MAST30027: Modern Applied Statistics (REVISED)

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Assignment 1, 2019
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All R code and working out is attached at the end of the assignment.

1. (a) Estimates for the parameters are:

$$\hat{\beta}_0 = 5.597882$$
$$\hat{\beta}_1 = -0.1059069.$$

(b) 95% CI for β_0 is $[2.243931, 8.951834]$
95% CI for β_1 is $[-0.1579897, -0.05382405]$.

(c) With a p-value = 5.18671169167522e-06 at the $\alpha = 0.05$ significance level, the LRT concludes that the β_1 parameter is significantly relevant.

(d) The estimate for the probability of damage when temperature is 31 Fahrenheit is:

$$t^T\beta_1 = 0.9896872, \tag{1}$$

and a 95% CI is $[0.7115098, 0.9999767]$.

(e) *Plot is provided in the R code attached.*

2. (a) Estimated increase in log(odds) is 0.997168404895279 when the bmi increase by 10.

(b) A 95% CI for the estimated increase is $[0.6976052, 1.296732]$.

3. (a) Take the log Gamma function:

$$\begin{aligned}
\log(f(x;v,\lambda)) &= v\log(\lambda) - \log(\Gamma(v)) + (v-1)\log(x) - \lambda x \\
&= \frac{\frac{-1}{v}\lambda x - (-\log(\frac{\lambda}{v}))}{\frac{1}{v}} + (v-1)\log(x) - \log(\Gamma(v)) \\
&= \frac{x(\frac{-\lambda}{v}) - \log(\frac{v}{\lambda})}{\frac{1}{v}} + (v-1)\log(x) - \log(\Gamma(v)).
\end{aligned}$$

Let $\theta = \frac{-\lambda}{v}, \phi = \frac{1}{v}$. Then,

$$\log(f(x;v,\lambda)) = \frac{x\theta - \log(\frac{-1}{\theta})}{\phi} + \left(\frac{1}{\phi} - 1\right)\log(x) - \log\left(\Gamma\left(\frac{1}{\phi}\right)\right).$$

Hence, we have shown Gamma as an exponential family with form:

$$
\begin{aligned}
b(\theta) &= \log\left(\frac{-1}{\theta}\right) \\
a(\phi) &= \phi \\
c(x, \phi) &= \left(\frac{1}{\phi} - 1\right)\log(x) - \log\left(\Gamma\left(\frac{1}{\phi}\right)\right).
\end{aligned}
$$

(b) The Canonical Link is defined as $g(\mu) = \theta$, where any Exponential Family will have $\mu = b'(\theta)$. Therefore the Canonical Link for Gamma is:

$$
\begin{aligned}
\mu &= b'(\theta) \\
&= \frac{-1}{\theta}.
\end{aligned}
$$

In addition, any Exponential Family will have $\text{var}(x) = v(\mu)a(\phi)$, and so

$$
\begin{aligned}
\text{var}(x) &= b''(\theta)a(\phi) \\
&= \frac{1}{\theta^2}a(\phi) \\
&= \mu^2 a(\phi), \quad \text{recall} \quad \mu = \frac{-1}{\theta} \quad \text{from above.}
\end{aligned}
$$

Hence the Variance Function for Gamma is $v(\mu) = \mu^2$.

Assignment 1 R

August 23, 2019

1 Question 1 a)

Estimation for $\hat{\beta}$ using log-likelihood.

$$\begin{aligned}\ell(\beta) &= \sum_{i=1}^{N} y_i \log\left(\frac{p_i}{1-p_i}\right) + m_i \log(1-p_i) \\ &= \sum_{i=1}^{N} y_i \log(p) + (m_i - y_i)\log(1-p)\end{aligned}$$

In [24]:
```
library(faraway)
data(orings)

N = 6
INITIAL_PARAMS = c(10, -0.05)
```

In [9]:
```
F = function(eta) pnorm(eta)
f = function(eta) dnorm(eta)
iprobit = function(p) qnorm(p)
```

