Least-Squares Filter Applied to a Temperature Control System using Simulink

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***Abstract*—Accurate temperature monitoring is crucial in numerous daily and industrial applications. To address this, the utilization of a least-squares filter to effectively mitigate noise in the measured variable is explored in this paper.**

For the study, a heating system was simulated using Simulink, incorporating Gaussian white noise into the output to mirror real-world disturbances. The system maintains a setpoint of 25°C, which represents the ambient temperature, and features an input pulse that activates the heating mechanism.

The findings indicate that the least-squares estimation technique successfully approximates the actual temperature with a high degree of accuracy, achieving results that closely match the true values.

***Keywords*—Control System, Least Squares Filter, Matlab, Simulink, Temperature**

I. Introduction

I-A. Heating and Temperature Control

Heating and temperature control systems have played a crucial role since the end of the 19th century. These systems have a wide range of applications from day-to-day tasks to industrial applications.

In 1895, Johnson made a significant breakthrough in temperature control with an automatic multi-zone temperature control system, designed to regulate the temperature in individual rooms or apartments [1]. Subsequentially, a variety of other applications have emerged. In the field of biotechnology, applications like Polymerase Chain Reaction (PCR)–such [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]– and Temperature Gradient Focusing (TGF) [17, 18] for Electrophoresis requires tight temperature control [19]. In biological science, precise temperature control is crucial, such as maintaining specific temperatures for cell viability or for PCR temperature cycling [19, 20]. Also, in the food supply chain is important to ensure product quality and customer satisfaction, in transportation modes like sea, land, and railway, and in the logistic clusterization process can significantly reduce costs [21]. On the other hand, in the automotive industry, thermal management in vehicle electrification enhances vehicle efficiency and battery performance [22, 23] and in specific areas like thermal management of lithium-ion batteries [24], electric machines [25], and novel thermal management systems for batteries [26].

In critical applications such as mentioned before, accurate temperature monitoring is imperative. Consequently, the utilization of a least-squares filter to effectively mitigate noise in the target variable is explored.

I-B. Least-Squares Filter

Least-squares estimation theory was introduced by Gauss and further developed by Kalman, focusing on minimizing the sum of the squares of the difference between observed and estimated values [27]. This methodology finds applications across a broad spectrum of categories, such as data curve fitting, parameter identification, and the realization of system models [28]. This technique is versatile, with applications spanning various domains. Examples include calculating the damping properties of a fluid-filled damper based on temperature, identifying aircraft dynamic and static aerodynamic coefficients, determining orbit and attitude, locating position using triangulation, identifying modes of vibratory systems, and modern control strategies like some adaptative controllers where least-squares method is used to refine model parameters within the control system [28].

Assuming a set or a batch of measured values, $\tilde{y}_j$, of a process $y(t)$, taken at known discrete instants of time t_j:

$$\{\tilde{y}_1, t_1; \tilde{y}_2, t_2; \ldots; \tilde{y}_m, t_m;\} \tag{1}$$

and a proposed mathematical model of the form

$$y(t) = \sum_{i=1}^{n} x_i h_i(t), \quad m \geq n \tag{2}$$

where

$$h_i(t) \in \{h_1(t), h_2(t), \ldots, h_n(t)\} \tag{3}$$

are a set of independently specified basis functions where the measurements $\tilde{y}_j$ and the estimated output $\hat{y}_j$ can be related to the true and the estimated x-values leading to the Eq 4 and 5 [28]:

$$\tilde{y}_j \equiv \tilde{y}(t_j) = \sum_{i=1}^{n} x_i h_i(t_j) + v_j, \quad j = 1, 2, \ldots, m \tag{4}$$

$$\hat{y}_j \equiv \hat{y}(t_j) = \sum_{i=1}^{n} \hat{x}_i h_i(t_j), \quad j = 1, 2, \ldots, m \tag{5}$$

where v_j is the measurement error. This leads to the following identity:

$$\tilde{y}_j = \sum_{i=1}^{n} \hat{x}_i h_i(t_j) + e_j, \quad j = 1, 2, \ldots, m \tag{6}$$

where residual error e_j is defined by

$$e_j \equiv \tilde{y}_j - \hat{y}_j \tag{7}$$

and this can be rewritten in compact matrix form as:

$$\tilde{\mathbf{y}} = H\hat{\mathbf{x}} + \mathbf{e} \tag{8}$$

where [28]

