MAST30027: Modern Applied Statistics (REVISED)

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Assignment 4, 2019
Tutorial: Qiuyi Li, Mon 12pm-1pm

October 28, 2019

All R code and working out is attached at the end of the assignment for full reproducability. The revised edition fixes the transition probability.

1. (a) To obtain the full conditional distributions, we require the joint density.
The joint density (for $n = 100$) is:

$$\begin{aligned}
p(\mathbf{x}, \mu, \tau) &\propto p(\mathbf{x}|\mu, \tau)p(\mu, \tau) \\
&\propto \prod_{i=1}^{n} \sqrt{\tau} \exp\left(\frac{-\sum_{i=1}^{n}(x_i - \mu)^2}{2(\frac{1}{\tau})}\right)\tau^{-1} \\
&\propto \tau^{\frac{n}{2}-1} \exp\left(\frac{-(\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i\mu + n\mu^2)}{2(\frac{1}{\tau})}\right).
\end{aligned}$$

Hence, the conditional posterior for μ is:

$$\begin{aligned}
p(\mu|\mathbf{x}, \tau) &\propto \exp\left(\frac{-(-2\bar{x}\mu - n\mu^2)}{2(\frac{1}{\tau})}\right) \\
&\propto \exp\left(\frac{-(\mu - \bar{x})^2}{2(\frac{1}{n\tau})}\right) \\
&\stackrel{d}{=} N\left(\bar{x}, \frac{1}{n\tau}\right),
\end{aligned}$$

and the conditional posterior for τ is:

$$\begin{aligned}
p(\tau|\mathbf{x}, \mu) &\propto \tau^{\frac{n}{2}-1} \exp\left(-\tau\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2}\right) \\
&\stackrel{d}{=} \gamma\left(\frac{n}{2}, \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2}\right).
\end{aligned}$$

(b) *Refer to Jupyter Notebook R Output for Gibbs Sampler code, trace plot and QQ-Plot.*

(c) For μ:

- The marginal posterior mean is 5.104502.
- A 90% Credible Interval is [4.782497, 5.341995].

- Estimated Distribution Plot: *Refer to Jupyter Notebook R Output.*

For τ:

- The marginal posterior mean is 0.251488.
- A 90% Credible Interval is [0.1903570, 0.3007045].
- Estimated Distribution Plot: *Refer to Jupyter Notebook R Output.*

2. (a) **Due to the issue of integer underflow in programming languages, we must take the log transformation in order to preserve precision.**

   So, the log joint density is:

$$\begin{aligned}\log p(\mathbf{x}, \mu, \tau) &\propto \log\left(\tau^{\frac{n}{2}-1}\exp\left(\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2(\frac{1}{\tau})}\right)\right)\\ &\propto \left(\frac{n}{2}-1\right)\log\tau + \left(\frac{-\tau\sum_{i=1}^{n}(x_i-\mu)^2}{2}\right).\end{aligned}$$

   Refer to Jupyter Notebook R Output for MH-Algorithm code, trace plot and QQ-Plot.

   (b) For μ:

   - The marginal posterior mean is 5.088475.
   - A 90% Credible Interval is [4.717057, 5.346193].
   - Estimated Distribution Plot: *Refer to Jupyter Notebook R Output.*

   For τ:

   - The marginal posterior mean is 0.2339575.
   - A 90% Credible Interval is [0.1789654, 0.2779468].
   - Estimated Distribution Plot: *Refer to Jupyter Notebook R Output.*

R

October 28, 2019

0.0.1 General

Code setup:

[1]:
```
set.seed(30027)
X = rnorm(100, 5, 2)
N = length(X)
x.bar = mean(X)
```

0.0.2 Question 1b)

- Code for the Gibbs Sampler:

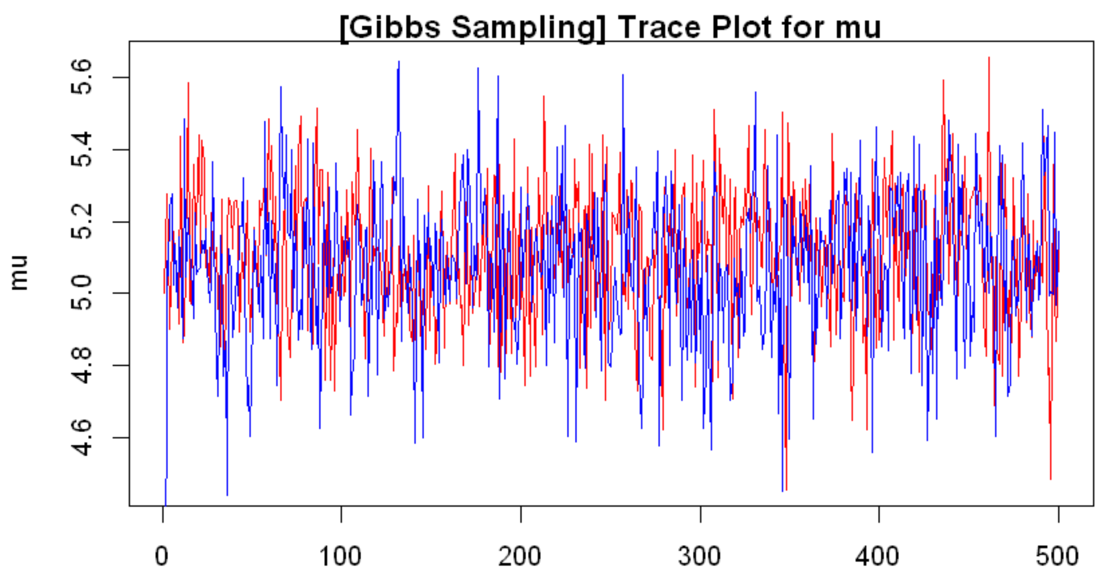
[2]:
```
gibbs.sample = function(mu0, tau0, niter=500) {
    # columns = mu, tau
    params = matrix(nrow = niter, ncol = 2)
    # initialize values
    params[1,] = c(mu0, tau0)

    for (t in 2:niter) {
        # the current parameters
        mu0 = params[t-1, 1]
        tau0 = params[t-1, 2]

        # the new parameters after resampling
        mu1 = rnorm(1, x.bar, sd=1/sqrt(N*tau0))
        tau1 = rgamma(1, shape=N/2, rate=(sum((X - mu1)^2))/2)
        params[t,] = c(mu1, tau1)
    }

    return(params)
}
```

