

# **Data Structures & Algorithms with C++ for Students in Computer Science**

Hubert Cecotti  
*Spring 2020 (v0.03)*



Department of Computer Science  
College of Science and Mathematics  
FRESNO STATE

# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
1.1	Symbols . . . . .	5
1.2	Sets . . . . .	6
1.3	Definitions . . . . .	8
1.3.1	Asymptotic notation . . . . .	8
1.3.2	Divide and conquer . . . . .	9
1.3.3	Loop invariant . . . . .	9
1.3.4	Master theorem . . . . .	11
1.3.5	Time complexity . . . . .	13
1.4	C++ examples . . . . .	14
1.4.1	Expression value categories . . . . .	14
1.4.2	NULL vs. nullptr . . . . .	14
1.4.3	Include . . . . .	15
1.4.4	Types . . . . .	15
1.4.5	Arrays . . . . .	16
1.4.6	Functions . . . . .	17
1.4.7	Sum and product . . . . .	21
1.4.8	No break, no continue! . . . . .	24
1.4.9	No new, no delete! . . . . .	26
1.5	Recursivity . . . . .	27
1.5.1	Factorial . . . . .	27
1.5.2	Fibonacci . . . . .	29
1.5.3	Anagram . . . . .	30
1.5.4	Hanoi Tower . . . . .	31
<b>2</b>	<b>Queues and Stacks</b>	<b>32</b>
2.1	Stacks . . . . .	32
2.2	Queues . . . . .	35
<b>3</b>	<b>Prototype pattern</b>	<b>38</b>
3.1	Main program . . . . .	38
3.2	Parent class . . . . .	39
3.3	Factory class . . . . .	40
<b>4</b>	<b>Array and sorting</b>	<b>41</b>
4.1	Array . . . . .	42
4.1.1	MyArray interface . . . . .	42

4.1.2	Constructors and destructor . . . . .	44
4.1.3	Access elements . . . . .	45
4.1.4	Find, Delete, Insert . . . . .	46
4.1.5	Binary Search . . . . .	47
4.1.6	Useful functions . . . . .	48
4.1.7	Maximum and Argmax . . . . .	49
4.1.8	Minimum and Argmin . . . . .	50
4.1.9	Mean and Standard Deviation . . . . .	51
4.1.10	FindMaximumSubarray . . . . .	52
4.2	Sorting algorithms . . . . .	55
4.2.1	Selection Sort: $O(n^2)$ . . . . .	56
4.2.2	Insertion Sort: $O(n^2)$ . . . . .	56
4.2.3	Bubble Sort: $O(n^2)$ . . . . .	57
4.2.4	Merge sort: $O(n \cdot \log(n))$ . . . . .	58
4.2.5	Quicksort: $O(n \cdot \log(n))$ . . . . .	60
4.2.6	Some main examples . . . . .	61
4.2.7	Time measurement . . . . .	63
<b>5</b>	<b>Lists</b>	<b>64</b>
5.1	Simple chained list . . . . .	65
5.1.1	Node definition . . . . .	65
5.1.2	Class interface . . . . .	65
5.1.3	Constructor and destructor . . . . .	66
5.1.4	Insert . . . . .	67
5.1.5	Delete . . . . .	72
5.1.6	Display . . . . .	74
5.2	Double chained list . . . . .	75
5.2.1	Node definition . . . . .	75
5.2.2	Insert . . . . .	77
5.2.3	Delete . . . . .	82
5.2.4	Display . . . . .	85
5.2.5	The Sieve of Eratosthenes . . . . .	88
5.3	Circular list . . . . .	90
5.3.1	Class definition . . . . .	90
5.3.2	Insert . . . . .	91
5.3.3	Delete . . . . .	92
5.3.4	Search and display . . . . .	93
5.4	Skip list . . . . .	95
5.4.1	Class definition . . . . .	95
5.4.2	Main functions . . . . .	96
5.4.3	Insert and delete . . . . .	97
5.4.4	Search and display . . . . .	98
5.4.5	Example . . . . .	99

<b>6</b>	<b>Hash tables</b>	<b>100</b>
6.1	Hash tables . . . . .	101
6.1.1	Class definition . . . . .	101
6.1.2	Main functions . . . . .	102
6.1.3	Insert . . . . .	103
6.1.4	Search . . . . .	104
6.1.5	Display . . . . .	106
<b>7</b>	<b>Trees</b>	<b>108</b>
7.1	Binary Search Trees . . . . .	109
7.1.1	Definitions . . . . .	109
7.1.2	TreeNode . . . . .	110
7.1.3	Tree Traversal . . . . .	112
7.1.4	Search . . . . .	116
7.1.5	Insert . . . . .	117
7.1.6	Delete . . . . .	121
7.1.7	Min, Max . . . . .	122
7.1.8	Max and Min Depth . . . . .	123
7.1.9	Comparisons . . . . .	124
7.1.10	MyBST . . . . .	126
7.2	AVL Trees . . . . .	128
7.2.1	Class interface . . . . .	128
7.2.2	Class main functions . . . . .	129
7.2.3	Rotations and balance . . . . .	130
7.2.4	Display . . . . .	133
7.2.5	Search, Insert, Delete . . . . .	134
7.3	23-Trees . . . . .	137
7.3.1	Class definition . . . . .	137
7.3.2	Search . . . . .	140
7.3.3	Insert . . . . .	141
7.3.4	Delete . . . . .	145
7.3.5	Display . . . . .	150
7.4	Red & Black Trees . . . . .	153
7.4.1	Definitions . . . . .	153
7.4.2	Family . . . . .	153
7.4.3	Class definition . . . . .	154
7.4.4	Search . . . . .	158
7.4.5	Insert . . . . .	158
7.4.6	Delete . . . . .	160
<b>8</b>	<b>Heaps</b>	<b>164</b>
8.1	Binary Heaps . . . . .	165
8.2	Priority Queue . . . . .	172
8.3	Fibonacci heaps . . . . .	178
8.3.1	Proof by induction . . . . .	188
8.3.2	Lemma 2 . . . . .	188
8.3.3	Lemma 3 . . . . .	189

<b>9</b>	<b>Graphs</b>	<b>190</b>
9.1	Definitions . . . . .	191
9.2	Adjacency lists . . . . .	194
9.2.1	Breadth First Search . . . . .	201
9.2.2	Depth First Search . . . . .	203
9.3	Shortest paths . . . . .	205
9.3.1	Bellman Ford algorithm . . . . .	206
9.3.2	Dijkstra algorithm . . . . .	208
9.4	Adjacency matrix . . . . .	210
9.4.1	Comparisons . . . . .	213
9.4.2	Display . . . . .	214
9.4.3	Breadth First Search and Depth First Search . . . . .	215
9.5	Dynamic programming . . . . .	217
9.5.1	Definitions . . . . .	217
9.5.2	Back to Fibonacci . . . . .	218
9.6	Shortest path - all pairs . . . . .	220
9.7	Minimum Spanning Tree . . . . .	224
9.7.1	Kruskal . . . . .	226
9.7.2	Prim . . . . .	227
<b>10</b>	<b>Matrix</b>	<b>229</b>
10.1	Class interface . . . . .	229
10.2	Main methods . . . . .	231
10.3	Matrix operations . . . . .	233
10.4	Multiplication of multiple matrices . . . . .	238
	<b>Bibliography</b>	<b>245</b>

## Abstract

The goal of this document is to provide some notes about the main definitions and the main algorithms related to the key data structures that are used in computer science. It includes: arrays, queues, stacks, lists (simple chained, double chained, circular, skip), hash tables, binary trees, ...

This document contains examples in C++ for the development of the main data structures that can be covered during an undergraduate course related to data structures in computer science. The chosen implementation for the different algorithms stresses the readability aspect. It may not represent proper practices for industrial codes, but it includes relevant algorithmic practices with some software engineering practices. In addition, the presented code is aimed at being presented in a document, therefore the choice of the name for the variables should be changed to respect industrial standards and conventions. It is worth noting that data structures such as queues and stacks are readily available in C++. Therefore, this document is for **instructional purposes**, highlighting some key functionalities of C++. It does not aim at presenting all the existing data structures can be found in the Standard Template Library (STL).

Warning: This book uses C++ to present state of the art data structures. It does NOT focus on object oriented programming concepts.

You should keep this document to get prepared for job interviews.

Hubert Cecotti (copyright).

Contact: [hcecotti@csufresno.edu](mailto:hcecotti@csufresno.edu)

## **CSCi 115 - Course outline**

The outline corresponds to the list topics covered in each class, with 3 classes per week. The order of the classes does not represent the order of the chapters and sessions in the book. Some advanced data structures are covered at the end of the course, so important elements that are necessary for the projects are covered in advance.

1. Syllabus introduction - outline and advice
2. C++ implementation guideline for labs and classes.
3. Examples in C++
4. Complexity definitions + Hospital rule
5. Recursivity (Fibonacci + Hanoi tower)
6. Divide and conquer + Master Theorem
7. Time measurements in C++ for performance evaluation in labs.
8. Arrays: insertion/deletion/creation/search
9. Arrays: Selection sort, insert sort, bubble sort
10. Arrays: mergesort and quicksort
11. Queues and Stacks
12. Lists: definition + simple chained
13. Lists: double chained
14. Lists: circular lists
15. Skip lists and applications
16. Hash tables - Definition + Keys
17. Hash Tables Part - Probe functions
18. Sets
19. Trees - Definitions
20. BST: invert tree, insert, delete, search
21. Midterm 1 - revision
22. AVL trees (rotations)
23. AVL trees (insert/delete)
24. 2-3 trees
25. Heaps + Heap sort

26. Definition of Graphs
27. Main graphs problems
28. Graph traversals (BFS)
29. Graph traversals (DFS + Parenthesis)
30. Dynamic programming
31. Greedy algorithms
32. Shortest paths methods - Single source
33. Shortest paths methods - All pairs
34. Minimum spanning tree (Kruskal and Prim) (ITA: Chapter 23)
35. Minimum spanning tree (implementation) + Discussion about Midterm 2
36. B-trees (part 1)
37. B-trees (part 2)
38. Quadtree in images, Octree in 3D
39. Red and Black trees (part 1)
40. Red and Black trees (part 2)
41. Fibonacci heap
42. Questions and answers related to the project
43. Preparation for the finals



# Chapter 1

## Introduction

### Contents

---

<b>1.1</b>	<b>Symbols</b> . . . . .	<b>5</b>
<b>1.2</b>	<b>Sets</b> . . . . .	<b>6</b>
<b>1.3</b>	<b>Definitions</b> . . . . .	<b>8</b>
1.3.1	Asymptotic notation . . . . .	8
1.3.2	Divide and conquer . . . . .	9
1.3.3	Loop invariant . . . . .	9
1.3.4	Master theorem . . . . .	11
1.3.5	Time complexity . . . . .	13
<b>1.4</b>	<b>C++ examples</b> . . . . .	<b>14</b>
1.4.1	Expression value categories . . . . .	14
1.4.2	NULL vs. nullptr . . . . .	14
1.4.3	Include . . . . .	15
1.4.4	Types . . . . .	15
1.4.5	Arrays . . . . .	16
1.4.6	Functions . . . . .	17
1.4.7	Sum and product . . . . .	21
1.4.8	No break, no continue! . . . . .	24
1.4.9	No new, no delete! . . . . .	26
<b>1.5</b>	<b>Recursivity</b> . . . . .	<b>27</b>
1.5.1	Factorial . . . . .	27
1.5.2	Fibonacci . . . . .	29
1.5.3	Anagram . . . . .	30
1.5.4	Hanoi Tower . . . . .	31

---

## 1.1 Symbols

Symbols are used in mathematical and logical expressions. They can also be used in pseudo-code to represent variables. For a better understanding, it is critical to know how to pronounce these different symbols, to establish a direct connection in your mind between their visual representation and their sound when you read them.

### Greek letters

#### Lower case symbols

Greek symbol	$\alpha$	$\beta$	$\delta$	$\epsilon$	$\phi$	$\varphi$	$\gamma$	$\eta$	$\iota$	$\kappa$
English	alpha	beta	delta	epsilon	phi	phi	gamma	eta	iota	kappa
Greek symbol	$\lambda$	$\mu$	$\nu$	$\pi$	$\theta$	$\rho$	$\sigma$	$\tau$	$\upsilon$	$\omega$
English	lambda	mu	nu	pi	theta	rho	sigma	tau	upsilon	omega
Greek symbol	$\xi$	$\psi$	$\zeta$							
English	xi	psi	zeta							

#### Upper case symbols

Greek symbol	$\Delta$	$\Phi$	$\Gamma$	$\Lambda$	$\Pi$	$\Theta$	$\Sigma$	$\Upsilon$	$\Omega$	$\Xi$	$\Psi$
English	delta	phi	gamma	lambda	pi	theta	sigma	upsilon	omega	xi	psi

#### Some useful symbols

- $\exists$  There exists...
- $\exists!$  There exists a unique ...
- $\nexists$  There does not exist ...
- $\forall$  For all...
- $\in$  belongs (example:  $x \in X$ , x belongs to X).
- $\notin$  does not belong (example:  $x \notin X$ , x does not belong to X).
- $\infty$  infinity
- $\emptyset$  empty set (the shape can be more rounded).

## 1.2 Sets

To define the elements in a set, we use curly brackets:  $\{ \}$ . These brackets should not be confused with the beginning and end of an instruction block in C++, or the definition of a static array in C++. Example:  $A = \{1, 2, 3\}$ . The set A contains the elements 1, 2, and 3.

We consider 2 sets A and B.

- A is a subset of B:  $A \subseteq B$

$$\forall a(a \in A \rightarrow a \in B) \quad (1.1)$$

- A is not a subset of B:  $A \not\subseteq B$ .

$$\exists a|a \in A \wedge a \notin B \quad (1.2)$$

- B is a superset of A:  $B \supseteq A$

$$\forall a(a \in A \rightarrow a \in B) \quad (1.3)$$

- B is a not superset of A:  $B \not\supseteq A$

$$\exists a|a \in A \wedge a \notin B \quad (1.4)$$

- If A is a subset of B but A is not equal to B, then A is a proper subset of B:  $A \subset B$ .

$$(\forall a, a \in A \rightarrow a \in B) \wedge (\exists b \in B|b \notin A) \quad (1.5)$$

- If B is a superset of A but B is not equal to A, then B is a proper superset of A:  $B \supset A$ .

$$(\forall a, a \in A \rightarrow a \in B) \wedge (\exists b \in B|b \notin A) \quad (1.6)$$

- A is not a proper subset of B:  $A \not\subset B$

$$(\exists a \in A|a \notin B) \vee (\forall b \in B|b \in A) \quad (1.7)$$

- B is not a proper superset of A:  $B \not\supset A$ .

$$(\exists a \in A|a \notin B) \vee (\forall b \in B|b \in A) \quad (1.8)$$

- Equality, both sets of the same elements:  $A = B$

$$(\forall a, a \in A \rightarrow a \in B) \wedge (a \notin A \rightarrow a \notin B) \quad (1.9)$$

- Complement, the set of elements that do not belong to A:  $C = A'$

$$C = \{c|c \notin A\} \quad (1.10)$$

- Union:  $C = A \cup B$  (or)

$$C = \{c|c \in A \vee c \in B\} \quad (1.11)$$

- Intersection:  $C = A \cap B$  (and)

$$C = \{c | c \in A \wedge c \in B\} \quad (1.12)$$

- Relative complement, objects that belong to A and not to B:  $C = A - B$

$$C = \{a | a \in A \wedge a \notin B\} \quad (1.13)$$

- Symmetric difference, objects that belong to A or B but not to their intersection:  $C = A \Delta B$

$$C = \{c | (c \in A \wedge c \notin B) \vee (c \notin A \wedge c \in B)\} \quad (1.14)$$

- Cardinality, number of elements in the set A:  $|A|$

Here, we define some common sets:

- Natural numbers with 0:  $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$
- Natural numbers without zero:  $\mathbb{N}_1 = \{1, 2, 3, 4, 5, \dots\}$
- Integer numbers set:  $\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$
- Rational numbers set:  $\mathbb{Q} = \{x | x = a/b, (a, b) \in \mathbb{Z}^2 \text{ and } b \neq 0\}$
- Real numbers set:  $\mathbb{R} = \{x | -\infty < x < +\infty\}$
- Complex numbers set:  $\mathbb{C} = \{z | z = a + bi, -\infty < a < +\infty, -\infty < b < +\infty, i = \sqrt{-1}\}$

## 1.3 Definitions

### 1.3.1 Asymptotic notation

Let  $f(n)$  and  $g(n)$  be two functions that map positive integers to positive real numbers.  $n$  is the main parameter of the function. Example:  $f$  is an algorithm to sort an array of size  $n$ ,  $f(n)$  represents the number of “operations” required to sort the array of size  $n$ .

- $\Theta$ : Big Theta, asymptotically tight bound.  $f(n)$  is  $\Theta(g(n))$  (or  $f(n) \in \Theta(g(n))$ ) if and only if  $f(n) \in O(g(n))$  and  $f(n) \in \Omega(g(n))$ .

$$\Theta(g(n)) = \{f(n) : \exists\{c_1, c_2, n_0\} | 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \forall n \geq n_0\} \quad (1.15)$$

- $O$ : Big O, asymptotic upper bound.

$$O(g(n)) = \{f(n) : \exists\{c, n_0\} | 0 \leq f(n) \leq c \cdot g(n) \forall n \geq n_0\} \quad (1.16)$$

- $\Omega$ : Big omega, asymptotic lower bound

$$\Omega(g(n)) = \{f(n) : \exists\{c, n_0\} | 0 \leq c \cdot g(n) \leq f(n) \forall n \geq n_0\} \quad (1.17)$$

- $o$ : little o, upper bound, not asymptotically tight.

$$o(g(n)) = \{f(n) : \forall c > 0, \exists n_0 | 0 \leq f(n) < c \cdot g(n) \forall n \geq n_0\} \quad (1.18)$$

- $\omega$ : little omega, lower bound, not asymptotically tight.

$$\omega(g(n)) = \{f(n) : \forall c > 0, \exists n_0 | 0 \leq c \cdot g(n) < f(n) \forall n \geq n_0\} \quad (1.19)$$

Given the definitions of Big  $\Theta$ , Big  $O$ , small  $o$ , Big  $\Omega$ , and little  $\omega$ , it is possible to resume their relationship in the following table:

Table 1.1: Definitions summary.

Notation	? $c > 0$	? $n_0 \geq 1$	$f(n)$ ? $c \cdot g(n)$
$O()$	$\exists$	$\exists$	$\leq$
$o()$	$\forall$	$\exists$	$<$
$\Omega()$	$\exists$	$\exists$	$\geq$
$\omega()$	$\forall$	$\exists$	$>$

### Examples

It is important to know the meaning of the Big  $O$  notations in relation to their meaning. It is important to separate the notion of complexity from the notion of “how long it takes” because  $O(1)$  that is constant may take a very long time but independent of  $n$ : no matter how much time it takes to do it, it  $n$  increases, it will still be the same amount of time.

- $O(1)$ : constant.
- $O(\log(n))$ : logarithmic (examples: finding an item in a sorted array with a binary search or a balanced search tree).

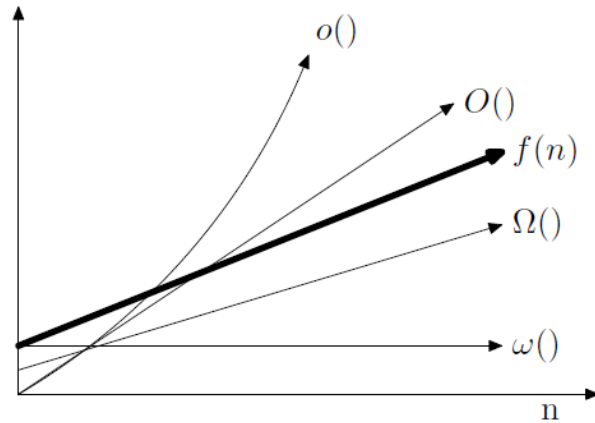


Figure 1.1: Relationships between the notations.

- $O(n)$ : linear (examples: finding an item in an unsorted list or in an unsorted array).
- $O(n \cdot \log(n))$ : log-linear (example: mergesort).
- $O(n^2)$ : quadratic (examples: selection sort and insertion sort).

$\forall n > 0$ , and  $c > 0$  we have:  $n^{c+1} > n^c$ ,  $n > \log(n)$ ,  $n \cdot \log(n) > \log(n) \cdot \log(n)$ .

$$\begin{aligned}
 T(n) &= 5n^3 + 3n \cdot \log(n) + 2n \\
 &\leq 5n^3 + 5n \cdot \log(n) + 5n \\
 &\leq 5n^3 + 5n^3 + 5n^3 \\
 &\leq 15n^3 \\
 &= O(n^3)
 \end{aligned}$$

### 1.3.2 Divide and conquer

The divide and conquer principle has 3 main steps that are resumed here:

1. Divide the problem into a number of sub-problems that are smaller instances of the same problem.
2. Conquer the sub-problems by solving them recursively.
  - If the sub-problems are large enough to solve recursively, solve the recursive case.
  - If the sub-problem sizes are small enough, solve the base case (solve the sub-problems in a straightforward manner).
3. Combine the solutions to the sub-problems into the solution for the original problem.

### 1.3.3 Loop invariant

Determining loop invariant is critical for your understanding of the algorithms and to select wisely where the code should be placed. While loops are basic structures in computer science, they must remain well thought and their goals should be clear. For sorting algorithms and other algorithms with nested loops, it is necessary to determine what will change after the execution of a loop and what set of statements will remain true.

1. Initialization: It is true prior to the first iteration of the loop.
2. Maintenance: If it is true before an iteration of the loop, it remains true before the next iteration.
3. Termination: When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.

### 1.3.4 Master theorem

The Master theorem is defined as follows:

$$T(n) = aT(n/b) + f(n) \quad (1.20)$$

with

- $n$ : size of an input problem (e.g. number of elements in an array).
- $a$ : number of sub-problems, with  $a \geq 1$ .
- $n/b$ : size of the sub problem, with  $b > 1$ .
- $f(n)$ : Divide + Combine operations.  $f(n)$  is an asymptotically positive function.

It means that for  $T(n)$ , we will do  $a$  times  $T$  with an input of size  $n/b$ , and each time we will run  $T$  we will have to run the function  $f$  as well.

The critical exponent is defined by:

$$c_{crit} = \log_b(a) \quad (1.21)$$

In relation to the balance between  $aT(n/b)$  and  $f(n)$ , we can distinguish 3 cases. In the first case, the first part will be more important, in the second case they have equal importance, and in the third case, the seconde part will be more important.

#### Case 1

Recursion tree is leaf heavy:  $aT(n/b) > f(n)$ .

If  $f(n) = O(n^c)$  where  $c < c_{crit}$  then  $T(n) = \Omega(n^{c_{crit}})$ .

#### Case 2

Split/recombine, same as sub-problems:  $aT(n/b) = f(n)$ .

If  $f(n) = \Omega(n^{c_{crit}} \cdot \log^k(n)) \forall k \geq 0$  then  $T(n) = \Omega(n^{c_{crit}} \cdot \log^{k+1}(n))$

#### Case 3

Recursion tree is root heavy:  $aT(n/b) < f(n)$ .

When  $f(n) = \Omega(n^c)$  where  $c > c_{crit}$  and  $a \cdot f(n/b) \leq k \cdot f(n)$  for a constant  $k < 1$  and  $n$  large enough then it is dominated by the splitting term  $f(n)$ , so  $T(n) = \Omega(f(n))$ .

**Remark:** the following expression must remain polynomial

$$\frac{f(n)}{n^{c_{crit}}} \quad (1.22)$$

Therefore:

$$\frac{n/\log(n)}{n} = \frac{1}{\log(n)} \quad (1.23)$$

will not work with the Master Theorem.



## Examples

Table 1.2 presents examples related to the master theorem with the 3 cases, and examples where any of the 3 cases can be applied.

Table 1.2: Examples for case 1, 2, and 3.

$T(n)$	a	b	Case	Notation
$16T(n/4) + n$	16	4	1	$\Theta(n^2)$
$3T(n/2) + n$	3	2	1	$\Theta(n^{\log_2(3)})$
$3T(n/3) + \sqrt{n}$	3	3	1	$\Theta(n)$
$4T(n/2) + cn$	4	2	1	$\Theta(n^2)$
$4T(n/2) + n/\log n$	4	2	1	$\Theta(n^2)$
$4T(n/2) + \log n$	4	2	1	$\Theta(n^2)$
$\sqrt{2}T(n/2) + \log n$	$\sqrt{2}$	2	1	$\Theta(\sqrt{n})$
$4T(n/2) + n^2$	4	2	2	$\Theta(n^2 \cdot \log(n))$
$2T(n/2) + n \log n$	2	2	2	$\Theta(n \cdot \log^2(n))$
$3T(n/3) + n/2$	3	3	2	$\Theta(n \cdot \log(n))$
$T(n/2) + 2^n$	1	2	3	$\Theta(2^n)$
$3T(n/2) + n^2$	3	2	3	$\Theta(n^2)$
$2T(n/4) + n^{0.51}$	2	4	3	$\Theta(n^{0.51})$
$3T(n/4) + n \log n$	4	4	3	$\Theta(n \cdot \log(n))$
$6T(n/3) + n^2 \log n$	6	3	3	$\Theta(n^2 \cdot \log(n))$
$7T(n/3) + n^2$	7	3	3	$\Theta(n^2)$
$16T(n/4) + n!$	16	4	3	$\Theta(n!)$
$2^n T(n/2) + n^n$	$2^n$	2	NO	a: not a constant
$2T(n/2) + n/\log n$	2	2	NO	$f(n)$ : not polynomial
$0.5T(n/2) + 1/n$	0.5	2	NO	$a < 1$
$64T(n/8) - n^2 \log n$	64	8	NO	$f(n)$ not positive

### 1.3.5 Time complexity

This subsection presents the time complexity of the main algorithms that are present in this book. The time complexity using the Big O notation for the main sorting algorithms is presented in Table 1.3. It is worth noting the bad performance in the worst case of the Quicksort as well as the good performance in the best case for the insertion sort and the bubble sort. The comparison between dynamic arrays and linked lists is given in Table 1.4. The time complexity for the different types of tree is given in Table 1.5. Finally, the time complexity for the Fibonacci heap is given in Table 1.6. These algorithms will be presented in the subsequent chapters.

Table 1.3: Sorting algorithms

Algorithm	Best case	Average case	Worst case
Quicksort	$n \cdot \log(n)$	$n \cdot \log(n)$	$n^2$
Mergesort	$n \cdot \log(n)$	$n \cdot \log(n)$	$n \cdot \log(n)$
Insertion	$n$	$n^2$	$n^2$
Selection	$n^2$	$n^2$	$n^2$
Bubble sort	$n$	$n^2$	$n^2$
Heap sort	$n \cdot \log(n)$	$n \cdot \log(n)$	$n \cdot \log(n)$

Table 1.4: Array vs. List

Data structure	Indexing	Insert/Delete		
		Beginning	Middle	End
Dynamic array	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Linked list	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	search + $\Theta(1)$

Table 1.5: Trees in O notation

Data structure	Space		Search		Insert		Delete	
	Average	Worst	Average	Worst	Average	Worst	Average	Worst
Skip list	$n$	$n \cdot \log(n)$	$\log(n)$	$n$	$\log(n)$	$n$	$\log(n)$	$n$
Binary Search Tree	$n$	$n$	$\log(n)$	$n$	$\log(n)$	$n$	$\log(n)$	$n$
AVL Tree	$n$	$n$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$
B Tree	$n$	$n$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$
Red-Black Tree	$n$	$n$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$	$\log(n)$

Table 1.6: Heap (average case)

Data structure	Insert	Find min	Delete min	Decrease key	Decrease key
Fibonacci heap	$\Theta(1)$	$\Theta(1)$	$O(\log(n))$	$\Theta(1)$	$\Theta(1)$

## 1.4 C++ examples

This section is about the C++ programming language. It will certainly not replace a book about C++ and all its specifications. The book defining the specifications of C++ 17 is around 1600 pages. It is worth noting that the present document stresses implementation using C++ and not C. Many algorithms are present in the Standard Template Library (STL), in algorithms, which defines a collection of functions especially designed to be used on ranges of elements. The present document will just highlight a few elements that are used in the next chapters to better understand the implementation of the data structures.

In the next chapters, we consider the definition of typedef double MyType. The goal is to avoid the definition of template in the code that is heavy, and to stress the type of the values that are manipulated in the data structures.

### 1.4.1 Expression value categories

The C++17 standard defines the following expression value categories:

- A **glvalue** is an expression whose evaluation determines the identity of an object, bit-field, or function.
- A **prvalue** is an expression whose evaluation initializes an object or a bit-field, or computes the value of the operand of an operator, as specified by the context in which it appears.
- An **xvalue** is a glvalue that denotes an object or bit-field whose resources can be reused, usually because it is near the end of its lifetime. For instance, some kinds of expressions involving rvalue references yield xvalues, such as a call to a function whose return type is an rvalue reference or a cast to an rvalue reference type.
- An **lvalue** is a glvalue that is not an xvalue.
- An **rvalue** is a prvalue or an xvalue.

Algorithm 1.1: Expression values.

---

```
1 void main() {
2     const int i = 12;
3     int j, l = 12;
4     // i = 13; // cannot change i because it was defined as a const
5     j = i; // j can change its value as it is not a const
6     // a literal such as 56 is a prvalue
7     // 56 = i; // it does not work
8     j = 56; // it works
9     // left side is an expression BUT it returns an lvalue ( j or l)
10    ((j < 30) ? j : l) = 7;
11    // It does not work as it returns an evaluated expression j*2 or l*2
12    // ((j < 30) ? j*2 : l*2) = 7;
13 }
```

---

### 1.4.2 NULL vs. nullptr

The C++09 nullptr keyword represents an rvalue constant that serves as a universal null pointer literal. It replaces the weakly-typed literal 0 and the infamous NULL macro. 0x or NULL is therefore replaced by nullptr. nullptr is a pointer literal of type std::nullptr\_t, it is a prvalue.

### 1.4.3 Include

The using namespace std is to not use std:: in front of cout and other methods of classes that are in the std namespace. Namespaces allow to cluster named entities that would have global scope into narrower scopes otherwise, giving them namespace scope. It provides a way of organizing the elements of programs into different logical scopes referred to by names. A namespace is a declarative region that gives a scope to the identifiers (names of the types, function, variables,...) inside it.

Algorithm 1.2: Includes.

---

```
1 using namespace std;
2 #include <chrono>    // to use functions related to the time
3 #include <string>    // to use strings
4 #include <tuple>     // to use tuple
5 #include <iostream>  // to use cout, cin
6 #include <fstream>   // to use files
7 #include <algorithm> // to use max, min
8 #include <stdlib.h>  // to use rand
9 #include <memory>    // to use unique_ptr
```

---

### 1.4.4 Types

Algorithm 1.3: Type conversion and promotion.

---

```
1 void main() {
2     int i = 5;
3     int j = 2;
4     double d = 2.1;
5     cout << i + d << endl; // 7.1 (double)
6     cout << i * d << endl; // 10.5 (double)
7     cout << d * i << endl; // 10.5 (double)
8     cout << i / d << endl; // 2.38095 (double)
9     cout << d / i << endl; // 0.42 (double)
10    cout << i / j << endl; // 2 (int)
11 }
```

---

### 1.4.5 Arrays

For static arrays, you can specify the maximum size between square brackets. You can determine the elements contained in the array using curly brackets. For dynamic arrays, you must use the keyword **new** to allocate the memory corresponding to the number of elements that are needed. When the array is not needed anymore you must use the keyword **delete[]** to deallocate the array from the memory. In the example, the array `primes` contains 7 elements and has the memory allocated for these elements, while the array `primes1` has only 5 elements in it but 10 elements are allocated.

Algorithm 1.4: Static and dynamic arrays.

---

```
1 void main() {
2     // Static array
3     int s[5]; // 5 elements only, cannot be changed
4     // primes: 7 elements only
5     int primes[] = { 1, 2, 3, 5, 7, 11, 13 };
6     // primes1: 10 elements, 5 first elements are initialized
7     int primes1[10] = { 1, 2, 3, 5, 7 };
8     // Dynamic array
9     int n = 10;
10    int *x = new int[n]; // an array of n int
11    cout << "Number of elements in primes: ";
12    cout << sizeof(primes) / sizeof(int) << endl; // 7
13    cout << "Number of elements in primes1: ";
14    cout << sizeof(primes1) / sizeof(int) << endl; // 10
15    cout << "Number of elements in x: ";
16    cout << sizeof(*x) / sizeof(double) << endl; // 1
17    delete[] x; // delete the array x
18 }
```

---

## 1.4.6 Functions

### Variables

In this example, you can see different types of function. f1 and f2 are functions that return nothing (void), while f3 is a function that return an int\*. In f1, a and n are just as input. In f2, a\*\* and n are inputs, but modify the link to \*a which is \*\*a, so \*a is an output. In f3, n is the input and the function returns a as an output.

Algorithm 1.5: C++ examples.

---

```
1 // void: no return keyword
2 void f1(int* a, int n) {
3     a = new int[n];
4     for (int i = 0; i < n; i++)
5         a[i] = i;
6 }
7 // void: no return keyword
8 void f2(int** a, int n) {
9     *a = new int[n];
10    for (int i = 0; i < n; i++)
11        (*a)[i] = i;
12 }
13 // f3 must return an int*
14 int* f3(int n) {
15     int *a = new int[n];
16     for (int i = 0; i < n; i++)
17         a[i] = i;
18     return a;
19 }
20 void DisplayArray(int* a, int n) {
21     for (int i = 0; i < n; i++)
22         cout << a[i] << ", ";
23     cout << endl;
24 }
```

---

Algorithm 1.6: Main test functions.

---

```

1 void main() {
2     int n = 10;
3     int* a = nullptr;
4     cout << "Evaluate f1" << endl;
5     f1(a,n);
6     //DisplayArray(a,n); // not working because a has nothing !
7     cout << "Evaluate f2" << endl;
8     f2(&a,n);
9     DisplayArray(a, n);
10    cout << "Evaluate f3" << endl;
11    a = nullptr;
12    a=f3(n);
13    DisplayArray(a, n);
14 }

```

---

## Arrays

Reference versus Pointers for arrays. The following examples illustrate the differences between `x++` and `++x`.

Algorithm 1.7: C++ functions.

---

```

1 int f0(int x) {
2     return x++; // done after the return
3 }
4 int f1(int x) {
5     return ++x; // done before the return
6 }
7 void f2(int* x) {
8     cout << x << " " << *x << endl;
9     // 00AFFB3C 6
10    *x++;
11    cout << x << " " << *x << endl;
12    // 00AFFB40 -858993460
13 }
14 void f3(int* x) { // Passing by pointer
15
16     (*x)++;
17 }
18 void f4(int& x) { // Passing by reference
19     x++;
20 }

```

---

Algorithm 1.8: C++ function calls.

---

```
1  int main() {  
2      int x = 6;  
3      cout << f0(x) << endl; // x=6, f0(x) returns 6  
4      cout << f1(x) << endl; // x=6, f1(x) returns 7  
5      f2(&x); // x=6, after f2, x=6  
6      cout << x << endl; // x=6  
7      f3(&x); // x=6, after f3, x=7  
8      cout << x << endl; // x=7  
9      f4(x); // x=7, after f4, x=8  
10     cout << x << endl; // x=8  
11 }
```

---



It is not because you have a \* that you pass the variable as input/output, you need to consider the type of the variable you are manipulating. In the following examples, we consider an array of int v of size n. In the function f8, we consider v as an integer.

Algorithm 1.9: C++ functions.

---

```

1 void f5(int** v, int n) { // Passing by pointer a variable v of type int*
2     *v = new int[n];
3     for (int i = 0; i < n; i++)
4         (*v)[i] = i;
5 }
6 void f6(int* &v, int n) { // Passing by reference a variable v of type int*
7     v = new int[n];
8     for (int i = 0; i < n; i++)
9         v[i] = i;
10 }
11 void f7(int* v, int n) { // Passing by value a variable v of type int*
12     v = new int[n];
13     for (int i = 0; i < n; i++)
14         v[i] = i;
15 }
16 void f8(int* v) { // Passing by pointer a variable v of type int
17     *v = 12;
18 }
19 void f9(int &v) { // Passing by reference a variable v of type int
20     v = 15;
21 }

```

---

The functions f5, f6, f7, and f8 are used with different inputs. You can notice that in f7, v as an int\* has not been modified within the function.

