

# Math 6040/7260 Linear Models

Mon/Wed/Fri 11:00am - 11:50am

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## 1 Lecture 1: Jan 26

### Today

- Introduction
- Introduce yourself
- Course logistics

### What is this course about?

The term “linear models” describes a wide class of methods for the statistical analysis of multivariate data. The underlying theory is grounded in linear algebra and multivariate statistics, but applications range from biological research to public policy. The objective of this course is to provide a solid introduction to both the theory and practice of linear models, combining mathematical concepts with realistic examples.

### Prerequisite

- **Must:** Introduction to Probability
- **Good to have:** Mathematical Statistics, Scientific Computation II

### A hierarchy of linear models

- The linear mean model:

$$\underset{n \times 1}{\mathbf{y}} = \underset{n \times p}{\mathbf{X}} \underset{p \times 1}{\boldsymbol{\beta}} + \underset{n \times 1}{\boldsymbol{\epsilon}}$$

where  $\mathbf{E}(\boldsymbol{\epsilon}) = \mathbf{0}$ . Only assumption is that errors have mean 0.

- Gauss-Markov model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where  $\mathbf{E}(\boldsymbol{\epsilon}) = \mathbf{0}$  and  $\mathbf{Var}(\boldsymbol{\epsilon}) = \sigma^2 \mathbf{I}$ . Uncorrelated errors with constant variance.

- Aitken model or general linear model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where  $\mathbf{E}(\boldsymbol{\epsilon}) = \mathbf{0}$  and  $\mathbf{Var}(\boldsymbol{\epsilon}) = \sigma^2 \mathbf{V}$ .  $\mathbf{V}$  is fixed and known.

- Variance components models:  $\mathbf{y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma_1^2 \mathbf{V}_1 + \sigma_2^2 \mathbf{V}_2 + \cdots + \sigma_r^2 \mathbf{V}_r)$  with  $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_r$  known.

- General mixed linear Model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where  $\mathbf{E}(\boldsymbol{\epsilon}) = \mathbf{0}$  and  $\mathbf{Var}(\boldsymbol{\epsilon}) = \boldsymbol{\Sigma}(\theta)$ .

- Generalized linear models (GLMs). Logistic regression, probit regression, log-linear model (Poisson regression), ... Note the difference from the general linear model. GLMs are generalization of the *concept* of linear models. They are covered in Math 7360 - Data Analysis class (<https://tulane-math-7360-2021.github.io/>).

## Syllabus

Check course website frequently for updates and announcements.

<https://tulane-math-7260-2022.github.io/>

## HW submission

Through Github with demo on Friday class.

## Presentations

Let me know your pick by the end of Friday (01/28/2022).

## Last year comments

### 1. Experience in this course

- Overall, I had a pretty good experience in this course. It moved quickly, but that is expected from this level of course. Sometimes it was hard to stay engaged with the lectures and to really absorb the course material. Because the lectures moved so fast, I really appreciated how Professor made the full notes available at the time of the lecture. I would have liked if there were a few more examples with the notes, as sometimes the homework felt disjoint from the notes.

*Response:* I will try to move slower this semester. I will start lab sessions earlier too.

- The professor is an extremely intelligent, kind, and understanding professor. He prioritizes in making sure that we understand the material and seeing how the material can be applied. His lecture notes were a godsend because the texts could be a bit ambiguous at times but he elucidated the material in such a comprehensible manner.

*Response:* I will try to fix the left-over typos.

- Mentioned in class from other students/internal evaluation, conveying the mathematical concepts through the presentation is not a good idea to follow the class in real-time. Prepared presentation can give rise to a distraction on what we have been going over.

*Response:* I am still delivering this class in hybrid-mode. I found the presentations fit online teaching better. I think the difficulty might be caused by (1) fast moving lecture (2) I only realized the need of reviewing basic concepts of probability almost a quarter into the semester...

- I found the setup of the course not very engaging. Additionally, many of the class notes came directly from the additional sources with no additional information or explanation, which I found to be not very helpful.

*Response:* I actually like them. I was the guinea pig to test them.

- Easily help us to understand the main course, and the notes and details are great.

