Computer Graphics

Lecture 03

Linear Algebra

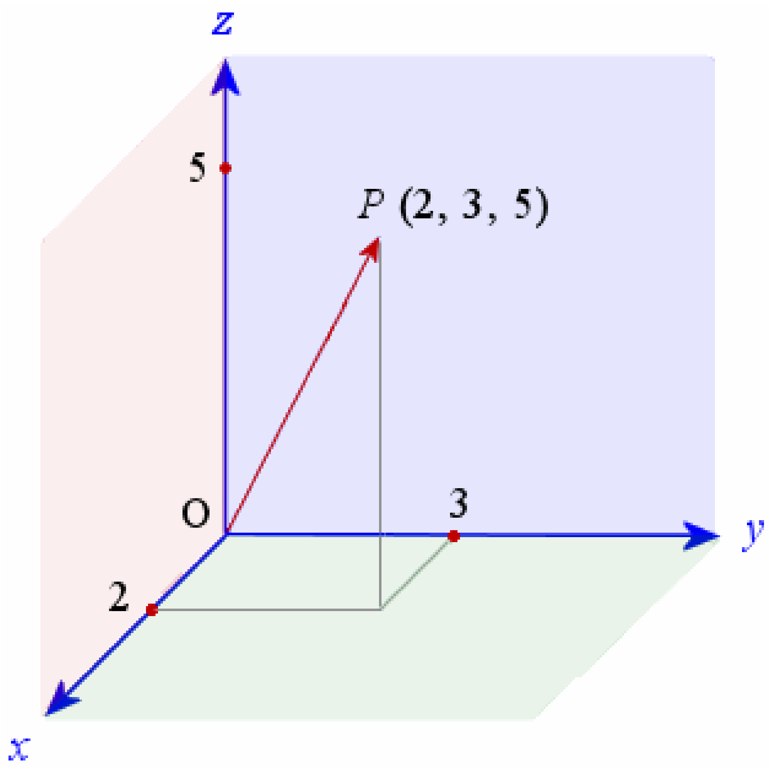
Euclidean Space

The n-dimensional real Euclidean space is denoted $\mathbb{R}^n$. A vector ***v*** in this space is an n-tuple, that is, an ordered list of real numbers:

$$\mathbf{v} \in \mathbb{R}^n \iff \mathbf{v} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \text{ with } v_i \in \mathbb{R},\ i = 0, \ldots, n-1.$$

Vector

Ex: The vector OP has initial point at the origin O (0, 0, 0) and terminal point at P (2, 3, 5). We can draw the vector OP as follows:



$\mathbf{v} = (v_0, v_1, v_2)^T \in R^3$

Vector Operations

For vectors in a Euclidean space there exist two operators, *addition* and *multiplication by a scalar*

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} u_0 + v_0 \\ u_1 + v_1 \\ \vdots \\ u_{n-1} + v_{n-1} \end{pmatrix} \in \mathbb{R}^n$$

(addition)

$$a\mathbf{u} = \begin{pmatrix} au_0 \\ au_1 \\ \vdots \\ au_{n-1} \end{pmatrix} \in \mathbb{R}^n$$

(multiplication by a scalar)

The "$\in R^n$" simply means that addition and multiplication by 'a' scalar yields vectors of the same space.

Euclidean space

Addition of vectors in a Euclidean

- ❖ $(u + v) + w = u + (v + w)$ (associativity)
- ❖ $u + v = v + u$ (commutativity)

In case of zero vector where $0 = (0,0,.....0)$ with n zeros

- ❖ $0 + v = v$ (zero identity)

In case of –v vector where $-\mathbf{v} = (-v_0, -v_1, \ldots, -v_{n-1})$

- ❖ $v + (-v) = 0$ (additive inverse)

Euclidean space

Vector multiplication by a scalar in a Euclidean

- $(ab)\mathrm{u} = a(b\mathrm{u})$
- $(a + b)\mathrm{u} = a\mathrm{u} + b\mathrm{u}$ (distributive law)
- $a(\mathrm{u} + \mathrm{v}) = a\mathrm{u} + a\mathrm{v}$ (distributive law)
- $1\mathrm{u} = \mathrm{u}$

Trigonometry

In an Euclidean space where **p** = (*px, py*) is a unit vector, such that ||**p**|| = 1,

the fundamental trigonometric functions, sin, cos, and tan, are defined by

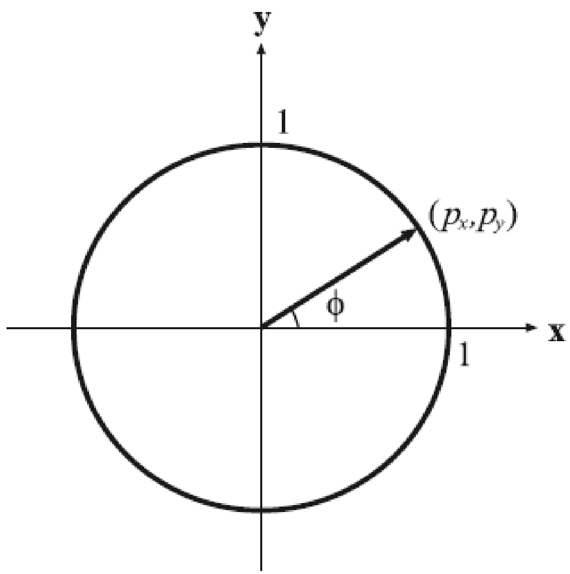
Fundamental trigonometric functions :