Algorithm 1.10: C++ function calls.

---

```

1 void main() {
2     int *v1= nullptr, *v2= nullptr, *v3= nullptr;
3     int v4, v5;
4     f5(&v2, 5);
5     f6(v1, 5);
6     f7(v3, 5);
7     f8(&v4, 5);
8     f9(v4, 6);
9     cout << v1[3] << endl; // 3
10    cout << v2[3] << endl; // 3
11    cout << v3[3] << endl; // read access violation
12    cout << v4; // 12
13    cout << v5; // 15
14 }

```

---

### 1.4.7 Sum and product

It is important to implement computations and to represent formally a piece of code. In the following examples, you have the description of equations and the corresponding evaluation for a particular value of  $n$ . Note the initialization of  $x$  to 0 for a sum ( $\sum$ ) and to 1 for a product ( $\prod$ ). As the iterator  $i$  is an integer, you have to change its type when it is evaluated when the expected output is a real number, i.e. a variable of type float or double. Summing the values in an array in C++, the values will range from  $i = 0$  (index of the first element) to  $i = n - 1$  (index of the last element, where  $n$  is the size of the array).

Algorithm 1.11: Variables declaration.

---

```

1  int n = 10;
2  double x;
3  double* a = new double[n];
4  double* b = new double[n];
5  for (int i = 0; i < n; i++) {
6      a[i] = (2*(double)i+1)/n;
7      b[i] = (5* (double)i+2)/n;
8  }
```

---

Sum 1: The sum of all the  $i$ , from  $i = 1$  to  $i = n$ .

$$x = \sum_{i=1}^n i \quad (1.24)$$

Sum 2: The sum of all the  $i/n$ , from  $i = 1$  to  $i = n$ .

$$x = \sum_{i=1}^n 1/i \quad (1.25)$$

Sum 3: The sum of all the  $a(i) * b(i)$ , with values of  $i$  ranging from  $i = 0$  to  $i = n - 1$ .

$$x = \sum_{i=0}^{n-1} a(i) \cdot b(i) \quad (1.26)$$

Product 1: The product of all the  $i$ , from  $i = 1$  to  $i = n$ .

$$x = \prod_{i=1}^n i \quad (1.27)$$

Product 1: The product of all the  $a(i) * b(i)$ , from  $i = 0$  to  $i = n - 1$ .

$$x = \prod_{i=0}^{n-1} a(i) \cdot b(i) \quad (1.28)$$

---

Algorithm 1.12: Sum and product.

---

```
1  // Sum 1
2  x = 0.0;
3  for (int i = 1; i <= n; i++)
4      x += i;
5  cout << x << endl; // 55
6  // Sum 2a
7  x = 0.0;
8  for (int i = 1; i <= n; i++)
9      x += 1/i;
10 cout << x << endl; // 1
11 // Sum 2b
12 x = 0.0;
13 for (int i = 1; i <= n; i++)
14     x += 1.0 / i;
15 cout << x << endl; // 2.93
16 // Sum 2c
17 x = 0.0;
18 for (int i = 1; i <= n; i++)
19     x += 1/(double)i;
20 cout << x << endl; // 2.93
21 // Sum 3
22 x = 0.0;
23 for (int i = 0; i < n; i++)
24     x += a[i]*b[i];
25 cout << x << endl; // 32.8
26 // Product 1
27 x = 1.0;
28 for (int i = 1; i <= n; i++)
29     x *= i;
30 cout << x << endl; // 3.63e+06
31 // Product 2
32 x = 1.0;
33 for (int i = 0; i < n; i++)
34     x *= a[i]*b[i];
35 cout << x << endl; // 26
36 }
```

---

This example illustrates the notion of reference, and pointer arithmetic.

Algorithm 1.13: C++ examples.

---

```
1 void main() {
2     // Access by the reference
3     int x = 25;
4     int &y = x;
5     y = 12;
6     cout << "x:" << x << endl;
7     cout << "y:" << x << endl;
8     // Increment example
9     int i = 3;
10    cout << i++ << endl;
11    int j = 3;
12    cout << ++j << endl;
13    // Pointer shifting example
14    int n = 10;
15    int* a1 = new int[n];
16    for (int i = 0; i < n; i++)
17        a1[i] = i * 2;
18    int* b1 = a1;
19    cout << b1[2] << endl;
20    cout << *(b1+3) << endl;
21 }
```

---

### 1.4.8 No break, no continue!

This examples illustrates different ways to find an element in an array, with a for loop using a break, a while loop, and a do while loop. In the remaining part of this document, there will be no break or continue.

Algorithm 1.14: For loop with break.

---

```
1 void Search_v01(int* a, int n, int x) {
2     bool find = false;
3     int argx = -1;
4     for (int i = 0; i < n; i++) {
5         if (a[i] > x) {
6             find = true;
7             argx = i;
8             break;
9         }
10    }
11    if (find)
12        cout << "Found " << a[argx] << endl;
13    else
14        cout << "Not found" << endl;
15 }
```

---

Algorithm 1.15: While loop.

---

```
1 void Search_v02(int* a, int n, int x) {
2     bool find = false;
3     int argx = -1, i = 0;
4     while ((!find) && (i < n)) {
5         if (a[i] > x) {
6             find = true;
7             argx = i;
8         }
9         i++;
10    }
11    if (find)
12        cout << "Found " << a[argx] << endl;
13    else
14        cout << "Not found" << endl;
15 }
```

---

Algorithm 1.16: Do While loop.

---

```
1 void Search_v03(int* a, int n, int x) {
2     bool find = false;
3     int argx = -1, i = 0;
4     do {
5         if (a[i] > x) {
6             find = true;
7             argx = i;
8         }
9         i++;
10    } while ((!find) && (i < n));
11    if (find)
12        cout << "Found " << a[argx] << endl;
13    else
14        cout << "Not found" << endl;
15
16 }
```

---

Algorithm 1.17: Main.

---

```
1 void main() {
2     int n = 10, x = 100;
3     int* a = new int[n];
4     for (int i = 0; i < n; i++)
5         a[i] = rand() % 200;
6     Search_v01(a, n, x);
7     Search_v02(a, n, x);
8     Search_v03(a, n, x);
9 }
```

---

### 1.4.9 No new, no delete!

In this section, we highlight different possibilities for the creation of objects. In particular, the use of `unique_ptr` allow to forget the `delete` once the function is finished.

Algorithm 1.18: Class definition.

---

```
1 class MyClass {
2 public:
3     MyClass() {
4         x = 0;
5         cout << "Constructor " << x << endl;
6     }
7     MyClass(int x1) {
8         x = x1;
9         cout << "Constructor " << x << endl;
10    }
11    ~MyClass() {
12        cout << "Destructor " << x << endl;
13    }
14    int x;
15 };
```

---

In this example, the constructor functions are called in the order for Object1, Object2, and Object3. For the destructor function, the calls are for Object2 (heap), Object3 (heap), and then Object1 (stack).

Algorithm 1.19: Example.

---

```
1 void main() {
2     MyClass Object1(1); // Memory allocated on the stack
3     MyClass *Object2 = new MyClass(2); // Memory allocated on the heap
4     unique_ptr<MyClass> Object3(new MyClass(3));
5     cout << "Object1:" << Object1.x << endl;
6     cout << "Object2:" << Object2->x << endl; // pointer
7     cout << "Object3:" << Object3->x << endl; // pointer
8     delete Object2; // Must use delete free the memory
9     // Object3 is free automatically thanks to the use of unique_ptr
10    // Object1 is free automatically at the end the function
11 }
```

---

## 1.5 Recursivity

Recursivity is a key concept in mathematic and computer science. Depending on the problems, some algorithms can be better formulated using recursivity while some others are better with an iterative version.

- **A simple base case (or simple cases):** It corresponds to a terminating scenario that does not use recursion to produce an output. If you get a stack overflow error in an algorithm, it may be because you have not properly defined the simple case, which cannot be reached, leading to an infinite recursion.
- **A recursive step:** It corresponds to a set of rules that reduces all other cases toward the base case.

### 1.5.1 Factorial

The factorial function is defined as follows:

$$Factorial(1) = 1 \quad (1.29)$$

$$Factorial(x) = x \cdot Factorial(x - 1) \quad (1.30)$$

It is possible to implement the factorial function in various ways. Factorial1 uses a for loop while Factorial2 uses a while loop, and Factorial3 uses a recursive approach in which the function Factorial3 is called.

Algorithm 1.20: Factorial functions - iterative version.

---

```
1 // iterative function with for a loop
2 int Factorial1(int x) {
3     int result = 1;
4     for (int i = 2; i <= x; i++)
5         result *= i;
6     return result;
7 }
8
9 // iterative function with for a while loop
10 int Factorial2(int x) {
11     int result = 1;
12     int i = 2;
13     while (i <= x) {
14         result *= i;
15         i++;
16     }
17     return result;
18 }
```

---

Algorithm 1.21: Factorial functions - recursive version.

---

```
1 // recursive function
2 int Factorial3(int x) {
3     if (x <= 1)
4         return 1; // base case
5     else
6         return x*Factorial3(x - 1); // recursive call
```

---



```

7  }
8
9  // recursive procedure, passing the input/output variable by pointer
10 void Factorial4(int x, int* y) {
11     if (x <= 1)
12         *y = 1; // base case
13     else {
14         int y1;
15         Factorial4(x - 1, &y1); // recursive call
16         *y = x*y1;
17     }
18 }
19
20 // recursive procedure, passing the input/output variable by reference
21 void Factorial5(int x, int &y) {
22     if (x <= 1)
23         y = 1; // base case
24     else {
25         int y1;
26         Factorial5(x - 1, y1); // recursive call
27         y = x*y1;
28     }
29 }

```

---

Algorithm 1.22: Factorial examples.

```

1  void main() {
2      cout << Factorial1(6) << endl; // 720
3      cout << Factorial2(6) << endl; // 720
4      cout << Factorial3(6) << endl; // 720
5      int y1;
6      Factorial4(6, &y1);
7      cout << y1 << endl; // 720
8      int y2;
9      Factorial5(6, y2);
10     cout << y2 << endl; // 720
11 }

```

---

## 1.5.2 Fibonacci

This famous sequence is from Leonardo Bonacci, Italian mathematician from the Republic of Pisa, commonly named Fibonacci. In this sequence, each number is the sum of the two preceding ones.

The sequence is defined as follows:

$$Fibonacci(0) = 0 \quad (1.31)$$

$$Fibonacci(1) = 1 \quad (1.32)$$

$$Fibonacci(x) = Fibonacci(x - 1) + Fibonacci(x - 2) \quad (1.33)$$

Algorithm 1.23: Fibonacci function.

---

```
1 int Fibonacci(int x) {
2     if (x == 0)
3         return 0;
4     else if (x == 1)
5         return 1;
6     else
7         return Fibonacci(x - 1) + Fibonacci(x - 2);
8 }
```

---

Algorithm 1.24: Fibonacci example.

---

```
1 void main() {
2     cout << Fibonacci(6) << endl; // 8
3 }
```

---

### 1.5.3 Anagram

Algorithm 1.25: Rotate.

---

```
1 // Method to rotate left all characters from position to end
2 void Rotate(char* str, int size, int newsize) {
3     int position = size - newsize;
4     char temp = str[position];
5     int i;
6     for (i = position + 1; i < size; i++) {
7         str[i-1] = str[i];
8     }
9     str[i-1] = temp;
10 }
```

---

Algorithm 1.26: Do anagram.

---

```
1 void DoAnagram(char* str, int size, int newsize) {
2     if (newsize > 1) {
3         for (int loop = 0; loop < newsize; loop++) {
4             DoAnagram(str, size, (newsize - 1));
5             if (newsize == 2) {
6                 for (int i = 0; i < size; i++) {
7                     cout << str[i];
8                 }
9                 cout << endl;
10            }
11            Rotate(str, size, newsize);
12        }
13    }
14 }
```

---

Algorithm 1.27: Anagram example.

---

```
1 void main() {
2     char mystr[] = { 'R', 'A', 'T', 'S' };
3     DoAnagram(mystr, 4, 4);
4     // RATS, RAST, RTSA, RTAS, RSAT, RSTA, ATSR, ATRS, ASRT, ASTR,
5     // ARTS, ARST, TSRA, TSAR, TRAS, TRSA, TASR, TARS, SRAT, SRTA,
6     // SATR, SART, STRA, STAR
```

---

### 1.5.4 Hanoi Tower

The Hanoi Tower is a famous puzzle game. It consists of three rods and a number of disks of different sizes. The disks can be placed onto any rod. The puzzle starts with the disks in a stack in ascending order of size on one stick, the smallest at the top. The goal is to move the whole stack to another rod, following these rules:

- One disk can be moved at a time.
- Each action consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.
- It is possible to place disk of diameter  $d_1$  on top of disk  $d_2$  only if  $d_1 < d_2$ .

Thinking of the Hanoi Tower solution in an iterative way is a difficult problem. It is a perfect example of thinking: “what if this part is already done, what else would I need to do”, where the base case, i.e. the easy case, has to be determined first. It is an example in which it is better to think about the last steps than the first steps to execute.

Algorithm 1.28: Solve Hanoi Tower.

---

```
1 void HanoiTower(int n, string start, string auxiliary, string end) {
2     if (n == 1)
3         cout << start << " -> " << end << endl;
4     else {
5         HanoiTower(n - 1, start, end, auxiliary);
6         cout << start << " -> " << end << endl;
7         HanoiTower(n - 1, auxiliary, start, end);
8     }
9 }
```

---

Algorithm 1.29: Hanoi tower example.

---

```
1 void main() {
2     int n = 5; // number of disks
3     HanoiTower(n, "A", "B", "C");
4     // A->C, B->A, B->C, A->C, A->B, C->B, C->A, B->A, C->B, A->C,
5     // A->B, C->B, A->C, B->A, B->C, A->C, B->A, C->B, C->A, B->A,
6     // B->C, A->C, A->B, C->B, A->C, B->A, B->C, A->C.
7 }
```

---

## Chapter 2

# Queues and Stacks

### Contents

2.1	Stacks . . . . .	32
2.2	Queues . . . . .	35

## 2.1 Stacks

A stack

Algorithm 2.1: Stack - class definition

```
1  typedef double MyType;
2  class MyStack {
3  public:
4      MyStack ();
5      MyStack (int capacity);
6      ~MyStack ();
7      bool IsFull ();
8      bool IsEmpty ();
9      MyType Pop ();
10     MyType Top ();
11     void Push (MyType x);
12     void Display ();
13 public:
14     MyType* s;
15     int capacity;
16     int size;
17 };
```

---

Algorithm 2.2: Stack - functions

---

```
1  MyStack::MyStack() {
2      s = nullptr;
3      capacity = 0;
4      size = 0;
5  }
6  MyStack::MyStack(int capacity1) {
7      capacity = capacity1;
8      s = new MyType[ capacity ];
9      size = 0;
10 }
11 MyStack::~MyStack() {
12     delete[] s;
13 }
14 bool MyStack::IsFull() {
15     return (size == capacity);
16 }
17 bool MyStack::IsEmpty() {
18     return (size == 0);
19 }
20 MyType MyStack::Pop() {
21     if (!IsEmpty()) {
22         size--;
23         return s[ size ];
24     }
25 }
26 MyType MyStack::Top() {
27     return s[ size - 1 ];
28 }
29 void MyStack::Push(MyType x) {
30     if (size < capacity) {
31         s[ size ] = x;
32         size++;
33     }
34 }
35 void MyStack::Display() {
36     cout << "Max capacity: " << capacity << endl;
37     cout << "Size: " << size << endl;
38     for (int i = 0; i < size; i++)
39         cout << "Element: " << s[i] << " at position " << i << endl;
40     cout << endl;
41 }
```

---

### Algorithm 2.3: Example

---

```
1 void main() {  
2     MyStack* S = new MyStack(5);  
3     S->Push(4);  
4     S->Push(6);  
5     S->Push(8);  
6     S->Push(10);  
7     S->Push(12);  
8     S->Push(16);  
9     S->Push(18);  
10    S->Display();  
11    cout << "Pop: " << S->Pop() << endl;  
12    cout << "Pop: " << S->Pop() << endl;  
13    cout << "Top: " << S->Top() << endl;  
14    S->Push(20);  
15    S->Display();  
16    delete s;  
17 }
```

---

## 2.2 Queues

Algorithm 2.4: Circular queue - class definition.

---

```
1 typedef double MyType;
2 class MyQueue {
3 public:
4     MyQueue();
5     MyQueue(int capacity);
6     ~MyQueue();
7     bool IsFull();
8     bool IsEmpty();
9     void Enqueue(MyType x);
10    MyType Dequeue();
11    MyType Front();
12    MyType Rear();
13    void DisplayQueue();
14    void DisplayAll();
15 public:
16     int front, rear, size;
17     int capacity;
18     MyType* q;
19 };
```

---

Notice how the beginning of the queue is at position front and not at position 0 of the array.

Algorithm 2.5: Display Queue.

---

```
1 // Display the elements in the queue in the order they should be dequeued
2 void MyQueue::DisplayQueue() {
3     cout << "Size: " << size << endl;
4     for (int i = 0; i < size; i++)
5         cout << "Element: " << q[(front+i)%capacity] << " at position " << i << endl;
6     cout << endl;
7 }
8
9 // Display all the elements in the array sorting the values of the queue
10 void MyQueue::DisplayAll() {
11     cout << "Max capacity: " << capacity << endl;
12     for (int i = 0; i < capacity; i++)
13         cout << "Element: " << q[i] << " at position " << i << endl;
14     cout << endl;
15 }
```

---



Algorithm 2.6: Circular queue - functions.

---

```
1  MyQueue::MyQueue() {
2      capacity = 0;
3      front=size=0;
4      rear=capacity-1;
5      q= nullptr;
6  }
7  MyQueue::~~MyQueue() {
8      delete[] q;
9  }
10 MyQueue::MyQueue(int capacity1) {
11     capacity=capacity1;
12     front=size=0;
13     rear=capacity-1; // important, see the enqueue
14     q=new MyType[ capacity ];
15 }
16 bool MyQueue::IsFull() {
17     return (size==capacity);
18 }
19 bool MyQueue::IsEmpty() {
20     return (size==0);
21 }
22 void MyQueue::Enqueue(MyType x) {
23     if (!IsFull()) {
24         rear = (rear + 1) % capacity;
25         q[rear] = x;
26         size = size + 1;
27     }
28 }
29 MyType MyQueue::Dequeue() {
30     if (IsEmpty())
31         return INT_MIN;
32     MyType item = q[front];
33     front=(front+1)%capacity;
34     size=size-1;
35     return item;
36 }
37 MyType MyQueue::Front() {
38     if (IsEmpty())
39         return INT_MIN;
40     return q[front];
41 }
42 MyType MyQueue::Rear() {
43     if (IsEmpty())
44         return INT_MIN;
45     return q[rear];
46 }
```

---

### Algorithm 2.7: Example

---

```
1 void main() {
2     MyQueue* Q = new MyQueue(4);
3     Q->Enqueue(4);
4     Q->Enqueue(6);
5     Q->Enqueue(8);
6     Q->Enqueue(10);
7     Q->Enqueue(12);
8     Q->Enqueue(16);
9     Q->Enqueue(18);
10    Q->Display();
11    cout << "Dequeue: " << Q->Dequeue() << endl;
12    cout << "Dequeue: " << Q->Dequeue() << endl;
13    Q->Enqueue(20);
14    Q->Enqueue(22);
15    Q->Enqueue(24);
16    Q->Enqueue(26);
17    Q->Display();
18    cout << "Dequeue: " << Q->Dequeue() << endl;
19    Q->Enqueue(28);
20    Q->Display();
21    delete Q;
22 }
```

---

## Chapter 3

# Prototype pattern

### Contents

---

3.1	Main program . . . . .	38
3.2	Parent class . . . . .	39
3.3	Factory class . . . . .	40

---

The goal of this section is to show how you can create the prototype pattern. It is one of the 23 well-known Gang of Four design patterns. The first part is the main program, which creates dynamically a data structure. In the present case, the code 1 indicates that we want to create a data structure array.

### 3.1 Main program

Algorithm 3.1: Main program

---

```
1  #include <iostream>
2  #include <tuple>
3  using namespace std;
4  #include "MyDataStructure.h"
5  #include "MyArray.h"
6  #include "MySCList.h"
7  #include "MyDCList.h"
8  #include "DataStructureFactory.h"
9
10 int main() {
11     MyDataStructure* ds;
12     ds = DataStructureFactory::makeDataStructure(1); // 1 = MyArray
13     int n = 10;
14     for (int i = 0; i < n; i++) {
15         ds->Insert(i * 10 + 2);
16     }
17     ds->Display();
18     delete ds;
19     return 0;
20 }
```

---

## 3.2 Parent class

This part corresponds to the class `MyDataStructure`, which is the parent class of the classes related to each data structure.

Algorithm 3.2: `MyDataStructure.h`

---

```
1 #include <tuple>
2 #include <iostream>
3 typedef double MyType;
4 extern void Swap(MyType *r, MyType *s);
5 class MyDataStructure {
6 public:
7     MyDataStructure();
8     ~MyDataStructure();
9     virtual MyDataStructure* clone() = 0;
10    virtual void MyDataStructure::Insert(MyType x) { }
11    virtual void MyDataStructure::Delete(MyType x) { }
12    virtual bool MyDataStructure::Search(MyType x) { return false; }
13    virtual void MyDataStructure::Display() {}
14};
```

---

Algorithm 3.3: `MyDataStructure.cpp`

---

```
1 #include "MyDataStructure.h"
2 void Swap(MyType *r, MyType *s) {
3     MyType tmp = *r;
4     *r = *s;
5     *s = tmp;
6 }
7 MyDataStructure::MyDataStructure() {}
8 MyDataStructure::~MyDataStructure() {}
```

---

### 3.3 Factory class

This part corresponds to the class `DataStructureFactory`, which is how we create data structures, children of the class `MyDataStructure`.

Algorithm 3.4: `DataStructureFactory.h`

---

```
1 #pragma once
2 #include "MyDataStructure.h"
3 const int N = 8; // number of data structures
4 class DataStructureFactory {
5 public:
6     static MyDataStructure* makeDataStructure(int choice);
7 private:
8     static MyDataStructure* mDataStructureTypes[N];
9 };
```

---

Algorithm 3.5: `DataStructureFactory.cpp`

---

```
1 #include "MySCList.h"
2 #include "MyDCList.h"
3 MyDataStructure* DataStructureFactory::mDataStructureTypes[] =
4 {
5     0, new MyArray, new MySCList, new MyDCList,
6     new MyCList, new MySkipList, new MyHashTable, new MyBST
7 };
8 MyDataStructure* DataStructureFactory::makeDataStructure(int choice) {
9     return mDataStructureTypes[choice] -> clone();
10 }
```

---

# Chapter 4

## Array and sorting

### Contents

---

<b>4.1</b>	<b>Array</b>	<b>42</b>
4.1.1	MyArray interface	42
4.1.2	Constructors and destructor	44
4.1.3	Access elements	45
4.1.4	Find, Delete, Insert	46
4.1.5	Binary Search	47
4.1.6	Useful functions	48
4.1.7	Maximum and Argmax	49
4.1.8	Minimum and Argmin	50
4.1.9	Mean and Standard Deviation	51
4.1.10	FindMaximumSubarray	52
<b>4.2</b>	<b>Sorting algorithms</b>	<b>55</b>
4.2.1	Selection Sort: $O(n^2)$	56
4.2.2	Insertion Sort: $O(n^2)$	56
4.2.3	Bubble Sort: $O(n^2)$	57
4.2.4	Merge sort: $O(n \cdot \log(n))$	58
4.2.5	Quicksort: $O(n \cdot \log(n))$	60
4.2.6	Some main examples	61
4.2.7	Time measurement	63

---

## 4.1 Array

### 4.1.1 MyArray interface

The class MyArray includes state of the art methods that are typically used with arrays. It includes insert, delete, search, sorting algorithms, and other useful algorithms.

Algorithm 4.1: MyArray.h

---

```
1 #pragma once
2 #include "MyDataStructure.h"
3 #include <chrono>
4 #include <string>
5
6 class MyArray : public MyDataStructure {
7 public:
8     // constructor
9     MyArray();
10    MyArray(int n);
11    // destructor
12    ~MyArray();
13    MyDataStructure* clone() { return new MyArray(); }
14    // Accessor + Modifiers
15    int GetSize() const;
16    MyType GetElement(int i) const;
17    void SetElement(int i, MyType x);
18    bool Find(MyType x);
19    pair<bool, int> BinarySearch1(MyType x);
20    pair<bool, int> BinarySearch2(MyType x);
21    void Delete(MyType x);
22    void Insert(MyType x);
23    bool Search(MyType x);
24    void Display();
25    void Display(int low, int high);
26    void DisplayFile();
27    void DisplayFileC();
28    void Invert();
29    MyArray* FindOdd();
30    MyType GetMax();
31    pair<MyType, int> GetMaxArg();
32    MyType GetMin();
33    pair<MyType, int> GetMinArg();
34    double GetAverage();
35    double GetStandardDeviation();
36    MyType& operator[] (unsigned i);
37    MyArray* operator+(const MyArray* a);
38    tuple<int, int, int> FindMaxCrossingSubarray(int low, int mid, int high);
39    tuple<int, int, int> FindMaximumSubarray(int low, int high);
40    // Sorting functions
41    bool IsSorted();
42    void SwapIndex(int i, int j);
43    void DisplayStep(string f);
44    // Init functions
45    void InitRandom(int v);
```

```
46     void InitSortedAscending(int v);
47     void InitSortedDescending(int v);
48     // Sorting algorithms
49     void SelectionSort();
50     void InsertionSort();
51     void BubbleSort();
52     void BubbleOptSort();
53     void MergeSort();
54     void QuickSort();
55 private:
56     MyType* a; // array
57     int n; // size of the array
58 };
```

---



### 4.1.2 Constructors and destructor

Algorithm 4.2: Constructors and destructor (MyArray.cpp)

---

```
1  // Number of steps
2  // to count how many main steps are done in an algorithm.
3  int step;
4
5  // Default constructor
6  MyArray::MyArray() {
7      a = nullptr;
8      n = 0;
9  }
10
11 // Basic constructor
12 // Create an array of size n1
13 MyArray::MyArray(int n1) {
14     n = n1;
15     a = new MyType[n];
16     for (int i = 0; i < n; i++)
17         a[i] = 0;
18 }
19
20 // Destructor
21 MyArray::~MyArray() {
22     delete[] a;
23 }
```

---

### 4.1.3 Access elements

Algorithm 4.3: Access elements (MyArray.cpp).

---

```
1  // Return the number of elements in the array
2  int MyArray::GetSize() const { return n; }
3
4  MyType MyArray::GetElement(int i) const {
5      if (i < 0) {
6          cout << "Too small index";
7          exit(EXIT_FAILURE);
8      } else if (i >= n) {
9          cout << "Too large index";
10         exit(EXIT_FAILURE);
11     } else
12         return a[i];
13 }
14
15 // a is private
16 // we have a function to set the value x at the position i
17 void MyArray::SetElement(int i, MyType x) {
18     a[i] = x;
19 }
20
21 MyType& MyArray::operator[] (unsigned i) {
22     try {
23         return a[i];
24     } catch (exception& e) {
25         cout << "Problem with index" << e.what() << endl;
26     }
27 }
28
29 // Concatenate two arrays with the + operator
30 MyArray* MyArray::operator+(const MyArray* a) {
31     MyArray* out = new MyArray(this->n + a->GetSize());
32     for (int i = 0; i < this->n; i++)
33         out->SetElement(i, this->GetElement(i));
34     for (int i = this->n; i < out->GetSize(); i++)
35         out->SetElement(i+this->n, a->GetElement(i));
36     return out;
37 }
```

---

#### 4.1.4 Find, Delete, Insert

// Determine if the value x is in the array

Algorithm 4.4: Find, Delete, Insert.

---

```
1  bool MyArray::Find(MyType x) {
2      for (int i = 0; i < n; i++) {
3          if (a[i] == x) {
4              return true;
5          }
6      }
7      return false;
8  }
9
10 void MyArray::Delete(MyType x) {
11     if (Find(x)) {
12         MyType* a1 = new MyType[n - 1];
13         int j = 0;
14         for (int i = 0; i < n; i++) {
15             if (a[i] != x) {
16                 a1[j] = a[i];
17                 j++;
18             }
19         }
20         delete[] a;
21         a = a1;
22     }
23 }
24
25 void MyArray::Insert(MyType x) {
26     MyType* a1 = new MyType[n + 1];
27     for (int i = 0; i < n; i++) {
28         a1[i] = a[i];
29     }
30     a1[n] = x;
31     delete[] a;
32     a = a1;
33     n++;
34 }
```

---

### 4.1.5 Binary Search

The binary search is presented in the iterative and recursive versions.

Algorithm 4.5: Binary search.

---

```
1  // Return if the value x is in the array or not, and the index of the value.
2  // Iterative version
3  pair<bool, int> MyArray::BinarySearch1(MyType x) {
4      bool found = false;
5      int mid, low = 0, high = n-1;
6      while ((low <= high) && (!found)) {
7          mid = (low + high) / 2;
8          if (x==a[mid])
9              return make_pair(true, mid);
10         else {
11             if (x < a[mid])
12                 high = mid - 1;
13             else
14                 low = mid + 1;
15         }
16     }
17     return make_pair(false, -1);
18 }
19 // Recursive version
20 pair<bool, int> BinarySearchRec(MyType* a, int x, int low, int high) {
21     int mid;
22     if (low>high) // not found
23         return make_pair(false, -1);
24     else {
25         mid= (low + high) / 2;
26         if (x == a[mid])
27             return make_pair(true, mid);
28         else {
29             if (x < a[mid])
30                 return BinarySearchRec(a, x, low, mid - 1);
31             else
32                 return BinarySearchRec(a, x, mid + 1, high);
33         }
34     }
35 }
36
37 pair<bool, int> MyArray::BinarySearch2(MyType x) {
38     return BinarySearchRec(a, x, 0, n - 1);
39 }
```

---

### 4.1.6 Useful functions

Algorithm 4.6: Invert (MyArray.cpp).

---

```
1 void MyArray::Invert() {
2     MyType tmp;
3     for (int i = 0; i < n/2; i++) {
4         tmp = a[i];
5         a[i] = a[n - 1 - i];
6         a[n - 1 - i] = tmp;
7     }
8 }
```

---

Algorithm 4.7: Array of odd numbers (MyArray.cpp).

---

```
1 MyArray* MyArray::FindOdd() {
2     int nodd = 0;
3     for (int i = 0; i < n; i++) {
4         if (((int)a[i] % 2) == 1)
5             nodd++;
6     }
7     nodd = 0;
8     MyArray* a1 = new MyArray(nodd);
9     for (int i = 0; i < n; i++) {
10        if (((int)a[i] % 2) == 1) {
11            a1->SetElement(nodd, a[i]);
12            nodd++;
13        }
14    }
15    return a1;
16 }
```

---

Algorithm 4.8: Search and return index (MyArray.cpp).

---

```
1 bool MyArray::Search(MyType x) {
2     for (int i = 0; i < n; i++) {
3         if (a[i] == x) {
4             return true;
5         }
6     }
7     return false;
8 }
```

---

### 4.1.7 Maximum and Argmax

Algorithm 4.9: Maximum (MyArray.cpp).

---

```
1 MyType MyArray::GetMax() {
2     if (n > 0) {
3         MyType max = a[0];
4         for (int i = 1; i < n; i++) {
5             if (a[i] > max)
6                 max = a[i];
7         }
8         return max;
9     }
10    else
11        return 0;
12 }
```

---

Algorithm 4.10: Maximum and argmax (MyArray.cpp).

---

```
1 pair<MyType, int> MyArray::GetMaxArg() {
2     if (n > 0) {
3         MyType max = a[0];
4         int argmax = 0;
5         for (int i = 1; i < n; i++) {
6             if (a[i] > max) {
7                 max = a[i];
8                 argmax = i;
9             }
10        }
11        return std::make_pair(max, argmax);
12    }
13    else
14        return std::make_pair(0, -1);
15 }
```

---

### 4.1.8 Minimum and Argmin

Algorithm 4.11: Minimum (MyArray.cpp).

---

```
1 MyType MyArray::GetMin() {
2     if (n > 0) {
3         MyType min = a[0];
4         for (int i = 1; i < n; i++) {
5             if (a[i] < min)
6                 min = a[i];
7         }
8         return min;
9     }
10    else
11        return 0;
12 }
```

---

Algorithm 4.12: Minimum and argmin (MyArray.cpp).

---

```
1 pair<MyType, int> MyArray::GetMinArg() {
2     if (n > 0) {
3         MyType min = a[0];
4         int argmin = 0;
5         for (int i = 1; i < n; i++) {
6             if (a[i] < min) {
7                 min = a[i];
8                 argmin = i;
9             }
10        }
11        return std::make_pair(min, argmin);
12    }
13    else
14        return std::make_pair(0, -1);
15 }
```

---

### 4.1.9 Mean and Standard Deviation

Algorithm 4.13: Basic statistic (MyArray.cpp).

---

```
1  double MyArray::GetAverage() {
2      double result = 0;
3      if (n > 0) {
4          for (int i = 0; i < n; i++)
5              result += (double)a[i];
6          result /= n;
7      }
8      return result;
9  }
10
11 double MyArray::GetStandardDeviation() {
12     double result = 0;
13     if (n > 0) {
14         double mean = GetAverage();
15         for (int i = 0; i < n; i++) {
16             double tmp = (double)a[i] - mean;
17             result += tmp*tmp;
18         }
19         result /= n;
20         result = sqrt(result);
21     }
22     return result;
23 }
```

---



#### 4.1.10 FindMaximumSubarray

The FindMaximumSubarray algorithm is a state of the art example of the divide and conquer approach.

Algorithm 4.14: FindMaxCrossingSubarray (MyArray.cpp).

---

```
1 tuple<int ,int ,int> MyArray::FindMaxCrossingSubarray(int low ,int mid ,int high) {
2     int left_sum = -INT_MAX, right_sum= -INT_MAX; // -infinity
3     int max_left = 0, max_right = 0, sum = 0;
4     for (int i=mid;i>=low;i--) {
5         sum=sum+(int)a[i];
6         if (sum>left_sum) {
7             left_sum = sum;
8             max_left = i;
9         }
10    }
11    sum = 0;
12    for (int j = mid + 1; j <= high; j++) {
13        sum=sum+ (int)a[j];
14        if (sum>right_sum) {
15            right_sum = sum;
16            max_right = j;
17        }
18    }
19    return make_tuple(max_left , max_right , left_sum + right_sum);
20 }
21
22 tuple<int , int , int> MyArray::FindMaximumSubarray(int low , int high) {
23     int mid;
24     int left_low , left_high , left_sum;
25     int right_low , right_high , right_sum;
26     int cross_low , cross_high , cross_sum;
27     if (high == low)
28         return make_tuple(low , high , a[low]);
29     // base case: only one element
30     else {
31         mid=(low+high)/2;
32         tie(left_low , left_high , left_sum)=
33         FindMaximumSubarray(low , mid);
34         tie(right_low , right_high , right_sum)=
35         FindMaximumSubarray(mid + 1, high);
36         tie(cross_low , cross_high , cross_sum)=
37         FindMaxCrossingSubarray(low , mid , high);
38         if ((left_sum >= right_sum) && (left_sum >= cross_sum))
39             return make_tuple(left_low , left_high , left_sum);
40         else if ((right_sum >= left_sum) && (right_sum >= cross_sum))
41             return make_tuple(right_low , right_high , right_sum);
42         else return make_tuple(cross_low , cross_high , cross_sum);
43     }
44 }
```

---

Algorithm 4.15: Display (C++) (MyArray.cpp).