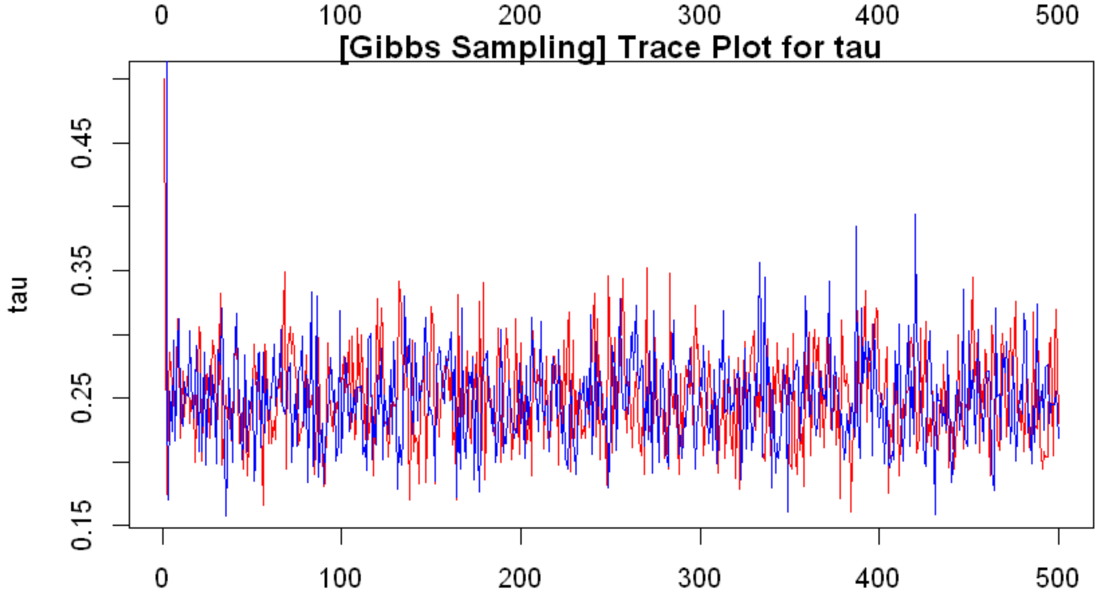
- Two chains with different initial starting values:

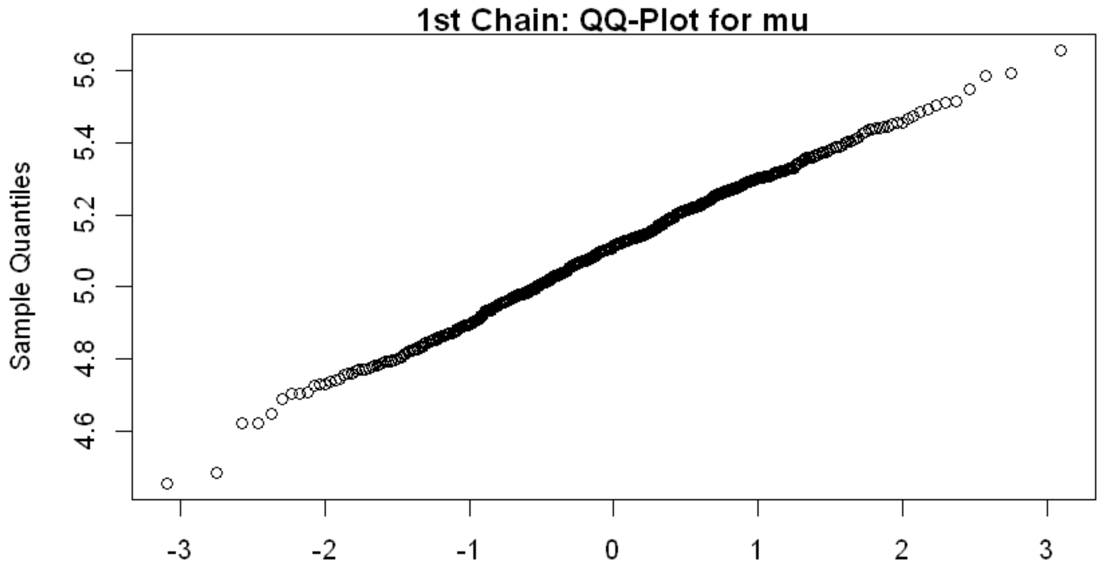
[3]:
```
run1 = gibbs.sample(mu=5, tau=0.5)
run2 = gibbs.sample(mu=2, tau=3)
```

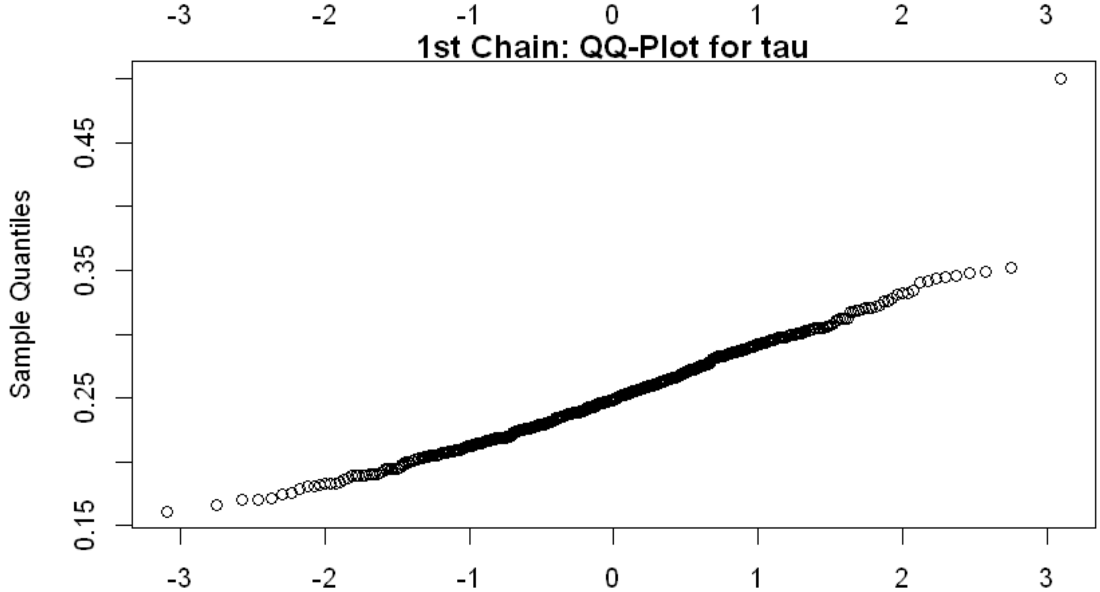
- Trace Plot for both chains (1st chain is red, 2nd chain is blue)
- QQ-Plot for both chains