*Response:* There will be notes.

- Moves very quickly and can be hard to keep up with. Sometimes instructions are unclear.

*Response:* I will try to slow down.

- Both the instructor and the TA were helpful. It was hard to follow along in class though.

*Response:* We don't have TA this time. Make use of the office hour. And I have to say, it needs effort to ace in this class.

## 2. Strong aspects of this course

- Having the lecture notes and labs available was very helpful. Professor was also always very nice and accommodating, and willing to meet with me when I needed help. He also always responded to student feedback, if we asked for an extra day or two on the homework or something like that.

*Response:* Here is an example of correctly using the office hours.

- His lecture notes and the lab sessions.

*Response:* They will be there again.

- Lab session is necessarily required to this class. A lot of computations in the class would be done by computer due to the complexity, and students are expected to handle with the computer programming properly at a desired level. The course can be an introduction to the statistical computation, which does not exist in the mathematics department.

*Response:* Hmm, there is a course Math 7360 Data Analysis that focuses more on the computational side.

- I appreciated the homework reviews in class and felt these helped clarify the material.

*Response:* Of course, the reviews will be there again. The purpose of the course is for you to learn.

- Grading was easy which made up for the rigor.

*Response:* Don't rely on this...

- Really appreciate that Professor Xiang made such a neat and tidy notes for us.

It is really helpful for me to review. And notes have a great interaction with us, Professor Xiang also leaves some questions to help us think about the logic behind.

*Response:* Well, Xiang is my first name. Please call me Prof. X.

- Prof. Xiang was highly organized and wanted his students to understand the course content more than he made them worry about grades. I learned a lot about Linear Models and feel confident applying the course content professionally and academically. I wish most of the Math department had his teaching style and implemented his course documents and organization structure. Prof. Xiang made the course content in class digestible and if I needed to review the material I could easily find it through his course notes and textbook. I wish I could say the same about my other courses.

*Response:* Hmm, I like Prof. X. better.

- I really appreciated the emphasis on learning. It allowed for most students to take it at the pace that was good for them.

*Response:* Please don't let your score rely on this comment.

3. There will be an internal mid-term-ish evaluation for this course. Will remember to go over them.

## 2 Lecture 2:Jan 28

### Last time

- Introduction
- Course logistics

### Today

- Reply to the “Presentation Dates” thread on Canvas by the end of today.
- Introduce yourself (remind remote students to record a short video)
  - basic info (name, department, year, ...)
  - why taking this course
- Git
- Linear algebra: vector and vector space, rank of a matrix (maybe)

### What is git?

Git is currently the most popular system for version control according to [Google Trend](#). Git was initially designed and developed by [Linus Torvalds](#) in 2005 for Linux kernel development. Git is the British English slang for unpleasant person.

### Why using git?

- [GitHub](#) is becoming a de facto central repository for open source development.
- **Advertise** yourself through GitHub (e.g., host a free personal webpage on GitHub).
- a skill that employers look for (according to [this AmStat article](#)).

### Git workflow

Figure [2.1](#) shows its basic workflow.

### What do I need to use Git?

- A **Git server** enabling multi-person collaboration through a centralized repository.
- A **Git client** on your own machine.
  - Linux: Git client program is shipped with many Linux distributions, e.g., Ubuntu and CentOS. If not, install using a package manager, e.g., `yum install git` on CentOS.
  - Mac: follow instructions at <https://www.atlassian.com/git/tutorials/install-git>.



Figure 2.1

– Windows: Git for Windows at <https://gitforwindows.org> (GUI) aka Git Bash.

- Do **not** totally rely on GUI or IDE. Learn to use Git on command line, which is needed for cluster and cloud computing.

## Git survival commands

- `git pull` synchronize local Git directory with remote repository.
- Modify files in local working directory.
- `git add FILES` add snapshots to staging area
- `git commit -m "message"` store snapshots permanently to (**local**) Git repository
- `git push` push commits to remote repository.

## Git basic usage

Working with your local copy.

- `git pull` : update local Git repository with remote repository (fetch + merge).