---

```
1 void MyArray::Display() {
2     for (int i = 0; i < n; i++) {
3         cout << "Element " << i << " with value " << a[i] << endl;
4     }
5 }
6
7 void Display(MyType *a, int start, int end) {
8     for (int i = start; i <= end; i++) {
9         cout << "Element " << i << " with value " << a[i] << endl;
10    }
11 }
12
13 void MyArray::Display(int low, int high) {
14     if (low < 0 || high > (n - 1))
15         exit(EXIT_FAILURE);
16     else
17         for (int i = low; i <= high; i++)
18             cout << "Element " << i << " with value " << a[i] << endl;
19 }
```

---

Algorithm 4.16: Print in a file (C++).

---

```
1 void MyArray::DisplayFile() {
2     ofstream myfile;
3     myfile.open("log.txt");
4     myfile << "Array of size " << n << endl;
5     for (int i = 0; i < n; i++) {
6         myfile << "Element " << i << " with value " << a[i] << endl;
7     }
8     myfile.close();
9 }
```

---

Algorithm 4.17: Print in a file (C) (MyArray.cpp).

---

```
1 void MyArray::DisplayFileC() {
2     FILE* f;
3     f=fopen("log.txt", "wt");
4     fprintf(f, "Array of size %d\n", n);
5     for (int i = 0; i < n; i++) {
6         fprintf(f, "Element %d with value %d\n", i, (int)a[i]);
7     }
8     fclose(f);
9 }
```

---

Algorithm 4.18: Array initialization (MyArray.cpp).

---

```
1 void MyArray::InitRandom(int v) {
2     for (int i = 0; i < n; i++)
```

```

3         a[i] = rand() % v;
4     }
5
6     void MyArray::InitSortedAscending(int v) {
7         if (n > 0) {
8             a[0] = rand() % v;
9             for (int i = 1; i < n; i++)
10                 a[i] = a[i - 1] + rand() % v;
11         }
12     }
13
14     void MyArray::InitSortedDescending(int v) {
15         if (n > 0) {
16             a[0] = rand() % v;
17             for (int i = 1; i < n; i++)
18                 a[i] = a[i - 1] - rand() % v;
19         }
20     }

```

---

Algorithm 4.19: Display steps (MyArray.cpp).

```

1 void MyArray::DisplayStep(string f) {
2     cout << "Array of size " << n << endl;
3     cout << "Number of steps for " << f << " is " << step << endl;
4 }

```

---

Algorithm 4.20: Swapping.

```

1 void MyArray::SwapIndex(int i, int j) {
2     MyType tmp = a[i];
3     a[i] = a[j];
4     a[j] = tmp;
5 }
6
7 void SwapIndex(MyType* a, int i, int j) {
8     MyType tmp = a[i];
9     a[i] = a[j];
10    a[j] = tmp;
11 }

```

---

## 4.2 Sorting algorithms

What does it mean for an array to be sorted?

We consider an array  $a_u$  of size  $n$ . We assume that this array is unsorted. The sorted array  $a_s$  represents the sorted elements of the array  $a_u$ . We consider the names  $a_u$  to denote the fact that  $a_u$  is unsorted, and  $a_s$  to show that  $a$  is sorted but for C++ variables, we may consider `a_unsorted` and `a_sorted` to denote the unsorted and sorted variables.

- $x \in a_u$  if and only if  $x \in a_s$ . If an element is in the initial array, then it must be in the sorted array. The output array may be sorted but if it misses elements it is not correct. In other words,  $\exists(i, j) | a[i] = b[j]$ .
- If all the elements of  $a_u$  are in  $a_s$ , then there are as many elements in  $a_s$  than in  $a_u$ . It means the size of  $a_s$  is also  $n$ .
- **Minimum:** We defined the minimum  $x_{min}$  by:

$$\forall i \in \{0, \dots, n-1\} \mid a_u[i] \geq x_{min} \text{ and } \exists i \in \{0, \dots, n-1\} \mid a_u[i] = x_{min} \quad (4.1)$$

- **Maximum:** We defined the maximum  $x_{max}$  by:

$$\forall i \in \{0, \dots, n-1\} \mid a_u[i] \leq x_{max} \text{ and } \exists i \in \{0, \dots, n-1\} \mid a_u[i] = x_{max} \quad (4.2)$$

- $a_u$  is sorted then  $\forall i \in \{1, \dots, n-1\}, a_u[i-1] \leq a_u[i]$ .
- $a_u$  is sorted then  $\forall i_1 \in \{0, \dots, k\}, \forall i_2 \in \{k+1, \dots, n-1\}, a_u[i_1] \leq a_u[i_2]$ .

The base case: if there is only one element, then the array is sorted.

Algorithm 4.21: Is the array sorted?.

---

```
1 bool MyArray :: IsSorted () {
2     bool output = true;
3     if ((n == 0) || (n==1))
4         return true;
5     else {
6         int i = 1;
7         while ((a[i - 1] < a[i]) && (i<n))
8             i++;
9         return (i==n);
10    }
11 }
```

---

### 4.2.1 Selection Sort: $O(n^2)$

Algorithm 4.22: Selection sort (MyArray.cpp).

---

```
1 void MyArray::SelectionSort() {
2     step = 0;
3     // Base case: the array with 1 element is already sorted
4     // Invariant: the whole array is sorted between position 0 and i
5     for (int i = 0; i < n; ++i) {
6         int min = i;
7         // min = index of the minimum value for the array between
8         for (int j = i + 1; j < n; ++j) {
9             // search for the minimum index j between i and n-1
10            if (a[j] < a[min]) {
11                min = j;
12            }
13            step++;
14        }
15        SwapIndex(i, min);
16    }
17    DisplayStep(--func--); // --func-- : name of the function
18 }
```

---

### 4.2.2 Insertion Sort: $O(n^2)$

Algorithm 4.23: Insertion sort (MyArray.cpp).

---

```
1 void MyArray::InsertionSort() {
2     step = 0;
3     // Base case: the array with 1 element is already sorted
4     // Invariant: the array defined between position 0 and i is sorted
5     bool done;
6     for (int i = 0; i < n-1; ++i) {
7         int j = i + 1;
8         done = false;
9         while ((j > 0) && (!done)) {
10            step++;
11            if (a[j] < a[j - 1]) {
12                SwapIndex(j, j - 1);
13            }
14            else
15                done=true;
16            j--;
17        }
18    }
19    DisplayStep(--func--);
20 }
```

---

### 4.2.3 Bubble Sort: $O(n^2)$

Algorithm 4.24: Bubble sort (MyArray.cpp).

---

```
1 void MyArray::BubbleSort() {
2     step = 0;
3     // Invariant: the whole array is sorted between n-1-i and n-1
4     for (int i = 0; i < n-1; ++i) {
5         for (int j = 0; j < n-i-1; ++j) {
6             step++;
7             if (a[j] > a[j + 1]) {
8                 SwapIndex(j, j + 1);
9             }
10        }
11    }
12    DisplayStep(--func--);
13 }
14
15 void MyArray::BubbleOptSort() {
16     step = 0;
17     // Invariant: the whole array is sorted between n-1-i and n-1
18     bool sorted=false;
19     int i = 0;
20     while ((i<n-1) && (!sorted)) {
21         sorted = true; // we assume the array is sorted between 0 and n-i-1
22         for (int j =0 ; j < n-i-1; j++) {
23             step++;
24             if (a[j]>a[j+1]) {
25                 SwapIndex(j,j+1);
26                 sorted = false;
27             }
28         }
29         i++;
30     }
31     DisplayStep(--func--);
32 }
```

---

#### 4.2.4 Merge sort: $O(n \cdot \log(n))$

Algorithm 4.25: Merge (MyArray.cpp).

---

```
1 void merge(MyType* a, int start, int mid, int end) {
2     // Init a1
3     int n1 = mid - start + 1;
4     MyType* a1 = new MyType[n1];
5     for (int i = 0; i < n1; i++)
6         a1[i] = a[i+start];
7     // Init a2
8     int n2 = end - mid;
9     MyType* a2 = new MyType[n2];
10    for (int i = 0; i < n2; i++)
11        a2[i] = a[i+mid+1];
12    int i1 = 0, i2 = 0, i3 = start;
13    while ((i1 < n1) && (i2 < n2)) {
14        if (a1[i1] < a2[i2]) {
15            a[i3] = a1[i1];
16            i1++;
17        }
18        else {
19            a[i3] = a2[i2];
20            i2++;
21        }
22        i3++;
23        step++;
24    }
25    if (i1 < n1) // -> we left the while loop because i2 >= n2
26        for (int i = i1; i < n1; i++) {
27            a[i3] = a1[i];
28            i3++;
29            step++;
30        }
31    else // -> we left the while loop because i1 >= n1
32        for (int i = i2; i < n2; i++) {
33            a[i3] = a2[i];
34            i3++;
35            step++;
36        }
37    delete[] a1;
38    delete[] a2;
39 }
```

---

Algorithm 4.26: Mergesort (MyArray.cpp).

---

```
1 void mergesort(MyType* a, int start, int end) {
2     if (start < end) {
3         int mid = (start + end) / 2;
4         mergesort(a, start, mid);
5         // a is sorted between start and mid
6         mergesort(a, mid + 1, end);
7         // a is sorted between mid+1 and end
8         merge(a, start, mid, end);
9         // a is sorted
10    }
11    // else there is 1 element and it is already sorted (base case)
12 }
13
14 // 1st call of the function with
15 // start=0 and end=n-1
16 void MyArray::MergeSort() {
17     step = 0;
18     mergesort(a, 0, n - 1);
19     DisplayStep(__func__);
20 }
```

---



## 4.2.5 Quicksort: $O(n \cdot \log(n))$

Algorithm 4.27: Quicksort (MyArray.cpp).

---

```
1 // Hoare partition
2 int partition(MyType* a, int start, int end) {
3     int i = start;
4     int j = end;
5     MyType pivot_value = a[(start+end)/2]; // in the middle
6     bool finished = false;
7     while (!finished) {
8         while ((i<end) && (a[i] <= pivot_value)) {
9             i++; // move to the right
10            step++;
11        }
12        while ((j>start) && (a[j] > pivot_value)) {
13            j--; // move to the left
14            step++;
15        }
16        if (i < j)
17            SwapIndex(a,i,j);
18        else
19            finished = true;
20    }
21    cout << "Pivot: " << pivot_value << " at position " << j << endl;
22    Display(a, start, end);
23    return j; // index of the pivot, which can move !
24 }
25
26 void quicksort(MyType* a, int start, int end) {
27     if (start < end) {
28         int pivot_index = partition(a, start, end);
29         // All the elements between start and pivot_index-1 are
30         // inferior to a[pivot_index].
31         // All the elements between pivot_index+1 and end are
32         // superior to a[pivot_index].
33         quicksort(a,start, pivot_index -1);
34         quicksort(a, pivot_index +1,end);
35     }
36     // else there is 1 element and it is already sorted (base case)
37 }
38
39 // 1st call of the function with
40 // start=0 and end=n-1
41 void MyArray::QuickSort() {
42     step = 0;
43     quicksort(a, 0, n - 1);
44     DisplayStep(--func--);
45 }
```

---

## 4.2.6 Some main examples

Algorithm 4.28: Example with Pairs.

---

```
1 void main() {
2     int n=10;
3     MyArray* B = new MyArray(n);
4     B->InitRandom(n);
5     pair<double, int> r=B->GetMaxArg();
6     cout << "vmax: " << r.first << endl;
7     cout << "argmax: " << r.second << endl;
8 }
```

---

Algorithm 4.29: Example with FindMaxSubArray.

---

```
1 void main() {
2     int nsize = 8;
3     int x1[] = { 8,7,6,5,4,3,2,1 };
4     MyArray* A = new MyArray(nsize);
5     for (int i = 0; i<nsize; i++) {
6         (*A)[i] = x1[i];
7     }
8     int low = 0;
9     int high = nsize - 1;
10    int sublow, subhigh, subsum;
11    tie(sublow, subhigh, subsum) = A->FindMaximumSubarray(low, high);
12    cout << "Sum: " << subsum << endl;
13 }
```

---

Algorithm 4.30: Application of Quicksort.

---

```
1 void main() {
2     int nsize = 12;
3     MyArray* A = new MyArray(nsize);
4     A->InitRandom(2);
5     cout << "Quicksort" << endl;
6     A->QuickSort();
7     A->Display();
8 }
```

---

Algorithm 4.31: Application of Binary Search.

---

```
1 void main() {
2   MyType x1[] = {2,5,9,13,18,23,27,33};
3   // number of elements = total space / space of 1 element
4   int nsize = sizeof(x1) / sizeof(x1[0]);
5   MyArray* A = new MyArray(nsize);
6   for (int i = 0; i<nsize; i++) {
7     (*A)[i] = x1[i];
8   }
9   pair<double, int> r = A->BinarySearch2(13);
10  cout << "Found: " << r.first << " at position " << r.second << endl;
11  r = A->BinarySearch2(2);
12  cout << "Found: " << r.first << " at position " << r.second << endl;
13  r = A->BinarySearch2(33);
14  cout << "Found: " << r.first << " at position " << r.second << endl;
15  r = A->BinarySearch2(12);
16  cout << "Found: " << r.first << " at position " << r.second << endl;
17 }
```

---

## 4.2.7 Time measurement

This function measures the time it takes to sort an array.

Algorithm 4.32: Example of benchmarks.

---

```
1 void main() {
2     int nsize = 10;
3     MyArray* A = new MyArray(nsize);
4     int type_array = 0;
5     for (int type_sort = 1; type_sort <=6; type_sort++) {
6         // Create a new array at each iteration!
7         if (type_array == 0) {
8             A->InitRandom(2);
9             cout << "Random array" << endl;
10        }
11        else if (type_array == 1) {
12            A->InitSortedAscending(nsize*nsize);
13            cout << "Sorted array" << endl;
14        }
15        if (nsize < 15) {
16            cout << "Input array:" << endl;
17            A->Display();
18        }
19        auto t1 = chrono::high_resolution_clock::now();
20        switch (type_sort) {
21            case 1: // Selection sort
22                A->SelectionSort(); break;
23            case 2: // Insertion sort
24                A->InsertionSort(); break;
25            case 3: // Bubble sort
26                A->BubbleSort(); break;
27            case 4: // Bubble Opt sort
28                A->BubbleOptSort(); break;
29            case 5: // Merge sort
30                A->MergeSort(); break;
31            case 6: // Quick sort
32                A->QuickSort(); break;
33        }
34        if (nsize < 15)
35            A->Display();
36        auto t2 = chrono::high_resolution_clock::now();
37        chrono::duration<double, milli> duration_ms = t2 - t1;
38        cout << "It took " << duration_ms.count() << " ms" << endl;
39    }
40 }
```

---

# Chapter 5

## Lists

### Contents

---

<b>5.1</b>	<b>Simple chained list . . . . .</b>	<b>65</b>
5.1.1	Node definition . . . . .	65
5.1.2	Class interface . . . . .	65
5.1.3	Constructor and destructor . . . . .	66
5.1.4	Insert . . . . .	67
5.1.5	Delete . . . . .	72
5.1.6	Display . . . . .	74
<b>5.2</b>	<b>Double chained list . . . . .</b>	<b>75</b>
5.2.1	Node definition . . . . .	75
5.2.2	Insert . . . . .	77
5.2.3	Delete . . . . .	82
5.2.4	Display . . . . .	85
5.2.5	The Sieve of Eratosthenes . . . . .	88
<b>5.3</b>	<b>Circular list . . . . .</b>	<b>90</b>
5.3.1	Class definition . . . . .	90
5.3.2	Insert . . . . .	91
5.3.3	Delete . . . . .	92
5.3.4	Search and display . . . . .	93
<b>5.4</b>	<b>Skip list . . . . .</b>	<b>95</b>
5.4.1	Class definition . . . . .	95
5.4.2	Main functions . . . . .	96
5.4.3	Insert and delete . . . . .	97
5.4.4	Search and display . . . . .	98
5.4.5	Example . . . . .	99

---

## 5.1 Simple chained list

### 5.1.1 Node definition

Algorithm 5.1: Node simple chained list.

---

```
1 class NodeSC {
2 public:
3     NodeSC(): data(0), next(nullptr) {}
4     NodeSC(MyType d) : data(d), next(nullptr) {}
5     NodeSC(MyType d, NodeSC* nxt) : data(d), next(nxt) {}
6     ~NodeSC() {}
7     MyType data;
8     NodeSC *next;
9 };
```

---

### 5.1.2 Class interface

Algorithm 5.2: Class simple chained list (MySCList.h).

---

```
1 class MySCList : public MyDataStructure {
2 public:
3     MySCList();
4     ~MySCList();
5     int GetSize() const { return n; }
6     MyDataStructure* clone() { return new MySCList(); }
7     void CreateNode(MyType value);
8     void Insert(MyType value);
9     void InsertFirst(MyType value);
10    void InsertLast(MyType value);
11    void InsertPosition(int pos, MyType value);
12    void DeleteFirst();
13    void DeleteLast();
14    void DeletePosition(int pos);
15    void DeleteMiddle(MyType value);
16    void InitRandom(int n, int v);
17    void InitSortedAscending(int n, int v);
18    void InitSortedDescending(int n, int v);
19    void Reverse();
20    void Display();
21    void DisplayFile();
22 private:
23     NodeSC *head; // pointer on the head
24     NodeSC *tail; // pointer on the tail
25     int n;
26 };
```

---

### 5.1.3 Constructor and destructor

Algorithm 5.3: Class simple chained list (MySCList.cpp).

---

```
1 MySCList::MySCList(): head(nullptr), tail(nullptr), n(0) { }
2
3 MySCList::~~MySCList() {
4     NodeSC *current = head;
5     while (current) {
6         NodeSC *next = current->next;
7         delete current;
8         current = next;
9     }
10 }
```

---

Algorithm 5.4: Reverse (MySCList.cpp).

---

```
1 void MySCList::Reverse() {
2     NodeSC *current = head;
3     NodeSC *prev = nullptr, *next = nullptr;
4     while (current) {
5         next = current->next;
6         current->next = prev;
7         prev = current;
8         current = next;
9     }
10     head = prev;
11 }
```

---

### 5.1.4 Insert

Algorithm 5.5: Insert.

---

```
1 void MySCList::CreateNode(MyType value) {
2     NodeSC *tmp = new NodeSC(value);
3     if (head==nullptr) {
4         head = tmp;
5         tail = tmp;
6     } else {
7         tail->next = tmp;
8         tail = tmp;
9     }
10    n++;
11 }
12
13 void MySCList::InsertFirst(MyType value) {
14     NodeSC *tmp = new NodeSC(value, head);
15     head = tmp;
16     n++;
17 }
18
19 void MySCList::InsertLast(MyType value) {
20     CreateNode(value);
21 }
22
23 void MySCList::Insert(MyType value) {
24     InsertMiddle(value);
25 }
```

---



In this function, we insert an element in a sorted list. We consider 3 cases: 1) if the list is empty, 2) if we insert at the beginning (we replace the head), or 3) if the element to insert is later in the list.

Algorithm 5.6: Insert (iterative version).

---

```

1 void MySCList::InsertMiddle(MyType value) {
2     NodeSC *tmp = new NodeSC(value);
3     NodeSC *cursor = head;
4     if (head == nullptr) { // insert to the head (empty list)
5         cout << "Insert to head/tail: " << value << endl;
6         head = tmp;
7         tail = tmp;
8     }
9     else if (head->data > value) { // insert to the head
10        cout << "Insert to head: " << value << endl;
11        tmp->next = head;
12        head = tmp;
13    }
14    else {
15        NodeSC *prev = head;
16        NodeSC *cursor = head->next;
17        while ((cursor != nullptr) && (cursor->data < value)) {
18            prev = prev->next;
19            cursor = cursor->next;
20        }
21        // Case 1: we arrive at the end of the list
22        if (cursor == nullptr) { // insert to the tail
23            cout << "Insert to tail: " << value << endl;
24            prev->next = tmp;
25            tail = tmp;
26        }
27        // Case 2: we arrive between 2 elements
28        else {
29            cout << "Insert middle: " << value << endl;
30            prev->next = tmp;
31            tmp->next = cursor;
32        }
33    }
34    n++;
35 }

```

---

In this function, we insert an element in a sorted list. It is the same as before, however we are considering a recursive function. In such a case, we need 2 functions: one function that is used for the main work, and another function that is used for calling the first function with the proper arguments, so we can hide the head within this function, i.e. there is no need to use the head as an input of the function. It is worth noting that cursor is as input/output, and it is passed by reference. This function is similar to the iterative function, but using a recursive function, the code is cleaner, there is just a call to the function, and no while.

Algorithm 5.7: Insert (recursive version) - Procedure.

---

```

1 void MySCList::InsertMiddleRec(NodeSC* &cursor, MyType value) {
2     if (cursor == nullptr) { // insert to the head (empty list)
3         cout << "Insert to head/tail: " << value << endl;
4         NodeSC* tmp = new NodeSC(value);
5         cursor = tmp;
6         tail = tmp;
7         n++; // increment the number of elements
8     }
9     else if (cursor->data > value) { // insert to the head
10        cout << "Insert to head: " << value << endl;
11        NodeSC* tmp = new NodeSC(value);
12        tmp->next = cursor;
13        cursor = tmp; // the "previous->next" is modified
14        n++; // increment the number of elements
15    }
16    else // cursor->next input/output
17        InsertMiddleRec(cursor->next, value);
18 }
19
20 void MySCList::InsertMiddleMainRec(MyType value) {
21     InsertMiddleRec(head, value);
22 }

```

---

In this function, we insert an element in a sorted list. It is similar as the previous function but we are using a function instead of a procedure (i.e. a function returning void).

Algorithm 5.8: Insert (recursive version) - Function.

---

```

1 NodeSC* MySCList::InsertMiddleRecf(NodeSC* cursor , MyType value) {
2     if (cursor == nullptr) { // insert to the head (empty list)
3         cout << "Insert to head/tail: " << value << endl;
4         NodeSC* tmp = new NodeSC(value);
5         tail = tmp;
6         n++;
7         return tmp;
8     }
9     else if (cursor->data > value) { // insert to the head
10        cout << "Insert to head: " << value << endl;
11        NodeSC* tmp = new NodeSC(value);
12        tmp->next = cursor;
13        n++;
14        return tmp;
15    }
16    else {
17        cursor->next=InsertMiddleRecf(cursor->next , value);
18        return cursor;
19    }
20 }
21
22 void MySCList::InsertMiddleMainRecf(MyType value) {
23     head=InsertMiddleRecf(head , value);
24 }

```

---

---

Algorithm 5.9: Insert.

---

```
1 void MySCList::InsertPosition(int pos, MyType value) {
2     if (pos == 0)
3         InsertFirst(value);
4     else if (pos == n - 1)
5         InsertLast(value);
6     else {
7         NodeSC *previous = nullptr;
8         NodeSC *current = head;
9         NodeSC *tmp = new NodeSC(value);
10        for (int i = 0; i < pos; i++) {
11            previous = current;
12            current = current->next;
13        }
14        previous->next = tmp;
15        tmp->next = current;
16        n++;
17    }
18 }
```

---

---

Algorithm 5.10: Initialization.

---

```
1 void MySCList::InitRandom(int n, int v) {
2     for (int i = 0; i < n; i++)
3         InsertLast(rand() % v);
4 }
5
6 void MySCList::InitSortedAscending(int n, int v) {
7     MyType tmp = rand() % v;
8     for (int i = 0; i < n; i++) {
9         MyType tmp1 = tmp + rand() % v;
10        InsertLast(tmp1);
11        tmp = tmp1;
12    }
13 }
14
15 void MySCList::InitSortedDescending(int n, int v) {
16     MyType tmp = 10e4+rand() % v;
17     for (int i = 0; i < n; i++) {
18         MyType tmp1 = tmp - rand() % v;
19         InsertLast(tmp1);
20         tmp = tmp1;
21    }
22 }
```

---

### 5.1.5 Delete

Algorithm 5.11: Delete.

---

```
1 void MySCList::DeleteFirst() {
2     NodeSC *tmp;
3     tmp = head;
4     head = head->next;
5     delete tmp;
6     n--;
7 }
8
9 void MySCList::DeleteLast() {
10    NodeSC *current=nullptr;
11    NodeSC *previous = nullptr;
12    current = head;
13    while (current->next!=nullptr) {
14        previous=current;
15        current=current->next;
16    }
17    tail=previous;
18    previous->next = nullptr;
19    delete current;
20    n--;
21 }
22
23 void MySCList::DeletePosition(int pos) {
24    NodeSC *current=head;
25    NodeSC *previous=nullptr;
26    for (int i = 1; i<pos; i++) {
27        previous = current;
28        current = current->next;
29    }
30    previous->next = current->next;
31    delete current;
32    n--;
33 }
```

---

Algorithm 5.12: Delete in a sorted list.

---

```
1 void MySCList::DeleteMiddle(MyType value) {
2     NodeSC* current = head;
3     if (current != nullptr) {
4         if (current->data == value) {
5             head = current->next;
6             if (tail == current)
7                 tail = head;
8             delete current;
9         }
10        else {
11            if (current->next != nullptr) {
12                NodeSC* prev = current;
13                current = current->next;
14                bool found = false;
15                while ((current != nullptr) && (!found)) {
16                    if (current->data == value) {
17                        prev->next = current->next;
18                        delete current;
19                        found = true;
20                        if (tail == current)
21                            tail = prev;
22                    }
23                    else {
24                        prev = current;
25                        current = current->next;
26                    }
27                }
28            }
29        }
30    }
31 }
```

---

## 5.1.6 Display

Algorithm 5.13: Display the list.

---

```
1 void MySCList::Display () {
2     NodeSC *tmp;
3     tmp = head;
4     cout << "List of size :" << n << endl;
5     cout << "Head:";
6     (head != nullptr) ? cout << head->data << endl : cout << "NULL" << endl;
7     cout << "Tail:";
8     (tail != nullptr) ? cout << tail->data << endl : cout << "NULL" << endl;
9     int i = 0;
10    while (tmp) {
11        cout << "(" << i << ") ";
12        if (tmp->next != nullptr)
13            cout << tmp->data << "->" << tmp->next->data << endl;
14        else
15            cout << tmp->data << "-> NULL" << endl;
16        tmp = tmp->next;
17        i++;
18    }
19 }
20
21 void MySCList::DisplayFile () {
22     ofstream myfile;
23     myfile.open("log.txt");
24     NodeSC *tmp;
25     tmp = head;
26     while (tmp) {
27         myfile << tmp->data << endl;
28         tmp = tmp->next;
29     }
30     myfile.close();
31 }
```

---

Algorithm 5.14: Example simple chained list.

---

```
1 void main () {
2     MySCList* L = new MySCList();
3     L->InitSortedAscending(12, 100);
4     L->Display();
5     cout << "Reverse the list" << endl;
6     L->Reverse();
7     L->Display();
8     delete L;
9 }
```

---

## 5.2 Double chained list

### 5.2.1 Node definition

Algorithm 5.15: Node simple chained list.

---

```
1  class NodeDC {
2  public:
3      NodeDC():
4          data(0), next(nullptr), previous(nullptr) {}
5      NodeDC(MyType d):
6          data(d), next(nullptr), previous(nullptr) {}
7      NodeDC(MyType d, NodeDC* nxt, NodeDC* prv):
8          data(d), next(nxt), previous(prv) {}
9      ~NodeDC() {}
10     MyType data;
11     NodeDC *next;
12     NodeDC *previous;
13 };
```

---

Algorithm 5.16: Class interface.

---

```
1  class MyDCList : public MyDataStructure {
2  public:
3      MyDCList();
4      ~MyDCList();
5      MyDataStructure* clone() { return new MyDCList(); }
6      void Insert(MyType value);
7      void InsertMiddle(MyType value);
8      void InsertHead(MyType value);
9      void InsertTail(MyType value);
10     void InsertPosition(int pos, MyType value);
11     void DeleteHead();
12     void DeleteTail();
13     void DeletePosition(int pos);
14     void DeleteMiddle(MyType value);
15     void Display();
16     void DisplayDC();
17     void DisplayFile();
18     int GetSize() { return n; }
19 private:
20     NodeDC *head, *tail;
21     NodeDC *iterator_start, *iterator_end;
22     int n;
23 };
```

---



Algorithm 5.17: Constructor and destructor.

---

```
1  MyDCList::MyDCList() {
2      head = nullptr;
3      tail = nullptr;
4      n = 0;
5  }
6
7  MyDCList::~~MyDCList() {
8      NodeDC *current = head;
9      while (current) {
10         NodeDC *next = current->next;
11         delete current;
12         current = next;
13     }
14 }
```

---

## 5.2.2 Insert

Algorithm 5.18: Insert tail.

---

```
1 void MyDCList::InsertTail(MyType value) {
2     NodeDC *tmp = new NodeDC(value);
3     if (head == nullptr) {
4         head = tmp;
5         tail = tmp;
6     } else {
7         tail->next = tmp;
8         tmp->previous = tail;
9         tail = tmp;
10    }
11    n++;
12 }
```

---

Algorithm 5.19: Insert head.

---

```
1 void MyDCList::InsertHead(MyType value) {
2     NodeDC *tmp = new NodeDC(value, head, nullptr);
3     if (head == nullptr) {
4         head = tmp;
5         tail = tmp;
6     }
7     else {
8         head->previous = tmp;
9         head = tmp;
10    }
11    n++;
12 }
13
14 // default insert
15 void MyDCList::Insert(MyType value) {
16     InsertHead(value);
17 }
```

---

Algorithm 5.20: Insert at position.

---

```
1 void MyDCList::InsertPosition(int pos, MyType value) {
2     NodeDC *pre = new NodeDC;
3     NodeDC *cur = new NodeDC;
4     NodeDC *tmp = new NodeDC;
5     cur = head;
6     for (int i = 1; i < pos; i++) {
7         pre = cur;
8         cur = cur->next;
9     }
10    tmp->data = value;
11    pre->next = tmp;
12    tmp->next = cur;
13    n++;
14 }
```

---



Algorithm 5.21: Insert anywhere.

---

```
1  // Iterative version
2  void MyDCList::InsertMiddle(MyType value) {
3      NodeDC *tmp = new NodeDC(value);
4      NodeDC *cursor = head;
5      if (head == nullptr) { // insert to the head
6          cout << "Insert to head/tail: " << value << endl;
7          head = tmp;
8          tail = tmp;
9      }
10     else if (head->data > value) { // insert to the head
11         cout << "Insert to head: " << value << endl;
12         head->previous = tmp;
13         tmp->next = head;
14         head = tmp;
15     }
16     else {
17         NodeDC *prev = head;
18         NodeDC *cursor = head->next;
19         while ((cursor != nullptr) && (cursor->data < value)) {
20             prev = prev->next;
21             cursor = cursor->next;
22         }
23         if (cursor == nullptr) { // insert to the tail
24             cout << "Insert to tail: " << value << endl;
25             prev->next = tmp;
26             tmp->previous = prev;
27             tail = tmp;
28         }
29         else {
30             cout << "Insert middle: " << value << endl;
31             prev->next = tmp;
32             tmp->previous = prev;
33             tmp->next = cursor;
34             cursor->previous = tmp;
35         }
36     }
37     n++;
38 }
```

---

---

Algorithm 5.22: Insert anywhere.

---

```
1  // Recursive version (with pointers)
2  void InsertMiddleDC1(NodeDC **head, NodeDC *prev, NodeDC **tail, MyType value) {
3      if ((*head) == nullptr) {
4          cout << "Insert to head/tail: " << value << endl;
5          NodeDC *tmp = new NodeDC(value, nullptr, prev);
6          *head = tmp;
7          *tail = tmp;
8      }
9      else if ((*head)->data > value) {
10         cout << "Insert to head: " << value << endl;
11         NodeDC *tmp = new NodeDC(value);
12         (*head)->previous = tmp;
13         tmp->next = (*head);
14         (*head) = tmp;
15         if (prev != nullptr) {
16             cout << "Insert to middle: " << value << endl;
17             tmp->previous = prev;
18             prev->next = tmp;
19         }
20     }
21     else
22         InsertMiddleDC1(&(*head)->next, (*head), tail, value);
23 }
24
25 void MyDCList::InsertMiddle1(MyType value) {
26     InsertMiddleDC1(&head, nullptr, &tail, value);
27     n++;
28 }
```

---

Algorithm 5.23: Insert anywhere.

---

```
1  // Recursive version (with references)
2  void InsertMiddleDC2(NodeDC* &head, NodeDC *prev, NodeDC* &tail, MyType value) {
3      if (head==nullptr) {
4          cout << "Insert to head/tail: " << value << endl;
5          NodeDC *tmp = new NodeDC(value, nullptr, prev);
6          head = tmp;
7          tail = tmp;
8      }
9      else if (head->data > value) {
10         cout << "Insert to head: " << value << endl;
11         NodeDC *tmp = new NodeDC(value);
12         head->previous = tmp;
13         tmp->next = (*head);
14         head = tmp;
15         if (prev != nullptr) {
16             cout << "Insert to middle: " << value << endl;
17             tmp->previous = prev;
18             prev->next = tmp;
19         }
20     }
21     else
22         InsertMiddleDC1(head->next, head, tail, value);
23 }
24
25 void MyDCList::InsertMiddle2(MyType value) {
26     InsertMiddleDC2(head, nullptr, tail, value);
27     n++;
28 }
```

---

### 5.2.3 Delete

Algorithm 5.24: Delete head and tail.

---

```
1 void MyDCList::DeleteHead() {
2     NodeDC *tmp = new NodeDC;
3     tmp = head;
4     head = head->next;
5     head->previous = nullptr;
6     delete tmp;
7     n--;
8 }
9
10 void MyDCList::DeleteTail() {
11     NodeDC *current=nullptr;
12     NodeDC *previous = nullptr;
13     current = head;
14     while (current->next != nullptr) {
15         previous = current;
16         current = current->next;
17     }
18     tail = previous;
19     previous->next = nullptr;
20     delete current;
21     n--;
22 }
```

---

Algorithm 5.25: Delete position.

---

```
1 void MyDCList::DeletePosition(int pos) {
2     if (pos == 0)
3         DeleteHead();
4     else if (pos==n-1)
5         DeleteTail();
6     else {
7         NodeDC *current = head;
8         NodeDC *previous = nullptr;
9         for (int i = 1; i <= pos; i++) {
10             previous = current;
11             current = current->next;
12         }
13         previous->next = current->next;
14         current->next->previous = previous;
15         delete current;
16         n--;
17     }
18 }
```

---

Algorithm 5.26: Delete anywhere.

---

```
1  // Iterative version
2  void MyDCList::DeleteMiddle(MyType value) {
3      if (head->data == value) { // remove to the head
4          cout << "Remove to head: " << value << endl;
5          NodeDC *tmp = head;
6          head = head->next;
7          head->previous = nullptr;
8          delete tmp;
9      }
10     else {
11         NodeDC *cursor = head;
12         NodeDC *prev = cursor->previous;
13         while ((cursor != nullptr) && (cursor->data != value)) {
14             prev = cursor;
15             cursor = cursor->next;
16         }
17         if (cursor != nullptr) { // remove
18             if (cursor->next == nullptr) { // it is the tail
19                 cout << "Remove tail: " << value << endl;
20                 prev->next = nullptr;
21                 tail = prev;
22             }
23             else
24             {
25                 cout << "Remove middle: " << value << endl;
26                 cursor->next->previous = prev;
27                 prev->next = cursor->next;
28             }
29             delete cursor;
30         }
31     }
32     n--;
33 }
```

---



Algorithm 5.27: Delete anywhere.

---

```
1 // Recursive version
2 void DeleteMiddleDC(NodeDC **head, NodeDC **tail, MyType value) {
3     if ((*head) != nullptr) {
4         if ((*head)->data == value) { // remove to the head
5             NodeDC *tmp = *head;
6             *head = (*head)->next;
7             if ((*head) == nullptr)
8                 *tail = tmp->previous;
9             else
10                 (*head)->previous = tmp->previous;
11             delete tmp;
12         }
13         else
14             DeleteMiddleDC(&(*head)->next, tail, value);
15     }
16 }
17
18 void MyDCList::DeleteMiddle1(MyType value) {
19     DeleteMiddleDC(&head, &tail, value);
20     n--;
21 }
```

---

## 5.2.4 Display

Algorithm 5.28: Display details.