In [5]:
```
logL = function(beta, data) {
    n = 6
    y = data$damage
    eta = cbind(1, data$temp) %*% beta
    p = probit(eta)
    return(sum(y * log(p) + (n - y)* log(1 - p)))
}

beta.hat = optim(INITIAL_PARAMS, logL, data=orings, control=list(fnscale=-1))$par
beta0 = beta.hat[1]
beta1 = beta.hat[2]
```

Answer to Q1 a)

In [6]:
```
cbind(beta0, beta1)
```

beta0	beta1
5.597882	-0.1059069

2 Question 1b)

Fisher Information Matrix Let $\eta_i = x_i^T\beta = \beta_0 + \beta_1 x_i$, where β is the parameter vector and x_i is a vector of observed variables.

Probit function is defined as the Standard Normal CDF $p = F(\eta)$, and we take note of the some properties of the Standard Normal CDF:

$$
\begin{aligned}
F(-x) &= 1 - F(x) \\
F'(x) &= f(x) \\
f'(x) &= -xf(x).
\end{aligned}
$$

The Score Vector (first derivative) is:

$$
\begin{aligned}
U(\eta_i) &= \frac{\partial \ell(\beta_0, \beta_1)}{\partial \eta_i} \\
&= \sum_{i=1}^{N} \left[y_i \frac{f(\eta_i)}{F(\eta_i)} + y_i \frac{f(\eta_i)}{1 - F(\eta_i)} - m_i \frac{f(\eta_i)}{1 - F(\eta_i)} \right] \eta_i' \\
&= \sum_{i=1}^{N} \left[y_i \frac{f(\eta_i)}{F(\eta_i)} - (m_i - y_i) \frac{f(\eta_i)}{1 - F(\eta_i)} \right] \eta_i',
\end{aligned}
$$

The Fisher Information (second derivative using quotient rule) is:

$$
\begin{aligned}
I(\eta_i) &= -\frac{\partial U(\eta_i)}{\partial \eta_i} \\
&= -\sum_{i=1}^{N} \left[y_i \frac{f(\eta)f(\eta) + \eta f(\eta)F(\eta)}{F(\eta)^2} - (m_i - y_i) \frac{-f(\eta)f(\eta) - \eta f(\eta)[1 - F(\eta)]}{[1 - F(\eta)]^2} \right] \eta_i' \eta_i'^T
\end{aligned}
$$

We also have,

$$
\begin{aligned}
\frac{\partial \eta_i}{\partial \beta_0} &= 1 \\
\frac{\partial \eta_i}{\partial \beta_1} &= x_i,
\end{aligned}
$$

where,

$$
\eta_i' \eta_i'^T = X_i X_i^T.
$$

Substitute $\hat{\eta}$ and $\hat{p} = E[y_i]$,

$$
I(\hat{\eta}) = -\sum_{i=1}^{N} \left[6\hat{p} \frac{f(\hat{\eta})f(\hat{\eta}) + \eta f(\hat{\eta})F(\hat{\eta})}{F(\hat{\eta})^2} - (m_i - 6\hat{p}) \frac{-f(\hat{\eta})f(\hat{\eta}) - \hat{\eta} f(\hat{\eta})[1 - F(\hat{\eta})]}{[1 - F(\hat{\eta})]^2} \right] X_i X_i^T \quad (1)
$$

In [7]:
```
X = cbind(1, orings$temp)
x = orings$temp
y = orings$damage
eta = X %*% beta.hat
```

```
f.dash = function(eta) {
    constants = -eta / sqrt(2 * pi)
    exponential = (-eta^2)/2
    return(constants * exp(exponential))
}
```

In [8]:
```
LEFT = (f.dash(eta) * F(eta) - f(eta)^2) / (F(eta)^2)
RIGHT = (f.dash(eta) * (1 - F(eta)) + f(eta)^2) / (1 - F(eta))^2

I11 = (N * F(eta) * LEFT) + (N * F(eta) - N) * RIGHT
I12 = I21 = x %*% I11
I22 = x^2 %*% I11

I = -matrix(c(sum(I11), sum(I21), sum(I12), sum(I22)), 2, 2)
```