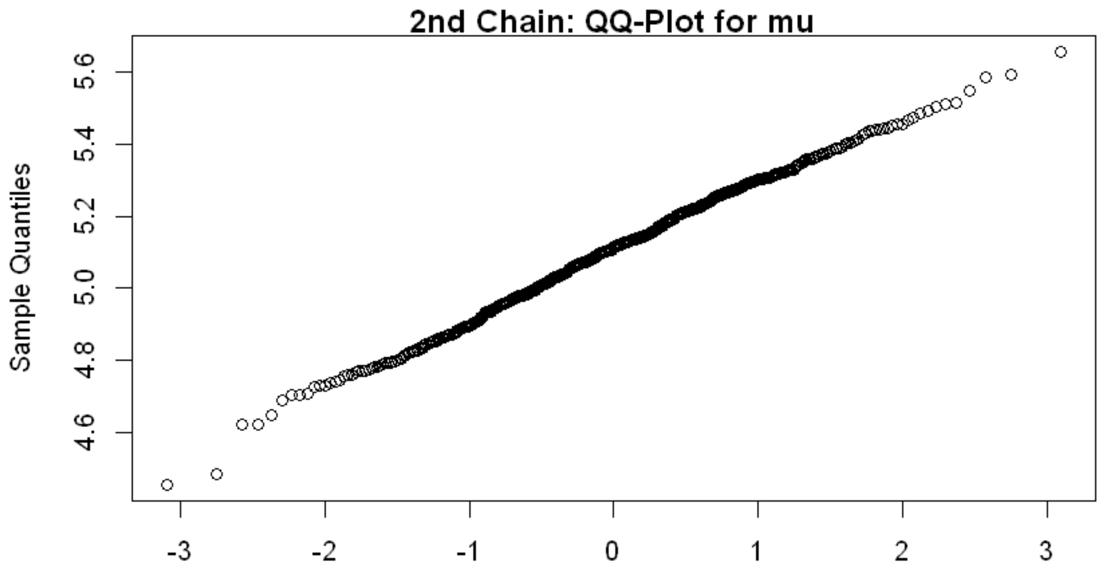
[4]:
```
par(mfrow=c(2,1), mar=c(2,4,1,1))
plot(run1[,1], type="l", xlab="iteration", ylab="mu", col="red", main="[Gibbs␣
 ↪Sampling] Trace Plot for mu")
lines(run2[,1], col="blue")

plot(run1[,2], type="l", xlab="iteration", ylab="tau", col="red", main="[Gibbs␣
 ↪Sampling] Trace Plot for tau")
lines(run2[,2], col="blue")

qqnorm(run1[,1], main="1st Chain: QQ-Plot for mu")
qqnorm(run1[,2], main="1st Chain: QQ-Plot for tau")
qqnorm(run1[,1], main="2nd Chain: QQ-Plot for mu")
qqnorm(run1[,2], main="2nd Chain: QQ-Plot for tau")
```

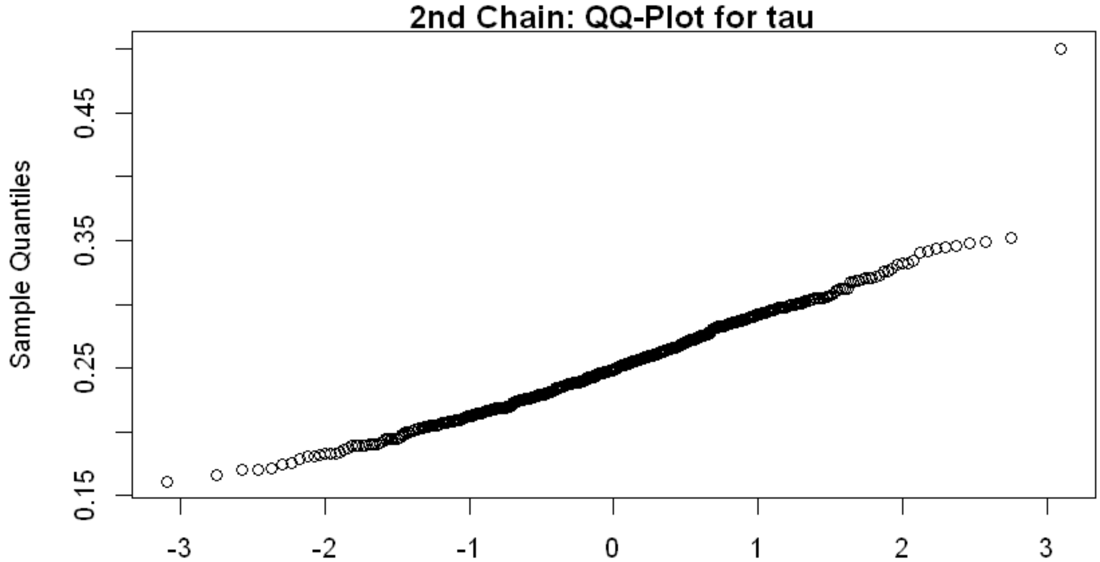





1st Chain: QQ-Plot for mu
Sample Quantiles
5.6
5.4
5.2
5.0
4.8
4.6
-3
-2
-1
0
1
2
3