$$\sin \phi = p_y$$

$$\cos \phi = p_x$$

$$\tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{p_y}{p_x}$$

Trigonometry Fundamentals



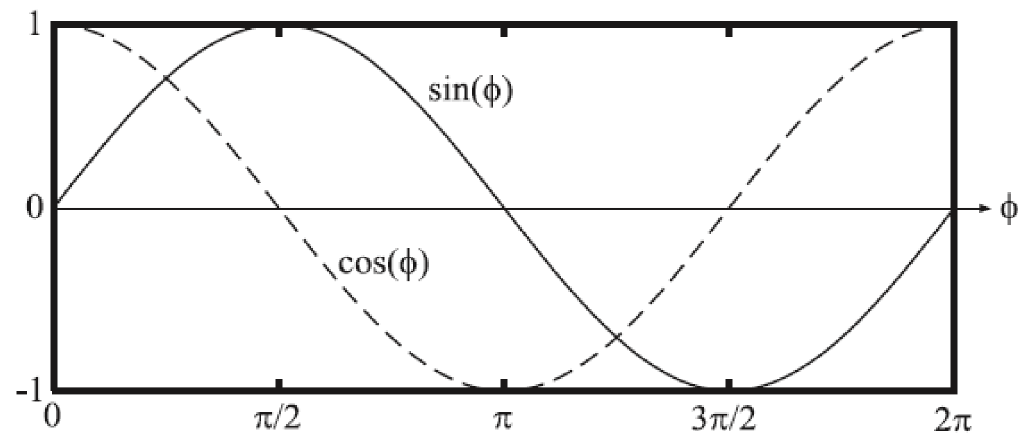


Figure B.1. The geometry for the definition of the sin, cos, and tan functions is shown to the left. The right-hand part of the figure shows $p_x = \cos\phi$ and $p_y = \sin\phi$, which together traces out the circle.

MacLaurin Series

The sin, cos, and tan functions can be expanded into MacLaurin series

MacLaurin series :

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots + (-1)^n \frac{\phi^{2n+1}}{(2n+1)!} + \ldots \qquad \text{hold for } -\infty < \phi < \infty$$

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \cdots + (-1)^n \frac{\phi^{2n}}{(2n)!} + \ldots \qquad \text{hold for } -\infty < \phi < \infty$$

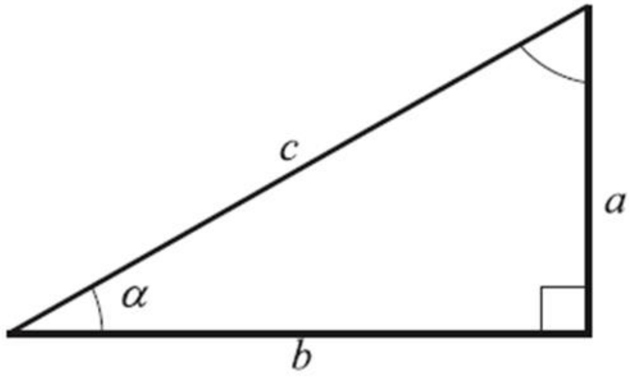
$$\tan\phi = \phi + \frac{\phi^3}{3} + \frac{2\phi^5}{15} + \cdots + (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n}\phi^{2n-1} + \ldots \qquad \text{hold for } -\pi/2 < \varphi < \pi/2$$

Where B_n is the n^{th} Bernoulli number that can be generated with a recursive formula,

where $B_0 = 1$ and then for $k > 1$, $\sum_{j=0}^{k-1} \binom{k}{j} B_j = 0$

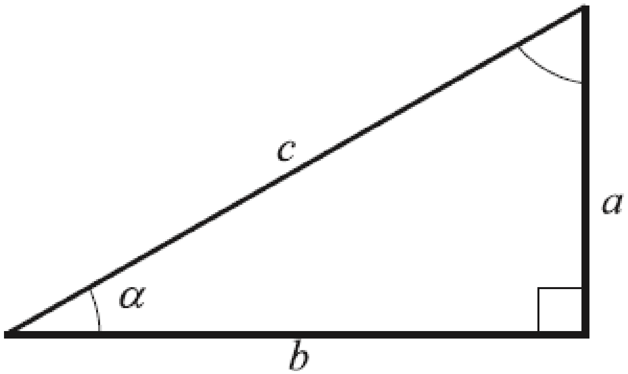
Pythagorean relation

Trigonometric identity $$\cos^2\phi + \sin^2\phi = 1$$



$$c^2 = a^2 + b^2$$

Trigonometry Revised



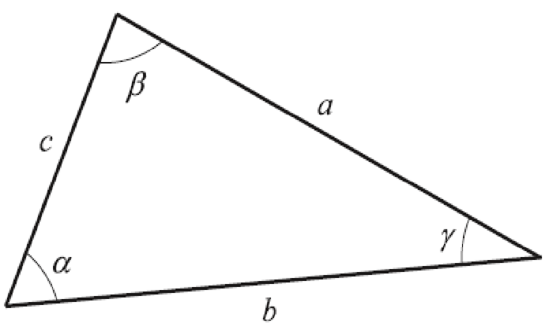
Right triangle laws :

$$\sin\alpha = \frac{a}{c}$$

$$\cos\alpha = \frac{b}{c}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{a}{b}$$

well-known rules are



An arbitrarily angled triangle and its notation.