In [9]:
```
I
```

30.16329	1931.384
1931.38421	125084.505

In [10]:
```
Iinv = solve(I)

beta0.error = sqrt(Iinv[1,1])
beta1.error = sqrt(Iinv[2,2])
```

In [11]:
```
Iinv
```

2.92831155	-0.0452149904
-0.04521499	0.0007061428

In [12]:
```
z.alpha = qnorm(0.975)

CI.beta0 = beta0 + c(-1, 1) * z.alpha * beta0.error
CI.beta1 = beta1 + c(-1, 1) * z.alpha * beta1.error
```

Answer to Q1 b)

In [13]:
```
rbind(CI.beta0)
rbind(CI.beta1)
```

CI.beta0	2.243931	8.951834
CI.beta1	-0.1579897	-0.05382405

3 Question 1c)

LRT for the temperature parameter β_1

```
In [14]: p.hat = sum(y) / sum(rep(n, length(y)))

         F.model = logL(beta.hat, orings)
         R.model = sum(y)*log(p.hat) + sum(N - y)*log(1 - p.hat)

         LR = -2 * (R.model - F.model)
```

Answer to Q1 c)

```
In [15]: pchisq(LR, df=1, lower=FALSE)
```

5.18671169167522e-06

At the $\alpha = 0.05$ significance level, the LRT concludes that the β_1 parameter is significantly relevant.

4 Question 1d)

Estimate when temeperature $= 31^o F$. Let $t^T = \begin{bmatrix} 1 & 31 \end{bmatrix}$.

```
In [16]: TEMP = 31
         eta = beta0 + beta1 * TEMP

         t = c(1, 31)
         damage.estimate = F(eta)

         CI.ttbeta = F(eta + c(-1, 1) * c(z.alpha * sqrt(t(t) %*% Iinv %*% t)))
```

Answer to Q1 d)

```
In [17]: rbind(damage.estimate)
         rbind(CI.ttbeta)
```

damage.estimate	0.9896872	
CI.ttbeta	0.7115098	0.9999767

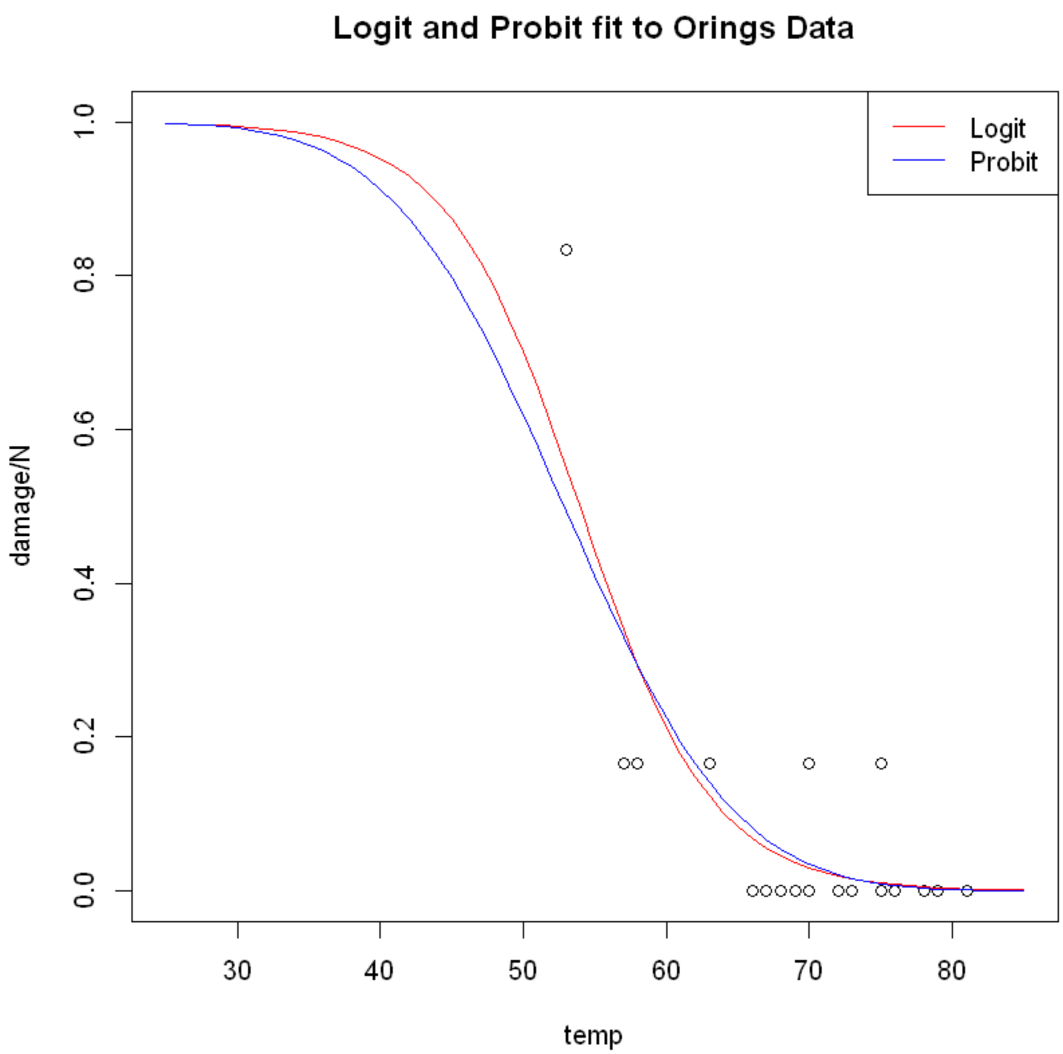
5 Question 1e)