1st Chain: QQ-Plot for tau
Sample Quantiles
0.45
0.35
0.25
0.15
-3
-2
-1
0
1
2
3





0.0.3 Question 1c)

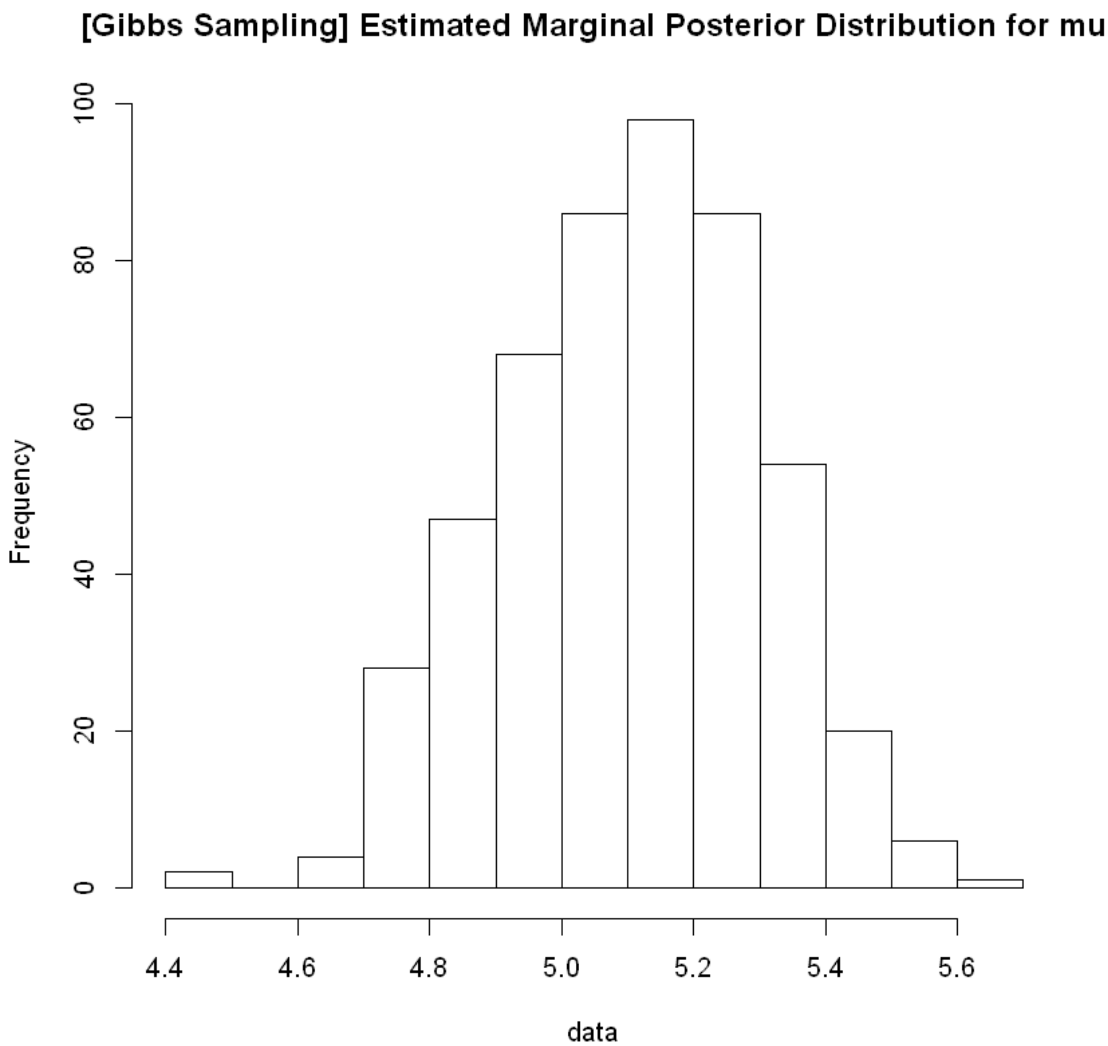
- Function to output estimated mean, a 90% Credible Interval and histogram plot of estimated distribution

[5]:
```
part.c = function(data, param.name="NONE") {
    # Histogram
    hist(data, main=param.name)
    print(c("The mean is:"))
    # Estimated Mean
    print(mean(data))
    print(c("The 90% Credible Interval is"))
    # 5th and 90th quantile
```

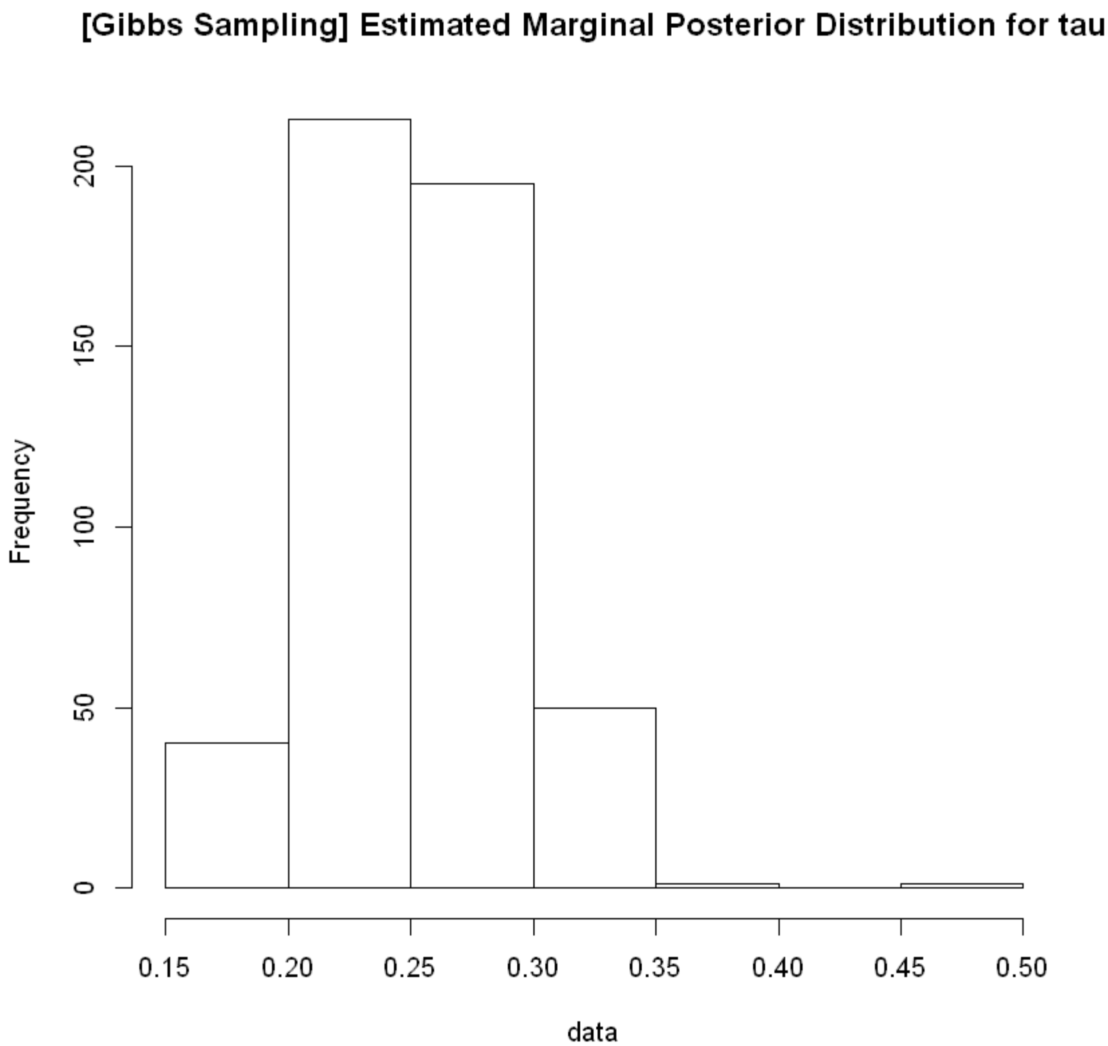
```
    print(quantile(data, probs=seq(0, 1, 0.05))[c(2,19)])
}
```