- `git log FILENAME` : display the current status of working directory.
- `git diff` : show differences (by default difference from the most recent commit).
- `git add file1 file2 ...` : add file(s) to the staging area.
- `git commit` : commit changes in staging area to Git directory.
- `git push` : publish commits in local Git repository to remote repository.
- `git reset --soft HEAD 1` : undo the last commit.
- `git checkout FILENAME` : go back to the last commit, discarding all changes made.
- `git rm FILENAME` : remove files from git control.

## Git demonstration

Show how to create a private git repository for HW and Exam submissions.

On [GitHub](#)

- Obtain [student developer pack](#).
- Create a private repository `math-6040-2022-spring` (please substitute 6040 by 7260 if you are taking the graduate level). Add `xji3` as your collaborators with write permission ([instruction](#)).

On your local machine:

- clone the repository: please refer to [this webpage](#) with instructions for your operating system.
- enter the folder: `cd math-6040-2022-spring`.
- after finishing the rest of the questions, save your file inside your git repository folder `math-6040-2022-spring` with name `hw1.pdf` (for example). Please make it human-readable.
- now using git commands to stage this change: `git add hw1.pdf`
- commit: `git commit -m "hw1 submission"` (remember to replace the quotation mark)
- push to remote server: `git push`
- tag version hw1: `git tag hw1` and push: `git push --tags`.

Take a look at the tags on GitHub ([instructions](#)).

When submitting your hw, please email your instructor ([xji4@tulane.edu](mailto:xji4@tulane.edu)) a link to your tag ([instructions](#)).

### 3 Lecture 3: Jan 25

Last time

- Git

Today

- Linear algebra: vector and vector space, rank of a matrix
- Column space and Nullspace (JM Appendix A)

Notations

$$\underset{n \times 1}{\mathbf{y}} = \underset{n \times p}{\mathbf{X}} \underset{p \times 1}{\boldsymbol{\beta}} + \underset{n \times 1}{\boldsymbol{\epsilon}}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

- All vectors are column vector
- Write dimensions underneath as in  $\underset{n \times p}{\mathbf{X}}$  or as  $\mathbf{X} \in \mathbb{R}^{n \times p}$
- Bold upper-case letters for Matrices. Bold lower-case letters for Vectors.

Vector and vector space

(from JM Appendix A)

- A set of vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are *linearly dependent* if there exist coefficients  $c_j$  for  $j = 1, 2, \dots, n$  such that  $\sum_{j=1}^n c_j \mathbf{x}_j = \mathbf{0}$  and  $\|\mathbf{c}\|_2 = \sum_{j=1}^n c_j^2 > 0$ . They are *linearly independent* if  $\sum_{j=1}^n c_j \mathbf{x}_j = \mathbf{0}$  implies (i.e.  $\implies$ )  $c_j = 0$  for all  $j$ .
- Two vectors are *orthogonal* to each other, written  $\mathbf{x} \perp \mathbf{y}$ , if their inner product is 0, that is  $\mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = \sum_j x_j y_j = 0$ .
- A set of vectors  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$  are mutually orthogonal iff (i.e.  $\iff$ )  $\mathbf{x}^{(i)T} \mathbf{x}^{(j)} = 0$  for  $\forall i \neq j$ .
- The most common set of vectors that are mutually orthogonal are the *elementary* vectors  $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \dots, \mathbf{e}^{(n)}$ , which are all zero, except for one element equal to 1, so that  $\mathbf{e}_i^{(i)} = 1$  and  $\mathbf{e}_j^{(i)} = 0, \forall j \neq i$ .
- A *vector space*  $\mathcal{S}$  is a set of vectors that are closed under addition and scalar multiplication, that is



- if  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  are in  $\mathcal{S}$ , then  $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$  is in  $\mathcal{S}$ .
- A vector space  $\mathcal{S}$  is *generated* or *spanned* by a set of vectors  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ , written as  $\mathcal{S} = \text{span}\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}\}$ , if any vector  $\mathbf{x}$  in the vector space is a linear combination of  $\mathbf{x}_i, i = 1, 2, \dots, n$ .
- A set of linearly independent vectors that generate or span a space  $\mathcal{S}$  is called a *basis* of  $\mathcal{S}$ .