---

```
1 void MyDCList::DisplayDC() {
2     NodeDC *tmp;
3     tmp = head;
4     MyType data_prev, data_next, data_head, data_tail;
5     cout << "List of size :" << n << endl;
6     cout << "Head:";
7     (head != nullptr) ? cout << head->data << endl : cout << "NULL" << endl;
8     cout << "Tail:";
9     (tail != nullptr) ? cout << tail->data << endl : cout << "NULL" << endl;
10    while (tmp != nullptr) {
11        if (tmp->previous != nullptr)
12            data_prev = tmp->previous->data;
13        else
14            data_prev = -1;
15        if (tmp->next != nullptr)
16            data_next = tmp->next->data;
17        else
18            data_next = -1;
19        cout << "(" << data_prev << ","
20            << tmp->data << ","
21            << data_next << ")" << endl;
22        tmp = tmp->next;
23    }
24 }
```

---

Algorithm 5.29: Display.

---

```
1 void MyDCList::Display() {
2     NodeDC *tmp = new NodeDC;
3     tmp = head;
4     while (tmp != nullptr) {
5         cout << tmp->data << "\n";
6         tmp = tmp->next;
7     }
8 }
9
10 void MyDCList::DisplayFile() {
11     ofstream myfile;
12     myfile.open("log.txt");
13     NodeDC *tmp = new NodeDC;
14     tmp = head;
15     while (tmp != nullptr) {
16         myfile << tmp->data << "\n";
17         tmp = tmp->next;
18     }
19     myfile.close();
20 }
```

---

Algorithm 5.30: Example.

---

```
1 void main() {
2     cout << "Test the Double Chained List" << endl;
3     MyDCList* LD = new MyDCList();
4     LD->DisplayDC ();
5     LD->InsertMiddle (5);
6     LD->InsertMiddle (10);
7     LD->InsertMiddle (15);
8     LD->InsertMiddle (20);
9     LD->InsertMiddle (3);
10    LD->InsertMiddle (2);
11    LD->InsertMiddle (1);
12    LD->InsertMiddle (11);
13    LD->InsertMiddle (50);
14    LD->InsertMiddle (17);
15    LD->InsertMiddle (-50);
16    LD->DeleteMiddle (10);
17    LD->DeleteMiddle (50);
18    LD->DeleteMiddle (-50);
19    LD->DeleteMiddle (5);
20    LD->DeleteMiddle (20);
21    LD->DisplayDC ();
22    delete LD;
23 }
```

---

## 5.2.5 The Sieve of Eratosthenes

### Iterator

We want to browse the elements of the list from a particular element to the end. We add `iterator_start` and `iterator_end` as properties. We add the following functions to the class:

Algorithm 5.31: Iterator.

---

```
1 void GetNext() {
2     iterator_start = iterator_start->next;
3 }
4 MyType GetIterator() {
5     return iterator_start->data;
6 }
7 void SetStartIterator(int start) {
8     iterator_start = Search(start);
9 }
10 void SetEndIterator(int end) {
11     iterator_end = Search(end);
12 }
13 void SetIterator(int start, int end) {
14     iterator_start = Search(start);
15     iterator_end = Search(end);
16 }
17 bool IsFinishedIterator() {
18     return (iterator_start == iterator_end);
19 }
```

---

Algorithm 5.32: Example.

---

```
1 void main();
2     MyDCList* L = new MyDCList();
3     L->InsertHead(4);
4     L->InsertHead(5);
5     L->InsertHead(6);
6     L->InsertHead(8);
7     for (L->SetIterator(0, 4); !L->IsFinishedIterator(); L->GetNext())
8         cout << L->GetIterator() << endl;
9     delete L;
```

---

## Find Prime numbers

The sieve of Eratosthenes is a simple and ancient algorithm to determine all the prime numbers up to any given number.

Algorithm 5.33: Prime numbers until n.

---

```
1 void FindPrime(int n) {
2     MyDCList* L=new MyDCList();
3     // L contains the current list of prime numbers from 2 to i
4     L->InsertTail(2);
5     for (int i = 3; i < n; i++) {
6         // determine if i is prime
7         bool prime = true;
8         // Start at the position 0 of the list
9         // Finish at the end of the list
10        L->SetIterator(0,L->GetSize());
11        cout << "For: " << i << " Check: " << L->GetSize() << " numbers ";
12        while ((!L->IsFinishedIterator()) &&
13                (pow(L->GetIterator(),2)<=i) &&
14                prime) {
15            if (i % (int)L->GetIterator() == 0)
16                prime = false;
17            // Go to the next element in the current list of prime numbers
18            L->GetNext();
19        }
20        if (prime) {
21            L->InsertTail(i);
22            cout << i << " is prime." << endl;
23        }
24        else
25            cout << i << " is not prime." << endl;
26    }
27    delete L;
28 }
```

---

## 5.3 Circular list

### 5.3.1 Class definition

Algorithm 5.34: Circular list - Class definition.

---

```
1  class Cnode {
2  public:
3      Cnode() : data(0), next(nullptr) {}
4      Cnode(MyType x, Cnode* next1) : data(x), next(next1) {}
5      MyType data;
6      Cnode *next;
7  };
8
9  class MyCList : public MyDataStructure {
10 public:
11     MyCList() : head(nullptr) {}
12     ~MyCList();
13     MyDataStructure* clone() { return new MyCList(); }
14     void Insert(MyType value);
15     void InsertBegin(MyType value);
16     void InsertAfter(MyType value, int position);
17     void Delete(MyType value);
18     bool Search(MyType value);
19     void Update(MyType value, int position);
20     void Display();
21 private:
22     Cnode* head;
23 };
```

---

Algorithm 5.35: Destructor.

---

```
1  MyCList::~~MyCList() {
2      if (head != nullptr) {
3          Cnode *current = head->next;
4          while (current != head) {
5              Cnode *next = current->next;
6              delete current;
7              current = next;
8          }
9          delete head;
10     }
11 }
```

---

### 5.3.2 Insert

Algorithm 5.36: Insert.

---

```
1 void MyCList::Insert(MyType value) {
2     Cnode *temp= new Cnode(value , nullptr);
3     if (head == nullptr) {
4         head = temp;
5         temp->next = head;
6     }
7     else {
8         temp->next = head->next;
9         head->next = temp;
10        head = temp;
11    }
12 }
13 // Insertion of element at beginning
14 void MyCList::InsertBegin(MyType value) {
15     if (head == nullptr)
16         cout << "First Create the list." << endl;
17     else {
18         Cnode *temp = new Cnode(value , head->next);
19         head->next = temp;
20     }
21 }
22 // Insertion of element at a particular place
23 void MyCList::InsertAfter(MyType value , int position) {
24     if (head == nullptr) {
25         cout << "First Create the list." << endl;
26         return;
27     }
28     Cnode *temp, *s=head->next;
29     for (int i = 0; i < position - 1; i++) {
30         s = s->next;
31         if (s == head->next) {
32             cout << "There are less than "
33                  << position << " in the list" << endl;
34             return;
35         }
36     }
37     temp = new Cnode(value , s->next);
38     s->next = temp;
39     // Element inserted at the end
40     if (s == head) {
41         head = temp;
42     }
43 }
```

---



### 5.3.3 Delete

Algorithm 5.37: Delete.

---

```
1  // Deletion of element from the list
2  void MyCList::Delete(MyType value) {
3      Cnode *temp, *s;
4      s = head->next;
5      // If List has only one element
6      if (head->next == head && head->data == value) {
7          temp = head;
8          head = nullptr;
9          delete temp;
10         cout << "Element " << value << " deleted" << endl;
11     }
12     else if (s->data == value) {
13         temp = s;
14         head->next = s->next;
15         delete temp;
16         cout << "Element " << value << " deleted" << endl;
17     }
18     else {
19         bool found = false;
20         while ((s->next != head) && (!found)) {
21             if (s->next->data == value) {
22                 temp = s->next;
23                 s->next = temp->next;
24                 delete temp;
25                 cout << "Element " << value << " deleted" << endl;
26                 found = true;
27             }
28             s = s->next;
29         }
30         if (!found) {
31             if (s->next->data == value) {
32                 temp = s->next;
33                 s->next = head->next;
34                 delete temp;
35                 cout << "Element " << value << " deleted" << endl;
36                 head = s;
37             }
38             else
39                 cout << "Element " << value
40                     << " is not found in the list" << endl;
41         }
42     }
43 }
```

---

### 5.3.4 Search and display

Algorithm 5.38: Search, Display.

---

```
1  bool MyCList::Search(MyType value) {
2      Cnode *s;
3      bool found = false;
4      int counter = 0;
5      s = head->next;
6      while ((s != head) && (!found)) {
7          counter++;
8          if (s->data == value) {
9              cout << "Element " << value
10                 << " found at position " << counter << endl;
11              found = true;
12          }
13          s = s->next;
14      }
15      if (!found) {
16          if (s->data == value) {
17              counter++;
18              cout << "Element " << value
19                 << " found at position " << counter << endl;
20          }
21          else
22              cout << "Element " << value
23                 << " not found in the list" << endl;
24      }
25      return found;
26  }
27  void MyCList::Display() {
28      Cnode *s;
29      if (head == nullptr) {
30          cout << "List is empty" << endl;
31      }
32      else {
33          s = head->next;
34          cout << "Circular Linked List: " << endl;
35          while (s != head) {
36              cout << s->data << "->";
37              s = s->next;
38          }
39          cout << s->data << endl;
40      }
41  }
```

---

Algorithm 5.39: Update.

---

```
1 void MyCList::Update(MyType value , int position) {
2     if (head == nullptr)
3         cout << "The list is empty." << endl;
4     else {
5         Cnode *s;
6         s = head->next;
7         int i = 0;
8         while ((i < position - 1) && (s != head)) {
9             s = s->next;
10            i++;
11        }
12        if (s != head)
13            s->data = value;
14    }
15 }
```

---

## 5.4 Skip list

### 5.4.1 Class definition

Algorithm 5.40: Skip list - Class definition.

---

```
1  class SkipNode {
2  public:
3      SkipNode() : next(nullptr), data(0) {};
4      SkipNode(MyType key, int level);
5      // Array of pointers to nodes of different levels
6      SkipNode **next;
7      MyType data;
8      int level;
9  };
10
11 class MySkipList : public MyDataStructure {
12 public:
13     MySkipList();
14     MySkipList(int MAXLVL, float P);
15     ~MySkipList();
16     MyDataStructure* clone() { return new MySkipList(); }
17     void Insert(MyType x);
18     void Delete(MyType x);
19     bool Search(MyType x);
20     void Display();
21     void DisplayFile();
22 private:
23     int RandomLevel();
24     int MaxLvl; // Maximum level for this skip list
25                 // P is the fraction of the nodes with level
26                 // i pointers also having level i+1 pointers
27     float P;
28     // current level of skip list
29     int level;
30     SkipNode *head; // pointer to header node
31 };
```

---

## 5.4.2 Main functions

Algorithm 5.41: Skip list - functions.

---

```
1 SkipNode::SkipNode(MyType x, int level) {
2     data = x;
3     level = level;
4     next = new SkipNode*[level + 1];
5     // Fill the next array with NULL
6     memset(next, 0, sizeof(SkipNode)*(level + 1));
7 }
8
9 MySkipList::MySkipList() {
10     MaxLvl = 4;
11     P = 0.5;
12     level = 0;
13     head = new SkipNode(-1, MaxLvl);
14 }
15
16 MySkipList::MySkipList(int MAXLVL1, float P1) {
17     MaxLvl = MAXLVL1;
18     P = P1;
19     level = 0;
20     // -1 for smallest value
21     head = new SkipNode(-1, MaxLvl);
22 }
23
24 MySkipList::~MySkipList() {
25     SkipNode *current = head;
26     while (current) {
27         SkipNode *next = current->next[0];
28         delete current;
29         current=next;
30     }
31 }
32
33 int MySkipList::RandomLevel() {
34     float r = (float)rand()/RAND_MAX;
35     int lvl = 0;
36     while (r < P && lvl < MaxLvl) {
37         lvl++;
38         r = (float)rand()/RAND_MAX;
39     }
40     return lvl;
41 }
```

---

### 5.4.3 Insert and delete

Algorithm 5.42: Insert and delete.

---

```
1 void MySkipList::Insert(MyType x) {
2     SkipNode *current = head;
3     SkipNode **update=new SkipNode*[MaxLvl + 1];
4     memset(update, 0, sizeof(SkipNode)*(MaxLvl + 1));
5     for (int i = level; i >= 0; i--) {
6         while (current->next[i] != nullptr &&
7             current->next[i]->data < x)
8             current = current->next[i];
9         update[i] = current;
10    }
11    current = current->next[0];
12    if (current == nullptr || current->data != x) {
13        int rlevel = RandomLevel();
14        if (rlevel > level) {
15            for (int i = level + 1; i < rlevel + 1; i++)
16                update[i] = head;
17            level = rlevel;
18        }
19        SkipNode* n = new SkipNode(x, rlevel);
20        for (int i = 0; i <= rlevel; i++) {
21            n->next[i] = update[i]->next[i];
22            update[i]->next[i] = n;
23        }
24    }
25 }
26 void MySkipList::Delete(MyType x) {
27     SkipNode *current = head;
28     SkipNode **update = new SkipNode*[MaxLvl + 1];
29     memset(update, 0, sizeof(SkipNode)*(MaxLvl + 1));
30     for (int i = level; i >= 0; i--) {
31         while (current->next[i] != nullptr &&
32             current->next[i]->data < x)
33             current = current->next[i];
34         update[i] = current;
35    }
36    if (current->next[0] != nullptr) {
37        current = current->next[0];
38        if (current->data == x) {
39            for (int i = 0; i <= current->level; i++) {
40                update[i]->next[i] = current->next[i];
41            }
42            delete current;
43        }
44    }
45 }
```

---

## 5.4.4 Search and display

Algorithm 5.43: Search and display.

---

```
1  bool MySkipList::Search(MyType x) {
2      SkipNode *current = head;
3      SkipNode **update = new SkipNode*[MaxLvl + 1];
4      memset(update, 0, sizeof(SkipNode)*(MaxLvl + 1));
5      for (int i = level; i >= 0; i--) {
6          while (current->next[i] != nullptr &&
7              current->next[i]->data < x)
8              current = current->next[i];
9          update[i] = current;
10     }
11     if (current->next[0] == nullptr)
12         return false; // not found
13     if (current->next[0]->data == x)
14         return true; // found
15     return false; // not found
16 }
17
18 void MySkipList::Display() {
19     for (int i = 0; i <= level; i++) {
20         SkipNode *node = head->next[i];
21         cout << "Level: " << i << endl;
22         while (node != nullptr) {
23             cout << node->data << " ";
24             cout << "(" << node->level << ")" ";
25             node = node->next[i];
26         }
27         cout << endl;
28     }
29 }
30
31 void MySkipList::DisplayFile() {
32     ofstream myfile;
33     myfile.open("log_skiplist.txt");
34     for (int i = 0; i <= level; i++) {
35         SkipNode *node = head->next[i];
36         myfile << "Level: " << i << endl;
37         while (node != nullptr) {
38             myfile << node->data << " ";
39             node = node->next[i];
40         }
41         myfile << endl;
42     }
43     myfile.close();
44 }
```

---

### 5.4.5 Example

Algorithm 5.44: Example.

---

```
1 void main() {
2     MySkipList* s = new MySkipList(2,0.5);
3     s->Insert(10);
4     s->Insert(20);
5     s->Insert(5);
6     s->Insert(7);
7     s->Insert(9);
8     s->Insert(8);
9     s->Insert(5);
10    s->Insert(15);
11    s->Insert(25);
12    s->Insert(16);
13    s->Insert(26);
14
15    cout << s->Search(25) << endl;
16    cout << s->Search(14) << endl;
17    cout << s->Search(26) << endl;
18    cout << s->Search(27) << endl;
19    cout << s->Search(3) << endl;
20    cout << s->Search(12) << endl;
21    cout << s->Search(5) << endl;
22    s->Display();
23
24    s->Delete(15);
25    s->Display();
26
27    delete s;
28 }
```

---



## Chapter 6

# Hash tables

### Contents

---

<b>6.1</b>	<b>Hash tables . . . . .</b>	<b>101</b>
6.1.1	Class definition . . . . .	101
6.1.2	Main functions . . . . .	102
6.1.3	Insert . . . . .	103
6.1.4	Search . . . . .	104
6.1.5	Display . . . . .	106

---

## 6.1 Hash tables

### 6.1.1 Class definition

Algorithm 6.1: Hash Table - Class definition.

---

```
1  class MyHashTable : public MyDataStructure {
2  public:
3      MyHashTable();
4      MyHashTable(int n, int m, int type);
5      ~MyHashTable();
6      MyDataStructure* clone() { return new MyHashTable(); }
7      void Insert(MyType x);
8      void Delete(MyType x);
9      bool Search(MyType x);
10     pair<bool, int> SearchKey(MyType x);
11     void Display();
12     void DisplayFile();
13 private:
14     int HashFunction(MyType x);
15     int n; // size
16     int m; // modulus value
17     MyType* ht; // hash table
18     bool* htd; // present or not
19     int type; // collision management: 0: linear, 1: quadratic probing
20 };
```

---

### 6.1.2 Main functions

Algorithm 6.2: Hash Table - functions.

---

```
1 MyHashTable::MyHashTable() : n(0), m(0), type(0) {}
2
3 MyHashTable::MyHashTable(int n1, int m1, int type1) {
4     n = n1;
5     m = m1;
6     type = type1;
7     ht = new MyType[n];
8     htd = new bool[n];
9     for (int i = 0; i < n; i++) {
10         ht[i] = -1;
11         htd[i] = false;
12     }
13 }
14
15 MyHashTable::~~MyHashTable() {
16     delete[] ht;
17     delete[] htd;
18 }
19
20 void MyHashTable::Delete(MyType x) {
21     pair<bool, int> r = SearchKey(x);
22     if (r.second != -1)
23         htd[r.second] = false;
24 }
25
26 int MyHashTable::HashFunction(MyType x) {
27     return (int)x % m;
28 }
```

---

### 6.1.3 Insert

Algorithm 6.3: Hash Table - Insert.

---

```
1 void MyHashTable::Insert(MyType x) {
2     int key = HashFunction(x);
3     if (htd[key]) { // collision
4         int probe = key + 1;
5         int step = 1;
6         int k = 1;
7         bool place = false;
8         while ((k < n) && (!place)) {
9             if (!htd[probe]) {
10                 place = true;
11                 ht[probe] = x;
12                 htd[probe] = true;
13             }
14             else {
15                 step++;
16                 if (type == 0) // linear probing
17                     probe = key + step;
18                 else // quadratic probing
19                     probe = key + step*step;
20                 probe = probe % m;
21             }
22             k++;
23         }
24     }
25     else {
26         ht[key] = x;
27         htd[key] = true;
28     }
29 }
```

---

### 6.1.4 Search

Algorithm 6.4: Hash Table - Search.

---

```
1  bool MyHashTable::Search(MyType x) {
2      int key = HashFunction(x);
3      if (ht[key] == x) {
4          return true;
5      }
6      else {
7          int probe = key + 1;
8          int step = 1;
9          int k = 1;
10         while (k < n) {
11             if ((ht[probe] == x) && (htd[probe])) { // found
12                 return true;
13             }
14             else if (!htd[probe]) // it is a hole
15                 return false;
16             else {
17                 step++;
18                 if (type == 0) // linear probing
19                     probe = key + step;
20                 else // quadratic probing
21                     probe = key + step*step;
22                 probe = probe % m;
23             }
24             k++;
25         }
26         return false;
27     }
28 }
```

---

Algorithm 6.5: Hash Table - SearchKey.

---

```

1 pair<bool, int> MyHashTable::SearchKey(MyType x) {
2     int key = HashFunction(x);
3     if (ht[key] == x) {
4         return make_pair(true, key);
5     }
6     else {
7         int probe = key + 1;
8         int step = 1;
9         int k = 1;
10        while (k < n) {
11            if ((ht[probe] == x) && (htd[probe])) { // found
12                return make_pair(true, probe);
13            }
14            else if (!htd[probe]) // it is a hole
15                return make_pair(false, -1);
16            else {
17                step++;
18                if (type == 0) // linear probing
19                    probe = key + step;
20                else // quadratic probing
21                    probe = key + step*step;
22                probe = probe % m;
23            }
24            k++;
25        }
26        return make_pair(false, -1);
27    }
28 }

```

---

### 6.1.5 Display

Algorithm 6.6: Hash Table - Display.

---

```
1 void MyHashTable::Display() {
2     cout << "Hash table of size " << n << endl;
3     for (int i = 0; i < n; i++) {
4         cout << "Key: " << i << " with value: " << ht[i]
5             << "(" << htd[i] << ")" << endl;
6     }
7     cout << endl;
8 }
9
10 void MyHashTable::DisplayFile() {
11     ofstream myfile;
12     myfile.open("log_hashtable.txt");
13     myfile << "Hash table of size " << n << endl;
14     for (int i = 0; i < n; i++) {
15         myfile << "Key: " << i << " with value: " << ht[i]
16             << "(" << htd[i] << ")" << endl;
17     }
18     myfile.close();
19 }
```

---

## 6.1.6 Examples

Algorithm 6.7: Example.

---

```
1 void main() {
2     MyHashTable* H = new MyHashTable(10, 10, 1);
3     H->Insert(39);
4     H->Insert(13);
5     H->Insert(23);
6     H->Insert(63);
7     H->Insert(30);
8     H->Insert(31);
9     H->Insert(49);
10    H->Delete(49);
11    H->Insert(59);
12    H->Display();
13    cout << "Search 39: " << H->Search(39) << endl;
14    cout << "Search 49: " << H->Search(49) << endl;
15    cout << "Search 59: " << H->Search(59) << endl;
16    cout << "Search 23: " << H->Search(23) << endl;
17    delete H;
18 }
```

---

Algorithm 6.8: Distribution of the characters in a string.

---

```
1 void main() {
2     string str = "ao_abc ! zozo za"; // input string
3     int n = 26; // number of letters in the alphabet
4     // out: output array containing the number of letters, for each letter
5     int* out = new int[n];
6     // initialization
7     for (int i = 0; i < n; i++)
8         out[i] = 0;
9     for (int i = 0; i < str.size(); i++) {
10
11         int codechar = str[i] - 'a';
12         if ((codechar >= 0) && (codechar < 26))
13             out[codechar]++;
14     }
15     // Display output array
16     for (int i = 0; i < n; i++) {
17         cout << (char)('a' + i) << " " << out[i] << endl;
18     }
19     delete[] out;
20 }
```

---



# Chapter 7

## Trees

### Contents

---

<b>7.1</b>	<b>Binary Search Trees</b>	<b>109</b>
7.1.1	Definitions	109
7.1.2	TreeNode	110
7.1.3	Tree Traversal	112
7.1.4	Search	116
7.1.5	Insert	117
7.1.6	Delete	121
7.1.7	Min, Max	122
7.1.8	Max and Min Depth	123
7.1.9	Comparisons	124
7.1.10	MyBST	126
<b>7.2</b>	<b>AVL Trees</b>	<b>128</b>
7.2.1	Class interface	128
7.2.2	Class main functions	129
7.2.3	Rotations and balance	130
7.2.4	Display	133
7.2.5	Search, Insert, Delete	134
<b>7.3</b>	<b>23-Trees</b>	<b>137</b>
7.3.1	Class definition	137
7.3.2	Search	140
7.3.3	Insert	141
7.3.4	Delete	145
7.3.5	Display	150
<b>7.4</b>	<b>Red &amp; Black Trees</b>	<b>153</b>
7.4.1	Definitions	153
7.4.2	Family	153
7.4.3	Class definition	154
7.4.4	Search	158
7.4.5	Insert	158
7.4.6	Delete	160

---

## 7.1 Binary Search Trees

### 7.1.1 Definitions

Notation:

- **Depth:** The depth of a node: the number of edges from the root to the node.
- **Height:** The height of a node corresponds to the number of edges from the node to the deepest leaf. The height of a tree is the height of the root.
- **Levels:** The level of a particular node represents how many generations the node is from the root. The root node is at Level 0 (start at 0), the root node's children are at Level 1, the root node's grandchildren are at Level 2, etc.
- **Keys:** One data field in an object is usually designated a key value. It is used to search for the item.
- **Traversing:** To traverse a tree means to visit all the nodes in a specified order.
- **Size:** the total number of nodes in that tree

Special types of binary trees:

- **Binary Search Tree (BST):** It is a binary tree in which a node's left child has a key less than its parent, and a node's right child has a key greater than or equal to its parent
- **Complete binary tree:** It is a binary tree in which all the nodes at one level must have values before starting the next level, and all the nodes in the last level must be completed from left to right.
- **Full binary tree:** It is a binary tree in which every node has either 0 or 2 children
- **Perfect binary trees:** It is a binary tree in which all interior nodes have 2 children and all leaves have the same depth or same level. Hence, it is a full binary tree and all leaf nodes are at the same level.

### 7.1.2 TreeNode

Algorithm 7.1: TreeNode definition.

---

```
1 class TreeNode {
2 public:
3     TreeNode() : data(0), left(nullptr), right(nullptr) { }
4     TreeNode(int d) : data(d), left(nullptr), right(nullptr) { }
5     ~TreeNode() {}
6     int data;           // data in this node
7     TreeNode *left;     // pointer to the left subtree
8     TreeNode *right;    // pointer to the right subtree
9 };
```

---

Algorithm 7.2: TreeNode functions.

---

```
1 void PrintNode(TreeNode* root);
2 int CountNodes(TreeNode* root);
3
4 // Tree traversal
5 void PreorderNode(TreeNode* root, void(*fct)(TreeNode* root));
6 void InorderNode(TreeNode* root, void(*fct)(TreeNode* root));
7 void PostorderNode(TreeNode* root, void(*fct)(TreeNode* root));
8 void PrintPreorderNode(TreeNode* root, int lvl);
9 void PrintInorderNode(TreeNode* root, int lvl);
10 void PrintPostorderNode(TreeNode* root, int lvl);
11 void PrintLevelOrder(TreeNode* root);
12
13 void GetNumberNodesLevel(TreeNode* root);
14 TreeNode* InvertTreeNode(TreeNode* root);
15 // Search
16 bool SearchNode(TreeNode* root, MyType data);
17 bool SearchNode1(TreeNode* root, MyType data);
18 // Insert
19 void InsertNode(TreeNode** root, MyType data);
20 void InsertNode1(TreeNode* &root, MyType data);
21 TreeNode* InsertNode2(TreeNode* root, MyType data);
22 void InsertNode3(TreeNode** root, MyType data);
23 void InsertNode4(TreeNode* &root, MyType data);
24 // Delete
25 TreeNode* DeleteNode(TreeNode *root, MyType data);
26 MyType FindMinTree(TreeNode *root);
27 MyType FindMaxTree(TreeNode *root);
28 TreeNode* FindMinNode(TreeNode *root);
29 TreeNode* FindMaxNode(TreeNode *root);
30 int MaxDepthTree2(TreeNode *root);
31 int MaxDepthTree(TreeNode* root);
32 int MinDepthTree(TreeNode *root);
33
34 void DestroyTree(TreeNode *root);
35
36 // Comparisons
37 bool SameTree(TreeNode* t1, TreeNode* t2);
```

---

```
38 bool IsBST(TreeNode* node, int min, int max);
39 bool IsCompleteTree(TreeNode* root, int index, int nnodes);
40 bool IsFullTree(TreeNode* root);
41 bool IsPerfectTree(TreeNode* root);
```

---

Algorithm 7.3: Destroy the nodes in the tree.

---

```
1 void DestroyTree(TreeNode *root) {
2     if (root != nullptr) {
3         DestroyTree(root->left);
4         DestroyTree(root->right);
5         delete root;
6     }
7 }
```

---

Algorithm 7.4: Count the nodes in the tree.

---

```
1 // Count the nodes in the binary tree to which root points.
2 int CountNodes(TreeNode* root) {
3     if (!root)
4         return 0; // The tree is empty.
5     else {
6         int count = 1; // Start by counting the root.
7         // Add the number of nodes in the left subtree
8         count += CountNodes(root->left);
9         // Add the number of nodes in the right subtree
10        count += CountNodes(root->right);
11        return count;
12    }
13 }
```

---

### 7.1.3 Tree Traversal

Algorithm 7.5: Tree traversal: Pre-In-Post.

---

```
1 void PrintNode(TreeNode* root) {
2     cout << root->data << " ";
3 }
4
5 void PreorderNode(TreeNode* root, void(*fct)(TreeNode* root)) {
6     if (root != nullptr) {
7         (*fct)(root);
8         PreorderNode(root->left, fct);
9         PreorderNode(root->right, fct);
10    }
11 }
12 void InorderNode(TreeNode* root, void(*fct)(TreeNode* root)) {
13     if (root != nullptr) {
14         InorderNode(root->left, fct);
15         (*fct)(root);
16         InorderNode(root->right, fct);
17    }
18 }
19 void PostorderNode(TreeNode* root, void(*fct)(TreeNode* root)) {
20     if (root != nullptr) {
21         PostorderNode(root->left, fct);
22         PostorderNode(root->right, fct);
23         (*fct)(root);
24    }
25 }
```

---

Algorithm 7.6: Tree traversal example.

---

```
1 void main() {
2     TreeNode* n = nullptr;
3     InsertNode1(&n, 50);
4     InsertNode1(&n, 25);
5     InsertNode(&n, 75);
6     InsertNode(&n, 5);
7     InsertNode(&n, 15);
8     InsertNode(&n, 65);
9     InsertNode(&n, 85);
10    void(*fct)(TreeNode*) = PrintNode;
11    PreorderNode(n, fct);
12 }
```

---

Algorithm 7.7: Tree traversal: Pre-In-Post.

---

```
1 void PrintPreorderNode(TreeNode* root, int lvl) {
2     if (root != nullptr) {
3         cout << root->data << " (" << lvl << ")" << endl;
4         PrintPreorderNode(root->left, lvl + 1);
5         PrintPreorderNode(root->right, lvl + 1);
6     }
7 }
8
9 void PrintInorderNode(TreeNode* root, int lvl) {
10    if (root != nullptr) {
11        PrintInorderNode(root->left, lvl + 1);
12        cout << root->data << " (" << lvl << ")" << endl;
13        PrintInorderNode(root->right, lvl + 1);
14    }
15 }
16
17 void PrintPostorderNode(TreeNode* root, int lvl) {
18    if (root != nullptr) {
19        PrintPostorderNode(root->left, lvl + 1);
20        PrintPostorderNode(root->right, lvl + 1);
21        cout << root->data << " (" << lvl << ")" << endl;
22    }
23 }
```

---

Algorithm 7.8: Tree traversal: level order.

---

```
1 // Print nodes at a given level
2 void PrintGivenLevel(TreeNode* root , int level) {
3     if (root != nullptr) {
4         if (level == 0)
5             cout << "-" << root->data << "-" ;
6         else {
7             PrintGivenLevel(root->left , level - 1);
8             PrintGivenLevel(root->right , level - 1);
9         }
10    }
11 }
12
13 void PrintLevelOrder(TreeNode* root) {
14     int h = MaxDepthTree(root);
15     int i;
16     for (i = 0; i <= h; i++) {
17         PrintGivenLevel(root , i);
18         // at each line we have max 2^h
19         cout << endl;
20     }
21 }
```

---

Algorithm 7.9: Tree traversal: Nodes per level.

---

```
1 // Nodes per level
2 int GetNumberNodesGivenLevel(TreeNode* root , int level) {
3     if (root != nullptr) {
4         if (level == 0)
5             return 1;
6         else
7             return GetNumberNodesGivenLevel(root->left , level - 1) +
8                    GetNumberNodesGivenLevel(root->right , level - 1);
9     }
10    else
11        return 0;
12 }
13
14 void GetNumberNodesLevel(TreeNode* root) {
15     int h = MaxDepthTree(root);
16     int i;
17     for (i = 0; i <= h; i++)
18         cout << "level:" << i << " with "
19              << GetNumberNodesGivenLevel(root , i) << " nodes" << endl;
20 }
```

---

Algorithm 7.10: Invert Tree.

---

```
1  TreeNode* InvertTreeNode(TreeNode* root) {  
2      if (root==nullptr) {  
3          return nullptr; // terminal condition  
4      }  
5      auto left = InvertTreeNode(root->left); // invert left sub-tree  
6      auto right = InvertTreeNode(root->right); // invert right sub-tree  
7      root->left = right; // put right on left  
8      root->right = left; // put left on right  
9      return root;  
10 }
```

---



### 7.1.4 Search

Algorithm 7.11: Search.

---

```
1  // Recursive version
2  bool SearchNode(TreeNode* root, MyType data) {
3      if (root == nullptr)
4          return false;
5      else if (root->data == data)
6          return true;
7      else if (data < root->data)
8          SearchNode(root->left, data);
9      else
10         SearchNode(root->right, data);
11 }
12
13 // Iterative version
14 bool SearchNode1(TreeNode* root, MyType data) {
15     if (root == nullptr)
16         return false;
17     else {
18         TreeNode* current = root;
19         bool found = false;
20         while ((!found) && (current != nullptr)) {
21             if (current->data == data)
22                 found = true;
23             else if (data < current->data)
24                 current = current->left;
25             else
26                 current = current->right;
27         }
28         return found;
29     }
30 }
```

---

### 7.1.5 Insert

There are multiple solutions to create the Insert method. It can be set as a function that returns a new root node, or a function that take the root as input/output. The solution can be recursive (V1, V2, V3), or iterative. In both cases, it is possible to use pointers or references. The following examples show some possible implementations of the insert method.

Algorithm 7.12: Insert (recursive version).

---

```
1  // Recursive version V1
2  // root as input/output (pointer)
3  void InsertNode(TreeNode** root, MyType data) {
4      if ((*root) == nullptr)
5          (*root) = new TreeNode(data);
6      else if ((*root)->data == data)
7          cout << "Value already in the tree." << endl;
8      else if (data < (*root)->data)
9          InsertNode(&((*root)->left), data);
10     else
11         InsertNode(&((*root)->right), data);
12 }
13
14 // Recursive version V2
15 // root as input/output (reference)
16 void InsertNode1(TreeNode* &root, MyType data) {
17     if (root == nullptr)
18         root = new TreeNode(data);
19     else if (root->data == data)
20         cout << "Value already in the tree." << endl;
21     else if (data < root->data)
22         InsertNode1(root->left, data);
23     else
24         InsertNode1(root->right, data);
25 }
```

---

Algorithm 7.13: Insert (recursive version).

---

```
1 // Recursive version V3
2 // root as input only
3 TreeNode* InsertNode2(TreeNode* root, MyType data) {
4     if (root == nullptr) {
5         return new TreeNode(data);
6     }
7     else if (root->data == data) {
8         cout << "Value already in the tree." << endl;
9         return root;
10    }
11    else if (data < root->data) {
12        root->left = InsertNode2(root->left, data);
13        return root;
14    }
15    else {
16        root->right = InsertNode2(root->right, data);
17        return root;
18    }
19 }
```

---

Algorithm 7.14: Insert (iterative version).

---

```
1  // With pointer
2  void InsertNode3(TreeNode** root, MyType data) {
3      if ((*root) == nullptr) {
4          (*root) = new TreeNode(data);
5      }
6      else {
7          TreeNode* current= (*root);
8          TreeNode* parent = nullptr;
9          int direction = 0;
10         while (current != nullptr) {
11             parent = current;
12             if (current->data == data) {
13                 cout << "Value already in the tree." << endl;
14                 current = nullptr;
15             }
16             else if (data < current->data)
17                 current = current->left;
18             else
19                 current = current->right;
20         }
21         current= new TreeNode(data);
22         if (data < parent->data)
23             parent->left = current;
24         else if (data > parent->data)
25             parent->right = current;
26     }
27 }
```

---

Algorithm 7.15: Insert (iterative version).

---

```
1  // With reference
2  void InsertNode4(TreeNode* &root, MyType data) {
3      if (root == nullptr) {
4          root = new TreeNode(data);
5      }
6      else {
7          TreeNode* current = root;
8          TreeNode* parent = nullptr;
9          int direction = 0;
10         while (current != nullptr) {
11             parent = current;
12             if (current->data == data) {
13                 cout << "Value already in the tree." << endl;
14                 current = nullptr;
15             }
16             else if (data < current->data)
17                 current = current->left;
18             else
19                 current = current->right;
20         }
21         current = new TreeNode(data);
22         if (data < parent->data)
23             parent->left = current;
24         else if (data > parent->data)
25             parent->right = current;
26     }
27 }
```

---

### 7.1.6 Delete

Algorithm 7.16: Delete.

