MATH 2605-G2

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$\vec{r}(t) = \frac{t}{t+1}\hat{i} + \frac{1}{t}\hat{j} - 2t\hat{k}$

Ex.) Find the velocity and acceleration vectors at $t=-\frac{1}{2}$. Then draw the path.

Take the derivative of each component separately:

$r'(t) = \frac{t+1-t}{(t+1)^2}\hat{i} + \frac{-1}{t^2}\hat{j} - 2\hat{k} = \frac{1}{(1+t)^2}\hat{i} - \frac{1}{t^2}\hat{j} - 2\hat{k}$

$$\boxed{\begin{array}{l} d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \\ d(u^n) = nu^{n-1}du \end{array}}$$

$\vec{v}(-\frac{1}{2}) = \vec{r}'(-\frac{1}{2}) = \frac{1}{(\frac{1}{2})^2}\hat{i} - \frac{1}{(-\frac{1}{2})^2}\hat{j} - 2\hat{k} = \langle 4, -4, -2\rangle$

$r''(t) = \frac{-2}{(1+t)^3}\hat{i} + \frac{2}{t^3}\hat{j} + 0\hat{k}$

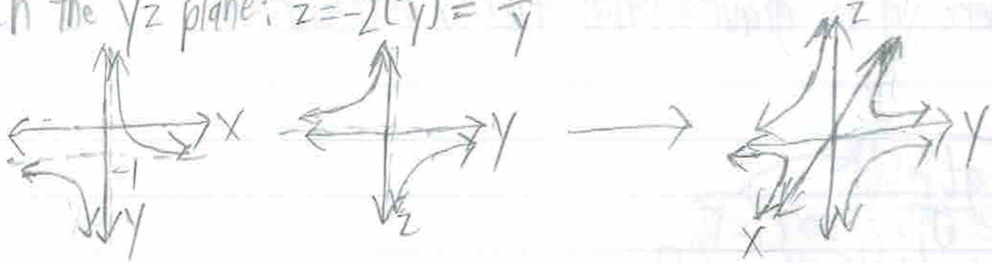
$\vec{a}(-\frac{1}{2}) = r''(-\frac{1}{2}) = \frac{-2}{(\frac{1}{2})^3}\hat{i} + \frac{2}{(-\frac{1}{2})^3}\hat{j} = \langle -16, -16, 0\rangle$

To draw the path:

$x = \frac{t}{t+1}$, $y = \frac{1}{t}$, $z = -2t$

In the xy plane, $t = \frac{1}{y}$, so $x = \frac{\frac{1}{y}}{\frac{1}{y}+1} = \frac{1}{1+y}$. Then $y = \frac{1}{x} - 1$.

In the yz plane: $z = -2(\frac{1}{y}) = -\frac{2}{y}$



<u>Differentiation Rules:</u> $\vec{u}(t)$, $\vec{v}(t)$ are vector functions, $\vec{c}$ is constant vector

1) $\frac{d}{dt}(\vec{c}) = \vec{0}$
2) $\frac{d}{dt}(c\vec{u}(t)) = c\frac{d\vec{u}}{dt}$, $c \in \mathbb{R}$
3) $\frac{d}{dt}[f(t)\vec{v}(t)] = f'(t)\vec{v}(t) + f(t)\vec{v}'(t)$
4) $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt}$
5) $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$
6) $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$
7) $\frac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

Integrals: If $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$, then

$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\hat{i} + \left(\int_a^b g(t)\,dt\right)\hat{j} + \left(\int_a^b h(t)\,dt\right)\hat{k}$.

Ex.) $\vec{a} = \langle -3\cos t, -3\sin t, 2\rangle$ Find $\vec{r}(t)$. (next page)

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$\vec{v}(t) = \int \vec{a}(t)dt = (\int -3\cos t\,dt)\hat{i} + (\int -3\sin t\,dt)\hat{j} + (\int 2\,dt)\hat{k}$

$= -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 2t\hat{k} + \vec{c}_1$ ← don't forget integration constant

$\vec{r}(t) = \int \vec{v}(t)dt = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + t^2\hat{k} + t\vec{c}_1 + \vec{c}_2$

$\vec{c}_1$ and $\vec{c}_2$ depend on initial conditions.