$\tilde{\mathbf{y}} = [\tilde{\mathbf{y}}_\mathbf{1} \ \tilde{\mathbf{y}}_\mathbf{2} \ \cdots \ \tilde{\mathbf{y}}_\mathbf{m}]$ = measured y-values

$\mathbf{e} = [\mathbf{e_1} \ \mathbf{e_2} \ \cdots \ \mathbf{e_m}]$ = resisdual errors

$\hat{\mathbf{x}} = [\hat{\mathbf{x}}_\mathbf{1} \ \hat{\mathbf{x}}_\mathbf{2} \ \cdots \ \hat{\mathbf{x}}_\mathbf{m}]$ = estimated x-values

In similar way, Eq 9 and 10 can be represented using its compact matrix form as [28]:

$$\tilde{\mathbf{y}} = H\mathbf{x} + \mathbf{v} \tag{9}$$

$$\hat{\mathbf{y}} = H\hat{\mathbf{x}} \tag{10}$$

where

$\mathbf{x} = [\mathbf{x_1} \ \mathbf{x_2} \ \cdots \ \mathbf{x_m}]$ = true x-values

$\mathbf{v} = [\mathbf{v_1} \ \mathbf{v_2} \ \cdots \ \mathbf{v_m}]$ = measurements errors

$\hat{\mathbf{y}} = [\hat{\mathbf{y}}_\mathbf{1} \ \hat{\mathbf{y}}_\mathbf{2} \ \cdots \ \hat{\mathbf{y}}_\mathbf{m}]$ = estimated y-values

$\tilde{\mathbf{y}} = [\tilde{\mathbf{y}}_\mathbf{1} \ \tilde{\mathbf{y}}_\mathbf{2} \ \cdots \ \tilde{\mathbf{y}}_\mathbf{m}]$ = measured y-values

I-B1. Linear Least Squares: The least squares principle selects particular $\hat{x}$ that minimizes the sum square of the residual errors as an optimum choice for the unknown parameters, given by [28]:

$$J = \frac{1}{2} e^T e \tag{11}$$

By substituting Eq 6 e into the Eq 11, applying the gradient of $\nabla_{\hat{x}} J$ and equaling to zero, the explicit solution for the optimal estimate is obtained:

$$\hat{\mathbf{x}} = (H_T H)^{-1} H_T \tilde{\mathbf{y}} \tag{12}$$

I-B2. Weighted Least Squares: In cases where measurements are taken with varying degrees of precision, the approach of assigning equal weight appears to be logically flawed. To incorporate proper weighting, a least squares criterion of the form is established:

$$J = \frac{1}{2} e^T W e \tag{13}$$

producing the solution for $\hat{x}$ given by

$$\hat{\mathbf{x}} = (H_T W H)^{-1} H_T \hat{W} \tilde{\mathbf{y}} \tag{14}$$

where W is an $m \times m$ symmetric matrix [28].

