a,b) is a critical point if $\frac{\partial f}{\partial x}(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) = 0$.

If $F(x,y,z) = f(x,y) - z$, then ∇F is normal to the surface $f(x,y) - z = 0$.

$\nabla F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$, and at $(a, b, f(a,b))$:

$\nabla F(a,b,f(a,b)) = (0,0,-1)$. So maxima and minima have a vertical normal vector.