[6]:
```
part.c(run1[,1], "[Gibbs Sampling] Estimated Marginal Posterior Distribution␣
 ↪for mu")
part.c(run1[,2], "[Gibbs Sampling] Estimated Marginal Posterior Distribution␣
 ↪for tau")
```

```
[1] "The mean is:"
[1] 5.104502
[1] "The 90% Credible Interval is"
      5%      90%
4.782497 5.341995
```



```
[1] "The mean is:"
[1] 0.251488
[1] "The 90% Credible Interval is"
       5%       90% 
0.1903570 0.3007045 
```



0.0.4 Question 2a)

- Code for the MH-Algorithm

[7]:
```
# Proposal function as given
proposal.func = function(mu.c, tau.c) {
    tau.n = rgamma(1, shape=5 * tau.c, rate=5)
```

```
    mu.n = rnorm(1, mu.c, tau.n)

    return (c(mu.n, tau.n))
}

# The log transition density probability from a given theta to theta.dash (and
↪vice-versa)
transition = function(theta, theta.dash) {
    prob1 = dgamma(theta.dash, shape=5 * theta, rate=5)
    prob2 = dnorm(theta.dash, theta)
    return (log(prob1) + log(prob2))
}

# The joint distribution with a log transformation to prevent underflow in R
joint = function(mu, tau) {
    return( (N/2 - 1)*log(tau) + (-tau * sum((X - mu)^2)/2) )
}

# Metropolis-Hastings algorithm
mh.sample = function(mu0=0, tau0=0, niter=10000) {
    # columns = mu, tau
    params = matrix(nrow = niter, ncol = 2)
    # initialize values
    params[1,] = c(mu0, tau0)

    for (i in 2:niter) {
        # theta
        mu0 = params[i-1,1]
        tau0 = params[i-1,2]

        # proposed theta.dash values
        proposal = proposal.func(mu0, tau0)
        mu1 = proposal[1]
        tau1 = proposal[2]

        # posterior = log(likelihood) + log(prior)
        pi = joint(mu0, tau0)
        pi.dash = joint(mu1, tau1)

        # transition probability between theta -> theta.dash and theta.dash ->
↪theta
        q.dash = transition(tau0, tau1)
        q = transition(tau1, tau0)

        # Acceptance Probability (with an exponential transformation for the
↪log)
        A = min(1, exp((pi.dash+q.dash) - (pi + q)))
```

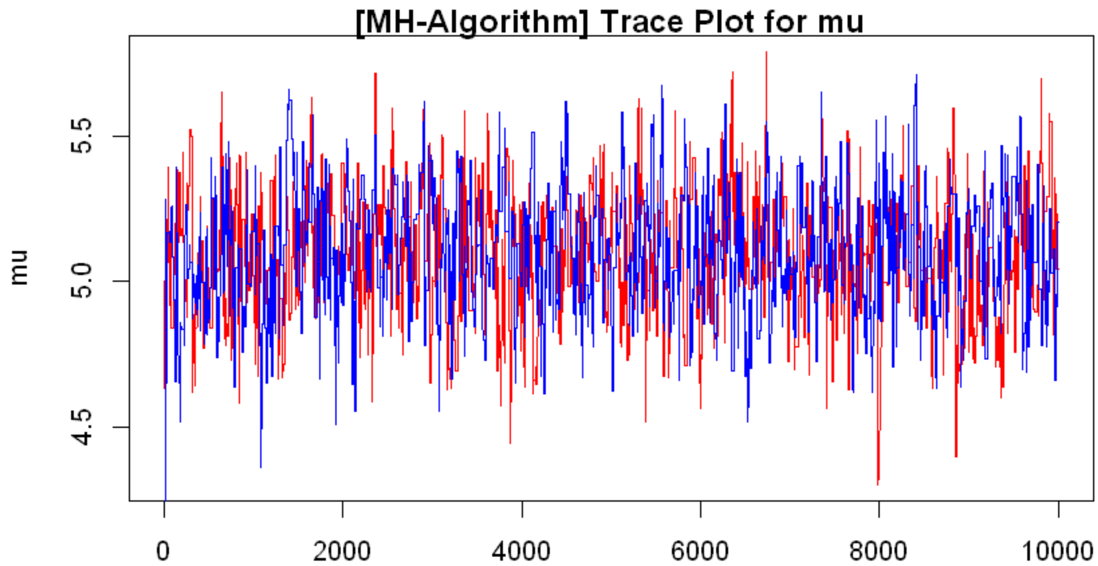
```
        # We accept the proposal
        if (runif(1) < A) {
            params[i,] = proposal
        }
        # We fail to accept the proposal
        else {
            params[i,] = params[i-1,]
        }
    }
    return(params)
}
```

