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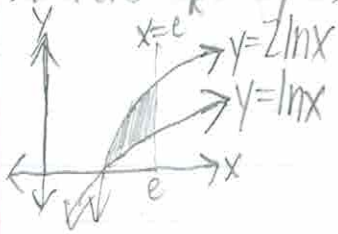
Find the area of the region bounded by $y=\ln x$, $y=2\ln x$, and $x=e$.

Ex.) $\text{Area}(R)=\iint_R dxdy=\int_1^e\int_{\ln x}^{2\ln x} dydx=\int_1^e(2\ln x-\ln x)dx=\int_1^e \ln x\, dx$



$u=\ln x \quad dv=dx$
$du=\frac{1}{x} \quad v=x$

$\Rightarrow x\ln x\Big|_1^e-\int_1^e 1dx$
$=e\ln e-0-(e-1)=1$

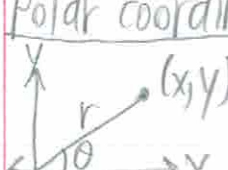
Average of $f(x,y)$ over a region R is $\frac{1}{\text{Area}(R)}\iint_R f(x,y)\,dxdy$. Intuitively, this can be thought of as the total value of all the points in the region divided by the number of points in the region.

Ex.) Average of $f(x,y)=x\cos(x,y)$ over $0\le x\le\pi$, $0\le y\le 1$:

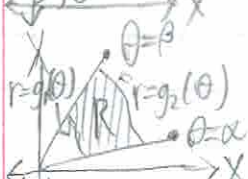
$V=\iint_R[x\cos(x,y)]dxdy$
$=\int_0^\pi\int_0^1 x\cos(x,y)dxdy$
$=\int_0^\pi \sin(xy)\Big|_{y=0}^{y=1}dx$
$=\int_0^\pi \sin x\,dx$
$=-\cos x\Big|_0^\pi$
$=2$

Average: $\frac{2}{(\pi-0)(1-0)}=\frac{2}{\pi}$

Polar coordinates:



$x=r\cos\theta \Rightarrow f(x,y)=f(r\cos\theta, r\sin\theta)=f(r,\theta)$
$y=r\sin\theta$

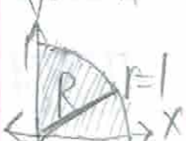


$R=\{(r,\theta): \alpha\le\theta\le\beta,\ g_1(\theta)\le r\le g_2(\theta)\}$

$\iint_R f(x,y)dxdy=\int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,\underline{r}\,drd\theta$

extra r: read book for more

Ex.) Evaluate $\iint_R xy\,dxdy$.



$=\int_0^{\pi/2}\int_0^1(r\cos\theta)(r\sin\theta)r\,drd\theta$
$=\int_0^{\pi/2}\int_0^1\frac{r^3\sin 2\theta}{2}drd\theta$
$=\int_0^{\pi/2}\frac{r^4}{8}\sin 2\theta\Big|_{r=0}^{r=1}d\theta$
$=\frac{1}{8}\left(-\frac{\cos 2\theta}{2}\right)\Big|_0^{\pi/2}$
$=\frac{1}{8}\left(\frac{1}{2}+\frac{1}{2}\right)=\frac{1}{8}$

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60) Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2+y^2=4$, and the plane $z+y=3$.

$\cancel{z=y-3}$

$V=\int_0^2\int_0^{\sqrt{4-x^2}}(3-y)\,dy\,dx$

$=\int_0^2\left(3y-\frac{y^2}{2}\right)\Big|_0^{\sqrt{4-x^2}}\,dy\,dx$

62) Find the volume of the solid cut from the first octant by the surface $z=4-x^2-y^2$.

$x^2+y^2=4-z$

$V=\int_0^2\int_0^{\sqrt{4-x^2}}(4-x^2-y^2)\,dy\,dx$

$=\int_0^2\left(4y-x^2y-\frac{y^3}{3}\right)\Big|_0^{\sqrt{4-x^2}}\,dx$

Polar coordinates:

$R:\{(r,\theta): 0\le r\le 2,\ 0\le\theta\le\frac{\pi}{2}\}$

$V=\int_0^{\pi/2}\int_0^2(4-r^2)\,r\,dr\,d\theta$

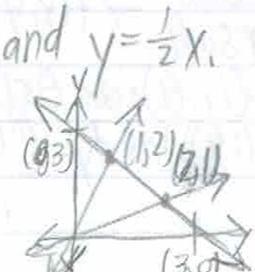
$=\int_0^{\pi/2}\left(2r^2-\frac{r^3}{3}\right)\Big|_0^2\,d\theta$

$=\int_0^{\pi/2}\left(8-\frac{8}{3}\right)d\theta$

$=\frac{16}{3}\theta\Big|_0^{\pi/2}$

$=\frac{8\pi}{3}$

Ex.) Find the area bounded by $y=2x$, $y=3-x$, and $y=\frac{1}{2}x$.



For $0\le x\le 1$, $\frac{1}{2}x\le y\le 2x$.

For $1\le x\le 2$, $\frac{1}{2}x\le y\le 3-x$.

$A=\int_0^1\int_{\frac{1}{2}x}^{2x}dy\,dx+\int_1^2\int_{\frac{1}{2}x}^{3-x}dy\,dx$.

Ex.) Find the average value of $f(x,y)=\sin(x+y)$ in $R:\{(x,y): 0\le x\le\frac{\pi}{2},\ 0\le y\le\frac{\pi}{2}\}$.