Answer to Q1 e)

```
In [18]: logit = function(eta) exp(eta) / (1 + exp(eta))

         logit.beta = glm(cbind(damage, N - damage) ~ temp, family=binomial(link='logit'), ori
         plot(damage/N ~ temp, orings, xlim=c(25,85), ylim=c(0,1), main="Logit and Probit fit 

         xrange = 25:85
         lines(xrange, logit(logit.beta[1] + logit.beta[2] * xrange), col='red')
         lines(xrange, F(beta0 + beta1 * xrange), col='blue')
         legend("topright", c("Logit","Probit"), col=c("red","blue"), lty=c(1,1))
```



6 Question 2a)

We have,

$$\log(o_i) = \log\left(\frac{p_i}{1-p_i}\right)$$
$$= \beta_0 + \beta_1 x_i.$$

Hence an increase in 10 for bmi is:

$$\log(o)_2 - \log(o)_1 = [\beta_0 + \beta_1(x_i + 10)] - [\beta_0 + \beta_1 x_i]$$
$$= 10\beta_1$$

```
In [24]: load("assign1.Robj", verbose=TRUE)

         INCREASE = 10
         odds = function(p) p / (1 - p)
```

```
Loading objects:
  pima_subset
```

```
In [25]: br = glm(cbind(test, 1 - test) ~ ., data=pima_subset, family=binomial)
         beta.hat = br$coeff

         beta0 = beta.hat[1]
         beta1 = beta.hat[2]
```

Answer to Question 2 a)

```
In [26]: INCREASE * beta1
```

bmi: 0.997168404895279
Estimated increase in log(odds) is 0.9972 when the bmi increase by 10.

7 Question 2b)

```
In [34]: # Standard Error is sqrt(variance)
         ilogit = function(eta) exp(eta) / (1 + exp(eta))
         x = pima_subset$bmi
         eta = beta0 + beta1 * x

         p.hat = ilogit(eta)

         I11 = sum(p.hat * (1 - p.hat))
         I12 = I21 = sum(x * p.hat *(1 - p.hat))
         I22 = sum(x^2 * p.hat * (1 - p.hat))

         Iinv = solve(matrix(c(I11, I12, I21, I22), 2, 2))

         beta1.error = sqrt(Iinv[2,2])
```

```
In [36]: CI.10beta1 = 10 * beta1 + c(-1, 1) * z.alpha * 10 * beta1.error
```

Answer to Question 2 b)

```
In [37]: rbind(CI.10beta1)
```

CI.10beta1	0.6976052	1.296732