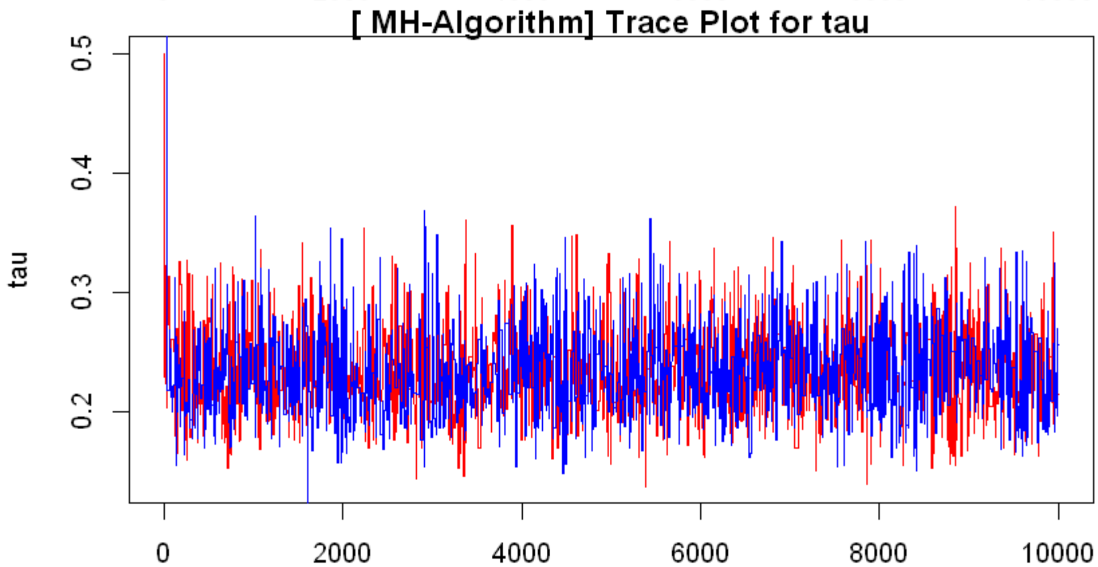
- 2 chains with different starting values:

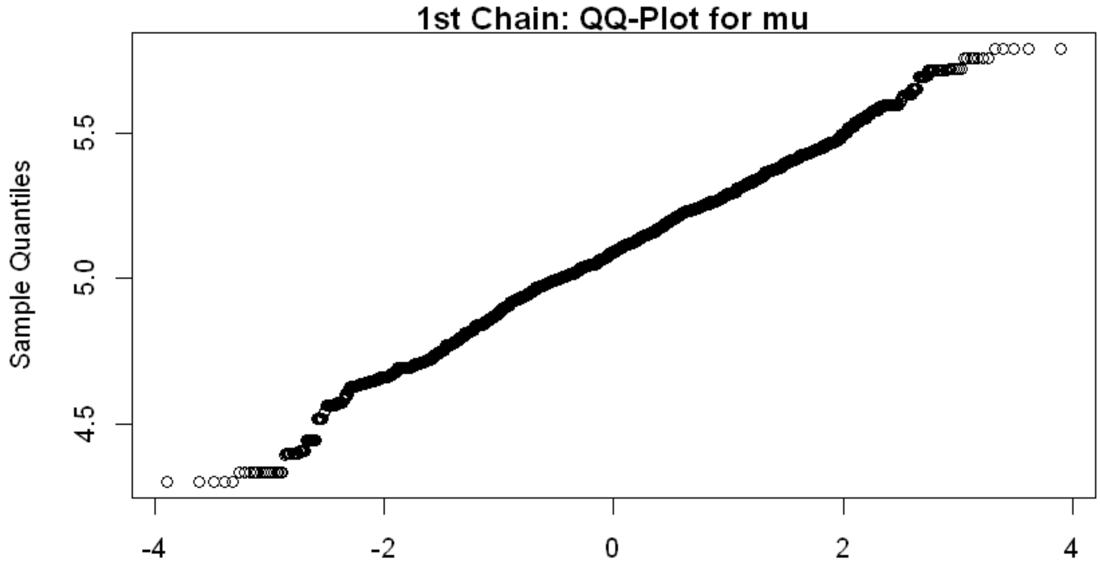
[8]:
```
run1.mh = mh.sample(mu=5, tau=0.5)
run2.mh = mh.sample(mu=2, tau=3)
```

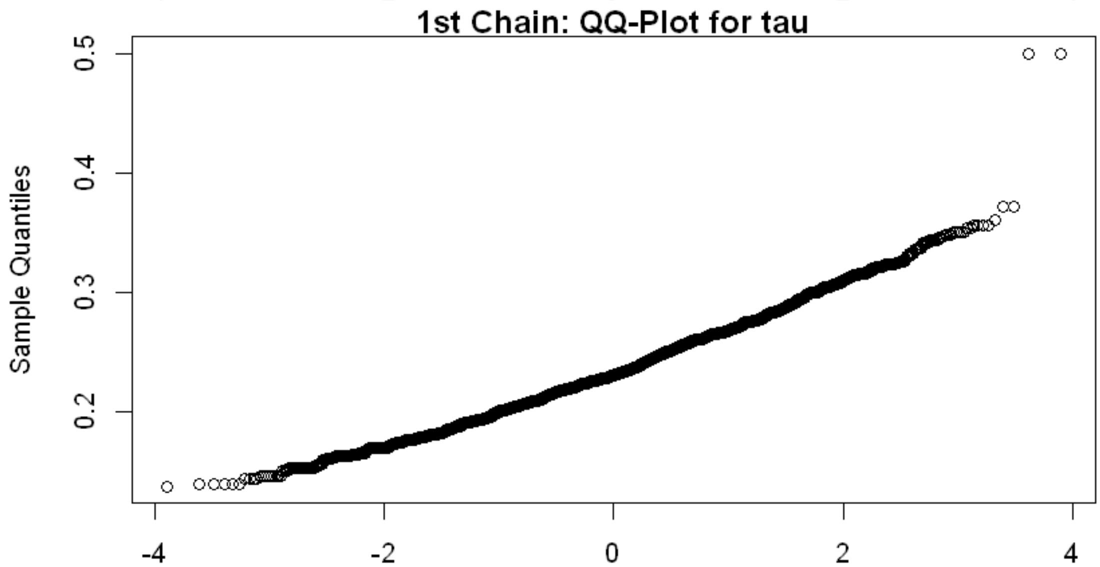
- Trace Plot for both chains (1st chain is red, 2nd chain is blue)
- QQ-Plot for both chains