When data is not available for batch processing–like in numerous real-world applications–but it is sequentially available, one might find it advantageous to calculate fresh estimates by considering all past measurements. For such cases, the covariance recursion form is used [28]:

$$\hat{x}(k+1) = \hat{x}(k) + K(k+1)\left[\tilde{y}(k+1) - H(k+1)\hat{x}(k)\right] \tag{15}$$

where

$$K(k+1) = P(k)H^t(k+1)[H(k+1)P(k)H^T(k+1) + W_{-1}(k+1)]^{-1} \tag{16}$$

$$P(k+1) = \left[I - K(k+1)H(k+1)\right] P(k) \tag{17}$$

with initial values of

$$P(0) = \left[\frac{1}{\alpha^2} I + H_T(0) W(0) H(0)\right]^{-1} \tag{18}$$

$$\hat{x}(0) = P(0)\left[\frac{1}{\alpha}\beta + H_T(0) W(0) \hat{y}(0)\right] \tag{19}$$

I-C. System Model

Given a basic heating house system that considers the outdoor and indoor temperature, the dynamics of the systems are described by a first-order differential equation with a time constant τ [29]:

$$\dot{T}(t) = \frac{1}{\tau}\left(T_{ambient} - T(t)\right) + \frac{K_p}{\tau} \cdot u(t) \tag{20}$$

where

- τ is the time constant of the system.
- $T_{ambient}$ is the temperature outside the house.
- $T(t)$ is the temperature in the house at time t.
- K_p is the proportional gain or the heat rating of the system.
- $u(t)$ is the swith, $= 1$ if the system is on and $= 0$ if the system is off.

In the continuous-time state-space representation of a first-order system, the dynamics are described by the following first-order differential equation [30]:

$$\dot{x} = a \cdot x(t) + b \cdot u(t) \tag{21}$$

meaning that $a = 1/\tau$ and $b = K_p/\tau$.

For a first-order system with a continuous-time transfer function, the discrete-time representation can be obtained using methods such as Euler's method [31], backward difference [32], or Tustin's method [33]. The solution using Euler's methods is:

$$y(k+1) = \phi(k)y(k) + \Gamma(k)u(k) \tag{22}$$

where

$$\phi(k) = e^{a\cdot\Delta t} \tag{23}$$

$$\Gamma(k) = \int_0^{\Delta t} b e^{a\cdot t} dt = \frac{b}{a}\left(e^{a\cdot\Delta t} - 1\right) \tag{24}$$

and Δt is the sampling time.

II. Methodology

The simulation is conducted through Simulink, as depicted in Fig 1 the heating or control temperature system. This system comprises four primary components: the heating system, the P_0 computation, the x_0 calculation, and the least-squares filter implemented as a Matlab function.

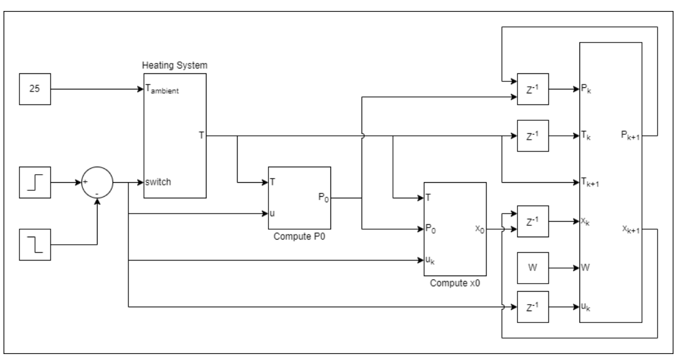

Figure 1. Diagram representing the complete system to be simulated

The heating system is responsible for replicating the temperature variation, using an ambient temperature of 25°C as a reference. Initially, the simulation begins with a transition from 0°C to 25°C. Subsequently, the heating system activates, initiating a temperature alteration over a 150-second interval. Following this, the system deactivates and the simulation runs for an additional 150-second period.

The component that computes P_0 uses the Eq 18. The following Matlab code shows its implementation:

```
function P0 = compute_p0(alpha, T, u, W)
I = eye(2);
H = [T u];
Ht = transpose(H);

P0 = inv([1/alpha^2 * I + Ht * W * H]);
```

In the other hand, Eq 19 is used to compute the value of x_0 and the implemented code is shown:

```
function x0 = compute_x0(alpha, beta, P0, T, u, W)
H = [T u];
Ht = transpose(H);

x0 = P0 * (1 / alpha * beta + Ht * W * T);
```

Finally, the least-squares estimation using the covariance recursion form based on Eq 15 is implemented as shown below:

```
function [Pk1, xk1] = compute_pk1(Pk, Tk, Tk1, xk,
    uk, W)
H = [Tk uk];
Ht = transpose(H);
I = eye(2);

K = Pk * Ht * inv(H * Pk * Ht + inv(W));
xk1 = xk + K * (Tk1 - H * xk);
Pk1 = (I - K * H) * Pk;
```

For the simulation, the following parameters are used:

- τ: the time constant of the system is set to 20s.
- K_p: the gain of the system is set to 0.5.
- σ: the standard deviation of the Gaussian white noise is set to 0.008.
- α: is set to 10^3
- β: is vector set to $[0.01\ 0.01]$.
- Δt: the sampling time is set to 0.1 seconds.

and the code used is shown, as well as the computation of the parameters a, b, and W:

```
% Contants
tau = 20;
Kp = 0.5;
sigma = 0.08;
alpha = 1E3;
beta = [1E-2; 1E-2];
delta_t = 0.1;

% Variables
a = -1 / tau;
b = Kp / tau;
W = 1 / sigma^2;
```

where:

$$a = \frac{-1}{\tau} = \frac{-1}{20} = -0.05 \tag{25}$$

$$b = \frac{K_p}{\tau} = \frac{0.5}{20} = 0.025 \tag{26}$$

$$W = \frac{1}{\sigma^2} = \frac{1}{0.08^2} = 156.25 \tag{27}$$

The system predicts the parameters ϕ and Γ which can be computed using the Eq 23 and 24:

$$\phi(k) = e^{a\cdot\Delta t} = e^{-0.05\cdot 0.1} = 0.9950 \tag{28}$$

$$\Gamma(k) = \frac{0.025}{-0.05}\left(e^{-0.05\cdot 0.1} - 1\right) = 0.0025 \tag{29}$$

III. Results and Discussion

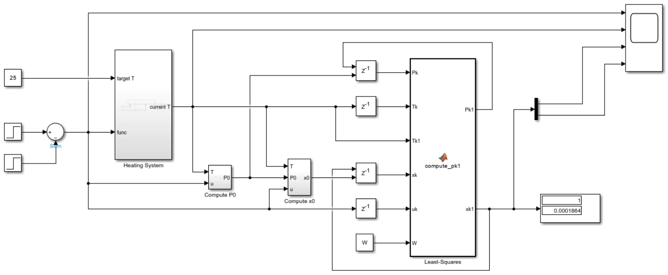

Figure 2.

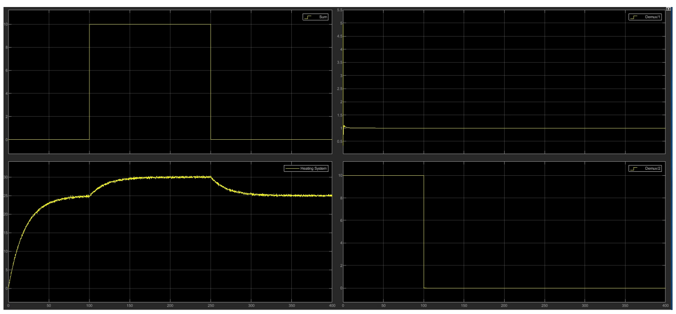

Figure 3.

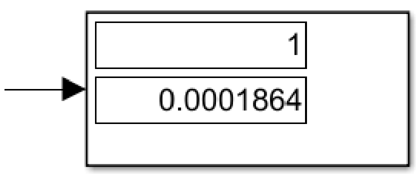

Figure 4.

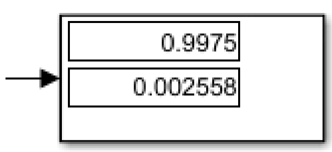

Figure 5.

IV. Conclusions

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