---

```
1  TreeNode* DeleteNode(TreeNode *root, MyType data) {
2      if (root == nullptr)
3          return nullptr;
4      if (data < root->data) // Data is in the left sub tree.
5          root->left = DeleteNode(root->left, data);
6      else if (data > root->data) // Data is in the right sub tree.
7          root->right = DeleteNode(root->right, data);
8      else {
9          // case 1: no children
10         if (root->left == nullptr && root->right == nullptr) {
11             delete root;
12             root = nullptr;
13         }
14         // case 2: 1 child (right)
15         else if (root->left == nullptr) {
16             TreeNode *temp = root; // Save current node as a backup
17             root = root->right;
18             delete temp;
19         }
20         // case 3: 1 child (left)
21         else if (root->right == nullptr) {
22             TreeNode *temp = root; // Save current node as a backup
23             root = root->left;
24             delete temp;
25         }
26         // case 4: 2 children
27         else {
28             // Find minimal value of right sub tree
29             TreeNode *temp = FindMinNode(root->right);
30             root->data = temp->data; // Duplicate the node
31             // Delete the duplicate node
32             root->right = DeleteNode(root->right, temp->data);
33         }
34     }
35     return root; // parent node can update reference
36 }
```

---

### 7.1.7 Min, Max

Algorithm 7.17: Find minimum and maximum values.

---

```
1 MyType FindMinTree(TreeNode *root) {
2     if (root==nullptr) {
3         return INT_MAX; // or undefined.
4     }
5     if (root->left != nullptr) {
6         return FindMinTree(root->left); // left tree is smaller
7     }
8     return root->data;
9 }
10
11 MyType FindMaxTree(TreeNode *root) {
12     if (root == nullptr) {
13         return INT_MAX; // or undefined.
14     }
15     if (root->right != nullptr) {
16         return FindMaxTree(root->right); // right tree is bigger
17     }
18     return root->data;
19 }
```

---

Algorithm 7.18: Find minimum and maximum nodes.

---

```
1 TreeNode* FindMinNode(TreeNode *root) {
2     if (root == nullptr) {
3         return nullptr;
4     }
5     if (root->left != nullptr) {
6         return FindMinNode(root->left); // left tree is smaller
7     }
8     return root;
9 }
10
11 TreeNode* FindMaxNode(TreeNode *root) {
12     if (root == nullptr) {
13         return nullptr;
14     }
15     if (root->right != nullptr) {
16         return FindMaxNode(root->right); // left tree is smaller
17     }
18     return root;
19 }
```

---

### 7.1.8 Max and Min Depth

Algorithm 7.19: MaxDepthTree.

---

```
1  int MaxDepthTree2(TreeNode *root) {
2      if (root == nullptr)
3          return 0;
4      else if ((root->left == nullptr) && (root->right == nullptr))
5          return 0;
6      else
7          return 1 + max(MaxDepthTree2(root->left),
8                          MaxDepthTree2(root->right));
9  }
10 int MaxDepthTree(TreeNode* root) {
11     if (root == nullptr) {
12         return 0;
13     }
14     else if ((root->left == nullptr) && (root->right == nullptr))
15         return 0;
16     else {
17         // compute the depth of each subtree
18         int leftDepth = MaxDepthTree(root->left);
19         int rightDepth = MaxDepthTree(root->right);
20         // use the larger subtree
21         if (leftDepth > rightDepth)
22             return leftDepth + 1;
23         else
24             return rightDepth + 1;
25     }
26 }
```

---

Algorithm 7.20: MinDepthTree.

---

```
1  int MinDepthTree(TreeNode *root) {
2      if (root == nullptr)
3          return 0;
4      // Base case : Leaf Node. This accounts for height = 1.
5      if (root->left == nullptr && root->right == nullptr)
6          return 1;
7      // If left subtree is NULL, recur for right subtree
8      if (!root->left)
9          return MinDepthTree(root->right) + 1;
10     // If right subtree is NULL, recur for right subtree
11     if (!root->right)
12         return MinDepthTree(root->left) + 1;
13     return min(MinDepthTree(root->left), MinDepthTree(root->right)) + 1;
14 }
```

---



## 7.1.9 Comparisons

Algorithm 7.21: Same tree?

---

```
1 bool SameTree(TreeNode* t1, TreeNode* t2) {
2     if (t1 == nullptr && t2 == nullptr) // both empty
3         return true;
4     if (t1 != nullptr && t2 != nullptr) { // both non-empty
5         return ((t1->data == t2->data) &&
6                 (SameTree(t1->left, t2->left)) &&
7                 (SameTree(t1->right, t2->right)))
8             );
9     }
10    return false; // one empty, one not -> false
11 }
```

---

Algorithm 7.22: Is it a BST?

---

```
1 // True if the tree is a BST and its values are >= min and <= max.
2 bool IsBST(TreeNode* node, int min, int max) {
3     if (!node)
4         return true;
5     if (node->data < min || node->data > max)
6         return false;
7     return (IsBST(node->left, min, node->data) &&
8             IsBST(node->right, node->data + 1, max))
9         );
10 }
```

---

---

Algorithm 7.23: Is it a complete tree?

---

```
1 // Array representation of a binary tree
2 // True if the tree is complete
3 bool IsCompleteTree(TreeNode* root, int index, int nnodes) {
4     if (root == nullptr)
5         return true; // An empty tree is complete
6     if (index >= nnodes)
7         return false;
8     return (IsCompleteTree(root->left, 2 * index + 1, nnodes) &&
9             IsCompleteTree(root->right, 2 * index + 2, nnodes));
10 }
```

---

---

Algorithm 7.24: Is it a full tree?

---

```
1 // True if the tree is full
2 bool IsFullTree(TreeNode* root) {
3     if ((root == nullptr) || ((root->left == nullptr) && (root->right == nullptr)))
4         return true; // An empty tree is full
5     else if ((root->left != nullptr) && (root->right != nullptr))
6         return (IsFullTree(root->left) && IsFullTree(root->right));
7     else
8         return false;
9 }
```

---

---

Algorithm 7.25: Is it a perfect tree?

---

```
1 // True if the tree is perfect
2 bool IsPerfectTree(TreeNode* root) {
3     if (root == nullptr)
4         return true; // An empty tree is perfect
5     else {
6         int h = MaxDepthTree(root);
7         int n = CountNodes(root);
8         return (n == pow(2, h + 1) - 1);
9     }
10 }
```

---

### 7.1.10 MyBST

Algorithm 7.26: BST interface.

---

```
1  class MyBST : public MyDataStructure {
2  public:
3      MyBST() { root = nullptr; }
4      ~MyBST() { DestroyTree(root); }
5
6      MyDataStructure* clone() { return new MyBST(); }
7
8      void Insert(int data) { InsertNode(&root, data); }
9      void Insert1(int data) { InsertNode1(root, data); }
10     void Insert2(int data) { root=InsertNode2(root, data); }
11     void Insert3(int data) { InsertNode3(&root, data); }
12     void Insert4(int data) { InsertNode4(root, data); }
13
14     void Delete(int data) { root=DeleteNode(root, data); }
15
16     void Preorder(void(*fct)(TreeNode* root)) { PreorderNode(root, fct); }
17     void Inorder(void(*fct)(TreeNode* root)) { InorderNode(root, fct); }
18     void Postorder(void(*fct)(TreeNode* root)) { PostorderNode(root, fct); }
19
20     void PrintPreorder() { PrintPreorderNode(root, 0); }
21     void PrintInorder() { PrintInorderNode(root, 0); }
22     void PrintPostorder() { PrintPostorderNode(root, 0); }
23     void PrintLevelorder() { PrintLevelOrder(root); }
24     void PrintGetNumberNodesLevel() { GetNumberNodesLevel(root); }
25
26     void InvertTree() { InvertTreeNode(root); }
27     int Height() { return MaxDepthTree(root); }
28     int Size() { return CountNodes(root); }
29
30     bool Search(MyType data) { return SearchNode(root, data); }
31     bool IsComplete() { return IsCompleteTree(root, 0, CountNodes(root)); }
32     bool IsFull() { return IsFullTree(root); }
33     bool IsPerfect() { return IsPerfectTree(root); }
34
35     int IsBSTv2() { return(IsBST(root, INT_MIN, INT_MAX)); }
36
37 private:
38     TreeNode* root; // pointer to the root
39 };
```

---

Algorithm 7.27: BST example 1.

---

```

1  main () {
2      MyBST* t = new MyBST();
3      t->Insert(5);
4      t->Insert(3);
5      t->Insert(8);
6      t->Insert(2);
7      t->Insert(4);
8      t->Insert(6);
9      t->Insert(9);
10     void(*fct)(TreeNode*) = PrintNode;
11     cout << "Print Pre-order: " << endl;
12     t->Preorder(fct);
13     cout << "MaxDepth: " << t->Height() << endl;
14     cout << "Print Pre-order: " << endl;
15     t->PrintPreorder();
16     cout << "Print In-order: " << endl;
17     t->PrintInorder();
18     cout << "Print Post-order: " << endl;
19     t->PrintPostorder();
20     cout << "Print Level-order: " << endl;
21     t->PrintLevelorder();
22     cout << "Number of nodes per level: " << endl;
23     t->PrintGetNumberNodesLevel();
24     cout << "IsComplete: " << t->IsComplete() << endl;
25     cout << "IsFull: " << t->IsFull() << endl;
26     cout << "IsPerfect: " << t->IsPerfect() << endl;
27     delete t;
28 }

```

---

Algorithm 7.28: BST example 2.

---

```

1  void main() {
2      MyBST* t1 = new MyBST();
3      t1->Insert(5);
4      t1->Insert(3);
5      t1->Insert(8);
6      t1->Insert(2);
7      t1->Insert(4);
8      cout << "IsComplete: " << t1->IsComplete() << endl;
9      cout << "IsFull: " << t1->IsFull() << endl;
10     cout << "IsPerfect: " << t1->IsPerfect() << endl;
11     delete t1;
12 }

```

---

## 7.2 AVL Trees

### 7.2.1 Class interface

Algorithm 7.29: AVL interface.

---

```
1  class AVLnode {
2  public:
3      MyType data;
4      int balance;
5      AVLnode *left , *right , *parent;
6      AVLnode(MyType k, AVLnode *p) : data(k), balance(0), parent(p),
7          left(nullptr), right(nullptr) {}
8  };
9
10 class MyAVL : public MyDataStructure {
11 public:
12     MyAVL();
13     ~MyAVL();
14     MyDataStructure* clone() { return new MyAVL(); }
15     void Insert(MyType x);
16     void Delete(MyType x);
17     bool Search(MyType x);
18     void Display ();
19     void Display1 ();
20 private:
21     void InsertNode(AVLnode** root , AVLnode* parent , MyType data);
22     AVLnode* DeleteNode(AVLnode *root , MyType data);
23     void rebalance(AVLnode *n);
24     AVLnode *root;
25 };
26
27 AVLnode* rotateLeft(AVLnode *a);
28 AVLnode* rotateRight(AVLnode *a);
29 AVLnode* rotateLeftThenRight(AVLnode *n);
30 AVLnode* rotateRightThenLeft(AVLnode *n);
31
32 int height(AVLnode *n);
33 void setBalance(AVLnode *n);
34 void printBalance(AVLnode *n);
```

---

## 7.2.2 Class main functions

Algorithm 7.30: Constructor and destructor.

---

```
1 void DestroyTree(AVLnode *root) {
2     if (root != nullptr) {
3         DestroyTree(root->left);
4         DestroyTree(root->right);
5         delete root;
6     }
7 }
8 MyAVL::MyAVL() : root(nullptr) {}
9 MyAVL::~MyAVL() { DestroyTree(root); }
```

---

Algorithm 7.31: Rebalance.

---

```
1 AVLnode* rebalance_node(AVLnode* n) {
2     setBalance(n);
3     if (n->balance == -2) {
4         if (height(n->left->left) >= height(n->left->right))
5             n = rotateRight(n);
6         else
7             n = rotateLeftThenRight(n);
8     }
9     else if (n->balance == 2) {
10        if (height(n->right->right) >= height(n->right->left))
11            n = rotateLeft(n);
12        else
13            n = rotateRightThenLeft(n);
14    }
15    return n;
16 }
17
18 void MyAVL::rebalance(AVLnode* n) {
19     if (n != nullptr) {
20         n=rebalance_node(n);
21         if (n->parent != nullptr)
22             rebalance(n->parent);
23         else
24             root = n; // very important to maintain the root
25     }
26 }
```

---

### 7.2.3 Rotations and balance

Algorithm 7.32: Rotate Left.

---

```
1 AVLnode* rotateLeft(AVLnode *a) {
2     if (a != nullptr) {
3         AVLnode *b = a->right;
4         b->parent = a->parent;
5         a->right = b->left;
6         if (a->right != nullptr)
7             a->right->parent = a;
8         b->left = a;
9         a->parent = b;
10        if (b->parent != nullptr) {
11            if (b->parent->right == a) {
12                b->parent->right = b;
13            }
14            else {
15                b->parent->left = b;
16            }
17        }
18        setBalance(a);
19        setBalance(b);
20        return b;
21    }
22    else
23        return nullptr;
24 }
```

---

---

Algorithm 7.33: Rotate Right.

---

```
1 AVLnode* rotateRight(AVLnode *a) {
2     if (a != nullptr) {
3         AVLnode *b = a->left;
4         b->parent = a->parent;
5         a->left = b->right;
6         if (a->left != nullptr)
7             a->left->parent = a;
8         b->right = a;
9         a->parent = b;
10        if (b->parent != nullptr) {
11            if (b->parent->right == a) {
12                b->parent->right = b;
13            }
14            else {
15                b->parent->left = b;
16            }
17        }
18        setBalance(a);
19        setBalance(b);
20        return b;
21    }
22    else
23        return nullptr;
24 }
```

---



---

Algorithm 7.34: Rotate Left then Right.

---

```
1 AVLnode* rotateLeftThenRight(AVLnode *n) {
2     n->left = rotateLeft(n->left);
3     return rotateRight(n);
4 }
```

---

---

Algorithm 7.35: Rotate Right then Left.

---

```
1 AVLnode* rotateRightThenLeft(AVLnode *n) {
2     n->right = rotateRight(n->right);
3     return rotateLeft(n);
4 }
```

---

---

Algorithm 7.36: Height.

---

```
1 int height(AVLnode *n) {
2     if (n == nullptr)
3         return -1;
4     return 1 + max(height(n->left), height(n->right));
5 }
```

---

---

Algorithm 7.37: Set Balance.

---

```
1 void setBalance(AVLnode *n) {
2     if (n != nullptr)
3         n->balance = height(n->right) - height(n->left);
4 }
```

---

---

Algorithm 7.38: Print balance.

---

```
1 void printBalance(AVLnode *n) {
2     if (n != nullptr) {
3         printBalance(n->left);
4         cout << n->balance << " ";
5         printBalance(n->right);
6     }
7 }
```

---

## 7.2.4 Display

Algorithm 7.39: Display.

---

```
1 void PrintGivenLevel(AVLnode* root , int level) {
2     if (root != nullptr) {
3         if (level == 0) {
4             if (root->parent != nullptr)
5                 cout << "_ " << root->data << "(" << root->parent->data << ") _ ";
6             else
7                 cout << "_ " << root->data << "( null ) _ ";
8         }
9         else {
10            PrintGivenLevel(root->left , level - 1);
11            PrintGivenLevel(root->right , level - 1);
12        }
13    }
14 }
15
16 void MyAVL::Display() {
17     int h = height(root);
18     int i;
19     for (i = 0; i <= h; i++) {
20         PrintGivenLevel(root , i);
21         // at each line we have max 2^h
22         cout << endl;
23     }
24 }
```

---

## 7.2.5 Search, Insert, Delete

Algorithm 7.40: Search.

---

```
1 bool SearchNode(AVLnode* root , MyType data) {
2     if (root == nullptr)
3         return false;
4     else
5     {
6         if (root->data == data)
7             return true;
8         else if (data < root->data)
9             SearchNode(root , data);
10        else
11            SearchNode(root , data);
12    }
13 }
14
15 bool MyAVL:: Search(MyType x) {
16     return SearchNode(root , x);
17 }
```

---

Algorithm 7.41: Insert.

---

```
1 void MyAVL:: InsertNode(AVLnode** root , AVLnode* parent , MyType data) {
2     if ((*root) == nullptr) {
3         (*root) = new AVLnode(data , parent);
4         rebalance(parent);
5     }
6     else if ((*root)->data == data) {
7         cout << "Value already in the tree." << endl;
8     }
9     else if (data < (*root)->data)
10        InsertNode(&((*root)->left) , (*root) , data);
11     else
12        InsertNode(&((*root)->right) , (*root) , data);
13 }
14
15 void MyAVL:: Insert(MyType x) {
16     InsertNode(&root , nullptr , x);
17 }
```

---

Algorithm 7.42: Find minimum.

---

```
1 AVLnode* FindMinNode(AVLnode *root) {  
2     if (root == nullptr) {  
3         return nullptr;  
4     }  
5     if (root->left != nullptr) {  
6         return FindMinNode(root->left); // left tree is smaller  
7     }  
8     return root;  
9 }
```

---

Algorithm 7.43: Delete.

---

```
1 void MyAVL::Delete(MyType x) {  
2     root = DeleteNode(root, x);  
3 }
```

---

---

Algorithm 7.44: Delete.

---

```

1  AVLnode* MyAVL::DeleteNode(AVLnode *n, MyType data) {
2      if (n == nullptr)
3          return nullptr;
4      if (data < n->data) // data is in the left sub tree.
5          n->left = DeleteNode(n->left, data);
6      else if (data > n->data) // data is in the right sub tree.
7          n->right = DeleteNode(n->right, data);
8      else { // data == n->data
9          // case 1: no children
10         if (n->left == nullptr && n->right == nullptr) {
11             delete n;
12             n = nullptr;
13         }
14         // case 2: 1 child (right)
15         else if (n->left == nullptr) {
16             AVLnode *temp = n; // save current node as a backup
17             n->right->parent = n->parent;
18             n = n->right;
19             delete temp;
20         }
21         // case 3: 1 child (left)
22         else if (n->right == nullptr) {
23             AVLnode *temp = n; // save current node as a backup
24             n->left->parent = n->parent;
25             n = n->left;
26             delete temp;
27         }
28         // case 4: 2 children
29         else {
30             // find minimal value of right sub tree
31             AVLnode *temp = FindMinNode(n->right);
32             // duplicate the node
33             n->data = temp->data;
34             // delete the duplicate node
35             n->right = DeleteNode(n->right, temp->data);
36         }
37     }
38     if (n == nullptr)
39         return nullptr;
40     else
41         return rebalance_node(n);
42 }

```

---

## 7.3 23-Trees

### 7.3.1 Class definition

Algorithm 7.45: Class interface.

---

```
1  class My23Node {
2  public:
3      My23Node() : key1(-1),key2(-1), nkeys(0),
4          left(nullptr), middle(nullptr), right(nullptr) { }
5      My23Node(int k1) : key1(k1), key2(-1), nkeys(1),
6          left(nullptr), middle(nullptr), right(nullptr) { }
7      My23Node(int k1,int k2) : key1(k1), key2(k2), nkeys(2),
8          left(nullptr), middle(nullptr), right(nullptr) { }
9      ~My23Node() {}
10     MyType key1,key2;    // data in this node
11     My23Node *left;     // pointer to the left subtree
12     My23Node *middle;   // pointer to the middle subtree
13     My23Node *right;    // pointer to the right subtree
14     int nkeys;
15 };
16
17 class My23Tree {
18 public:
19     My23Tree() { root = nullptr; }
20     ~My23Tree();
21     tuple<bool, int> Search(MyType data);
22     void Insert(MyType data);
23     void Delete(MyType data);
24     void Display();
25     void DisplayLevel();
26 private:
27     My23Node* root;
28 };
```

---

---

```
1 void destroy(My23Node* root) {
2     if (root != nullptr) {
3         if (root->nkeys == 1) {
4             destroy(root->left);
5             destroy(root->middle);
6         }
7         else {
8             destroy(root->left);
9             destroy(root->middle);
10            destroy(root->right);
11        }
12        delete root;
13    }
14 }
15
16 My23Tree::~My23Tree() {
17     if (root != nullptr)
18         destroy(root);
19 }
20
21 tuple<bool, int> My23Tree::Search(MyType data) {
22     return Search23(root, data, 0);
23 }
24
25 void My23Tree::Insert(MyType data) {
26     Insert23(root, data);
27 }
28
29 void My23Tree::Delete(MyType data) {
30     Delete23(root, data);
31 }
32
33 void My23Tree::Display() {
34     Display23(root, 0);
35 }
36
37 void My23Tree::DisplayLevel() {
38     PrintLevelOrder23(root);
39 }
```

---

Algorithm 7.47: Useful functions.

---

```

1  bool IsLeaf(My23Node* root) {
2      return ((root->left == nullptr)
3              && (root->middle == nullptr)
4              && (root->right == nullptr));
5  }
6
7  My23Node* minValue(My23Node* root) {
8      if (root != nullptr) {
9          if (root->left == nullptr)
10             return root;
11         else
12             return minValue(root->left);
13     }
14     else
15         return nullptr;
16 }
17
18 My23Node* FindInOrderSuccessor(My23Node* root, MyType data) {
19     if (root != nullptr) {
20         if (root->nkeys == 1) // 2 node
21             return minValue(root->middle);
22         else if ((root->nkeys == 2) && (root->key1 == data)) // 3 node
23             return minValue(root->middle);
24         else if ((root->nkeys == 2) && (root->key2 == data)) // 3 node
25             return minValue(root->right);
26     }
27 }

```

---

Algorithm 7.48: Order keys.

---

```

1  tuple<MyType, MyType, MyType> OrderKeys(My23Node* current, MyType k) {
2      MyType a, b, c; // with a < b
3      a = current->key1;
4      b = current->key2;
5      c = k;
6      if (c < a)
7          return make_tuple(c, a, b);
8      else if (c > b)
9          return make_tuple(a, b, c);
10     else
11         return make_tuple(a, c, b);
12 }

```

---



### 7.3.2 Search

Algorithm 7.49: Search.

---

```
1 tuple<bool,int> Search23(My23Node* root,MyType data,int level) {
2     if (root != nullptr) {
3         if (root->nkeys == 1) { // 2 Node
4             if (data==root->key1)
5                 return make_tuple(true,level);
6             else if (data<root->key1)
7                 return Search23(root->left,data,level+1);
8             else
9                 return Search23(root->middle,data,level+1);
10        }
11        else { // 3 Node
12            if (data==root->key1)
13                return make_tuple(true,level);
14            else if (data==root->key2)
15                return make_tuple(true,level);
16            else if (data<root->key1)
17                return Search23(root->left,data,level+1);
18            else if (data<root->key2)
19                return Search23(root->middle,data,level+1);
20            else
21                return Search23(root->right,data,level+1);
22        }
23    }
24    else
25        return make_tuple(false,-1);
26 }
```

---

### 7.3.3 Insert

Algorithm 7.50: Insert.

---

```
1  // Always inserting into a leaf node!!!
2  int Insert23(My23Node* &root, MyType &data) {
3      int r;
4      //Case 1: Empty tree
5      if (root == nullptr) {
6          My23Node* tmp = new My23Node(data);
7          root = tmp;
8          return 1;
9      }
10     //Case 2: Leaf Nodes
11     else if (root->left == nullptr) {
12         if (root->nkeys == 1) { //root = 2 node
13             if (data < root->key1) {
14                 root->key2 = root->key1;
15                 root->key1 = data;
16             }
17             else
18                 root->key2 = data;
19             root->nkeys = 2;
20             return 1;
21         }
22         else { //root = 3 node
23             r = FixInsert(root, data, nullptr);
24             return r;
25         }
26     }
27     //Case 3: Non-Leaf Nodes
28     else { // root = non-leaf node
29         if (root->nkeys == 1) { // root = 2 node
30             if (data < root->key1) {
31                 r = Insert23(root->left, data);
32                 if (r == 2)
33                     return FixInsert(root, data, root->left);
34             }
35             else if (data > root->key1) {
36                 r = Insert23(root->middle, data);
37                 if (r == 2)
38                     return FixInsert(root, data, root->middle);
39             }
40         }
41         else { //root is a 3 node
42             if (data < root->key1) {
43                 r = Insert23(root->left, data);
44                 if (r == 2)
45                     return FixInsert(root, data, root->left);
46             }
47             else if (data < root->key2) {
48                 r = Insert23(root->middle, data);
49                 if (r == 2)
50                     return FixInsert(root, data, root->middle);
```

```
51         }
52         else {
53             r = Insert23(root->right, data);
54             if (r == 2)
55                 return FixInsert(root, data, root->right);
56         }
57     }
58 }
59 }
```

---

Algorithm 7.51: Insert Upward.

---

```

1  int FixInsert(My23Node* &current, MyType &k, My23Node* child) {
2      if (current->nkeys == 1) { // 2 node parent
3          if (current->left == child) { // left child of parent
4              current->key2 = current->key1;
5              current->key1 = k;
6              current->nkeys = 2;
7              current->right = current->middle;
8              current->middle = child->middle;
9              current->left = child->left;
10             delete child;
11             return 1;
12         }
13         else { // middle child of parent
14             current->key2 = k;
15             current->nkeys = 2;
16             current->middle = child->left;
17             current->right = child->middle;
18             delete child;
19             return 1;
20         }
21     }
22     else { // 3 parent node
23         if (current->left == nullptr) {
24             MyType S, M, L;
25             tie(S, M, L) = OrderKeys(current, k);
26             My23Node* tmp1 = new My23Node(S);
27             My23Node* tmp2 = new My23Node(L);
28             current->key1 = M;
29             current->nkeys = 1;
30             current->left = tmp1;
31             current->middle = tmp2;
32             current->right = nullptr;
33             k = M; // update the value to insert for the upper level
34             return 2;
35         }
36         else {
37             if (current->left == child) {
38                 My23Node* tmp1 = new My23Node(current->key2);
39                 tmp1->left = current->middle;
40                 tmp1->middle = current->right;
41                 current->nkeys = 1;
42                 current->left = child;
43             }
44             else if (current->middle == child) {
45                 My23Node* tmp1 = new My23Node(current->key2);
46                 tmp1->left = child->middle;
47                 tmp1->middle = current->right;
48                 current->nkeys = 1;
49                 current->middle = child->left;
50                 child->left = current;
51                 child->middle = tmp1;

```

```

52         current = child;
53     }
54     else { // right child
55         My23Node* tmp1 = new My23Node( current->key1 );
56         current->key1 = current->key2;
57         current->nkeys = 1;
58         tmp1->middle = current->middle;
59         tmp1->left = current->left;
60         current->middle = child;
61         current->left = tmp1;
62     }
63 }
64 }
65 return 0;
66 }

```

---

### 7.3.4 Delete

Algorithm 7.52: Delete.

---

```
1 My23Node* Delete23(My23Node* &root, MyType data) {
2     My23Node* child=nullptr;
3     //Case 1: Empty tree
4     if (root == nullptr) {
5         cout << "Empty tree." << endl;
6         return nullptr;
7     }
8     //Case 2: Leaf Nodes
9     else if (root->left == nullptr) {
10        if (root->nkeys == 2) { //root = 3 node
11            if (data == root->key1) {
12                root->key1 = root->key2;
13                root->nkeys = 1;
14            }
15            else if (data == root->key2) {
16                root->nkeys = 1;
17            }
18            else
19                cout << "Key:" << data << " unfound." << endl;
20        }
21        else { //root = 2 node : only 1 element !!
22            if (data == root->key1) {
23                return root;
24            }
25            else {
26                cout << "Key:" << data << " unfound." << endl;
27                return nullptr;
28            }
29        }
30    }
31    //Case 3: Non-Leaf Nodes
32    else { // root = non-leaf node
33        if (root->nkeys == 1) { // root = 2 node
34            if (data < root->key1)
35                child=Delete23(root->left, data);
36            else if (data > root->key1) {
37                child=Delete23(root->middle, data);
38            }
39            else { // found
40                My23Node* successor = FindInOrderSuccessor(root, data);
41                MyType tmp = successor->key1;
42                child = Delete23(root->middle, tmp);
43                root->key1 = tmp;
44                return nullptr;
45            }
46        }
47        else { //root is a 3 node
48            if (data == root->key1) {
49                My23Node* successor = FindInOrderSuccessor(root, data);
50                MyType tmp = successor->key1;
```

```

51         child = Delete23(root->middle, tmp);
52         root->key1 = tmp;
53         return nullptr;
54     }
55     else if (data == root->key2) {
56         My23Node* successor = FindInOrderSuccessor(root, data);
57         MyType tmp = successor->key1;
58         child = Delete23(root->right, tmp);
59         root->key2 = tmp;
60         return nullptr;
61     }
62     else if (data < root->key1)
63         child=Delete23(root->left, data);
64     else if (data < root->key2)
65         child=Delete23(root->middle, data);
66     else
67         child=Delete23(root->right, data);
68     }
69 }
70 if (child != nullptr) FixDelete(root, child);
71 return child;
72 }

```

---

Algorithm 7.53: Fix delete.

---

```

1 void FixDelete(My23Node* &current , My23Node* &child) {
2     if (current->nkeys == 1) {
3         // Redistribute right
4         if ((current->middle == child) &&
5             (current->left->nkeys==2)) {
6             child->key1 = current->key1;
7             current->key1 = current->left->key2;
8             current->left->nkeys = 1;
9         }
10        // Redistribute left
11        else if ((current->left == child) &&
12                (current->middle->nkeys == 2)) {
13            child->key1 = current->key1;
14            current->key1 = current->middle->key1;
15            current->middle->key1 = current->middle->key2;
16            current->middle->nkeys = 1;
17        }
18        // Merge right
19        else if ((current->middle == child) &&
20                (current->left->nkeys == 1)) {
21            current->key2 = current->key1;
22            current->key1 = current->left->key1;
23            current->nkeys = 2;
24            current->left = nullptr;
25            current->middle = nullptr;
26            delete child;
27            delete current->left;
28        }
29        // Merge left
30        else if ((current->left == child) &&
31                (current->middle->nkeys == 1)) {
32            current->key2 = current->middle->key1;
33            current->nkeys = 2;
34            current->left = nullptr;
35            current->middle = nullptr;
36            delete child;
37            delete current->middle;
38        }
39    }
40    else { // (parent->nkeys == 2)
41        // child to delete is on the left
42        if ((current->left == child) &&
43            (current->middle->nkeys == 1)) {
44            child->key1 = current->key1;
45            current->middle->key2 = current->right->key1;
46            current->key1 = current->middle->key1;
47            current->middle->key1 = current->key2;
48            current->middle->nkeys = 2;
49            current->nkeys = 1;
50            delete current->right;
51        }

```



```

52     else if ((current->left == child) &&
53             (current->middle->nkeys == 2)) {
54         child->key1 = current->key1;
55         current->key1 = current->middle->key1;
56         current->middle->key1 = current->middle->key2;
57         current->middle->nkeys = 1;
58     }
59     // child to delete is on the right
60     if ((current->right == child) &&
61         (current->middle->nkeys == 1)) {
62         current->left->key2 = current->key1;
63         current->left->nkeys = 2;
64         current->key1 = current->middle->key1;
65         current->middle->key1 = current->key2;
66         current->nkeys = 1;
67         delete current->right;
68     }
69     else if ((current->right == child) &&
70             (current->middle->nkeys == 2)) {
71         current->right->key1 = current->key2;
72         current->key2 = current->middle->key2;
73         current->middle->nkeys = 1;
74     }
75     // child to delete is in the middle
76     else if ((current->middle == child) &&
77             (current->left->nkeys == 1) &&
78             (current->right->nkeys == 1)) {
79         current->left->key2 = current->key1;
80         current->left->nkeys = 2;
81         delete current->middle;
82         current->middle = current->right;
83         current->key1 = current->key2;
84         current->nkeys = 1;
85     }
86     else if ((current->middle == child) &&
87             (current->left->nkeys == 2) &&
88             (current->right->nkeys == 2)) {
89         child->key1 = current->key1;
90         child->key2 = current->key2;
91         current->key1 = current->left->key2;
92         current->key2 = current->right->key1;
93         current->right->key1 = current->right->key2;
94         current->left->nkeys = 1;
95         current->right->nkeys = 1;
96     }
97     else if ((current->middle == child) &&
98             (current->left->nkeys == 2) &&
99             (current->right->nkeys == 1)) {
100         current->middle->key1 = current->key1;
101         current->key1 = current->left->key2;
102         current->left->nkeys = 1;
103     }
104     else if ((current->middle == child) &&
105             (current->left->nkeys == 1) &&

```

```
106         (current->right->nkeys == 2)) {
107         current->middle->key1 = current->key2;
108         current->key2 = current->right->key1;
109         current->right->key1 = current->right->key2;
110         current->right->nkeys = 1;
111     }
112 }
113 }
```

---

### 7.3.5 Display

Algorithm 7.54: Print nodes at a given level.

---

```
1 void PrintGivenLevel23(My23Node* root , int level) {
2     if (root != nullptr) {
3         if (level == 0) {
4             if (root->nkeys==1)
5                 cout << "-" << root->key1 << "-";
6             else
7                 cout << "-" << root->key1 << ","
8                 << root->key2 << "-";
9         }
10        else {
11            if (root->nkeys == 1) {
12                PrintGivenLevel23(root->left , level - 1);
13                PrintGivenLevel23(root->middle , level - 1);
14            }
15            else {
16                PrintGivenLevel23(root->left , level - 1);
17                PrintGivenLevel23(root->middle , level - 1);
18                PrintGivenLevel23(root->right , level - 1);
19            }
20        }
21    }
22 }
```

---

Algorithm 7.55: MaxDepthTree.

---

```

1  int MaxDepthTree23(My23Node* root) {
2      if (root == nullptr) {
3          return 0;
4      }
5      else if ((root->left == nullptr) && (root->middle == nullptr)
6              && (root->right == nullptr))
7          return 0;
8      else {
9          if (root->nkeys == 1) {
10             int leftDepth = MaxDepthTree23(root->left);
11             int middleDepth = MaxDepthTree23(root->middle);
12             if (leftDepth > middleDepth)
13                 return leftDepth + 1;
14             else
15                 return middleDepth + 1;
16         }
17         else {
18             int leftDepth = MaxDepthTree23(root->left);
19             int middleDepth = MaxDepthTree23(root->middle);
20             int rightDepth = MaxDepthTree23(root->right);
21             if ((leftDepth >= middleDepth) && (leftDepth >= rightDepth))
22                 return leftDepth + 1;
23             else if ((middleDepth >= leftDepth) && (middleDepth >= rightDepth))
24                 return middleDepth + 1;
25             else
26                 return rightDepth + 1;
27         }
28     }
29 }

```

---

Algorithm 7.56: PrintLevelOrder.

---

```
1 void PrintLevelOrder23(My23Node* root) {
2     int h = MaxDepthTree23(root);
3     cout << "Height:" << h << endl;
4     int i;
5     for (i = 0; i <= h; i++) {
6         PrintGivenLevel23(root, i);
7         // at each line we have max 2^h
8         cout << endl;
9     }
10 }
```

---

Algorithm 7.57: Display.

---

```
1 void Display23(My23Node* root, int level) {
2     if (root != nullptr) {
3         if (root->nkeys == 1) { // 2 node
4             Display23(root->left, level+1);
5             cout << " (" << root->key1 << ", " << level << ")";
6             Display23(root->middle, level+1);
7         }
8         else { // 3 Node
9             Display23(root->left, level+1);
10            cout << " (" << root->key1 << ", " << level << ")";
11            Display23(root->middle, level + 1);
12            cout << " (" << root->key2 << ", " << level << ")";
13            Display23(root->right, level + 1);
14        }
15    }
16    else
17        cout << " NULL ";
18 }
```

---

## 7.4 Red & Black Trees

### 7.4.1 Definitions

Red-Black Trees (RBTs) are a type of Binary Search Tree (BST) with one additional bit of storage per node, its color that can be defined as RED or BLACK. There is a constraint of the node colors on any simple path from the root to a leaf, to ensure that no such path is more than two times as long as any other, so the tree is more or less balanced. Therefore, each node of an RBT has the following attributes: Color, Key, a pointer to the Left subtree, a pointer to the Right subtree, and a pointer to the Parent node. If a child or the parent of a node does not exist, then the corresponding pointer attribute of the node contains the value NULL. The black-height of a node  $x$  is defined by  $bh(x)$ . It corresponds to the number of black nodes on any simple path from  $x$  (excluded) down to a leaf in the tree.