### Example A.1

Let

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \text{ and } \mathbf{x}^{(3)} = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

Then  $\mathbf{x}^{(1)}$  and  $\mathbf{x}^{(2)}$  are linearly independent, but  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ , and  $\mathbf{x}^{(3)}$  are linearly dependent since  $5\mathbf{x}^{(1)} - 2\mathbf{x}^{(2)} + \mathbf{x}^{(3)} = \mathbf{0}$

## Rank

Some matrix concepts arise from viewing columns or rows of the matrix as vectors. Assume  $\mathbf{A} \in \mathbb{R}^{m \times n}$ .

- $\text{rank}(\mathbf{A})$  is the maximum number of linearly independent rows or columns of a matrix.
- $\text{rank}(\mathbf{A}) \leq \min\{m, n\}$ .
- A matrix is *full rank* if  $\text{rank}(\mathbf{A}) = \min\{m, n\}$ . It is *full row rank* if  $\text{rank}(\mathbf{A}) = m$ . It is *full column rank* if  $\text{rank}(\mathbf{A}) = n$ .
- a square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is *singular* if  $\text{rank}(\mathbf{A}) < n$  and *non-singular* if  $\text{rank}(\mathbf{A}) = n$ .
- $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) = \text{rank}(\mathbf{A}^T \mathbf{A}) = \text{rank}(\mathbf{A} \mathbf{A}^T)$ . (Show this in HW.)
- $\text{rank}(\mathbf{A} \mathbf{B}) \leq \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$ . (Hint: Columns of  $\mathbf{A} \mathbf{B}$  are spanned by columns of  $\mathbf{A}$  and rows of  $\mathbf{A} \mathbf{B}$  are spanned by rows of  $\mathbf{B}$ .)
- if  $\mathbf{A} \mathbf{x} = \mathbf{0}_m$  for some  $\mathbf{x} \neq \mathbf{0}_n$ , then  $\text{rank}(\mathbf{A}) \leq n - 1$ .

### Idempotent matrix

Assume  $\mathbf{A} \in \mathbb{R}^{n \times n}$ .

- A matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is idempotent if and only if  $\mathbf{A}^2 (= \mathbf{A} \mathbf{A}) = \mathbf{A}$ .
- Any idempotent matrix  $\mathbf{A}$  is a generalized inverse of itself.
- The only idempotent matrix of full rank is  $\mathbf{I}$ .  
*Proof.* Since  $\mathbf{A}$  has full rank, the inverse  $\mathbf{A}^{-1}$  exists. Then  $\mathbf{A} = \mathbf{A}^{-1} \mathbf{A} \mathbf{A} = \mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$ . Interpretation: all idempotent matrices are singular except for the identity matrix.

- $\mathbf{A}$  is idempotent if and only if  $\mathbf{A}^T$  is idempotent if and only if  $\mathbf{I}_n - \mathbf{A}$  is idempotent.
- For a general matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , the matrices  $\mathbf{A}^-\mathbf{A}$  and  $\mathbf{A}\mathbf{A}^-$  are idempotent and

$$\begin{aligned}\text{rank}(\mathbf{A}) &= \text{rank}(\mathbf{A}^-\mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}^-) \\ \text{rank}(\mathbf{I}_n - \mathbf{A}^-\mathbf{A}) &= n - \text{rank}(\mathbf{A}) \\ \text{rank}(\mathbf{I}_m - \mathbf{A}\mathbf{A}^-) &= m - \text{rank}(\mathbf{A}).\end{aligned}$$

## Column space

*Definition:* The column space of a matrix, denoted by  $C(\mathbf{A})$  is the vector space spanned by the columns of the matrix, that is,

$$C(\mathbf{A}) = \{\mathbf{x} : \text{there exists a vector } \mathbf{c} \text{ such that } \mathbf{x} = \mathbf{A}\mathbf{c}\}.$$

This means that if  $\mathbf{x} \in C(\mathbf{A})$ , we can find coefficients  $c_j$  such that

$$\mathbf{x} = \sum_j c_j \mathbf{a}^{(j)}$$

where  $\mathbf{a}^{(j)} = \mathbf{A}_{.j}$  denotes the  $j^{\text{th}}$  column of matrix  $\mathbf{A}$ .