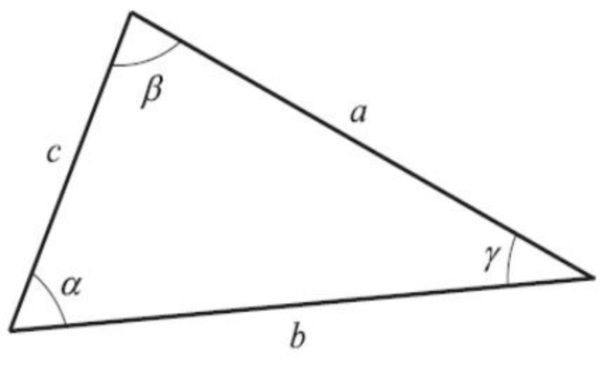
Law of sines : $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$

Law of cosines : $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Law of tangents : $\dfrac{a+b}{a-b} = \dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$

Arbitrarily Angled Triangles

Named after their inventors, the following two formulas are also valid for arbitrarily angled triangles.



An arbitrarily angled triangle and its notation.

$$\text{Newton's formula}: \quad \frac{b+c}{a} = \frac{\cos\frac{\beta-\gamma}{2}}{\sin\frac{\alpha}{2}}$$

$$\text{Mollweide's formula}: \quad \frac{b-c}{a} = \frac{\sin\frac{\beta-\gamma}{2}}{\cos\frac{\alpha}{2}}$$

Angle Relations

Angle sum relations

$$\sin(\phi+\rho) = \sin\phi\cos\rho + \cos\phi\sin\rho$$

$$\cos(\phi+\rho) = \cos\phi\cos\rho - \sin\phi\sin\rho$$

$$\tan(\phi+\rho) = \frac{\tan\phi + \tan\rho}{1-\tan\phi\tan\rho}$$

Angle difference relations

$$\sin(\phi-\rho) = \sin\phi\cos\rho - \cos\phi\sin\rho$$

$$\cos(\phi-\rho) = \cos\phi\cos\rho + \sin\phi\sin\rho$$

$$\tan(\phi-\rho) = \frac{\tan\phi - \tan\rho}{1+\tan\phi\tan\rho}$$

Angle Relations

Double angle relations

$$\sin 2\phi = 2\sin\phi\cos\phi = \frac{2\tan\phi}{1+\tan^2\phi}$$

$$\cos 2\phi = \cos^2\phi - \sin^2\phi = 1 - 2\sin^2\phi = 2\cos^2\phi - 1 = \frac{1-\tan^2\phi}{1+\tan^2\phi}$$

$$\tan 2\phi = \frac{2\tan\phi}{1-\tan^2\phi}$$

Angle Relations

Multiple angle relations

$$\sin(n\phi) = 2\sin((n-1)\phi)\cos\phi - \sin((n-2)\phi)$$

$$\cos(n\phi) = 2\cos((n-1)\phi)\cos\phi - \cos((n-2)\phi)$$

$$\tan(n\phi) = \frac{\tan((n-1)\phi) + \tan\phi}{1 - \tan((n-1)\phi)\tan\phi}$$

Angle Relations

Half-angle relations

$$\sin\frac{\phi}{2} = \pm\sqrt{\frac{1-\cos\phi}{2}}$$

$$\cos\frac{\phi}{2} = \pm\sqrt{\frac{1+\cos\phi}{2}}$$

$$\tan\frac{\phi}{2} = \pm\sqrt{\frac{1-\cos\phi}{1+\cos\phi}} = \frac{1-\cos\phi}{\sin\phi} = \frac{\sin\phi}{1+\cos\phi}$$

Angle Relations

Function sums and differences

$$\sin\phi + \sin\rho = 2\sin\frac{\phi+\rho}{2}\cos\frac{\phi-\rho}{2}$$

$$\cos\phi + \cos\rho = 2\cos\frac{\phi+\rho}{2}\cos\frac{\phi-\rho}{2}$$

$$\tan\phi + \tan\rho = \frac{\sin(\phi+\rho)}{\cos\phi\cos\rho}$$

$$\sin\phi - \sin\rho = 2\cos\frac{\phi+\rho}{2}\sin\frac{\phi-\rho}{2}$$

$$\cos\phi - \cos\rho = -2\sin\frac{\phi+\rho}{2}\sin\frac{\phi-\rho}{2}$$

$$\tan\phi - \tan\rho = \frac{\sin(\phi-\rho)}{\cos\phi\cos\rho}$$