There are 5 key properties in an RBT:

1. All the nodes are Red or Black.
2. The color of the root is Black.
3. All the leaves are Black.
4. If the color of a node is Red then both children of this node are Black.
5. For each node of an RBT, all simple paths from a node to descendant leaves has the same number of black nodes.

### 7.4.2 Family

If we consider a node  $x$ , then the parent is  $x.p$ , the grand-parent is  $x.p.p$ . The uncle of  $x$  is  $x.p.p.left$  if  $x.p.p.right == x.p$ , otherwise if it is  $x.p.p.right$ .

### 7.4.3 Class definition

Algorithm 7.58: Class interface.

---

```
1  enum class Color {
2      RED,
3      BLACK
4  };
5
6  class MyRBNode {
7  public:
8      MyRBNode(): left(nullptr), right(nullptr), parent(nullptr),
9                  color(Color::RED), key(-1) {}
10     // Default color is RED for insert
11     MyRBNode(MyType k): left(nullptr), right(nullptr), parent(nullptr),
12                    color(Color::RED), key(k) {}
13     MyRBNode* left;
14     MyRBNode* right;
15     MyRBNode* parent;
16     Color color;
17     MyType key;
18 };
19
20 class MyRBTree
21 {
22 public:
23     MyRBTree() { root = nullptr; }
24     ~MyRBTree();
25     tuple<bool, Color, MyType> Search(MyType key);
26     void Insert(MyType key);
27     void Delete(MyType key);
28     void Display();
29 private:
30     void TransplantRB(MyRBNode* u, MyRBNode* v);
31     void InsertFixup(MyRBNode* z);
32     void DeleteFixup(MyRBNode* x);
33     void LeftRotation(MyRBNode* x);
34     void RightRotation(MyRBNode* x);
35     MyRBNode* root;
36 };
```

---

Algorithm 7.59: Family business.

---

```
1  bool IsRed(MyRBNode* z) {
2      if (z == nullptr)
3          return false;
4      else
5          return (z->color == Color::RED);
6  }
7  bool IsBlack(MyRBNode* z) {
8      return z->color == Color::BLACK;
9  }
10 void SetRed(MyRBNode* z) {
11     z->color = Color::RED;
12 }
13 void SetBlack(MyRBNode* z) {
14     z->color = Color::BLACK;
15 }
16 MyRBNode* GrandParent(MyRBNode* z) {
17     return z->parent->parent;
18 }
19 bool IsLeftFamily(MyRBNode* z) {
20     return (z->parent == z->parent->parent->left);
21 }
22 bool IsLeftChild(MyRBNode* x) {
23     return (x == x->parent->left);
24 }
25 MyRBNode* GetRightSibling(MyRBNode* x) {
26     return x->parent->right;
27 }
28 MyRBNode* GetLeftSibling(MyRBNode* x) {
29     return x->parent->left;
30 }
31 MyRBNode* GetRightUncle(MyRBNode* z) {
32     return z->parent->parent->right;
33 }
34 MyRBNode* GetLeftUncle(MyRBNode* z) {
35     return z->parent->parent->left;
36 }
```

---



Algorithm 7.60: Destructor.

---

```
1 void destroyRB(MyRBNode* root) {
2     if (root != nullptr) {
3         destroyRB(root->left);
4         destroyRB(root->right);
5         delete root;
6     }
7 }
8 MyRBTree::~MyRBTree() {
9     destroyRB(root);
10 };
```

---

Algorithm 7.61: Transplant and TreeMinimum.

---

```
1 void MyRBTree::TransplantRB(MyRBNode* u, MyRBNode* v) {
2     if (u->parent == nullptr)
3         root = v;
4     else if (IsLeftChild(u))
5         u->parent->left = v;
6     else
7         u->parent->right = v;
8     if (v != nullptr)
9         v->parent = u->parent;
10 }
11
12 MyRBNode* TreeMinimum(MyRBNode* x) {
13     MyRBNode* y = x;
14     while (y->left != nullptr)
15         y = y->left;
16     return y;
17 }
```

---

Algorithm 7.62: Rotations again.

---

```
1 void MyRBTree::LeftRotation(MyRBNode* x) {
2     if (x->right != nullptr) {
3         MyRBNode* y = x->right;
4         x->right = y->left;
5         if (y->left != nullptr)
6             y->left->parent = x;
7         y->parent = x->parent;
8         if (x->parent == nullptr)
9             root = y;
10        else if (IsLeftChild(x))
11            x->parent->left = y;
12        else
13            x->parent->right = y;
14        y->left = x;
15        x->parent = y;
16    }
17 }
18
19 void MyRBTree::RightRotation(MyRBNode* x) {
20     if (x->left != nullptr) {
21         MyRBNode* y = x->left;
22         x->left = y->right;
23         if (y->right != nullptr)
24             y->right->parent = x;
25         y->parent = x->parent;
26         if (x->parent == nullptr)
27             root = y;
28         else if (IsLeftChild(x))
29             x->parent->left = y;
30         else
31             x->parent->right = y;
32         y->right = x;
33         x->parent = y;
34     }
35 }
```

---

## 7.4.4 Search

Algorithm 7.63: Search.

---

```
1 tuple<bool, Color, MyType> SearchRB(MyRBNode* root, MyType data) {
2     if (root == nullptr)
3         return make_tuple(false, Color::RED, 0);
4     else if (root->key == data)
5         return make_tuple(true, root->color, root->key);
6     else if (data < root->key)
7         SearchRB(root->left, data);
8     else
9         SearchRB(root->right, data);
10 }
11
12 tuple<bool, Color, MyType> MyRBTree::Search(MyType key)
13 { return SearchRB(root, key); }
```

---

## 7.4.5 Insert

Algorithm 7.64: Insert.

---

```
1 void MyRBTree::Insert(MyType v) {
2     MyRBNode* z = new MyRBNode(v);
3     MyRBNode* y = nullptr;
4     MyRBNode* x = root;
5     while (x != nullptr) {
6         y = x;
7         if (z->key < x->key)
8             x = x->left;
9         else
10            x = x->right;
11    }
12    z->parent = y;
13    if (y == nullptr)
14        root = z;
15    else if (z->key < y->key)
16        y->left = z;
17    else
18        y->right = z;
19    InsertFixup(z);
20 }
```

---

Algorithm 7.65: InsertFixup.

---

```

1 void MyRBTree::InsertFixup(MyRBNode* z) {
2     while ((z->parent != nullptr) && IsRed(z->parent)) {
3         if (IsLeftFamily(z)) {
4             MyRBNode* y = GetRightUncle(z);
5             if (IsRed(y)) { // Case 1
6                 SetBlack(z->parent);
7                 SetBlack(y);
8                 SetRed(z->parent->parent);
9                 z = z->parent->parent;
10            }
11            else {
12                if (z == z->parent->right) {
13                    z = z->parent;
14                    LeftRotation(z);
15                }
16                SetBlack(z->parent);
17                SetRed(z->parent->parent);
18                RightRotation(z->parent->parent);
19            }
20        }
21        else {
22            MyRBNode* y = GetLeftUncle(z);
23            if (IsRed(y)) { // Case 1
24                SetBlack(z->parent);
25                SetBlack(y);
26                SetRed(z->parent->parent);
27                z = z->parent->parent;
28            }
29            else {
30                if (IsLeftChild(z)) { // Case 2
31                    z = z->parent;
32                    RightRotation(z);
33                } // Case 3
34                SetBlack(z->parent);
35                SetRed(z->parent->parent);
36                LeftRotation(z->parent->parent);
37            }
38        }
39    }
40    SetBlack(root);
41 }

```

---

## 7.4.6 Delete

Algorithm 7.66: Delete.

---

```
1 void MyRBTree::Delete(MyType v) {
2     if (root == nullptr)
3         cout << "Empty Tree." << endl;
4     else {
5         MyRBNode *z=root;
6         MyRBNode* y = nullptr;
7         MyRBNode* x = nullptr;
8         bool found = false;
9         while (z!= nullptr && !found) {
10             if (z->key == v)
11                 found = true;
12             if (!found) {
13                 if (z->key < v)
14                     z = z->right;
15                 else
16                     z = z->left;
17             }
18         }
19         if (!found)
20             cout << "Element not found" << endl;
21         else {
22             y = z;
23             Color yoriginal = y->color;
24             if (z->left == nullptr) {
25                 x = z->right;
26                 TransplantRB(z, z->right);
27             }
28             else if (z->right == nullptr) {
29                 x = z->left;
30                 TransplantRB(z, z->left);
31             }
32             else { // 2 children
33                 y = TreeMinimum(z->right);
34                 yoriginal = y->color;
35                 x = y->right;
36                 if (y->parent == z)
37                     x->parent = y;
38                 else {
39                     TransplantRB(y,y->right);
40                     y->right = z->right;
41                     y->right->parent = y;
42                 }
43                 TransplantRB(z,y);
44                 y->left = z->left;
45                 y->left->parent = y;
46                 y->color = z->color;
47             }
48             if ((yoriginal == Color::BLACK) && (x!=nullptr))
49                 DeleteFixup(x);
50         }
```

51        }  
52    }

---

Algorithm 7.67: DeleteFixup.

---

```

1 void MyRBTree::DeleteFixup(MyRBNode* x) {
2     while ((x != root) && IsBlack(x)) {
3         if (IsLeftChild(x)) {
4             MyRBNode* w = GetRightSibling(x);
5             if (IsRed(w)) {
6                 SetBlack(w);
7                 SetRed(x->parent);
8                 LeftRotation(x->parent);
9                 w = GetRightSibling(x);
10            }
11            if (IsBlack(w->left) && IsBlack(w->right)) {
12                SetRed(w);
13                x = x->parent;
14            }
15            else {
16                if (IsBlack(w->right)) {
17                    SetBlack(w->left);
18                    SetRed(w);
19                    RightRotation(w);
20                    w = GetRightSibling(x);
21                }
22                else {
23                    w->color = x->parent->color;
24                    SetBlack(x->parent);
25                    SetBlack(w->right);
26                    LeftRotation(x->parent);
27                    x = root;
28                }
29            }
30        }
31        else {
32            MyRBNode* w = GetLeftSibling(x);
33            if (IsRed(w)) {
34                SetBlack(w);
35                SetRed(x->parent);
36                RightRotation(x->parent);
37                w = GetLeftSibling(x);
38            }
39            if (IsBlack(w->left) && IsBlack(w->right)) {
40                SetRed(w);
41                x = x->parent;
42            }
43            else {
44                if (IsBlack(w->left)) {
45                    SetBlack(w->right);
46                    SetRed(w);
47                    LeftRotation(w);
48                    w = GetLeftSibling(x);
49                }
50                else {
51                    w->color = x->parent->color;

```

```

52         SetBlack(x->parent);
53         SetBlack(w->left);
54         RightRotation(x->parent);
55         x = root;
56     }
57 }
58 }
59 }
60     SetBlack(x);
61 }

```

---



## Chapter 8

# Heaps

### Contents

---

<b>8.1</b>	<b>Binary Heaps . . . . .</b>	<b>165</b>
<b>8.2</b>	<b>Priority Queue . . . . .</b>	<b>172</b>
<b>8.3</b>	<b>Fibonacci heaps . . . . .</b>	<b>178</b>
8.3.1	Proof by induction . . . . .	188
8.3.2	Lemma 2 . . . . .	188
8.3.3	Lemma 3 . . . . .	189

---

## 8.1 Binary Heaps

A binary heap is a binary tree with two constraints:

- **Shape property:** A binary heap is a complete binary tree. All the levels of the tree (except possibly the last one) are fully filled. If the last level is not complete, then the level is filled from left to right.
- **Heap property:** The key stored in each node is either  $\geq$  or  $\leq$  to the keys in the node's children - according to some total order.
- **Sift-Down:** Swaps a node that is too small with its largest child (moving it down) until it is at least as large as both nodes below it.
- **Sift-Up:** Swaps a node that is too large with its parent (moving it up) until it is no larger than the node above it.
- **BuildHeap:** Array of unsorted items and moves them until they all satisfy the heap property.

The members of the class MyMaxHeap are:

- a: A pointer to a MyType, corresponding to an array of elements of type MyType.
- n: The size of the array that corresponds to the number of elements in the associated binary heap that is equivalent to a complete tree.
- heap: A boolean that indicates if the array a is a heap or not. It is useful after sorting the array to know if we have a sorted array or if it remains a max heap.

Algorithm 8.1: Class interface.

---

```
1  class MyMaxHeap {
2  public:
3      MyMaxHeap();
4      ~MyMaxHeap();
5      void InitArray(int n1);
6      void InitHeap(int n1);
7      bool IsMaxHeap();
8      bool IsSorted();
9      void MaxHeapify(int i, int size);
10     void BuildMaxHeap();
11     void HeapSort();
12     void InsertMaxHeap(int x);
13     void DeleteMaxHeap(int x);
14     void Display();
15 private:
16     MyType* a;
17     int n;
18     bool heap;
19 };
```

---

Algorithm 8.2: Heap constructor and destructor.

---

```

1 MyMaxHeap::MyMaxHeap() {
2     n = 0;
3     a = nullptr;
4     heap = true;
5 }
6
7 MyMaxHeap::~MyMaxHeap() {
8     delete[] a;
9 }

```

---

Algorithm 8.3: Heap methods.

---

```

1 void MyMaxHeap::MaxHeapify(int i, int size) {
2     int left, right, largest;
3     left = 2 * i + 1; // left child of a[i]
4     right = 2 * i + 2; // right child of a[i]
5     largest = i; // original largest element
6     if ((left < size) && a[left] > a[i])
7         largest = left;
8     if ((right < size) && (a[right] > a[largest]))
9         largest = right;
10    if (largest != i) { // there was a swap
11        swap(a[i], a[largest]);
12        MaxHeapify(largest, size);
13    }
14 }
15
16 // Determine if the array corresponds to a max heap
17 bool MyMaxHeap::IsMaxHeap() {
18     bool isheap = true;
19     int left, right;
20     int i = 0;
21     for (int i = 0; (i < n) && isheap; i++) {
22         left = 2 * i + 1;
23         right = 2 * i + 2;
24         if ((left < n) && a[left] > a[i])
25             isheap = false;
26         if ((right < n) && (a[right] > a[i]))
27             isheap = false;
28     }
29     heap = isheap;
30     return isheap;
31 }
32
33 // Determine if the elements in the array are sorted
34 bool MyMaxHeap::IsSorted() {
35     if (n <= 1)
36         return true;
37     else {
38         int i = 1;

```

```

39         while ((a[i - 1] <= a[i]) && (i < n))
40             i++;
41         return (i == n);
42     }
43 }
44
45 void MyMaxHeap::BuildMaxHeap() {
46     // Traverse the complete tree backward
47     // (right to left, bottom to top)
48     for (int i = n / 2 - 1; i >= 0; i--) {
49         MaxHeapify(i, n);
50     }
51     heap = true;
52 }

```

---

Algorithm 8.4: Init heap or array.

---

```

1  // Create a array of size n1 with random values
2  void MyMaxHeap::InitArray(int n1) {
3      if (n1 > 0) {
4          delete[] a;
5          n = n1;
6          a = new MyType[n];
7          for (int i = 0; i < n; i++)
8              a[i] = rand() % 200;
9      }
10     else
11         PositiveNumberCheck(--func--);
12 }
13
14
15 // Create a heap with n1 random elements
16 void MyMaxHeap::InitHeap(int n1) {
17     InitArray(n1);
18     BuildMaxHeap();
19 }

```

---

The first step is to build a Max Heap with the elements that are in the array. The maximum element will be therefore at the first position. Then, it will be moved at its final place in the array. The element that was at the end of the array will then come at the beginning. This swap is likely to break the max heap property of the data structure. Hence, we have to rebuild the heap in relation to this change.

---

Algorithm 8.5: Heapsort.

---

```
1 void MyMaxHeap::HeapSort() {
2     // First build the MaxHeap
3     BuildMaxHeap();
4     // Go backward as the Max is placed at the end
5     // Invariant: the elements between i to n-1 are at their correct places
6     // i is the size of the heap
7     for (int i = n - 1; i > 0; i--) {
8         // a[0] always contains the max value
9         swap(a[0], a[i]);
10        // the heap constraint must be maintained
11        MaxHeapify(0, i);
12    }
13    heap = false;
14 }
```

---

Algorithm 8.6: Heap insert, delete, display.

---

```
1 void MyMaxHeap::InsertMaxHeap(int x) {
2     MyType* a1 = new MyType[n + 1];
3     a1[0] = x;
4     for (int i = 0; i < n; i++) {
5         a1[1 + i] = a[i];
6     }
7     delete[] a;
8     a = a1;
9     MaxHeapify(0, n + 1);
10    n++;
11 }
12
13 void MyMaxHeap::DeleteMaxHeap(int x) {
14     int i = 0, indx = 0;
15     bool found = false;
16     while ((i < n) && (!found)) {
17         if (a[i] == x) {
18             found = true;
19             indx = i;
20         }
21         i++;
22     }
23     if (found) {
24         MyType* a1 = new MyType[n - 1];
25         a[indx] = a[n - 1];
26         for (int i = 0; i < n - 1; i++) {
27             a1[i] = a[i];
28         }
29         delete[] a;
30         a = a1;
31         MaxHeapify(indx, n - 1);
32         n--;
33     }
34 }
35
36 void MyMaxHeap::Display() {
37     for (int i = 0; i < n; i++) {
38         cout << a[i] << " ";
39     }
40     cout << endl;
41 }
```

---

Algorithm 8.7: Heapsort example.

---

```
1 void main() {
2     int n = 10;
3     MyMaxHeap* H = new MyMaxHeap();
4     H->InitArray(n);
5     cout << "Sorted: " << H->IsSorted() << endl;
6     H->HeapSort();
7     cout << "Sorted: " << H->IsSorted() << endl;
8     H->BuildMaxHeap(); // Need to build the heap first
9     cout << "IsHeap:" << H->IsMaxHeap() << endl;
10    H->Display();
11    cout << "Insert.." << endl;
12    H->InsertMaxHeap(99); // Insert 99 in the heap
13    H->Display();
14    cout << "IsHeap:" << H->IsMaxHeap() << endl;
15    cout << "Insert.." << endl;
16    H->DeleteMaxHeap(99); // Delete 99 from the heap
17    H->Display();
18    cout << "IsHeap:" << H->IsMaxHeap() << endl;
19    delete H;
20 }
```

---



## 8.2 Priority Queue

The Priority Queue is implemented as a Min Heap.

Algorithm 8.8: MyPriorityQueue class interface.

---

```
1 struct MyData {
2     int index;
3     double value;
4 };
5
6 class MyPriorityQueue
7 {
8 public:
9     MyPriorityQueue();
10    MyPriorityQueue(int c);
11    ~MyPriorityQueue();
12    int GetNElements() { return nelements; }
13    bool IsEmpty();
14    bool IsFull();
15    int GetParent(int child);
16    int GetLeftChild(int parent);
17    int GetRightChild(int parent);
18    void Push(int index, double value);
19    MyData Pop();
20    void DecreaseKey(int index, double value);
21    void BuildMinHeap();
22    void Display();
23 private:
24    void MinHeapify(int i, int size);
25    int capacity;
26    int nelements;
27    MyData* queue;
28 };
```

---

---

Algorithm 8.9: Constructors and destructor.

---

```
1 MyPriorityQueue :: MyPriorityQueue () {
2     capacity = 0;
3     nelements = 0;
4     queue = nullptr;
5 }
6
7 MyPriorityQueue :: MyPriorityQueue (int c) {
8     capacity = c;
9     nelements = 0;
10    queue = new MyData[ capacity ];
11 }
12
13 MyPriorityQueue :: ~ MyPriorityQueue () {
14     delete [] queue;
15 }
```

---

---

Algorithm 8.10: Methods.

---

```
1 bool MyPriorityQueue :: IsEmpty () {
2     return (nelements == 0);
3 }
4
5 bool MyPriorityQueue :: IsFull () {
6     return (nelements == capacity);
7 }
8
9 int MyPriorityQueue :: GetParent (int child) {
10    if (child % 2 == 0)
11        return (child / 2) - 1;
12    else
13        return child / 2;
14 }
15
16 int MyPriorityQueue :: GetLeftChild (int parent) {
17     return (2 * parent + 1);
18 }
19
20 int MyPriorityQueue :: GetRightChild (int parent) {
21     return (2 * parent + 2);
22 }
```

---

Algorithm 8.11: Heap methods.

---

```
1 void MyPriorityQueue::MinHeapify(int i, int size) {
2     int left, right, smallest;
3     left = 2 * i + 1; // left child of a[i]
4     right = 2 * i + 2; // right child of a[i]
5     smallest = i; // original smallest element
6     if ((left < size) && queue[left].value < queue[i].value)
7         smallest = left;
8     if ((right < size) && (queue[right].value < queue[smallest].value))
9         smallest = right;
10    if (smallest != i) { // there was a swap
11        swap(queue[i], queue[smallest]);
12        MinHeapify(smallest, size);
13    }
14 }
15
16 void MyPriorityQueue::BuildMinHeap() {
17     // Traverse the complete tree backward
18     // (right to left, bottom to top)
19     for (int i = nelements / 2 - 1; i >= 0; i--) {
20         MinHeapify(i, nelements);
21     }
22 }
```

---

Algorithm 8.12: Push/Enqueue.

---

```
1 void MyPriorityQueue::Push(int index, double value) {
2     if (nelements < capacity) {
3         MyData x;
4         x.index = index;
5         x.value = value;
6
7         int i = nelements;
8         while ((i != 0) && (x.value < queue[i / 2].value)) {
9             queue[i] = queue[i / 2]; // move down
10            i /= 2;
11        } // move to parent
12        queue[i] = x;
13        nelements++;
14        cout << "Added (" << index
15              << ", " << value << ") size="
16              << nelements << endl;
17    }
18    else
19        cout << "Out of capacity" << endl;
20 }
```

---

Algorithm 8.13: Pop/Dequeue.

---

```
1 MyData MyPriorityQueue::Pop() {
2     if (nelements > 0) {
3         MyData tmp = queue[0];
4         queue[0] = queue[nelements - 1];
5         MinHeapify(0, nelements);
6         nelements--;
7         return tmp;
8     }
9     else
10    {
11        MyData tmp;
12        tmp.value = 0;
13        tmp.index = -1;
14        return tmp;
15    }
16 }
```

---

Algorithm 8.14: Display.

---

```
1 void MyPriorityQueue::Display() {
2     cout << "Priority Queue" << endl;
3     cout << "\t capacity: " << capacity << endl;
4     cout << "\t nelements: " << nelements << endl;
5     for (int i = 0; i < nelements; i++) {
6         cout << i << ": (" << queue[i].index << ", "
7             << queue[i].value << ")" << endl;
8     }
9     cout << endl;
10 }
```

---

Algorithm 8.15: Decrease key.

---

```
1 void MyPriorityQueue::DecreaseKey(int index, double value) {
2     int i = 0;
3     bool found = false;
4     while ((!found) && (i < nelements)) {
5         if (queue[i].index == index) {
6             queue[i].value = value;
7             found = true;
8         }
9         i++;
10    }
11    if (found) {
12        int child = i - 1;
13        int parent = GetParent(child);
14        while ((queue[child].value < queue[parent].value) &&
15              (child >= 0 && parent >= 0)) {
16            swap(queue[child], queue[parent]);
17            child = parent;
18            parent = GetParent(child);
19        }
20    }
21 }
```

---

## 8.3 Fibonacci heaps

Algorithm 8.16: Node definition.

---

```
1 class MyFibNode {
2 public:
3     MyFibNode(MyType k)
4         :key(k), mark(false), p(nullptr), left(nullptr),
5         right(nullptr), child(nullptr), degree(-1) {}
6     ~MyFibNode() {}
7     MyType key;
8     bool mark;
9     MyFibNode *p,* left ,* right ,* child;
10    int degree;
11 };
```

---

Algorithm 8.17: Class interface.

---

```
1 class MyFibonacciHeap {
2 public:
3     MyFibonacciHeap();
4     ~MyFibonacciHeap();
5     MyFibNode* Minimum();
6     int GetSize() const;
7     void Insert(MyFibNode *x);
8     void Insert(MyType k);
9     void DecreaseKey(MyFibNode* x, int k);
10    MyFibonacciHeap* Union(MyFibonacciHeap *H1, MyFibonacciHeap *H2);
11    void Link(MyFibNode* y, MyFibNode* x);
12    MyFibNode* ExtractMin();
13    void Consolidate();
14    void Cut(MyFibNode* x, MyFibNode* y);
15    void CascadingCut(MyFibNode* y);
16    void Delete(MyFibNode* x);
17    bool Empty() const;
18    void DisplayChild(MyFibNode* p);
19    void Display();
20 private:
21     int n;
22     MyFibNode *min;
23 };
```

---

---

Algorithm 8.18: Constructor and destructor.

---

```

1  MyFibonacciHeap::MyFibonacciHeap() : n(0), min(nullptr) {}
2
3  void DeleteFibnodes(MyFibNode *x) {
4      if (x) {
5          MyFibNode *current = x;
6          bool finished = false;
7          while (true) {
8              if (current->left && current->left != x) {
9                  MyFibNode *tmp = current;
10                 current = current->left;
11                 if (tmp->child)
12                     DeleteFibnodes(tmp->child);
13                 delete tmp;
14             }
15             else {
16                 if (current->child)
17                     DeleteFibnodes(current->child);
18                 delete current;
19                 finished=true;
20             }
21         }
22     }
23 }
24
25 MyFibonacciHeap::~MyFibonacciHeap() {
26     DeleteFibnodes(min);
27 }

```

---



---

Algorithm 8.19: Accessors.

---

```

1  // The minimum node of the heap.
2  MyFibNode* MyFibonacciHeap::Minimum() {
3      return min;
4  }
5
6  int MyFibonacciHeap::GetSize() const {
7      return n;
8  }
9
10 bool MyFibonacciHeap::Empty() const {
11     return n == 0;
12 }

```

---



Algorithm 8.20: Insert.

---

```
1 void MyFibonacciHeap::Insert(MyFibNode *x) {
2     x->degree = 0;
3     x->p = nullptr;
4     x->child = nullptr;
5     x->mark = false;
6     if (min == nullptr)
7         min = x->left = x->right = x;
8     else {
9         min->left->right = x;
10        x->left = min->left;
11        min->left = x;
12        x->right = min;
13        if (x->key < min->key) {
14            min = x;
15        }
16    }
17    n++;
18 }
19
20 void MyFibonacciHeap::Insert(MyType k) {
21     MyFibNode *x = new MyFibNode(k);
22     Insert(x);
23 }
```

---

Algorithm 8.21: Delete.

---

```
1 void MyFibonacciHeap::Delete(MyFibNode* x) {
2     DecreaseKey(x,0); // set to a minimum value
3     MyFibNode *m = ExtractMin();
4     delete m;
5 }
```

---

Algorithm 8.22: Union.

---

```
1 MyFibonacciHeap* MyFibonacciHeap::Union(MyFibonacciHeap *H1, MyFibonacciHeap *H2) {
2     MyFibonacciHeap* H = new MyFibonacciHeap();
3     H->min = H1->min;
4     if (H->min != nullptr && H2->min != nullptr) {
5         H->min->right->left = H2->min->left;
6         H2->min->left->right = H->min->right;
7         H->min->right = H2->min;
8         H2->min->left = H->min;
9     }
10    if (H1->min == nullptr ||
11        (H2->min != nullptr && H2->min->key < H1->min->key))
12        H->min = H2->min;
13    H->n = H1->n + H2->n;
14    return H;
15 }
```

---

Algorithm 8.23: Extract Min.

---

```

1  MyFibNode* MyFibonacciHeap::ExtractMin() {
2      MyFibNode *z, *x, *next;
3      MyFibNode ** childList;
4      z = min;
5      if (z != nullptr) {
6          x = z->child;
7          if (x != nullptr) {
8              childList = new MyFibNode*[z->degree];
9              next = x;
10             for (int i = 0; i < (int)z->degree; i++) {
11                 childList[i] = next;
12                 next = next->right;
13             }
14             for (int i = 0; i < (int)z->degree; i++) {
15                 x = childList[i];
16                 min->left->right = x;
17                 x->left = min->left;
18                 min->left = x;
19                 x->right = min;
20                 x->p = nullptr;
21             }
22             delete[] childList;
23         }
24         z->left->right = z->right;
25         z->right->left = z->left;
26         if (z == z->right)
27             min = nullptr;
28         else {
29             min = z->right;
30             Consolidate();
31         }
32         n--;
33     }
34     return z;
35 }

```

---

---

```

1 void MyFibonacciHeap::Consolidate() {
2     MyFibNode* w, *next, *x, *y;
3     MyFibNode** A, ** rootList;
4     // Max degree <= log base golden ratio of n
5     int d, rootSize;
6     double goldenratio = (1.0 + sqrt(5.0))/2.0;
7     int max_degree = static_cast<int>(floor(log(static_cast<double>(n)) /
8                                             log(goldenratio)));
9
10    A = new MyFibNode*[max_degree + 2];
11    // + 2 : max degree and A[max_degree+1] == NULL
12    for (int i = 0; i < max_degree + 2; i++)
13        A[i] = nullptr;
14    w = min;
15    rootSize = 0;
16    next = w;
17    do {
18        rootSize++;
19        next = next->right;
20    } while (next != w);
21    rootList = new MyFibNode*[rootSize];
22    for (int i = 0; i < rootSize; i++) {
23        rootList[i] = next;
24        next = next->right;
25    }
26    for (int i = 0; i < rootSize; i++) {
27        w = rootList[i];
28        x = w;
29        d = x->degree;
30        while (A[d] != nullptr) {
31            y = A[d];
32            if (x->key > y->key)
33                swap(x, y);
34            Link(y, x);
35            A[d] = nullptr;
36            d++;
37        }
38        A[d] = x;
39    }
40    delete[] rootList;
41    min = nullptr;
42    for (int i = 0; i < max_degree + 2; i++) {
43        if (A[i] != nullptr) {
44            if (min == nullptr) {
45                min = A[i]->left = A[i]->right = A[i];
46            }
47            else {
48                min->left->right = A[i];
49                A[i]->left = min->left;
50                min->left = A[i];
51                A[i]->right = min;

```

```
52         if (A[i]->key < min->key) {
53             min = A[i];
54         }
55     }
56 }
57 }
58 delete[] A;
59 }
```

---

---

Algorithm 8.25: Link.

---

```

1 void MyFibonacciHeap::Link(MyFibNode* y, MyFibNode* x) {
2     // Remove y from the root list of heap
3     y->left->right = y->right;
4     y->right->left = y->left;
5     // Make y a child of x, incrementing x.degree
6     if (x->child != nullptr) {
7         x->child->left->right = y;
8         y->left = x->child->left;
9         x->child->left = y;
10        y->right = x->child;
11    }
12    else {
13        x->child = y;
14        y->right = y;
15        y->left = y;
16    }
17    y->p = x;
18    x->degree++;
19    y->mark = false;
20 }

```

---



---

Algorithm 8.26: Decrease Key.

---

```

1 void MyFibonacciHeap::DecreaseKey(MyFibNode* x, int k) {
2     MyFibNode* y;
3     if (k > x->key) {
4         cout << "The new key is greater than current key" << endl;
5     }
6     else {
7         x->key = k;
8         y = x->p;
9         if (y != nullptr && x->key < y->key) {
10            Cut(x, y);
11            CascadingCut(y);
12        }
13        if (x->key < min->key) {
14            min = x;
15        }
16    }
17 }

```

---

---

Algorithm 8.27: Cut.

---

```
1 void MyFibonacciHeap::Cut(MyFibNode* x, MyFibNode* y) {
2     if (x->right == x) {
3         y->child = nullptr;
4     }
5     else {
6         x->right->left = x->left;
7         x->left->right = x->right;
8         if (y->child == x) {
9             y->child = x->right;
10        }
11    }
12    y->degree--;
13    min->right->left = x;
14    x->right = min->right;
15    min->right = x;
16    x->left = min;
17    x->p = nullptr;
18    x->mark = false;
19 }
```

---

---

Algorithm 8.28: CascadingCut.

---

```
1 void MyFibonacciHeap::CascadingCut(MyFibNode* y) {
2     MyFibNode* z;
3     z = y->p;
4     if (z != nullptr) {
5         if (y->mark == false)
6             y->mark = true;
7         else {
8             Cut(y, z);
9             CascadingCut(z);
10        }
11    }
12 }
```

---

Algorithm 8.29: Display.

---

```

1 void MyFibonacciHeap::DisplayChild(MyFibNode* p) {
2     MyFibNode* copy = p;
3     do {
4         cout << p->key;
5         p = p->right;
6         if (p != copy) {
7             cout << "->";
8         }
9     } while (p != copy);
10    cout << endl;
11    p = copy;
12    do {
13        if (p->child != nullptr) {
14            cout << "Display child of :" << p->key << endl;
15            DisplayChild(p->child);
16        }
17        p = p->right;
18    } while (p != copy);
19    cout << endl;
20 }
21
22 void MyFibonacciHeap::Display() {
23     cout << "Minimum value : " << min->key << endl;
24     MyFibNode* p = min;
25     if (p == nullptr)
26         cout << "The Heap is Empty" << endl;
27     else {
28         cout << "Root nodes: " << endl;
29         do {
30             cout << p->key;
31             p = p->right;
32             if (p != min)
33                 cout << "->";
34         } while (p != min);
35         cout << endl;
36         p = min;
37         do {
38             if (p->child != nullptr) {
39                 cout << "Display child of :" << p->key << endl;
40                 DisplayChild(p->child);
41             }
42             p = p->right;
43         } while (p != min);
44     }
45 }

```

---



### 8.3.1 Proof by induction

What has to be done:

- To prove the default case (basic case). The first case is typically obvious. Yet, you need to tell which rule, or which definition has been used, if it is not a simple arithmetic operation.
- To prove the case for the next step (n+1) by using **only** the set of rules given by original definition, and the hypothesis.

**Remark:**

The way you organize the sequence on your draft is up to you, you can derive the expression from both sides. It is more a pattern matching game where you have to decompose the terms in order to retrieve the hypothesis and use it. When you have an equality (Prove: A=B), you may go with the development of A to get B, or to get A from B. Therefore, on your working sheet you may work on both sides to search where it is easier for you to extract the hypothesis. Keep the expected target in mind, so you don't derive one side in such a way that you go too far from the target. Some additional rules (from the definitions) may be needed to reach the target, hence it must be kept in mind in order to consider the right rules. You may go with A-B to arrive to 0.

There are 2 parts:

- The part on your working sheet, to find the sequence that leads to the proof. You may attack the problem in the side of the equality that is the most complex (with a  $\sum$  sign) to extract the hypothesis. In Lemma 2, we use directly the definition and the inductive step is hidden in the sum. In Lemma 3, the inductive step is direct, but the definition of  $\Phi$  must be used.
- The clean part you should write on the final or midterm, that contains the proper sequence that leads to the proof. In the sequence, each line may be justified by the application of a rule (for example, commutativity of the addition, decomposition of the sum,...). Depending on the problem, some rules are totally implied. Once the induction hypothesis is used, then you can present the sequence of the development of the proof in the proper order, going from A to B, or from B to A.

### 8.3.2 Lemma 2

With the sequence of Fibonacci, we have:

$$F_0 = 0 \quad (8.1)$$

$$F_1 = 1 \quad (8.2)$$

$$F_k = F_{k-1} + F_{k-2} \forall k \geq 2 \quad (8.3)$$

We want to prove that:

$$F_{k+2} = 1 + \sum_{i=0}^k F_i \forall k \geq 0 \quad (8.4)$$

**Step 1:** We verify it is true for k=0.

$$F_{0+2} = F_2 = F_1 + F_0 = 1 + 0 = 1 \quad (8.5)$$

$$1 + \sum_{i=0}^0 F_i = 1 + F_0 = 1 + 0 = 1 \quad (8.6)$$

**Step 2:** We want to show that the statement holds for  $(k+1)$  when we use the hypothesis for  $k$ . (You can start from any side of the equality, you can also keep the target you want to reach on the side, so you know where you need to go).

