MATH 2605-G2 10/17/14

Similarly (if x is held constant), $\frac{\partial f}{\partial y}(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$.

Ex.) Calculate the partial derivatives of $f(x,y)=2x^2y^3+x^4-y^2+3y\sin x$.

$\frac{\partial f}{\partial x}(x,y)=4xy^3+4x^3+3y\cos x$ $\qquad$ $\frac{\partial f}{\partial y}(x,y)=6x^2y^2-2y+3\sin x$

treat y as constant $\qquad$ treat x as constant

10/20/14

Ex.) $f(x,y)=xy^2\arctan(xy)$

$\frac{\partial f}{\partial x}=y^2\left[x\left(\frac{1}{1+(xy)^2}\right)\frac{\partial}{\partial x}(xy)+\arctan(xy)\frac{\partial}{\partial x}(x)\right]=y^2\left[\frac{xy}{1+(xy)^2}+\arctan(xy)\right]$

$\frac{\partial f}{\partial y}=y^2(x)\left(\frac{x}{1+(xy)^2}\right)+2xy\arctan(x,y)$ $\leftarrow$ Product rule, $\frac{\partial}{\partial u}\arctan(u)=\frac{1}{1+u^2}$

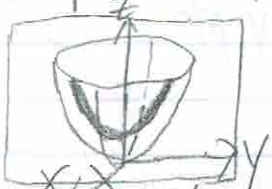
Ex.) Find $\frac{\partial z}{\partial x}$ if $z(x,y)$ is defined by $yz-\ln z=xy$.

$\frac{\partial}{\partial x}(yz-\ln z)=\frac{\partial}{\partial x}(xy)$ $\leftarrow$ Implicit differentiation

$y\frac{\partial z}{\partial x}-\frac{1}{z}\frac{\partial z}{\partial x}=y$

$\frac{\partial z}{\partial x}=\frac{y}{y-\frac{1}{z}}=\frac{yz}{yz-1}$

Ex.) The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola:



Find the slope of the tangent to the parabola at $(1,2,5)$.
x is constant, so the slope is $\frac{\partial z}{\partial y}(1,2)=2y|_{(1,2)}=4$.

With the addition of more variables, more partial derivatives are possible.

Ex.) $f(x,y,z)=g(x,y)h(y,z)$

$\frac{\partial f}{\partial x}=\frac{\partial g}{\partial x}(x,y)h(y,z)$

$\frac{\partial f}{\partial y}=\frac{\partial g}{\partial y}(x,y)h(y,z)+g(x,y)\frac{\partial h}{\partial z}(y,z)$

$\frac{\partial f}{\partial z}=g(x,y)\frac{\partial h}{\partial z}(y,z)$

MATH 2605-G2

10/20/14

Second order partials:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = (f_x)_x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x$$

Theorem: If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined on an open set containing (x_0, y_0) and continuous on (x_0, y_0), then

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

Higher order partials:

Ex.) $f(x,y,z) = 1 - 2xy^2z^3 + x^3y$. Find $\frac{\partial^4 f}{\partial z \partial y \partial x \partial y}$.

$$\frac{\partial f}{\partial y} = -4xyz^3 + x^3$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -4yz^3 + 3x^2$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = -4z^3$$

$$\frac{\partial}{\partial z}\left(\frac{\partial^3 f}{\partial y \partial x \partial y}\right) = -12z^2.$$

Since $f(x,y,z)$ is a continuous polynomial function, the theorem states that the partial derivatives can be taken in any order

Chain rule:

for $f: \mathbb{R} \to \mathbb{R}$: $\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$] review (one variable)

if $w = f(x(t))$, $\frac{dw}{dt} = f'(x(t))\,x'(t)$.

for functions of two variables:

$w = f(x,y)$ $\quad x = x(t)$ $\quad y = y(t)$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Ex.) $w = xy$, $x = \cos t$, $y = \sin t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = y(-\sin t) + x\cos t = \sin t(-\sin t) + \cos t(\cos t) = \cos 2t$$

MATH 2605-G2 10/20/14

Directional derivatives and gradient:

Consider $f(x,y)$ and (x_0,y_0) in its domain. Let $\vec{u}$ be a unit vector.

The rate of change of f at (x_0,y_0) in the direction of $\vec{u}$ is

$$\left(\frac{df}{ds}\right)\Big|_{\vec{u}}^{(x_0,y_0)} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2)-f(x_0,y_0)}{s}$$

The line that goes through (x_0,y_0) and has $\vec{u}$ as direction vector is

$$\vec{r}=(x(s), y(s)) = (x_0,y_0)+s\vec{u} = (x_0+su_1, y_0+su_2)$$

Then, from the chain rule:

$$\left(\frac{df}{ds}\right)_{\vec{u}}^{(x_0,y_0)} = \frac{\partial f}{\partial x}\frac{dx}{ds}+\frac{\partial f}{\partial y}\frac{dy}{ds} = \frac{\partial f}{\partial x}u_1+\frac{\partial f}{\partial y}u_2 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot(u_1,u_2)$$

Gradient of f: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Directional derivative in direction of $\vec{u}$: $\left(\frac{\partial f}{\partial s}\right)\Big|_{(x_0,y_0)}^{\vec{u}} = \nabla f\cdot\vec{u}$