Initial conditions: $\vec{r}(0) = (3,0,0)$ and $\vec{v}_0 = (0,3,0)$

Then: $\vec{v}(0) = 0\hat{i} + 3\hat{j} + 0\hat{k} + \vec{c}_1 = (0,3,0) \Rightarrow \vec{c}_1 = \vec{0}$

$\vec{r}(0) = 3\hat{i} + 0\hat{j} + 0\hat{k} + \vec{c}_2 = (3,0,0) \Rightarrow \vec{c}_2 = \vec{0}$

Therefore: $\vec{r}(t) = (3\cos t, 3\sin t, t^2)$. The graph is a helix with increasing pitch.

Ex.) Projectile motion: A particle at the origin moves with an initial velocity $\vec{v}_0$.

$\vec{r}(0) = \vec{0}$

$\vec{v}(0) = \vec{v}_0 = \|\vec{v}_0\|\cos\alpha\,\hat{i} + \|\vec{v}_0\|\sin\alpha\,\hat{j}$

$\|\vec{v}_0\|$ will now be denoted by v_0.

From Newton's 2nd Law:

$m\vec{r}\,'' = -mg\hat{j}$, where g is gravitational field/acceleration

$\vec{r}\,'' = -g\hat{j}$

Integrating:

$\vec{r}\,' = -gt\hat{j} + \vec{c}_1 = \boxed{-gt\hat{j} + \vec{v}_0}$

$\vec{r}\,'(0) = \vec{v}(0) = \vec{v}_0 = 0\hat{j} + \vec{c}_1 \Rightarrow \vec{c}_1 = \vec{v}_0$

$\vec{r}(t) = -\frac{gt^2}{2}\hat{j} + \vec{v}_0 t + \vec{c}_2 = \boxed{-\frac{gt^2}{2}\hat{j} + \vec{v}_0 t}$

$\vec{r}(0) = \vec{0} = \vec{c}_2$

Expanding $\vec{v}_0 t$:

$\vec{r}(t) = -\frac{gt^2}{2}\hat{j} + tv_0\cos\alpha\,\hat{i} + tv_0\sin\alpha\,\hat{j} = tv_0\cos\alpha\,\hat{i} + (tv_0\sin\alpha - \frac{gt^2}{2})\hat{j}$

Curve in xy plane (eliminate parameter t)

$x = tv_0\cos\alpha \quad y = tv_0\sin\alpha - \frac{gt^2}{2}$

$\Rightarrow t = \frac{x}{v_0\cos\alpha}$

$\Rightarrow y = x\tan\alpha - \frac{1}{2}g\left(\frac{x}{v_0\cos\alpha}\right)^2$

$= (\tan\alpha)x - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2$

Max. height: $y_{max} = \frac{(v_0\sin\alpha)^2}{2g}$ Flight time: $t = \frac{2v_0\sin\alpha}{g}$ Range: $R = \frac{v_0^2\sin\alpha}{g}$	← If proj. ← starts at ← $\vec{0}$

If projectile is fired from (x_0, y_0) instead of $\vec{0}$:

$\vec{r}(t) = (x_0, y_0) + (tv_0\cos\alpha)\hat{i} + (tv_0\sin\alpha - \frac{gt^2}{2})\hat{j}$

Ex.) $v_0 = 500$ m/s, $\alpha = 60°$, $\vec{r}(10) = ?$ (starts at origin)

$\vec{r}(10) = (10)(500)\cos(60°)\hat{i} + [(10)(500)\sin(60°) - \frac{1}{2}(9.8)(10^2)]\hat{j} = (2500, 3840)$ m