In the first line, if we develop  $F_{k+1+2}$ , we just use the definition. In the second line, we use the inductive step.

$$F_{(k+1)+2} = F_{k+2} + F_{k+1} \quad (8.7)$$

$$= 1 + \sum_{i=0}^k F_i + F_{k+1} \quad (8.8)$$

$$= 1 + \sum_{i=0}^{k+1} F_i \quad (8.9)$$

We can start the other way:

$$1 + \sum_{i=0}^{k+1} F_i = 1 + \sum_{i=0}^k F_i + F_{k+1} \quad (8.10)$$

$$= F_{k+2} + F_{k+1} \quad (8.11)$$

$$= F_{(k+1)+2} \quad (8.12)$$

In the last line, we change the variables to match the definition.

### 8.3.3 Lemma 3

We want to prove that  $\forall k \geq 0$ , the  $(k+2)^{nd}$  Fibonacci number satisfies  $F_{k+2} \geq \Phi^k$ . The most difficult part in proof of this lemma is to think about going back to the definition of the golden ratio, which is a solution to:

$$x^2 - x - 1 = 0 \quad (8.13)$$

So we have:  $\Phi^2 = \Phi + 1$ .

**Step 1:** We verify it is true for  $k=0$ . If we have  $k = 0$ ,  $F_2 = 1 = \Phi^0$ . If we have  $k = 1$ ,  $F_3 = 2 > 1.62 > \Phi^1$ .

**Step 2:** We want to show that the statement holds for  $(k+1)$  when we use the hypothesis for  $k$ . In the first line, we will just consider the definition. In the second line, we use the inductive step. In fourth line, we use the expression of  $\Phi^2 = \Phi + 1$ .

$$F_{k+2} = F_{k+1} + F_k \quad (8.14)$$

$$\geq \Phi^{k-1} + \Phi^{k-2} \quad (8.15)$$

$$= \Phi^{k-2} * (\Phi + 1) \quad (8.16)$$

$$= \Phi^{k-2} * (\Phi^2) \quad (8.17)$$

$$= \Phi^k \quad (8.18)$$

# Chapter 9

## Graphs

### Contents

---

<b>9.1</b>	<b>Definitions</b>	<b>191</b>
<b>9.2</b>	<b>Adjacency lists</b>	<b>194</b>
9.2.1	Breadth First Search	201
9.2.2	Depth First Search	203
<b>9.3</b>	<b>Shortest paths</b>	<b>205</b>
9.3.1	Bellman Ford algorithm	206
9.3.2	Dijkstra algorithm	208
<b>9.4</b>	<b>Adjacency matrix</b>	<b>210</b>
9.4.1	Comparisons	213
9.4.2	Display	214
9.4.3	Breadth First Search and Depth First Search	215
<b>9.5</b>	<b>Dynamic programming</b>	<b>217</b>
9.5.1	Definitions	217
9.5.2	Back to Fibonacci	218
<b>9.6</b>	<b>Shortest path - all pairs</b>	<b>220</b>
<b>9.7</b>	<b>Minimum Spanning Tree</b>	<b>224</b>
9.7.1	Kruskal	226
9.7.2	Prim	227

---

## 9.1 Definitions

A graph  $G = (V, E)$  is a couple of two sets defined by  $V$  corresponding to the set of vertices, and  $E$  corresponding to the set of edges.

Main types of graphs:

- **Undirected:** edge  $(u, v) = (v, u) \forall v, (v, v) \notin E$  (there are no self loops.)
- **Directed:**  $(u, v)$  is a edge from  $u$  to  $v$ , denoted as  $u \rightarrow v$ . Self loops are allowed.
- **Weighted:** Each edge has an associated weight, given by a weight function  $w : E \rightarrow \mathbb{R}$ .
- **Mixed:** some edges may be directed and some may be undirected
- **Multigraph:** multiple edges are two or more edges that connect the same two vertices.
- We say a graph is **dense** if  $|E| \approx |V|^2$ .
- We say a graph is **sparse**:  $|E| \ll |V|^2$ .

where  $|V|$  represents the number of vertices and  $|E|$  represents the number of edges. If  $(u, v) \in E$ , then vertex  $v$  is adjacent to vertex  $u$ . It is worth noting that the adjacency relationship is symmetric if  $G$  is undirected. However, it is not necessarily if  $G$  is directed. If  $G$  is connected then there is a path between every pair of vertices.  $|E| \geq |V| - 1$ . Furthermore, if  $|E| = |V| - 1$ , then  $G$  is a tree. The **degree** of a vertex  $v$  is the number of edges attached to the vertex  $v$ . **Simple** graph is undirected; both multiple edges and loops are disallowed. Each edge is an unordered pair of distinct vertices. Hence the degree of every vertex is at most  $n - 1$  with  $n$  vertices.

Types of graphs:

- **Connected** graphs: Every unordered pair of vertices in the graph is connected, there is a path from any point to any other point in the graph.
- **Bipartite** graphs: Vertices can be divided into 2 disjoint and independent sets  $U$  and  $V$ . Every edge connects a vertex in  $U$  to one in  $V$
- **Planar** graph: Vertices and edges can be drawn in a plane such that no two of the edges intersect
- **Cycle** graphs: Connected graphs in which the degree of all vertices is 2.
- **Tree:** A connected graph with no cycles.
- **Regular** graphs: Each vertex has the same number of neighbors every vertex has the same degree.
- **Complete** graphs: A simple undirected graph, where every pair of distinct vertices is connected by a unique edge.
- **Finite** graphs: The vertex set and the edge set are finite sets.
- **Eulerian** graphs: If the graph is both connected and has a closed trail containing all edges of the graph. It corresponds to a walk with no repeated edges.

More definitions:

- **Path:** It is a sequence of vertices  $v_1, \dots, v_k$  where each  $(v_i, v_{i+1})$  is an edge.

- **Simple path:** A path that does not repeat vertices.
- **Path Length:** Number of edges in the path.
- **Circuit:** It is a path that begins and ends at the same vertex.
- **Cycle:** It is a circuit that doesn't repeat vertices.
- **Euler path:** It is a path that travels through all edges of a connected graph.
- **Euler circuit:** It is a circuit that visits all edges of a connected graph.

### Hamiltonian graphs

- **Hamiltonian graph:** it is a graph possessing a Hamiltonian cycle. A graph that is not Hamiltonian is said to be non-hamiltonian.
- **Hamiltonian cycle:** Hamiltonian path that is acyclic.
- **Hamiltonian path:** A path in an undirected or directed graph that visits each vertex exactly once

### Euler's theorem

1. **Circuit:** If a graph has any vertex of odd degree, then it cannot have an Euler circuit. If a graph is connected and every vertex is of even degree, it has at least 1 Euler circuit.
2. **Path:** If a graph has more than 2 vertices of odd degree, then it cannot have an Euler path. If a graph is connected and has just 2 vertices of odd degree, then it has at least one Euler path. Any such path must start at one of the odd-vertices and end at the other odd vertex.

### Fleury's algorithm

With the Fleury's algorithm, we need to check first that the graph is connected and all vertices are of even degree. It starts at any vertex, travels through an edge, if it is not a bridge for the untraveled part or there is no other alternative (a bridge is an edge that if removed, it produces a disconnected graph). It then labels the edges in the order in which they have been visited. When it is not possible to travel further then it stops.

**Main idea:** expand a trail  $C_i$  while avoiding bridges in  $G - C_i$  until no other choices are possible.

### Hierholzer's algorithm

The Hierholzer's algorithm produces circuits in a graph  $G$  that are pairwise edge disjoint. When these circuits are combined properly, they form an Eulerian circuit of  $G$ . This patching together of circuits hinges on the circuits having a common vertex following from the connectivity of the graph. Once a circuit is formed, if all edges have not been used, then there must be an edge that is incident to a vertex of the circuit, and we use this edge to begin the next circuit. These circuits then share a common vertex.

**Main idea:** Combining circuits.

---

**Algorithm 1:** Fleury's algorithm.

---

```
1: Input: A connected graph  $G = (V, E)$ 
2: Output: An eulerian circuit  $C$  of  $G$ 
3: Pick any  $v_0 \in V$ 
4: Let  $C_0 = v_0$ 
5: Let  $i \leftarrow 0$ 
6: while  $(i! = |E|)$  do
7:   Assume the trail  $C_i = v_0, e_1, v_1, \dots, e_i, v_i$  has already been chosen.
8:   At  $v_i$  pick any edge  $e_{i+1}$  that is not on  $C_i$  and
9:   that is not a bridge of the graph  $G_i = G - E(C_i)$  unless there is no other choice.
10:  Define  $C_{i+1} = C_i, e_{i+1}, v_{i+1}$ .
11:   $i \leftarrow i + 1$ 
12: end while
```

---

---

**Algorithm 2:** Hierholzer's algorithm.

---

```
1: Input: A connected graph  $G = (V, E)$ 
2: Output: An eulerian circuit  $C$  of  $G$ 
3: Pick any  $v \in V$ 
4: Create a circuit  $C_0$  starting with  $v$  by traversing at each step,
5: any edge not yet included in the circuit.
6: Let  $i \leftarrow 0$ 
7: while  $(E(C_i) \neq E(G))$  do
8:   Pick a vertex  $v_i$  on  $C_i$  that is incident to an edge not on  $C_i$ 
9:   Create a circuit  $C_i^*$  starting with  $v_i$  in the graph  $G - E(C_i)$ 
10:  Create a circuit  $C_{i+1}$  that contains  $C_i$  and  $C_i^*$  by starting at  $v_{i-1}$ 
11:  traversing  $C_i$  until reaching  $v_i$ , then traversing completely  $C_i^*$ 
12:  and then completing the traversal of  $C_i$ 
13:   $i \leftarrow i + 1$ 
14: end while
```

---

## 9.2 Adjacency lists

Algorithm 9.1: Class interface.

---

```
1  class NodeGAL {
2  public:
3      NodeGAL() :
4          v(0), weight(0), next(nullptr) {}
5      NodeGAL(int v1, double weight1) :
6          v(v1), weight(weight1), next(nullptr) {}
7      ~NodeGAL() {}
8      int v; // vertex index
9      double weight; // weight of the edge to reach v
10     NodeGAL *next;
11 };
12
13 class MyGraphAL {
14 public:
15     MyGraphAL();
16     MyGraphAL(int n1);
17     ~MyGraphAL();
18     int GetNumberVertices() { return n; }
19     int GetDegree(int u) { return degree[u]; }
20     bool ExistEdge(int u, int v);
21     double GetEdgeWeight(int u, int v);
22     void SetDirectedEdge(int u, int v, double w);
23     void SetDirectedEdge(int u, int v);
24     void SetUndirectedEdge(int u, int v, double w);
25     void SetUndirectedEdge(int u, int v);
26     void RemoveDirectedEdge(int u, int v);
27     void RemoveUndirectedEdge(int u, int v);
28     bool HasSelfLoops();
29     bool IsUndirected();
30
31     bool ExistAdjacent(int u);
32     void SetCurrentVertex(int u);
33     int GetNextAdjacent(int u);
34     void Display();
35     void BFS(int s);
36     void DFS();
37 private:
38     int n; // number of vertices
39     NodeGAL** l; // array of lists
40     NodeGAL** current; // iterators
41     int* degree; // degree for each vertex
42 };
```

---

---

Algorithm 9.2: Constructor and Destructor.

---

```
1  MyGraphAL::MyGraphAL() {
2      int n = 0;
3      l = nullptr;
4      current=nullptr;
5      degree=nullptr;
6  }
7
8  MyGraphAL::MyGraphAL(int n1) {
9      n = n1;
10     l = new NodeGAL*[n];
11     degree = new int[n];
12     for (int u = 0; u < n; u++) {
13         l[u] = nullptr;
14         current[u]=nullptr;
15         degree[u] = 0;
16     }
17 }
18
19 MyGraphAL::~~MyGraphAL() {
20     for (int i = 0; i < n; i++) {
21         NodeGAL *current = l[i];
22         while (current) {
23             NodeGAL* next = current->next;
24             delete current;
25             current = next;
26         }
27     }
28     delete[] l;
29     delete[] current;
30     delete[] degree;
31 }
```

---



Algorithm 9.3: Exist and get edges.

---

```
1  bool MyGraphAL::ExistEdge(int u, int v) {
2      if (u < n) {
3          NodeGAL* cursor = l[u];
4          while (cursor != nullptr) {
5              if (cursor->v == v)
6                  return true;
7              else
8                  cursor = cursor->next;
9          }
10         return false;
11     }
12     else
13         return false;
14 }
15
16 double MyGraphAL::GetEdgeWeight(int u, int v) {
17     if (u < n) {
18         NodeGAL* cursor = l[u];
19         while (cursor != nullptr) {
20             if (cursor->v == v)
21                 return cursor->weight;
22             else
23                 cursor = cursor->next;
24         }
25         return 0;
26     }
27     else
28         return 0;
29 }
```

---

Algorithm 9.4: Set edges.

---

```
1 void MyGraphAL::SetDirectedEdge(int u, int v, double w) {
2     if (!ExistEdge(u, v)) {
3         NodeGAL* tmp = new NodeGAL(v, w);
4         tmp->next = l[u];
5         l[u] = tmp;
6         degree[u]++;
7     }
8 }
9
10 void MyGraphAL::SetDirectedEdge(int u, int v) {
11     SetDirectedEdge(u, v, 1);
12 }
13
14 void MyGraphAL::SetUndirectedEdge(int u, int v) {
15     SetDirectedEdge(u, v, 1);
16     SetDirectedEdge(v, u, 1);
17 }
18
19 void MyGraphAL::SetUndirectedEdge(int u, int v, double w) {
20     SetDirectedEdge(u, v, w);
21     SetDirectedEdge(v, u, w);
22 }
```

---

Algorithm 9.5: Remove edges.

---

```

1 void MyGraphAL::RemoveDirectedEdge(int u, int v) {
2     NodeGAL* cursor = l[u];
3     if (cursor!=nullptr) {
4         if (cursor->v == v) {
5             l[u] = cursor->next;
6             delete cursor;
7         }
8         else {
9             if (cursor->next != nullptr) {
10                NodeGAL* prev = cursor;
11                cursor = cursor->next;
12                bool found = false;
13                while ((cursor != nullptr) && (!found) ){
14                    if (cursor->v == v) {
15                        prev->next = cursor->next;
16                        delete cursor;
17                        found = true;
18                    }
19                    else {
20                        prev = cursor;
21                        cursor = cursor->next;
22                    }
23                }
24            }
25        }
26    }
27    degree[u]--;
28 }
29
30 void MyGraphAL::RemoveUndirectedEdge(int u, int v) {
31     RemoveDirectedEdge(u,v);
32     RemoveDirectedEdge(v,u);
33 }

```

---

---

Algorithm 9.6: Comparisons.

---

```
1 bool MyGraphAL::HasSelfLoops() {
2     int u = 0;
3     while (u < n) {
4         if (GetEdgeWeight(u,u) != 0)
5             return true;
6         u++;
7     }
8     return false;
9 }
10
11 bool MyGraphAL::IsUndirected() {
12     int v, u = 0;
13     while (u < n) {
14         v = u;
15         while (v < n) {
16             if (GetEdgeWeight(u,v) != GetEdgeWeight(v,u))
17                 return false;
18             v++;
19         }
20         u++;
21     }
22     return !HasSelfLoops();
23 }
```

---

---

Algorithm 9.7: Display.

---

```
1 void MyGraphAL::Display() {
2     for (int u = 0; u < n; u++) {
3         cout << u << ":";
4         NodeGAL* cursor = l[u];
5         while (cursor != nullptr) {
6             cout << cursor->v << "(" << cursor->weight << ")" ";
7             cursor = cursor->next;
8         }
9         cout << endl;
10    }
11 }
12 }
```

---

Algorithm 9.8: Iterator functions.

---

```
1  bool MyGraphAL::ExistAdjacent(int u) {  
2      return current[u] != nullptr;  
3  }  
4  
5  void MyGraphAL::SetCurrentVertex(int u) {  
6      current[u] = l[u];  
7  }  
8  
9  int MyGraphAL::GetNextAdjacent(int u) {  
10     int v = current[u]—>v;  
11     current[u] = current[u]—>next;  
12     return v;  
13 }
```

---

## 9.2.1 Breadth First Search

Algorithm 9.9: Color of Nodes.

---

```
1 enum class ColorNode {
2     Visited ,
3     Unvisited ,
4     Finished
5 };
```

---

Algorithm 9.10: BFS main function.

---

```
1 void BFS1(int s, MyGraphAL* G, ColorNode* &color , int* &distance , int* &pi) {
2     int n = G->GetNumberVertices();
3     if (!(s >= 0) && (s < n))
4         cout << "Bad source !" << endl;
5     else {
6         MyQueue* Q = new MyQueue(n);
7         color = new ColorNode[n];
8         distance = new int[n];
9         pi = new int[n];
10        for (int u = 0; u < n; u++) {
11            color[u] = ColorNode::Unvisited;
12            distance[u] = 0;
13            pi[u] = 0;
14        }
15        color[s] = ColorNode::Visited;
16        distance[s] = 0;
17        pi[s] = -1;
18        Q->Enqueue(s);
19        while (!Q->IsEmpty()) {
20            int u = Q->Dequeue();
21            for (int v = 0; v < n; v++) {
22                if (G->ExistEdge(u,v)) { // Adjacent to u
23                    if (color[v] == ColorNode::Unvisited) {
24                        color[v] = ColorNode::Visited;
25                        distance[v] = distance[u] + 1;
26                        pi[v] = u;
27                        Q->Enqueue(v);
28                    }
29                }
30            }
31            color[u] = ColorNode::Finished;
32        }
33        delete Q;
34    }
35 }
```

---

Algorithm 9.11: Breadth First Search Display.

---

```
1 void MyGraphAL::BFS(int s) {
2     cout << "BFS" << endl;
3     ColorNode *color;
4     int *distance , *pi;
5     BFS1(s, this , color , distance , pi);
6     for (int u = 0; u < n; u++)
7         cout << "from " << s << " to " << u
8             << " Distance=" << distance[u]
9             << " Pi=" << pi[u] << endl;
10    delete [] color;
11    delete [] distance;
12    delete [] pi;
13 }
```

---

## 9.2.2 Depth First Search

Algorithm 9.12: DFS functions.

---

```
1 void DFSVisit(int u, MyGraphAL* G, ColorNode* color ,
2             int* discovery, int* finished, int* pi, int t) {
3     color[u] = ColorNode::Visited;
4     t++;
5     discovery[u] = t;
6     G->SetCurrentVertex(u);
7     while (G->ExistAdjacent(u)) {
8         int v = G->GetNextAdjacent(u);
9         if (color[v] == ColorNode::Unvisited) { // not already visited
10             pi[v] = u;
11             DFSVisit(v, G, color, discovery, finished, pi, t);
12         }
13     }
14     color[u] = ColorNode::Finished;
15     t++;
16     finished[u] = t;
17 }
18
19 void DFS1(MyGraphAL* G, ColorNode* &color ,
20          int* &discovery, int* &finished, int* &pi) {
21     int n = G->GetNumberVertices();
22     color = new ColorNode[n];
23     discovery = new int[n];
24     finished = new int[n];
25     pi = new int[n];
26     for (int u = 0; u < n; u++) {
27         color[u] = ColorNode::Unvisited;
28         discovery[u] = 0;
29         finished[u] = 0;
30         pi[u] = 0;
31     }
32     int t = 0;
33     for (int u = 0; u < n; u++) {
34         if (color[u] == ColorNode::Unvisited)
35             DFSVisit(u, G, color, discovery, finished, pi, t);
36     }
37 }
```

---



Algorithm 9.13: Depth First Search Display.

---

```
1 void MyGraphAL::DFS() {
2     cout << "DFS" << endl;
3     ColorNode *color;
4     int *discovery,*finished,*pi;
5     DFS1(this, color, discovery, finished, pi);
6     for (int u = 0; u < n; u++)
7         cout << u << ": (" << discovery[u] << ", "
8             << finished[u] << ") Pi="
9             << pi[u]<< endl;
10    delete[] color;
11    delete[] discovery;
12    delete[] finished;
13    delete[] pi;
14 }
```

---

Algorithm 9.14: Example.

---

```
1 void main() {
2     int n = 5;
3     MyGraphAL* G = new MyGraphAL(n);
4     G->SetUndirectedEdge(0, 1);
5     G->SetUndirectedEdge(1, 2);
6     G->SetUndirectedEdge(2, 3);
7     G->SetUndirectedEdge(3, 0);
8     G->SetUndirectedEdge(0, 4);
9     G->SetUndirectedEdge(1, 4);
10    G->Display();
11    delete G;
12 }
```

---

### 9.3 Shortest paths

This section contains the algorithm of Bellman Ford and Dijkstra. For Dijkstra, we consider a Queue in which we add at each iteration the completed vertex, and a Priority Queue (see Heap section) that provides both the index of the vertex containing the minimum value and its minimum value. In the relaxation part, note that it is possible to decrease the key or it is possible to not update the value of the element in the priority queue and just add a new element in it. In the latter case, the Priority Queue should be able to get all these new elements.

Algorithm 9.15: Display path information.

---

```
1 void DisplayShortestPath(double* d, int* pi, int n) {
2     for (int i = 0; i < n; i++) {
3         cout << "Vertex:" << i
4             << " : d=" << d[i]
5             << " : pi:" << pi[i]
6             << endl;
7     }
8     cout << endl;
9 }
```

---

### 9.3.1 Bellman Ford algorithm

Algorithm 9.16: Bellman Ford.

---

```
1  bool MyGraphAL::BellmanFord(int s, double* &d, int* &pi) {
2      // Initialize Single Source
3      d = new double[n];
4      pi = new int[n];
5      for (int u = 0; u < n ; u++) {
6          d[u] = DBLMAX;
7          pi[u] = -1;
8      }
9      d[s] = 0;
10     for (int i = 0; i < n - 1; i++) {
11         for (int u = 0; u < n; u++) { // For each edge ...
12             SetCurrentVertex(u);
13             while (ExistAdjacent(u)) {
14                 int v = GetNextAdjacent(u);
15                 // Relaxation
16                 double w = GetEdgeWeight(u, v);
17                 if (d[v] > d[u] + w) {
18                     d[v] = d[u] + w;
19                     pi[v] = u;
20                 }
21             }
22         }
23         DisplayShortestPath(d, pi, n);
24     }
25     for (int u = 0; u < n; u++) { // For each edge
26         SetCurrentVertex(u);
27         while (ExistAdjacent(u)) {
28             int v = GetNextAdjacent(u);
29             double w = GetEdgeWeight(u, v);
30             if (d[v] > d[u] + w)
31                 return false;
32         }
33     }
34     return true;
35 }
```

---

Algorithm 9.17: Bellman Ford example.

---

```
1 void main() {
2     int n = 5;
3     MyGraphAL G(n);
4     G.SetDirectedEdge(0, 1, 6);
5     G.SetDirectedEdge(0, 4, 7);
6     G.SetDirectedEdge(1, 2, 5);
7     G.SetDirectedEdge(1, 3, -4);
8     G.SetDirectedEdge(1, 4, 8);
9     G.SetDirectedEdge(2, 1, -2);
10    G.SetDirectedEdge(3, 2, 7);
11    G.SetDirectedEdge(3, 0, 2);
12    G.SetDirectedEdge(4, 2, -3);
13    G.SetDirectedEdge(4, 3, 9);
14    G.Display();
15    cout << "BellmanFord" << endl;
16    int s = 0;
17    double* d;
18    int* pi;
19    bool out=G.BellmanFord(s,d,pi);
20    DisplayShortestPath(d, pi, n);
21    delete d;
22    delete pi;
23 }
```

---

### 9.3.2 Dijkstra algorithm

Algorithm 9.18: Dijkstra.

---

```
1 void MyGraphAL::Dijkstra(int s, double* &d, int* &pi) {
2     // Initialize Single Source
3     d = new double[n];
4     pi = new int[n];
5     MyPriorityQueue* PQ = new MyPriorityQueue(n);
6     MyQueue* Q = new MyQueue(n);
7     for (int u = 0; u < n; u++) {
8         d[u] = DBLMAX;
9         pi[u] = -1;
10    }
11    d[s] = 0;
12    for (int u = 0; u < n; u++)
13        PQ->Push(u, d[u]);
14    PQ->Display();
15    while (!Q->IsFull()) {
16        MyData out = PQ->Pop();
17        PQ->Display();
18        Q->Enqueue(out.index);
19        int u = out.index;
20        cout << "Use: " << out.index << " with d:" << out.value << endl;
21        SetCurrentVertex(u);
22        while (ExistAdjacent(u)) {
23            int v = GetNextAdjacent(u);
24            // Relaxation
25            double w = GetEdgeWeight(u, v);
26            if (d[v] > d[u] + w) {
27                d[v] = d[u] + w;
28                pi[v] = u;
29                PQ->DecreaseKey(v, d[v]);
30            }
31        }
32        DisplayShortestPath(d, pi, n);
33        PQ->Display();
34    }
35    delete PQ;
36    delete Q;
37 }
```

---

Algorithm 9.19: Dijkstra example.

---

```
1 void main() {
2     int n = 5;
3     MyGraphAL G(n);
4     G.SetDirectedEdge(0, 1, 10);
5     G.SetDirectedEdge(0, 4, 5);
6     G.SetDirectedEdge(1, 2, 1);
7     G.SetDirectedEdge(1, 4, 2);
8     G.SetDirectedEdge(2, 3, 4);
9     G.SetDirectedEdge(3, 0, 7);
10    G.SetDirectedEdge(3, 2, 6);
11    G.SetDirectedEdge(4, 1, 3);
12    G.SetDirectedEdge(4, 2, 9);
13    G.SetDirectedEdge(4, 3, 2);
14    G.Display();
15    cout << "Dijkstra" << endl;
16    int s = 0;
17    double* d;
18    int* pi;
19    G.Dijkstra(s, d, pi);
20    DisplayShortestPath(d, pi, n);
21    delete d;
22    delete pi;
23 }
```

---

## 9.4 Adjacency matrix

Algorithm 9.20: Class interface.

---

```
1  class MyGraphAM {
2  public:
3      MyGraphAM();
4      MyGraphAM(int n);
5      ~MyGraphAM();
6      int GetNumberVertices() { return n; }
7      int GetDegree(int u);
8      int GetIndex(int u, int v);
9      bool ExistEdge(int u, int v);
10     void SetDirectedEdge(int u, int v, double w);
11     void SetDirectedEdge(int u, int v);
12     void SetUndirectedEdge(int u, int v, double w);
13     void SetUndirectedEdge(int u, int v);
14     void RemoveDirctedEdge(int u, int v);
15     void RemoveUndirectedEdge(int u, int v);
16     bool HasSelfLoops();
17     bool IsUndirected();
18     void Display();
19     void DisplayDirectedEdge();
20     void DisplayUndirectedEdge();
21     void SetCurrentVertex(int u);
22     bool GetNextAdjacent(int u, int &vout);
23     void BFS(int s);
24     void DFS();
25     void PrintAllPairs(MyMatrix* pi, int i, int j);
26     void FloydWarshall(MyMatrix* &dout, MyMatrix* &piout);
27     void TransitiveClosure(MyMatrix* &TCout);
28     void MSTKruskal();
29     void MSTPrim();
30     int minKey(int* key, bool* mstSet);
31     void DisplayMST(int* parent);
32 private:
33     int n; // number of vertices
34     double* M;
35     int* current;
36 };
```

---

Algorithm 9.21: Constructors and Destructor.

---

```
1 MyGraphAM::MyGraphAM() {
2     n = 0;
3     M=nullptr;
4     current = nullptr;
5 }
6
7 MyGraphAM::MyGraphAM(int n1) {
8     n = n1;
9     M = new double[n*n];
10    for (int i = 0; i < n*n; i++)
11        M[i] = 0;
12    current = new int[n];
13    for (int i = 0; i < n; i++)
14        current[i] = 0;
15 }
16
17 MyGraphAM::~MyGraphAM() {
18     delete[] M;
19     delete[] current;
20 }
```

---



Algorithm 9.22: Get, Exist, and Set edges.

---

```
1  int MyGraphAM::GetIndex(int u, int v) {
2      // u: source, v: destination
3      return u*n + v;
4  }
5
6  bool MyGraphAM::ExistEdge(int u, int v) {
7      return (M[GetIndex(u, v)] != 0);
8  }
9
10 int MyGraphAM::GetDegree(int u) {
11     int degree = 0;
12     for (int v = 0; v < n; v++)
13         if (ExistEdge(u, v))
14             degree++;
15     return degree;
16 }
17
18 void MyGraphAM::SetDirectedEdge(int u, int v, double w) {
19     M[GetIndex(u, v)] = w;
20 }
21
22 void MyGraphAM::SetDirectedEdge(int u, int v) {
23     M[GetIndex(u, v)] = 1;
24 }
25
26 void MyGraphAM::SetUndirectedEdge(int u, int v, double w) {
27     M[GetIndex(u, v)] = w;
28     M[GetIndex(v, u)] = w;
29 }
30
31 void MyGraphAM::SetUndirectedEdge(int u, int v) {
32     M[GetIndex(u, v)] = 1;
33     M[GetIndex(v, u)] = 1;
34 }
35
36 void MyGraphAM::RemoveDirctedEdge(int u, int v) {
37     M[GetIndex(u, v)] = 0;
38 }
39
40 void MyGraphAM::RemoveUndirectedEdge(int u, int v) {
41     M[GetIndex(u, v)] = 0;
42     M[GetIndex(v, u)] = 0;
43 }
```

---

### 9.4.1 Comparisons

Algorithm 9.23: Comparisons.

---

```
1  bool MyGraphAM::HasSelfLoops() {
2      int u = 0;
3      while (u < n) {
4          if (M[GetIndex(u, u)] != 0)
5              return true;
6          u++;
7      }
8      return false;
9  }
10
11 bool MyGraphAM::IsUndirected() {
12     int v, u = 0;
13     while (u < n) {
14         v = u;
15         while (v < n) {
16             if (M[GetIndex(u, v)] != M[GetIndex(v, u)])
17                 return false;
18             v++;
19         }
20         u++;
21     }
22     return !HasSelfLoops();
23 }
```

---

## 9.4.2 Display

Algorithm 9.24: Display.

---

```
1 void MyGraphAM::Display () {
2     int k = 0;
3     cout << " :";
4     for (int v = 0; v < n; v++)
5         cout << v << "\t";
6     cout << endl;
7     for (int u = 0; u < n; u++) {
8         cout << u << " :";
9         for (int v = 0; v < n; v++) {
10            cout << M[k] << "\t";
11            k++;
12        }
13        cout << endl;
14    }
15 }
16
17 void MyGraphAM::DisplayDirectedEdge () {
18     int k = 0;
19     cout << "List of edges :" << endl;
20     for (int u = 0; u < n; u++) {
21         for (int v = 0; v < n; v++) {
22             if (ExistEdge(u, v))
23                 cout << "(" << u << ", "
24                     << v << ") w:" << GetEdgeWeight(u, v)
25                     << endl;
26         }
27     }
28 }
29
30 void MyGraphAM::DisplayUndirectedEdge () {
31     int k = 0;
32     cout << "List of edges :" << endl;
33     for (int u = 0; u < n; u++) {
34         for (int v = u+1; v < n; v++) {
35             if (ExistEdge(u, v))
36                 cout << "(" << u << ", "
37                     << v << ") w:" << GetEdgeWeight(u, v)
38                     << endl;
39         }
40     }
41 }
```

---

### 9.4.3 Breadth First Search and Depth First Search

The functions are similar to the the graph using the adjacency lists implementation. However, for DFS, there are slight changes in the way we access the different adjacent vertices to a given vertex.

Algorithm 9.25: Adjacent iterator.

---

```
1 void MyGraphAM:: SetCurrentVertex(int u) {
2     current[u] = -1;
3 }
4
5 bool MyGraphAM:: GetNextAdjacent(int u, int &vout) {
6     int v= current[u] + 1;
7     vout = -1;
8     bool found = false;
9     while ((!found) && (v < n)) {
10         if (ExistEdge(u, v)) {
11             found = true;
12             vout=v;
13         }
14         v++;
15     }
16     current[u] = vout;
17     return found;
18 }
19
20 void DFSVisit(int u, MyGraphAM* G, ColorNode* color ,
21             int* discovery , int* finished , int* pi , int t) {
22     color[u] = ColorNode:: Visited;
23     t++;
24     discovery[u] = t;
25     G->SetCurrentVertex(u);
26     int v;
27     while (G->GetNextAdjacent(u,v)) {
28         if (v!=-1)
29             if (color[v] == ColorNode:: Unvisited) {
30                 pi[v] = u;
31                 DFSVisit(v, G, color , discovery , finished , pi , t);
32             }
33     }
34     color[u] = ColorNode:: Finished;
35     t++;
36     finished[u] = t;
37 }
```

---

Algorithm 9.26: Example.

---

```
1 void main() {  
2     int n = 5;  
3     MyGraphAM* G = new MyGraphAM(n);  
4     G->SetUndirectedEdge(0, 1);  
5     G->SetUndirectedEdge(1, 2);  
6     G->SetUndirectedEdge(2, 3);  
7     G->SetUndirectedEdge(3, 0);  
8     G->SetUndirectedEdge(0, 4);  
9     G->SetUndirectedEdge(1, 4);  
10    G->Display();  
11    delete G;  
12 }
```

---

## 9.5 Dynamic programming

### 9.5.1 Definitions

Dynamic programming (DP) is about solving problems by combining the solution to subproblems. Contrary to the Divide & Conquer approach, the partition of the problem is into disjoint subproblems. Dynamic programming is when the subproblems overlap. In such a case, we solve each subproblem one time and then save its answer in a table. The goal is to avoid the work of recomputing the answer every time it solves each sub-sub-problem.

DP solves problems by combining solutions to subproblems. To apply DP: subproblems are not independent and subproblems may share subsubproblems. However, a solution to one subproblem may not affect the solutions to other subproblems of the same problem. DP reduces computation by:

- Solving subproblems in a bottom-up fashion.
- Storing solution to a subproblem the first time it is solved.
- Looking up the solution when subproblem is encountered again.

DP involves the following 4 main steps:

1. Characterize the structure of an optimal solution.
2. Recursively define the value of an optimal solution.
3. Compute the value of an optimal solution.
  - Bottom-up with a table
  - Top-down with caching
4. Construct an optimal solution from computed information.

Memoization is an optimization technique used primarily to speed up computer programs. It is used by storing the results of expensive function calls, and by returning the cached result when the same inputs occur again. It ensures that a method doesn't run for the same inputs more than once by keeping a record of the results for the given inputs. It is a common strategy for DP problems.

### 9.5.2 Back to Fibonacci

The variable `step` is used to count the number of calls to the Fibonacci function.

Algorithm 9.27: Default Fibonacci.

---

```
1 int Fibonacci1(int x, int &step) {
2     if (x == 0) {
3         step++;
4         return 0;
5     }
6     else if (x == 1) {
7         step++;
8         return 1;
9     }
10    else {
11        step++;
12        return Fibonacci1(x - 1, step) + Fibonacci1(x - 2, step);
13    }
14 }
```

---

Algorithm 9.28: Fibonacci with DP.

---

```
1 int Fibonacci2(int x, int* t, int &step) {
2     if (x == 0) {
3         t[x] = 0;
4         step++;
5         return 0;
6     }
7     else if (x == 1) {
8         t[x] = 1;
9         step++;
10        return 1;
11    }
12    else {
13        int a, b;
14        a = (t[x-1] != -1) ? t[x - 1] : Fibonacci2(x-1, t, step);
15        b = (t[x-2] != -1) ? t[x - 2] : Fibonacci2(x-2, t, step);
16        t[x] = a+b;
17        step++;
18        return t[x];
19    }
20 }
```

---

---

Algorithm 9.29: Comparisons.

