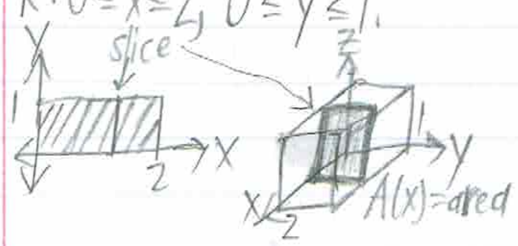
MATH 2605-G2 10/31/14

Ex.) Calculate the volume under the plane $z=4-x-y$ over the rectangle $R: 0 \le x \le 2,\ 0 \le y \le 1$.



Consider a slice with a region perpendicular to x-axis. If the area of the slice for each fixed x is A, then the volume is $V=\int_0^2 A(x)\,dx$.

Now, $A(x)=\int_0^1 f(x,y)\,dy$ for each fixed x.

Finally, $V=\int_0^2\left(\int_0^1 f(x,y)\,dy\right)dx$
$=\int_0^2\int_0^1 (4-x-y)\,dy\,dx$
$=\int_0^2 \left(4y-xy-\frac{y^2}{2}\right)\Big|_{y=0}^{y=1} dx$
$=\int_0^2 \left(4-x-\frac{1}{2}-0\right)dx$
$=\left(4x-\frac{x^2}{2}-\frac{x}{2}\right)\Big|_{x=0}^{x=2}$
$=5.$

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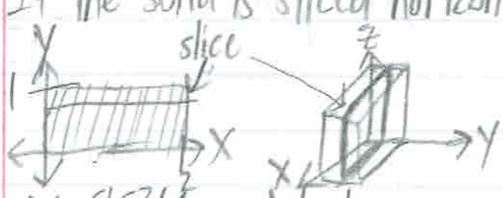
If the solid is sliced horizontally into regions perpendicular to the y-axis:

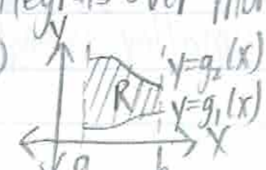


If the area of the slice for each fixed y is A, then the volume is $V=\int_0^1 A(y)\,dy$. Since $A(y)=\int_0^2 f(x,y)\,dy$,

$V=\int_0^1\int_0^2 (4-x-y)\,dx\,dy$
$=\int_0^1 \left(4x-\frac{x^2}{2}-xy\right)\Big|_{x=0}^{x=2} dy$
$=\int_0^1 (8-2-2y)\,dy$
$=(6y-y^2)\Big|_0^1$
$=5.$

<u>Fubini's Theorem</u>: If $f(x,y)$ is continuous through a rectangle $R: a\le x\le b$, $c\le y\le d$:
$$\iint_R f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy$$

Integrals over more general regions:

I 

$R=\{(x,y): a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$ ← vertical slices
$$\iint_R f(x,y)\,dx\,dy=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

II 

$R=\{(x,y): c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$ ← horizontal slices
$$\iint_R f(x,y)\,dx\,dy=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

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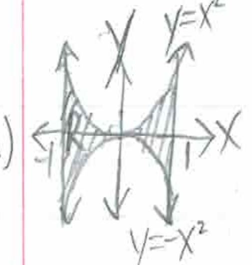
Ex.)



$\iint_R (xy+5)\,dx\,dy$ $R=\{(x,y): -1\le x\le 1, -x^2\le y\le x^2\}$

$= \int_{-1}^{1}\int_{-x^2}^{x^2} (xy+5)\,dy\,dx$

$= \int_{-1}^{1} \left(\frac{xy^2}{2}+5y\right)\Big|_{y=-x^2}^{y=x^2} dx$

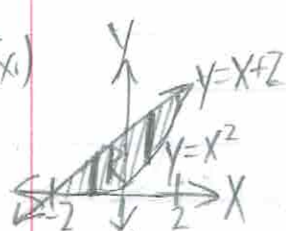
$= \int_{-1}^{1} \left[\frac{x(x^2)^2}{2}+5(x^2)-\left(x\frac{(-x^2)^2}{2}+5(-x^2)\right)\right] dx$

$= \int_{-1}^{1} 10x^2\,dx$

$= \frac{10x^3}{3}\Big|_{-1}^{1}$

$= \frac{20}{3}$

Ex.)

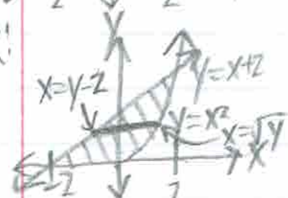


$\iint_R (xy-y^3)\,dy\,dx$ $x^2=x+2 \iff x^2-x-2=0 \iff (x-2)(x+1)=0$

$= \int_{-2}^{0}\int_{0}^{x+2} (xy-y^3)\,dy\,dx + \int_{0}^{2}\int_{x^2}^{x+2} (xy-y^3)\,dy\,dx$

$R=\{(x,y): -2\le x\le 0, 0\le y\le x+2\} \cup \{(x,y): 0\le x\le 2, x^2\le y\le x+2\}$

OR:



$\iint_R (xy-y^3)\,dx\,dy$ $R=\{(x,y): 0\le y\le 4, y-2\le x\le \sqrt{y}\}$

$= \int_{0}^{4}\int_{y-2}^{\sqrt{y}} (xy-y^3)\,dx\,dy$

$= \int_{0}^{4} \left(\frac{x^2}{2}y-xy^3\right)\Big|_{x=y-2}^{x=\sqrt{y}} dy$

$= \int_{0}^{4} \left[\frac{y}{2}y-\sqrt{y}\,y^3-\frac{(y-2)^2}{2}y-(y-2)y^3\right] dy$