$\dfrac{\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dx\,dy}{\int_0^{\pi/2}\int_0^{\pi/2}dx\,dy}$

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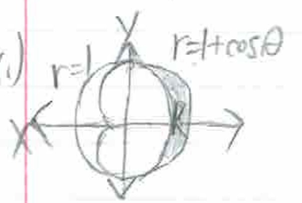
Ex.)



Find the area of R.

$A=\iint_R dA=\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}r\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\frac{r^2}{2}\Big|_{r=1}^{r=1+\cos\theta}d\theta$

$=\int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}(1+2\cos\theta+\cos^2\theta)-\frac{1}{2}\right]d\theta$

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$$= \int_{-\pi/2}^{\pi/2} \left(\cos\theta + \frac{1}{2}\cdot\frac{(1+\cos 2\theta)}{2}\right) d\theta$$
$$= \left(\sin\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/2}$$
$$= 1-(-1)+\frac{\pi}{8}-\left(-\frac{\pi}{8}\right)$$
$$= 2+\frac{\pi}{4}$$

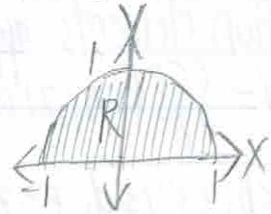
Ex.) $\iint_R e^{(x^2+y^2)}\,dx\,dy$, $R: \{(r,\theta): 0 \le \theta \le \pi, 0 \le r \le 1\}$

$= \int_0^{\pi}\int_0^1 e^{r^2} r\,dr\,d\theta$ because $r^2 = x^2+y^2$

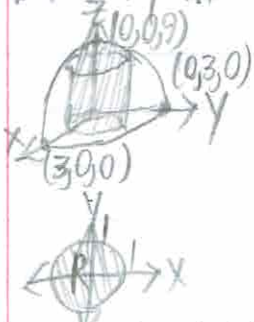
$= \int_0^{\pi} \frac{1}{2}e^{r^2}\Big|_{r=0}^{1} d\theta$

$= \int_0^{\pi} \frac{1}{2}(e-1)\,d\theta$

$= \frac{\pi}{2}(e-1)$



Ex.) Find the volume of the solid region bounded above by $z=9-x^2-y^2$ and below by the unit circle in the xy-plane.



$V = \iint_R (9-x^2-y^2)\,dx\,dy$, $R: \{(r,\theta): 0 \le \theta \le 2\pi, 0 \le r \le 1\}$

$= \int_0^{2\pi}\int_0^1 (9-r^2)\,r\,dr\,d\theta$

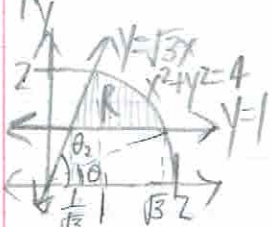
$= \int_0^{2\pi}\left(\frac{9r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 d\theta$

$= \frac{17}{4}\theta\Big|_0^{2\pi}$

$= \frac{17\pi}{2}$

Need to decide when switching from rectangular to polar coordinates is useful on a problem-by-problem basis.

Ex.) Find the area enclosed by the circle $x^2+y^2=4$, above $y=1$ and below $y=\sqrt{3}x$.



In Cartesian, this would be adding the integration from $x=\frac{1}{\sqrt{3}}$ to $x=1$ and the integration from $x=1$ to $x=\sqrt{3}$. However, only one integration is needed in polar coordinates.

$\tan\theta_1 = \frac{1}{\sqrt{3}} \Rightarrow \theta_1 = \frac{\pi}{6}$ $\quad \tan\theta_2 = \sqrt{3} \Rightarrow \theta_2 = \frac{\pi}{3}$

For $y=1$: $r\sin\theta = y = 1 \Rightarrow r = \frac{1}{\sin\theta} = \csc\theta$

$\Rightarrow R: \{(r,\theta): \frac{\pi}{6} \le \theta \le \frac{\pi}{3}, \csc\theta \le r \le 2\}$

$A = \iint_R dA = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta = \int_{\pi/6}^{\pi/3} \frac{r^2}{2}\Big|_{r=\csc\theta}^{r=2} d\theta = \int_{\pi/6}^{\pi/3}\left(2-\frac{\csc^2\theta}{2}\right)d\theta = 2\theta+\frac{1}{2}\cot\theta\Big|_{\pi/6}^{\pi/3}$

$= 2\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+\frac{1}{2}\left(\cot\left(\frac{\pi}{3}\right)-\cot\left(\frac{\pi}{6}\right)\right) = 2\left(\frac{\pi}{6}\right)+\frac{1}{2}\left(\frac{1}{\sqrt{3}}-\sqrt{3}\right) = \frac{\pi}{3}-\frac{1}{\sqrt{3}}$

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<u>Triple Integrals:</u>

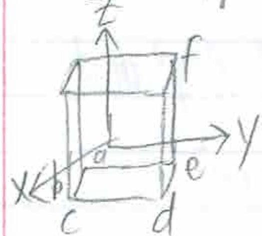
Let $f(x,y,z)$ be defined over a closed, bounded region D.

We take a partition element ΔV_k and a point (x_k, y_k, z_k) in ΔV_k. Then we calculate the sum

$\sum_k (x_k, y_k, z_k)\Delta V_k$ with the limit as the size of the partition elements approaches zero to get

$$\iiint_D f(x,y,z)\,dV = \iiint_D f(x,y,z)\,dx\,dy\,dz.$$

If $D=\{(x,y,z) : (a \le x \le b,\ c \le y \le d,\ e \le z \le f)$, (shaped like a box)



$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$