---

```
1 void main() {
2     int step, n=12;
3     for (int v = 0; v < n; v++) {
4         step = 0;
5         cout << "F1:" << v << ":"
6             << Fibonacci1(v, step);
7         cout << " with " << step << " steps " << endl;
8         // F1:11:89 with 287 steps
9         int* table = new int[n];
10        for (int i = 0; i < n; i++)
11            table[i] = -1;
12        step = 0;
13        cout << "F2:" << v << ":"
14            << Fibonacci2(v, table, step);
15        cout << " with " << step << " steps " << endl;
16        // F2:11:89 with 12 steps
17        delete[] table;
18    }
19 }
```

---



## 9.6 Shortest path - all pairs

This section is about the Floyd Warshall algorithm, which is based on dynamic programming. The method is added to the Graph with the Adjacency matrix implementation. In the presented code, it computes the matrix of distances, the matrix of predecessors (II), and the transitive closure (TC), i.e., to determine whether the graph G contains a path from i to j for all vertex pairs (i,j). The algorithm is based on 3 nested loops. However, it is critical to pay attention to the indices. In C++, we start the array at 0, and the original algorithm goes from 1 to n. Therefore, we allocate the space for n+1, where n is the number of vertices in G. In the algorithm, we consider the values of k from 1 to n, and as the indices in the matrix go from 0 to n-1, we access the values of k-1 in the main loop.

Algorithm 9.30: Print all pairs.

---

```
1 void MyGraphAM::PrintAllPairs(MyMatrix* pi, int i, int j) {
2     if (i == j)
3         cout << i ;
4     else if (pi->GetCell(i, j) == -1)
5         cout << "No path from " << i << " to " << j << endl;
6     else {
7         PrintAllPairs(pi, i, pi->GetCell(i, j));
8         cout << "," << j;
9     }
10 }
```

---

Algorithm 9.31: Floyd Warshall.

---

```

1 void MyGraphAM::FloydWarshall(MyMatrix* &dout, MyMatrix* &piout) {
2     MyMatrix** d = new MyMatrix*[n+1]; // 0 + n steps
3     MyMatrix** pi = new MyMatrix*[n+1];
4     d[0] = new MyMatrix(n, n); // matrix of distances
5     pi[0] = new MyMatrix(n, n); // matrix predecessors
6     // Initialization
7     for (int i = 0; i < n; i++)
8         for (int j = 0; j < n; j++) {
9             double w = GetEdgeWeight(i, j);
10            if (i==j)
11                d[0]->SetCell(i, j, 0);
12            else {
13                if (w != 0)
14                    d[0]->SetCell(i, j, w);
15                else
16                    d[0]->SetCell(i, j, DBLMAX);
17            }
18            if ((i == j) || (w == 0))
19                pi[0]->SetCell(i, j, -1); // No Predecessor
20            else
21                pi[0]->SetCell(i, j, i);
22        }
23    // Dynamic programming
24    for (int k = 1; k <= n; k++) {
25        d[k]=new MyMatrix(n, n);
26        pi[k]=new MyMatrix(n, n);
27        for (int i = 0; i < n; i++)
28            for (int j = 0; j < n; j++) {
29                d[k]->SetCell(i, j, min(d[k-1]->GetCell(i, j),
30                    d[k-1]->GetCell(i, k-1) + d[k-1]->GetCell(k-1, j)));
31                if (d[k]->GetCell(i, j) == d[k-1]->GetCell(i, j))
32                    pi[k]->SetCell(i, j, pi[k-1]->GetCell(i, j));
33                else
34                    pi[k]->SetCell(i, j, pi[k-1]->GetCell(k-1, j));
35            }
36        }
37    dout = d[n]; piout = pi[n];
38    for (int k = 0; k < n; k++) {
39        delete d[k]; delete pi[k];
40    }
41    delete[] d; delete[] pi;
42 }

```

---

Algorithm 9.32: Transitive closure.

---

```

1 void MyGraphAM::TransitiveClosure(MyMatrix* &TCout) {
2     MyMatrix** TC = new MyMatrix*[n + 1];
3     TC[0] = new MyMatrix(n, n);
4     for (int i = 0; i < n; i++)
5         for (int j = 0; j < n; j++) {
6             double w = GetEdgeWeight(i, j);
7             if ((i == j) || (w != 0))
8                 TC[0]→SetCell(i, j, 1);
9             else
10                TC[0]→SetCell(i, j, 0);
11        }
12    for (int k = 1; k <= n; k++) {
13        TC[k] = new MyMatrix(n, n);
14        for (int i = 0; i < n; i++)
15            for (int j = 0; j < n; j++)
16                TC[k]→SetCell(i, j, min(TC[k - 1]→GetCell(i, j),
17                                         TC[k - 1]→GetCell(i, k - 1) +
18                                         TC[k - 1]→GetCell(k - 1, j)));
19    }
20    TCout = TC[n];
21    for (int k = 0; k < n; k++)
22        delete TC[k];
23    delete[] TC;
24 }

```

---

Algorithm 9.33: Floyd Warshall Example.

---

```
1 void main() {
2     cout << "Floyd Warshall test" << endl;
3     int n = 5;
4     MyGraphAM* G = new MyGraphAM(n);
5     G->SetDirectedEdge(0, 1, 3);
6     G->SetDirectedEdge(0, 2, 8);
7     G->SetDirectedEdge(0, 4, -4);
8     G->SetDirectedEdge(1, 3, 1);
9     G->SetDirectedEdge(1, 4, 7);
10    G->SetDirectedEdge(2, 1, 4);
11    G->SetDirectedEdge(3, 0, 2);
12    G->SetDirectedEdge(3, 2, -5);
13    G->SetDirectedEdge(4, 3, 6);
14    cout << "Display graph:" << endl;
15    G->Display();
16    MyMatrix *d,*pi,*TC;
17    G->FloydWarshall(d, pi);
18    G->TransitiveClosure(TC);
19    cout << "Matrix with distances:" << endl;
20    d->Display();
21    cout << "Matrix with predecessor:" << endl;
22    pi->Display();
23    cout << "Matrix with Transitive Closure:" << endl;
24    TC->Display();
25    for (int i = 0; i < n; i++)
26        for (int j = 0; j < n; j++) {
27            cout << "Path from " << i << " to " << j << endl;
28            G->PrintAllPairs(pi, i, j);
29            cout << endl;
30        }
31 }
```

---

## 9.7 Minimum Spanning Tree

Algorithm 9.34: minKey (Prim).

---

```
1 int MyGraphAM::minKey(int* key, bool* mstSet) {
2     int min = INT_MAX;
3     int min_index = INT_MAX;
4     for (int v = 0; v < n; v++)
5         if ((mstSet[v] == false) && (key[v] < min)) {
6             min = key[v];
7             min_index = v;
8         }
9     return min_index;
10 }
```

---

Algorithm 9.35: Display MST.

---

```
1 void MyGraphAM::DisplayMST(int* parent) {
2     cout << "MST Edge Weight" << endl;
3     for (int i = 1; i < n; i++)
4         cout << "(" << parent[i] << "-> " << i << ") w="
5         << GetEdgeWeight(i, parent[i]) << endl;
6 }
```

---

Algorithm 9.36: Acyclic.

---

```

1 void Acyclic(int u, MyGraphAM &G, int* color, int* pi, int* back, int parent) {
2     color[u] = 1; // visited;
3     int v, n = G.GetNumberVertices();
4     G.SetCurrentVertex(u);
5     while (G.GetNextAdjacent(u, v)) {
6         if ( G.ExistEdge(u,v) && (v!=u) && (v!=parent)) {
7             if (color[v] == 0) { // not already visited
8                 pi[v] = u;
9                 Acyclic(v, G, color, pi, back, u);
10            }
11            else
12                back[v] = u;
13        }
14    }
15    color[u] = 2;
16 }

```

---

Algorithm 9.37: IsAcyclic.

---

```

1 bool IsAcyclic(int u, MyGraphAM &G) {
2     int n = G.GetNumberVertices();
3     int* color = new int[n];
4     int* back = new int[n];
5     int* pi = new int[n];
6     for (int i = 0; i < n; i++) {
7         color[i] = 0;
8         back[i] = -1;
9         pi[i] = -1;
10    }
11    Acyclic(u, G, color, pi, back, -1);
12    int i = 0;
13    bool acyclic = true;
14    bool found = false;
15    while ((i < n) && (!found)) {
16        if ((back[i] == u) && (pi[i] != u)) {
17            found = true;
18            acyclic = false;
19        }
20        i++;
21    }
22    delete[] color;      delete[] back;
23    return acyclic;
24 }

```

---

### 9.7.1 Kruskal

Algorithm 9.38: Kruskal.

---

```
1 void MyGraphAM::MSTKruskal() {
2     MyGraphAM A(n); // edges we will keep
3     MyMatrix S(n, n); // copy edges
4     S.Init(0);
5     for (int i = 0; i < n; i++)
6         for (int j = i + 1; j < n; j++)
7             S(i, j) = GetEdgeWeight(i, j);
8     int total = 0; // number of edges
9     for (int i = 0; i < n; i++)
10         for (int j = i + 1; j < n; j++)
11             if (S(i, j) != 0)
12                 total++;
13     int argi, argj;
14     double value;
15     tie(value, argi, argj) = S.GetMin();
16     S(argi, argj) = 0;
17     A.SetUndirectedEdge(argi, argj, value);
18     for (int i = 0; i < total; i++) { // for each edge
19         tie(value, argi, argj) = S.GetMin();
20         S(argi, argj) = 0;
21         A.SetUndirectedEdge(argi, argj, value);
22         if (!IsAcyclic(argi, A))
23             A.RemoveUndirectedEdge(argi, argj);
24     }
25     cout << "Display graph" << endl;
26     A.DisplayUndirectedEdge();
27 }
```

---

### 9.7.2 Prim

Algorithm 9.39: Prim.

---

```
1 void MyGraphAM::MSTPrim() {
2     int* key = new int[n];
3     int* pi = new int[n];
4     bool* mstSet = new bool[n];
5     for (int u = 0; u < n; u++) {
6         key[u] = INT_MAX; // infinite value
7         mstSet[u] = false;
8     }
9     key[0] = 0;
10    pi[0] = -1;
11    for (int i = 0; i < n - 1; i++) {
12        int u = minKey(key, mstSet);
13        mstSet[u] = true;
14        // Consider only those vertices which are not yet included in MST
15        SetCurrentVertex(u);
16        int v;
17        while (GetNextAdjacent(u, v)) {
18            double w = GetEdgeWeight(u, v);
19            if ((mstSet[v] == false) && (w < key[v])) {
20                pi[v] = u;
21                key[v] = w;
22            }
23        }
24    }
25    DisplayMST(pi);
26    delete[] pi;
27    delete[] key;
28    delete[] mstSet;
29 }
```

---



Algorithm 9.40: Example MST.

---

```
1 void main() {
2     int n = 9;
3     MyGraphAM* G = new MyGraphAM(n);
4     G->SetUndirectedEdge(0, 1, 4);
5     G->SetUndirectedEdge(1, 2, 8);
6     G->SetUndirectedEdge(2, 3, 7);
7     G->SetUndirectedEdge(3, 4, 9);
8     G->SetUndirectedEdge(4, 5, 10);
9     G->SetUndirectedEdge(5, 6, 2);
10    G->SetUndirectedEdge(6, 7, 1);
11    G->SetUndirectedEdge(7, 0, 8);
12    G->SetUndirectedEdge(7, 8, 7);
13    G->SetUndirectedEdge(8, 2, 2);
14    G->SetUndirectedEdge(8, 6, 6);
15    G->SetUndirectedEdge(2, 5, 4);
16    G->SetUndirectedEdge(3, 5, 14);
17    cout << "Kruskal" << endl;
18    G->MSTKruskal();
19    cout << "Prim" << endl;
20    G->MSTPrim();
```

---

# Chapter 10

## Matrix

### Contents

---

<b>10.1 Class interface</b>	<b>229</b>
<b>10.2 Main methods</b>	<b>231</b>
<b>10.3 Matrix operations</b>	<b>233</b>
<b>10.4 Multiplication of multiple matrices</b>	<b>238</b>

---

### 10.1 Class interface

The class `MyMatrix` is a useful class with a large number of different implementations. The proposed implementation highlight how to overload operators and it provides a state of the art algorithm related to dynamic programming with the multiplication of a sequence of matrices. In this implementation, we consider arrays of arrays, contrary to the implementation of the adjacency matrix that was used with the graphs.

Algorithm 10.1: Class interface.

---

```
1  class MyMatrix {
2  public:
3      MyMatrix ();
4      MyMatrix(int n, int m);
5      MyMatrix(const MyMatrix&);
6      ~MyMatrix ();
7      void Init(double x);
8      void Identity ();
9      void Display ();
10     int GetRows() const { return n; }
11     int GetCols() const { return m; }
12     double GetCell(int i, int j);
13     void SetCell(int i, int j, double x);
14     double operator()(int i, int j) const;
15     double& operator()(int i, int j);
16     void operator=(MyMatrix &M1);
17     MyMatrix Add(MyMatrix &M1);
18     MyMatrix Sub(MyMatrix &M1);
19     MyMatrix operator+(MyMatrix& a);
20     MyMatrix operator+(double x);
21     MyMatrix operator-(MyMatrix& a);
22     MyMatrix operator-(double x);
23     MyMatrix operator*(MyMatrix& a);
24     MyMatrix operator*(double x);
25     tuple<double, int, int> GetMin();
26 private:
27     int n, m; // size of the matrix
28     double** A;
29     void Clean();
30 };
```

---

## 10.2 Main methods

Algorithm 10.2: Constructors and destructor

---

```
1 MyMatrix::MyMatrix() {
2     n = 0;
3     m = 0;
4     A = nullptr;
5 }
6 MyMatrix::MyMatrix(int n1, int m1) {
7     n = n1; m = m1;
8     A = new double*[n];
9     for (int i = 0; i < n; i++) {
10        A[i] = new double[m];
11        for (int j = 0; j < m; j++)
12            A[i][j] = 0;
13    }
14 }
15 MyMatrix::MyMatrix(const MyMatrix& M) {
16     n = M.n; m = M.m;
17     A = new double*[n];
18     for (int i = 0; i < n; i++) {
19        A[i] = new double[m];
20        for (int j = 0; j < m; j++)
21            A[i][j] = M.A[i][j];
22    }
23 }
24 void MyMatrix::Clean() {
25     for (int i = 0; i < n; i++)
26         delete[] A[i];
27     delete[] A;
28 }
29 MyMatrix::~MyMatrix() {
30     Clean();
31 }
```

---

---

Algorithm 10.3: Accessors and modifiers

---

```
1 double MyMatrix::GetCell(int i, int j) {
2     try {
3         return A[i][j];
4     }
5     catch (exception& e) {
6         cout << "Problem with index" << e.what() << endl;
7     }
8 }
9
10 // Get value
11 double MyMatrix::operator()(int i, int j) const {
12     return A[i][j];
13 }
14 // Set value
15 double& MyMatrix::operator()(int i, int j) {
16     return A[i][j];
17 }
18
19 void MyMatrix::SetCell(int i, int j, double x) {
20     try {
21         A[i][j]=x;
22     }
23     catch (exception& e) {
24         cout << "Problem with index" << e.what() << endl;
25     }
26 }
```

---

---

Algorithm 10.4: Display

---

```
1 void MyMatrix::Display() {
2     cout << "size: " << n << "x" << m << endl;
3     for (int i = 0; i < n; i++) {
4         for (int j = 0; j < m; j++)
5             cout << A[i][j] << " ";
6         cout << endl;
7     }
8
9 }
```

---

## 10.3 Matrix operations

Algorithm 10.5: Initialization

---

```
1 void MyMatrix::Init(double x) {
2     for (int i = 0; i < n; i++)
3         for (int j = 0; j < m; j++)
4             A[i][j] = x;
5 }
```

---

Algorithm 10.6: Identity

---

```
1 void MyMatrix::Identity() {
2     if (n == m) {
3         Init(0);
4         for (int i = 0; i < n; i++)
5             A[i][i] = 1;
6     }
7 }
```

---

Algorithm 10.7: Assignment

---

```
1 void MyMatrix::operator=(MyMatrix &M1) {
2     Clean();
3     n=M1.n;
4     m=M1.m;
5     A = new double*[n];
6     for (int i = 0; i < n; i++) {
7         A[i] = new double[m];
8         for (int j = 0; j < m; j++)
9             A[i][j] = M1.A[i][j];
10    }
11 }
```

---

#### Algorithm 10.8: Addition

---

```
1  MyMatrix MyMatrix::Add(MyMatrix &M1) {
2      MyMatrix out(n, m);
3      for (int i = 0; i < n; i++)
4          for (int j = 0; j < m; j++)
5              out(i,j)=this->A[i][j] + M1.A[i][j];
6      return out;
7  }
8
9  MyMatrix MyMatrix::operator+(MyMatrix &M1) {
10     MyMatrix out(n,m);
11     for (int i = 0; i < n; i++)
12         for (int j = 0; j < m; j++)
13             out(i,j)=A[i][j] + M1.A[i][j];
14     return out;
15 }
16
17 MyMatrix MyMatrix::operator+(double x) {
18     MyMatrix out(n, m);
19     for (int i = 0; i < n; i++)
20         for (int j = 0; j < m; j++)
21             out(i,j)=A[i][j] + x;
22     return out;
23 }
```

---

#### Algorithm 10.9: Subtraction

---

```
1 MyMatrix MyMatrix::Add(MyMatrix &M1) {
2     MyMatrix out(n, m);
3     for (int i = 0; i < n; i++)
4         for (int j = 0; j < m; j++)
5             out(i, j) = A[i][j] - M1.A[i][j];
6     return out;
7 }
8 MyMatrix MyMatrix::operator-(MyMatrix &M1) {
9     MyMatrix out(n, m);
10    for (int i = 0; i < n; i++)
11        for (int j = 0; j < m; j++)
12            out.SetCell(i, j, A[i][j] - M1.A[i][j]);
13    return out;
14 }
15 MyMatrix MyMatrix::operator-(double x) {
16     MyMatrix out(n, m);
17     for (int i = 0; i < n; i++)
18         for (int j = 0; j < m; j++)
19             out.SetCell(i, j, A[i][j] - x);
20     return out;
21 }
```

---



---

Algorithm 10.10: Multiplication

---

```
1 MyMatrix MyMatrix::operator*(MyMatrix &M1) {
2     if (this->m != M1.n) {
3         cout << "Problem with the dimensions." << endl;
4         return *this;
5     }
6     else
7     {
8         double tmp;
9         MyMatrix out(n,M1.m);
10        for (int i = 0; i < n; i++)
11            for (int j = 0; j < M1.m; j++) {
12                tmp = 0;
13                for (int k = 0; k < m; k++)
14                    tmp += A[i][k] * M1(k,j);
15                out.A[i][j] = tmp;
16            }
17        return out;
18    }
19 }
20
21 MyMatrix MyMatrix::operator*(double x) {
22     MyMatrix out(n, m);
23     for (int i = 0; i < n; i++)
24         for (int j = 0; j < m; j++)
25             out.SetCell(i, j,A[i][j] * x);
26     return out;
27 }
```

---

The goal of the following function is to return the minimum value of the matrix with its corresponding indices. The element is then set to 0.

Algorithm 10.11: GetMin

---

```
1 tuple<double, int, int> MyMatrix::GetMin() {
2     double minvalue = DBLMAX;
3     int argi = 0;
4     int argj = 0;
5     for (int i = 0; i < n; i++)
6         for (int j = 0; j < m; j++) {
7             if ((A[i][j] < minvalue) && (A[i][j] > 0)) {
8                 minvalue = A[i][j];
9                 argi = i;
10                argj = j;
11            }
12        }
13    A[argi][argj] = 0; // set the min to 0
14    return make_tuple(minvalue, argi, argj);
15 }
```

---

## 10.4 Multiplication of multiple matrices

This algorithm uses the concept of dynamic programming that was seen in Chapter 9.

Algorithm 10.12: Print brackets

---

```
1 void PrintBrackets(MyMatrix &s, int i, int j) {
2     if (i == j)
3         cout << "A" << i;
4     else {
5         cout << "(";
6         PrintBrackets(s, i, s(i, j));
7         PrintBrackets(s, s(i, j)+1, j);
8         cout << ")";
9     }
10 }
```

---

Algorithm 10.13: Multiplication

---

```
1 MyMatrix Multiply(MyMatrix** sequence, MyMatrix &s, int i, int j) {
2     if (i == j)
3         return *(sequence[i]);
4     else
5         return Multiply(sequence, s, i, s(i, j))*
6             Multiply(sequence, s, s(i, j) + 1, j);
7 }
```

---

Algorithm 10.14: MatrixChainOrder

---

```
1 void MatrixChainOrder(int* p, int n, MyMatrix &m, MyMatrix &s) {
2     m = MyMatrix(n, n);
3     s = MyMatrix(n-1, n);
4     for (int i = 0; i < n; i++)
5         m(i, i) = 0; // base case
6     for (int l = 2; l <= n; l++) { // chain length
7         for (int i = 0; i < n - l + 1; i++) {
8             int j = i + l - 1;
9             m(i, j) = INT.MAX;
10            for (int k = i; k <= j - 1; k++) {
11                int q = m(i, k) + m(k+1, j) + p[i] * p[k+1] * p[j+1];
12                if (q < m(i, j)) {
13                    m(i, j) = q; // cost Ai*...*Aj
14                    s(i, j) = k;
15                }
16            }
17        }
18    }
19 }
```

---

Algorithm 10.15: Example 1.

---

```

1 void MatrixChainOrderMult(MyMatrix** sequence, int n) {
2     int* p = new int[n + 1];
3     p[0] = sequence[0]->GetRows();
4     for (int i = 1; i <= n; i++) {
5         p[i] = sequence[i-1]->GetCols();
6     }
7     for (int i = 0; i <= n; i++)
8         cout << p[i] << ", ";
9     cout << endl;
10    MyMatrix M, S;
11    MatrixChainOrder(p, n, M, S);
12    cout << "M:" << endl;
13    M.Display();
14    cout << "S:" << endl;
15    S.Display();
16    PrintBrackets(S, 0, n-1);
17    MyMatrix result=Multiply(sequence, S, 0, n - 1);
18    result.Display();
19    delete[] p;
20 }

```

---

# List of Algorithms and Examples

1.1	Expression values. . . . .	14
1.2	Includes. . . . .	15
1.3	Type conversion and promotion. . . . .	15
1.4	Static and dynamic arrays. . . . .	16
1.5	C++ examples. . . . .	17
1.6	Main test functions. . . . .	18
1.7	C++ functions. . . . .	18
1.8	C++ function calls. . . . .	19
1.9	C++ functions. . . . .	20
1.10	C++ function calls. . . . .	20
1.11	Variables declaration. . . . .	21
1.12	Sum and product. . . . .	22
1.13	C++ examples. . . . .	23
1.14	For loop with break. . . . .	24
1.15	While loop. . . . .	24
1.16	Do While loop. . . . .	25
1.17	Main. . . . .	25
1.18	Class definition. . . . .	26
1.19	Example. . . . .	26
1.20	Factorial functions - iterative version. . . . .	27
1.21	Factorial functions - recursive version. . . . .	27
1.22	Factorial examples. . . . .	28
1.23	Fibonacci function. . . . .	29
1.24	Fibonacci example. . . . .	29
1.25	Rotate. . . . .	30
1.26	Do anagram. . . . .	30
1.27	Anagram example. . . . .	30
1.28	Solve Hanoi Tower. . . . .	31
1.29	Hanoi tower example. . . . .	31
2.1	Stack - class definition . . . . .	32
2.2	Stack - functions . . . . .	33
2.3	Example . . . . .	34
2.4	Circular queue - class definition. . . . .	35
2.5	Display Queue. . . . .	35
2.6	Circular queue - functions. . . . .	36
2.7	Example . . . . .	37
3.1	Main program . . . . .	38
3.2	MyDataStructure.h . . . . .	39

3.3	MyDataStructure.cpp . . . . .	39
3.4	DataStructureFactory.h . . . . .	40
3.5	DataStructureFactory.cpp . . . . .	40
4.1	MyArray.h . . . . .	42
4.2	Constructors and destructor (MyArray.cpp) . . . . .	44
4.3	Access elements (MyArray.cpp). . . . .	45
4.4	Find, Delete, Insert. . . . .	46
4.5	Binary search. . . . .	47
4.6	Invert (MyArray.cpp). . . . .	48
4.7	Array of odd numbers (MyArray.cpp). . . . .	48
4.8	Search and return index (MyArray.cpp). . . . .	48
4.9	Maximum (MyArray.cpp). . . . .	49
4.10	Maximum and argmax (MyArray.cpp). . . . .	49
4.11	Minimum (MyArray.cpp). . . . .	50
4.12	Minimum and argmin (MyArray.cpp). . . . .	50
4.13	Basic statistic (MyArray.cpp). . . . .	51
4.14	FindMaxCrossingSubarray (MyArray.cpp). . . . .	52
4.15	Display (C++) (MyArray.cpp). . . . .	53
4.16	Print in a file (C++). . . . .	53
4.17	Print in a file (C) (MyArray.cpp). . . . .	53
4.18	Array initialization (MyArray.cpp). . . . .	53
4.19	Display steps (MyArray.cpp). . . . .	54
4.20	Swapping. . . . .	54
4.21	Is the array sorted?. . . . .	55
4.22	Selection sort (MyArray.cpp). . . . .	56
4.23	Insertion sort (MyArray.cpp). . . . .	56
4.24	Bubble sort (MyArray.cpp). . . . .	57
4.25	Merge (MyArray.cpp). . . . .	58
4.26	Mergesort (MyArray.cpp). . . . .	59
4.27	Quicksort (MyArray.cpp). . . . .	60
4.28	Example with Pairs. . . . .	61
4.29	Example with FindMaxSubArray. . . . .	61
4.30	Application of Quicksort. . . . .	61
4.31	Application of Binary Search. . . . .	62
4.32	Example of benchmarks. . . . .	63
5.1	Node simple chained list. . . . .	65
5.2	Class simple chained list (MySCList.h). . . . .	65
5.3	Class simple chained list (MySCList.cpp). . . . .	66
5.4	Reverse (MySCList.cpp). . . . .	66
5.5	Insert. . . . .	67
5.6	Insert (iterative version). . . . .	68
5.7	Insert (recursive version) - Procedure. . . . .	69
5.8	Insert (recursive version) - Function. . . . .	70
5.9	Insert. . . . .	71
5.10	Initialization. . . . .	71
5.11	Delete. . . . .	72
5.12	Delete in a sorted list. . . . .	73
5.13	Display the list. . . . .	74

5.14	Example simple chained list. . . . .	74
5.15	Node simple chained list. . . . .	75
5.16	Class interface. . . . .	75
5.17	Constructor and destructor. . . . .	76
5.18	Insert tail. . . . .	77
5.19	Insert head. . . . .	77
5.20	Insert at position. . . . .	77
5.21	Insert anywhere. . . . .	79
5.22	Insert anywhere. . . . .	80
5.23	Insert anywhere. . . . .	81
5.24	Delete head and tail. . . . .	82
5.25	Delete position. . . . .	82
5.26	Delete anywhere. . . . .	83
5.27	Delete anywhere. . . . .	84
5.28	Display details. . . . .	85
5.29	Display. . . . .	86
5.30	Example. . . . .	87
5.31	Iterator. . . . .	88
5.32	Example. . . . .	88
5.33	Prime numbers until n. . . . .	89
5.34	Circular list - Class definition. . . . .	90
5.35	Destructor. . . . .	90
5.36	Insert. . . . .	91
5.37	Delete. . . . .	92
5.38	Search, Display. . . . .	93
5.39	Update. . . . .	94
5.40	Skip list - Class definition. . . . .	95
5.41	Skip list - functions. . . . .	96
5.42	Insert and delete. . . . .	97
5.43	Search and display. . . . .	98
5.44	Example. . . . .	99
6.1	Hash Table - Class definition. . . . .	101
6.2	Hash Table - functions. . . . .	102
6.3	Hash Table - Insert. . . . .	103
6.4	Hash Table - Search. . . . .	104
6.5	Hash Table - SearchKey. . . . .	105
6.6	Hash Table - Display. . . . .	106
6.7	Example. . . . .	107
6.8	Distribution of the characters in a string. . . . .	107
7.1	TreeNode definition. . . . .	110
7.2	TreeNode functions. . . . .	110
7.3	Destroy the nodes in the tree. . . . .	111
7.4	Count the nodes in the tree. . . . .	111
7.5	Tree traversal: Pre-In-Post. . . . .	112
7.6	Tree traversal example. . . . .	112
7.7	Tree traversal: Pre-In-Post. . . . .	113
7.8	Tree traversal: level order. . . . .	114
7.9	Tree traversal: Nodes per level. . . . .	114

7.10	Invert Tree. . . . .	115
7.11	Search. . . . .	116
7.12	Insert (recursive version). . . . .	117
7.13	Insert (recursive version). . . . .	118
7.14	Insert (iterative version). . . . .	119
7.15	Insert (iterative version). . . . .	120
7.16	Delete. . . . .	121
7.17	Find minimum and maximum values. . . . .	122
7.18	Find minimum and maximum nodes. . . . .	122
7.19	MaxDepthTree. . . . .	123
7.20	MinDepthTree. . . . .	123
7.21	Same tree? . . . . .	124
7.22	Is it a BST? . . . . .	124
7.23	Is it a complete tree? . . . . .	125
7.24	Is it a full tree? . . . . .	125
7.25	Is it a perfect tree? . . . . .	125
7.26	BST interface. . . . .	126
7.27	BST example 1. . . . .	127
7.28	BST example 2. . . . .	127
7.29	AVL interface. . . . .	128
7.30	Constructor and destructor. . . . .	129
7.31	Rebalance. . . . .	129
7.32	Rotate Left. . . . .	130
7.33	Rotate Right. . . . .	131
7.34	Rotate Left then Right. . . . .	132
7.35	Rotate Right then Left. . . . .	132
7.36	Height. . . . .	132
7.37	Set Balance. . . . .	132
7.38	Print balance. . . . .	132
7.39	Display. . . . .	133
7.40	Search. . . . .	134
7.41	Insert. . . . .	134
7.42	Find minimum. . . . .	135
7.43	Delete. . . . .	135
7.44	Delete. . . . .	136
7.45	Class interface. . . . .	137
7.46	Methods. . . . .	138
7.47	Useful functions. . . . .	139
7.48	Order keys. . . . .	139
7.49	Search. . . . .	140
7.50	Insert. . . . .	141
7.51	Insert Upward. . . . .	143
7.52	Delete. . . . .	145
7.53	Fix delete. . . . .	147
7.54	Print nodes at a given level. . . . .	150
7.55	MaxDepthTree. . . . .	151
7.56	PrintLevelOrder. . . . .	152
7.57	Display. . . . .	152



7.58	Class interface. . . . .	154
7.59	Family business. . . . .	155
7.60	Destructor. . . . .	156
7.61	Transplant and TreeMinimum. . . . .	156
7.62	Rotations again. . . . .	157
7.63	Search. . . . .	158
7.64	Insert. . . . .	158
7.65	InsertFixup. . . . .	159
7.66	Delete. . . . .	160
7.67	DeleteFixup. . . . .	162
8.1	Class interface. . . . .	166
8.2	Heap constructor and destructor. . . . .	167
8.3	Heap methods. . . . .	167
8.4	Init heap or array. . . . .	168
8.5	Heapsort. . . . .	169
8.6	Heap insert, delete, display. . . . .	170
8.7	Heapsort example. . . . .	171
8.8	MyPriorityQueue class interface. . . . .	172
8.9	Constructors and destructor. . . . .	173
8.10	Methods. . . . .	173
8.11	Heap methods. . . . .	174
8.12	Push/Enqueue. . . . .	175
8.13	Pop/Dequeue. . . . .	176
8.14	Display. . . . .	176
8.15	Decrease key. . . . .	177
8.16	Node definition. . . . .	178
8.17	Class interface. . . . .	178
8.18	Constructor and destructor. . . . .	179
8.19	Accessors. . . . .	179
8.20	Insert. . . . .	180
8.21	Delete. . . . .	180
8.22	Union. . . . .	181
8.23	Extract Min. . . . .	182
8.24	Consolidate. . . . .	183
8.25	Link. . . . .	185
8.26	Decrease Key. . . . .	185
8.27	Cut. . . . .	186
8.28	CascadingCut. . . . .	186
8.29	Display. . . . .	187
9.1	Class interface. . . . .	194
9.2	Constructor and Destructor. . . . .	195
9.3	Exist and get edges. . . . .	196
9.4	Set edges. . . . .	197
9.5	Remove edges. . . . .	198
9.6	Comparisons. . . . .	199
9.7	Display. . . . .	199
9.8	Iterator functions. . . . .	200
9.9	Color of Nodes. . . . .	201

9.10	BFS main function.	201
9.11	Breadth First Search Display.	202
9.12	DFS functions.	203
9.13	Depth First Search Display.	204
9.14	Example.	204
9.15	Display path information.	205
9.16	Bellman Ford.	206
9.17	Bellman Ford example.	207
9.18	Dijkstra.	208
9.19	Dijkstra example.	209
9.20	Class interface.	210
9.21	Constructors and Destructor.	211
9.22	Get, Exist, and Set edges.	212
9.23	Comparisons.	213
9.24	Display.	214
9.25	Adjacent iterator.	215
9.26	Example.	216
9.27	Default Fibonacci.	218
9.28	Fibonacci with DP.	218
9.29	Comparisons.	219
9.30	Print all pairs.	220
9.31	Floyd Warshall.	221
9.32	Transitive closure.	222
9.33	Floyd Warshall Example.	223
9.34	minKey (Prim).	224
9.35	Display MST.	224
9.36	Acyclic.	225
9.37	IsAcyclic.	225
9.38	Kruskal.	226
9.39	Prim.	227
9.40	Example MST.	228
10.1	Class interface.	230
10.2	Constructors and destructor	231
10.3	Accessors and modifiers	232
10.4	Display	232
10.5	Initialization	233
10.6	Identity	233
10.7	Assignment	233
10.8	Addition	234
10.9	Subtraction	235
10.10	Multiplication	236
10.11	GetMin	237
10.12	Print brackets	238
10.13	Multiplication	238
10.14	MatrixChainOrder	238
10.15	Example 1.	239

# Bibliography

- [1] A. V. Aho, J. E. Hopcroft, and J. D. Ullman, “The design and analysis of computer algorithms.” Addison-Wesley, 1974, pp. 145–147.
- [2] R. Bayer, “Symmetric binary b-trees: Data structure and maintenance algorithms,” *Acta Informatica*, vol. 1, no. 4, pp. 290–306, 1972.
- [3] D. A. Bell, “The principles of sorting,” *The Computer Journal*, vol. 1, pp. 71–77, 1958.
- [4] L. Bonacci, *The Book of Calculation*, 1202.
- [5] A. D. Booth and A. J. T. Colin, “On the efficiency of a new method of dictionary construction,” *Information and control*, vol. 3, pp. 327–334, 1960.
- [6] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms - 3rd Edition*. MIT Press and McGraw-Hill, 2009.
- [7] A. S. Douglas, “Techniques for the recording of, and reference to data in a computer,” *The Computer Journal*, vol. 2, pp. 1–9, 1959.
- [8] E. W. Dijkstra, “A note on two problems in connection with graphs,” *Numerische Mathematik*, vol. 1, pp. 269–271, 1959.
- [9] M. L. Fredman and R. E. Tarjan, “Fibonacci heaps and their uses in improved network optimization algorithms,” *Journal of the Association for Computing Machinery*, vol. 34, no. 3, pp. 596–615, 1987.
- [10] L. J. Guibas and R. Sedgewick, “A dichromatic framework for balanced trees,” in *Proc. of the 19th Annual Symposium on Foundations of Computer Science*, 1978, pp. 8–21.
- [11] T. Hibbard, “Some combinatorial properties of certain trees,” *Assoc. Comp. Mach.*, vol. 9, p. 13, 1962.
- [12] J. B. Kruskal, “On the shortest spanning subtree of a graph and the traveling salesman problem,” in *Proc. of the American Mathematical Society*, vol. 7, 1956, pp. 48–50.
- [13] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*. Teubner, 1909, 2 volumes. Reprinted by Chelsea, New York, 1953.
- [14] R. C. Prim, “Shortest connection networks and some generalizations,” *Bell System Technical Journal*, vol. 36, no. 6, pp. 1389–1401, 1957.
- [15] B. Stroustrup, *The C++ Programming Language, 4th Edition*. Addison-Wesley Professional, Aug. 2013.
- [16] J. Vuillemin, “A data structure for manipulating priority queues,” *Communications of the ACM*, vol. 21, no. 4, pp. 309–315, 1978.

- [17] P. F. Windley, "Trees, forests, and rearranging," *The Computer Journal*, vol. 3, p. 84, 1960.