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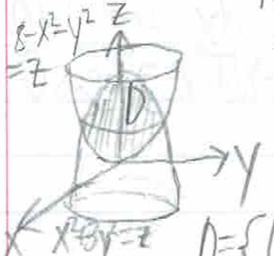
EX.) Find the volume of the region D enclosed by the surfaces $z=x^2+3y^2$ and $z=8-x^2-y^2$.

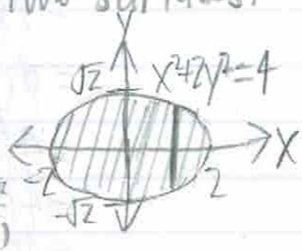


Intersection between the two surfaces:

$z=x^2+3y^2=8-x^2-y^2$

$\Leftrightarrow x^2+2y^2=4$

$\Leftrightarrow y=\pm\sqrt{\frac{4-x^2}{2}}$



$D=\{(x,y,z): -2\le x\le 2, -\sqrt{\frac{4-x^2}{2}}\le y\le\sqrt{\frac{4-x^2}{2}}, x^2+3y^2\le z\le 8-x^2-y^2\}$

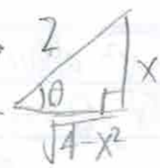
$V=\iiint_D dV=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$

$=\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}(8-2x^2-4y^2)\,dy\,dx$

$=\int_{-2}^{2}\left(8y-2x^2y-\frac{4y^3}{3}\right)\Big|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}dx$

$=\int_{-2}^{2}\left(4\sqrt{4-x^2}-x^2\sqrt{4-x^2}-\frac{1}{6}\sqrt{4-x^2}^3+4\sqrt{4-x^2}-x^2\sqrt{4-x^2}-\frac{1}{6}\sqrt{4-x^2}^3\right)dx$

$=\frac{4\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{3/2}dx \longrightarrow$



$\sin\theta=\frac{x}{2}\Rightarrow\theta=\sin^{-1}\left(\frac{x}{2}\right)$

$dx=2\cos\theta\,d\theta$

$\sqrt{4-x^2}=2\cos\theta$

$=\frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}(2\cos\theta)^3(2\cos\theta\,d\theta)\longleftarrow$

$=\frac{4\sqrt{2}}{3}(16)\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$

$=\frac{64\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$

$=\frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$

$=\frac{16\sqrt{2}}{3}\left(\frac{3\theta}{2}+\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\Big|_{-\pi/2}^{\pi/2}$

$=\frac{16\sqrt{2}}{3}\left(\frac{3}{2}\right)\left(\frac{\pi}{2}+\frac{\pi}{2}\right)$

$=8\sqrt{2}\pi$

Recitation

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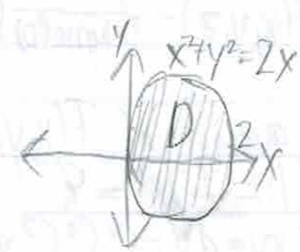
Parametrize the region by polar coordinates and find the area.

1) The region enclosed by $x^2+y^2=2x$.

$\Leftrightarrow (x-1)^2+y^2=1$ $\qquad \Leftrightarrow r^2=2r\cos\theta\Rightarrow r=2\cos\theta$

$A=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}2\cos^2\theta\,d\theta=\pi$



2) The region enclosed by $x^2+y^2=4$ and $x=1$.

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$r^2=4 \Rightarrow r=2;\ x=1 \Rightarrow r\cos\theta=1$

$A=\int_1^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx=\int_1^2 2\sqrt{4-x^2}\,dx$

$\sin\theta=\frac{x}{2}$
$dx=2\cos\theta\, d\theta$
$\sqrt{4-x^2}=2\cos\theta$

$=2\int_{\sin^{-1}\frac{1}{2}}^{\sin^{-1}1} 2\cos\theta(2\cos\theta\, d\theta)$

$=8\int_{\pi/6}^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)d\theta$

$=4\left(\theta+\frac{\sin2\theta}{2}\right)\Big|_{\pi/6}^{\pi/2}$

$=4\left(\frac{\pi}{2}+0-\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$

$=\frac{4\pi}{3}-\sqrt{3}$

OR: $\cos\theta=\frac{1}{r_{max}}=\frac{1}{2}\Rightarrow -\frac{\pi}{3}\le\theta\le\frac{\pi}{3}$ $\quad r_{min}=\frac{1}{\cos\theta}\Rightarrow\frac{1}{\cos\theta}\le r\le 2$