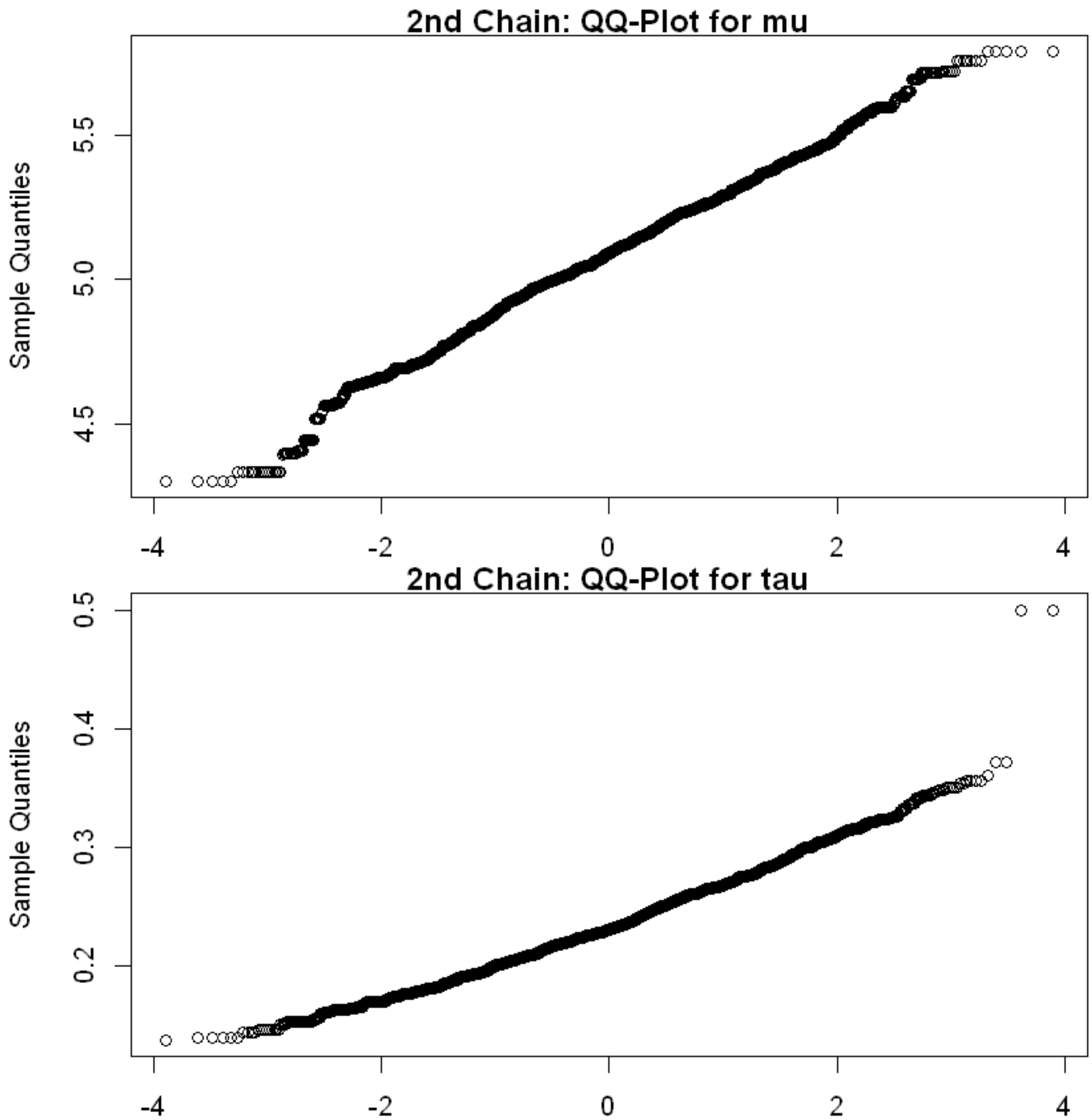
[9]:
```
par(mfrow=c(2,1), mar=c(2,4,1,1))
plot(run1.mh[,1], type="l", xlab="iteration", ylab="mu", col="red",␣
 ↪main="[MH-Algorithm] Trace Plot for mu")
lines(run2.mh[,1], col="blue")

plot(run1.mh[,2], type="l", xlab="iteration", ylab="tau", col="red", main="[␣
 ↪MH-Algorithm] Trace Plot for tau")
lines(run2.mh[,2], col="blue")

qqnorm(run1.mh[,1], main="1st Chain: QQ-Plot for mu")
qqnorm(run1.mh[,2], main="1st Chain: QQ-Plot for tau")
qqnorm(run1.mh[,1], main="2nd Chain: QQ-Plot for mu")
qqnorm(run1.mh[,2], main="2nd Chain: QQ-Plot for tau")
```


[MH-Algorithm] Trace Plot for mu
mu
5.5
5.0
4.5
0
2000
4000
6000
8000
10000


[MH-Algorithm] Trace Plot for tau
tau
0.5
0.4
0.3
0.2
0
2000
4000
6000
8000
10000