- The column space of a matrix consists of all vectors formed by multiplying that matrix by any vector.
- The number of basis vectors for  $C(\mathbf{A})$  is then the number of linearly independent columns of the matrix  $\mathbf{A}$ , and so,  $\dim(C(\mathbf{A})) = \text{rank}(\mathbf{A})$ .
- The dimension of a space is the number of vectors in its basis.

### Example A.2

Let  $\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 4 & 3 \end{bmatrix}$  and  $\mathbf{c} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ . Show that  $\mathbf{A}\mathbf{c}$  is a linear combination of columns

in  $\mathbf{A}$ . *solution:*

$$\mathbf{A}\mathbf{c} = \begin{bmatrix} 1 \times 5 + 1 \times 4 + (-3) \times 3 \\ 1 \times 5 + 2 \times 4 + (-1) \times 3 \\ 1 \times 5 + 3 \times 4 + 1 \times 3 \\ 1 \times 5 + 4 \times 4 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}.$$

You could recognize that

$$\mathbf{A}\mathbf{c} = 5 \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 4 \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + 3 \times \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix} = 5\mathbf{a}^{(1)} + 4\mathbf{a}^{(2)} + 3\mathbf{a}^{(3)} = \begin{bmatrix} 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}.$$

### Result A.1

$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B}))$ . *proof:* Each column of  $\mathbf{AB}$  is a linear combination of columns of  $\mathbf{A}$  (i.e.  $(\mathbf{AB})_{\cdot j} = \mathbf{A}\mathbf{b}^{(j)}$ ), so the number of linearly independent columns of  $\mathbf{AB}$  cannot be greater than that of  $\mathbf{A}$ . Similarly,  $\text{rank}(\mathbf{AB}) = \text{rank}(\mathbf{B}^T \mathbf{A}^T)$ , the same argument gives  $\text{rank}(\mathbf{B}^T)$  as an upper bound.

### Result A.2

- (a) If  $\mathbf{A} = \mathbf{BC}$ , then  $C(\mathbf{A}) \subseteq C(\mathbf{B})$ .
- (b) If  $C(\mathbf{A}) \subseteq C(\mathbf{B})$ , then there exists a matrix  $\mathbf{C}$  such that  $\mathbf{A} = \mathbf{BC}$ .

*proof:* For (a), any vector  $\mathbf{x} \in C(\mathbf{A})$  can be written as  $\mathbf{x} = \mathbf{A}\mathbf{d} = \mathbf{B}(\mathbf{C}\mathbf{d})$ .

For (b),  $\mathbf{A}_{\cdot j} \in C(\mathbf{B})$ , so that there exists a vector  $\mathbf{c}^{(j)}$  such that  $\mathbf{A}_{\cdot j} = \mathbf{B}\mathbf{c}^{(j)}$ . The matrix  $\mathbf{C} = (\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \dots, \mathbf{c}^{(n)})$  satisfies that  $\mathbf{A} = \mathbf{BC}$ .

### Null space

*Definition:* The null space of a matrix, denoted by  $\mathcal{N}(\mathbf{A})$ , is  $\mathcal{N}(\mathbf{A}) = \{\mathbf{y} : \mathbf{A}\mathbf{y} = \mathbf{0}\}$ .

### Result A.3

If  $\mathbf{A}$  has full-column rank, then  $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$ . *proof:* Matrix  $\mathbf{A}$  has full-column rank means its columns are linearly independent, which means that  $\mathbf{A}\mathbf{c} = \mathbf{0}$  implies  $\mathbf{c} = \mathbf{0}$ .

### Theorem A.1

Assume  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , then  $\dim(C(\mathbf{A})) = r$  and  $\dim(\mathcal{N}(\mathbf{A})) = n - r$ , where  $r = \text{rank}(\mathbf{A})$ .

See JM Appendix Theorem A.1 for the proof.

Interpretation: “dimension of column space + dimension of null space = # columns”

*Mis-Interpretation:* Columns space and null space are orthogonal complement to each other. They are of different orders in general! Next result gives the correct statement.