$A=\int_{-\pi/3}^{\pi/3}\int_{1/\cos\theta}^{2} r\,dr\,d\theta=\int_{-\pi/3}^{\pi/3}\left(2-\frac{1}{2\cos^2\theta}\right)d\theta$

$=\left(2\theta-\frac{1}{2}\tan\theta\right)\Big|_{-\pi/3}^{\pi/3}$

$=\frac{2\pi}{3}-\frac{1}{2}(\sqrt{3})+\frac{2\pi}{3}-\frac{1}{2}(\sqrt{3})$

$=\frac{4\pi}{3}-\sqrt{3}$

3) Find the average height of $z=\sqrt{a^2-x^2-y^2}$ above the disk $x^2+y^2\le a^2$.
$D:\{(x,y,z): -a\le x\le a,\ -\sqrt{a^2-x^2}\le y\le\sqrt{a^2-x^2},\ 0\le z\le\sqrt{a^2-x^2-y^2}\}$

$\text{Average}=\dfrac{\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx}{\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx}$

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Review of test:
A set that is both open and closed would be $\mathbb{R}^n$ in $\mathbb{R}^n$ (the entire space).
A set that is neither open nor closed would contain some but not all of the boundary points.

Average of a function in Space:
Avg. of $f(x,y,z)=\frac{1}{\text{Volume}(D)}\iiint_D f(x,y,z)\,dV$

Ex.) Find the average of $f(x,y,z)=xyz$ throughout $0\le x\le 2,\ 0\le y\le 2,\ 0\le z\le 2$.
Volume$(D)=2\cdot2\cdot2=8$
and $\iiint_D xyz\,dV=\int_0^2\int_0^2\int_0^2 xyz\,dz\,dy\,dx$

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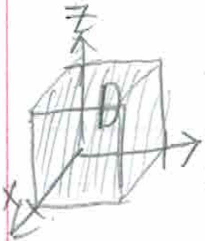


$= \int_0^2\int_0^2 xy\frac{z^2}{2}\Big|_{z=0}^{z=2}\,dy\,dx$ $\quad = \left(\int_0^2 x\,dx\right)\left(\int_0^2 y\,dx\right)\left(\int_0^2 z\,dz\right)$

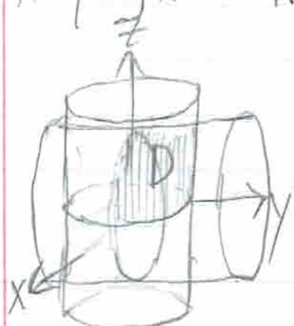
$= \int_0^2\int_0^2 2xy\,dy\,dx$

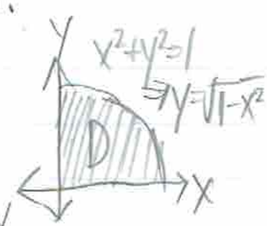
$= \int_0^2 2x(2)\,dx$ following the same pattern.

$= 2(2)(2) = 8$

Then the average is $\frac{8}{8} = 1$.

Ex) Find the volume of the region common to the interior of the cylinders $x^2+y^2=1$, $x^2+z^2=1$.





First find volume of first octant.

$D: \{(x,y,z): 0\le x\le 1,\ 0\le y\le\sqrt{1-x^2},\ 0\le z\le\sqrt{1-x^2}\}$

$\text{Volume}(D) = \iiint_D 1\,dV$

$= \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx$

$= \int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy\,dx$

$= \int_0^1 (1-x^2)\,dx$

$= x - \frac{x^3}{3}\Big|_0^1 = 1-\frac{1}{3} = \frac{2}{3}$

Since $\frac{2}{3}$ is the volume of the first octant, the total volume is $8\left(\frac{2}{3}\right) = \frac{16}{3}$.