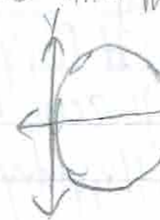
MATH 2605-G2

11/18/14

1) D is the right circular cylinder whose base is $r=3\cos\theta$ and whose top lies in the plane $z=5-x$. Find the volume of D.



$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $0\le r\le 3\cos\theta$

$$V=\int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} r\,dz\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta}(5r-r^2\cos\theta)\,dr\,d\theta$$ (cont. below)

2) Find the volume of the unit sphere.

$$V=\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}\frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{1}d\phi\,d\theta=\frac{1}{3}\int_0^{2\pi}-\cos\phi\Big|_0^{\pi}=\frac{4\pi}{3}$$

3) Find the volume of the region bounded below by $z=x^2+y^2$, laterally by $x^2+y^2=1$, and above by $z=x^2+y^2+1$.

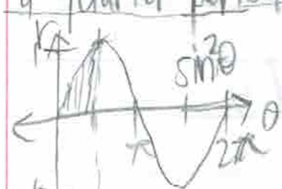
$$V=\int_0^1\int_0^{2\pi}\int_{x^2+y^2}^{x^2+y^2+1}dz\,d\theta\,dr=\pi \Longleftrightarrow$$

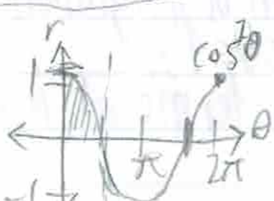
$\leftarrow V=\pi(1)^2=\pi$

1) (cont.) $$V=\int_{-\pi/2}^{\pi/2}\left(\frac{5r^2}{2}-\frac{r^3}{3}\cos\theta\right)\Big|_{r=0}^{3\cos\theta}d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left(\frac{45\cos^2\theta}{2}-9\cos^4\theta\right)d\theta$$

Technique for integrals of powers of $\sin\theta$ and $\cos\theta$ when the range of θ is a quarter period:





$$\int_0^{\pi/2}\sin^2\theta\,d\theta=\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$\int_{-\pi/2}^{0}\sin^2\theta\,d\theta=-\int_{-\pi/2}^{0}\cos^2\theta\,d\theta$$

$$\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta=\int_0^{\pi/2}\cos^2\theta\,d\theta=\frac{\pi}{4}\leftarrow$$

$$\int_0^{\pi/2}\cos^2\theta\,d\theta=\frac{1}{2}\int_0^{\pi/2}(\sin^2\theta+\cos^2\theta)\,d\theta=\frac{1}{2}\int_0^{\pi/2}d\theta=\frac{\pi}{4}$$

$$=2\int_0^{\pi/2}\left(\frac{45}{2}\cos^2\theta-9\cos^4\theta\right)d\theta$$

$$=2\left(\frac{45}{2}\cdot\frac{\pi}{4}-\int_0^{\pi/2}\left(\frac{1-\cos^2 2\theta}{2}\right)d\theta\right)$$

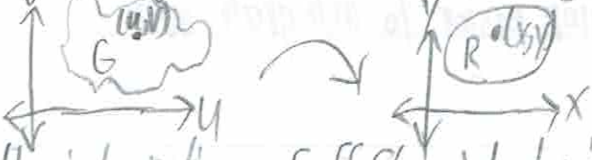
$$=\frac{45\pi}{4}-\int_0^{\pi/2}1-\cos^2 2\theta\,d\theta$$

$$=\frac{45\pi}{4}-\theta\Big|_0^{\pi/2}+0=\frac{43\pi}{4}$$

11/19/14

Substitutions in multiple integrals:

If a change of coordinates $(x=g(u,v),\ y=h(u,v))$ is introduced:



the integration of $\iint_R f(x,y)\,dx\,dy$ in terms of (u,v) is as follows:

Define the Jacobian of the coordinate transformation $x=g(u,v)$, $y=h(u,v)$ as

MATH 2605-G2 11/19/14

$$J(u,v) = \det\begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

<u>Theorem</u>: If $f(x,y)$, $g(u,v)$, $h(u,v)$ have continuous partial derivatives and $J(u,v)$ is zero only at isolated points, then

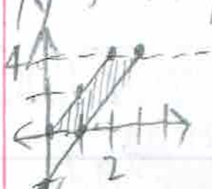
$$\iint_R f(x,y)\,dxy = \iint_G f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

Ex.) Change to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

This is why $\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

Ex.) Evaluate $\iint_D \frac{2x-y}{2}\,dx\,dy$, where D is the region bounded by $y=0$, $y=2x-2$, $y=2x$, and $y=4$.



Boundary: $y=0$, $y=4$, $2x-y=0$, $2x-y=2$

Propose the change of coordinates $(u=2x-y,\ v=y)$.

In these coordinates, the region of integration is

$$G = \{(u,v) : 0 \le u \le 2,\ 0 \le v \le 4\}.$$

Now solve for x and y in terms of u and v.

$$y = v,\quad x = \frac{u+v}{2}$$

$$J(u,v) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \frac{1}{2}$$

Then: $\iint_G uv\,|J(u,v)|\,du\,dv = \int_0^2\int_0^4 \frac{u}{2}\left|\frac{1}{2}\right| dv\,du$

$$= \frac{1}{4}\int_0^2 uv\Big|_0^4 du$$

$$= \frac{1}{4}\int_0^2 4u\,du$$

$$= \frac{u^2}{2}\Big|_0^2$$

$$= 2.$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

How to know the correct transformation (u,v) to use? Come up with a linear transformation matrix that makes the region easier to integrate over.

Ex.) $\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

Use $(u,v) = (x+y,\ y-2x)$.

$$\Rightarrow (x,y) = \left(\frac{u-v}{3}, \frac{2u+v}{3}\right)$$

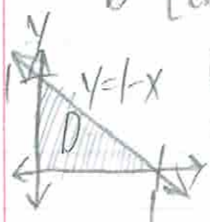
MATH 2605 - G2

11/19/14

$J(u,v) = \det\begin{pmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{pmatrix} = \frac{1}{3}$

To find the boundary, transform the original D:

$D = \{(x,y): 0 \le x \le 1, 0 \le y \le 1-x\} \Rightarrow \underset{①}{x=0}, \underset{②}{y=0}, \underset{③}{x=1}, \underset{④}{y=1-x}$



① $\frac{u-v}{3} = 0 \Rightarrow u = v$

② $\frac{2u+v}{3} = 0 \Rightarrow u = \frac{-v}{2}$

③ $1 = \frac{u-v}{3} \Rightarrow u = 3+v$

④ $\frac{2u+v}{3} = 1 - \left(\frac{u-v}{3}\right) \Rightarrow u = 1$