1st Chain: QQ-Plot for mu
Sample Quantiles
4.5
5.0
5.5
-4
-2
0
2
4


1st Chain: QQ-Plot for tau
Sample Quantiles
0.2
0.3
0.4
0.5
-4
-2
0
2
4



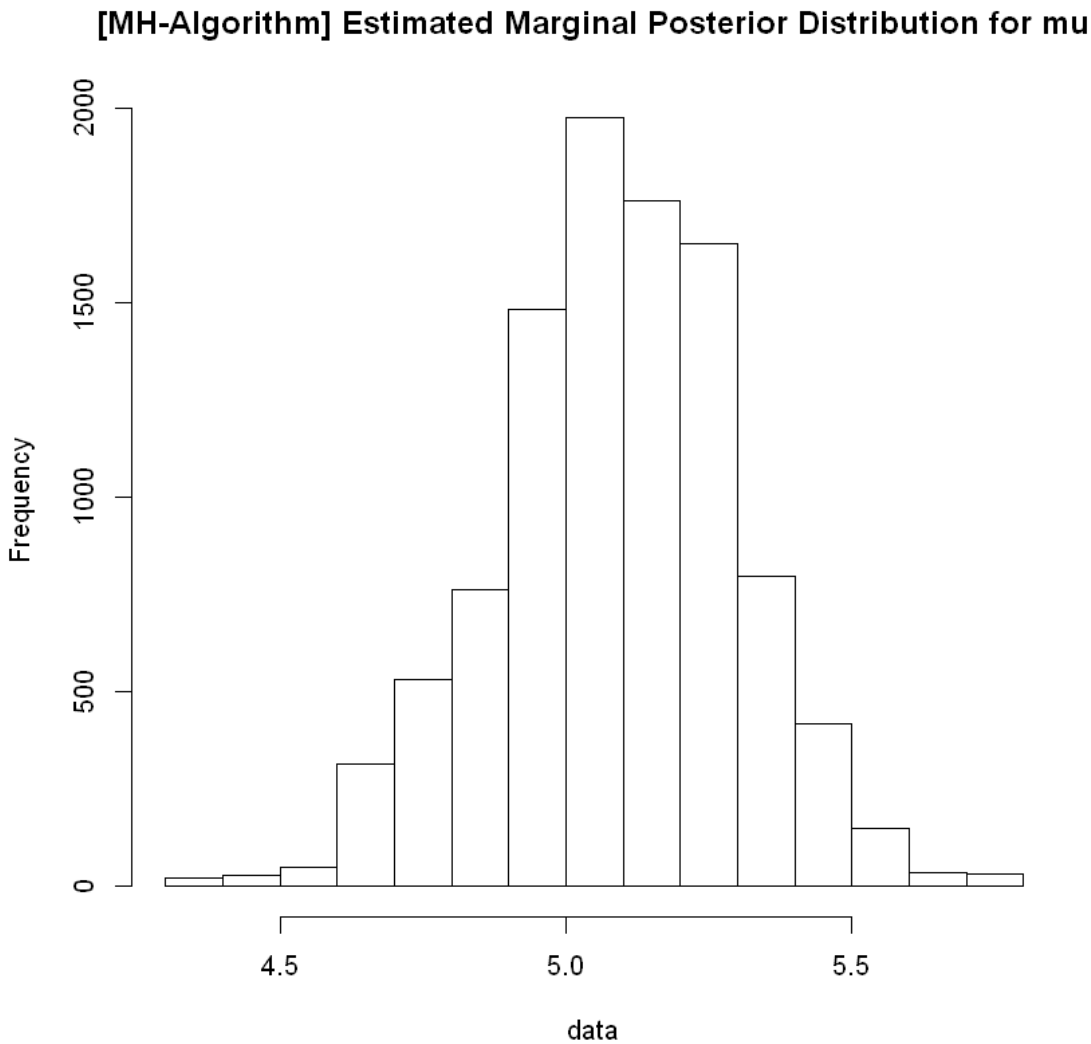
0.0.5 Question 2c)

- Function to output estimated mean, a 90% Credible Interval and histogram plot of estimated distribution

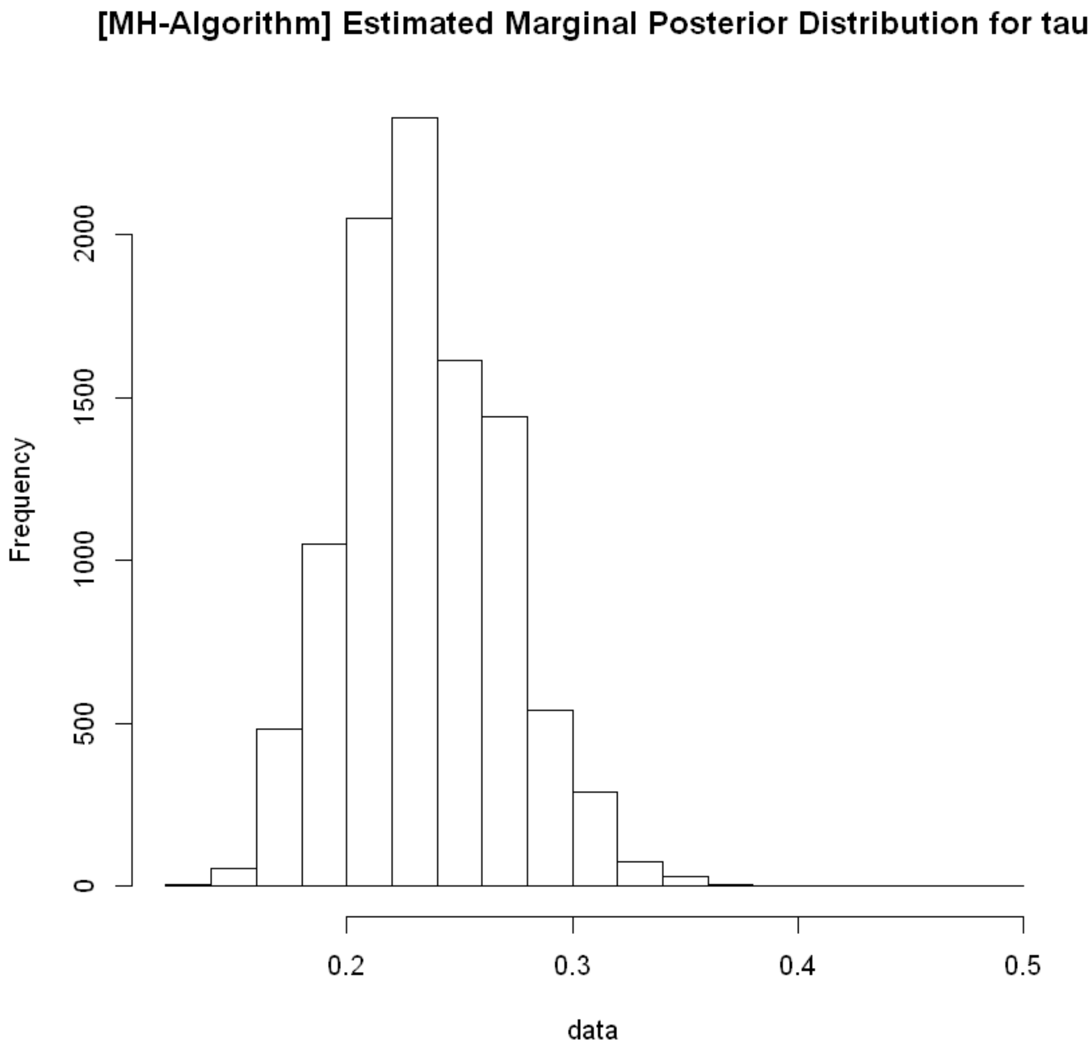
[10]:
```
part.c(run1.mh[,1], "[MH-Algorithm] Estimated Marginal Posterior Distribution␣
  ↪for mu")
part.c(run1.mh[,2], "[MH-Algorithm] Estimated Marginal Posterior Distribution␣
  ↪for tau")
```

```
[1] "The mean is:"
[1] 5.088475
[1] "The 90% Credible Interval is"
```

```
      5%      90%
4.717057 5.346193
```




```
[1] "The mean is:"
[1] 0.2339575
[1] "The 90% Credible Interval is"
       5%       90%
0.1789654 0.2779468
```


[MH-Algorithm] Estimated Marginal Posterior Distribution for tau
Frequency
0
500
1000
1500
2000
0.2
0.3
0.4
0.5
data

[ ]:

[ ]: