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Stability analysis of an electromagnetically levitated sphere

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We present a combined numerical and analytical approach to analyze the static and dynamic stabilities of an electromagnetically levitated spherical body depending on the ac frequency and the configuration of a three-dimensional (3D) coil made of thin winding which is modeled by linear current filaments. First, we calculate numerically the magnetic vector potential in grid points on the surface of the sphere and then use Legendre and fast Fourier transforms to find the expansion of the magnetic field in terms of spherical harmonics. Second, we employ a previously developed gauge transformation to solve analytically the 3D electromagnetic problem in terms of the numerically obtained expansion coefficients. Using this solution, we obtain the electromagnetic reaction force due to both a small displacement of the sphere from its equilibrium position and its velocity of motion which are defined by symmetric stiffness and damping matrices, respectively. Eigenvalues and eigenvectors of the stiffness matrix yield three principal stiffness coefficients, which all have to be positive for the equilibrium state to be statically stable, and three mutually orthogonal directions of principal oscillations. Dynamic instabilities are characterized by critical ac frequencies which, when exceeded, may result either in a spin up or oscillations with increasing amplitude. The effective electromagnetic damping coefficients are found by using a classical eigenvalue perturbation theory. A theoretical approach based on the vector field transformation by a small rotation in combination with a parametric frequency derivative is introduced to find the electromagnetic reaction torque due to a slow rotation of the sphere in a 3D ac magnetic field.
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I. INTRODUCTION

Electromagnetic levitation melting (ELM) is a well-known technique for the containerless processing of molten metals and their alloys.[1] The operational principle of ELM is the following. A charge of metal, which may weigh from few grams up to several kilograms, is placed in a coil fed by an ac current with a typical frequency ranging from about 10 kHz up to several hundreds of kilohertz. The eddy currents induced by the ac magnetic field cause two effects. On one hand, the eddy currents interact with those in the coil giving rise to a Lorentz force which repels the body from the coil. On the other hand, the eddy currents cause Ohmic dissipation which heats the metal. In such a way, the metal can be levitated and also melted, provided that the coil configuration and the current in it are properly chosen. ELM is particularly useful for the processing of high melting point metals which are often aggressive and react with the crucible material thus getting polluted by the latter.[1] ELM avoids the contamination of the melt and allows one to carry out solidification from deeply undercooled states which is of interest for certain material science applications.[2] Besides, ELM is widely used for measurements of thermophysical properties of molten metals.[3,4]

The equilibrium of electromagnetic force and gravity is necessary but not sufficient for levitation. In addition, the equilibrium state has to be stable at least to sufficiently small perturbations. First, the reaction force due to a displacement of the body from its equilibrium position has to act against the displacement to restore the equilibrium. Otherwise, the equilibrium will be statically unstable. The static stability alone may also be insufficient because sometimes the equilibrium is overstable. It means that there is a restoring force that causes the levitated body to oscillate with increasing amplitude which eventually results in the failure of levitation.[1] Similarly, sometimes the levitated body is observed to spin up.[1,5,6]

The static stability of a metal sphere with respect to axial disturbances in an axisymmetric coil has been analyzed by Holmes.[7] Clemente[8] and Clemente and Tessarotto[9] showed that under terrestrial conditions the dissipated power and the temperature of a levitated body may attain a minimum at a certain current amplitude in the coil. Giovachini used a Green's function technique to find the electromagnetic force on moving bodies depending on their velocity and acceleration.[10] In our previous work, we have developed a purely electromagnetic theory trying to account for dynamic instabilities of electromagnetically levitated bodies in ac magnetic fields of simple configurations.[11,12] Recently, we applied this theory to analyze the static and the dynamic stability of electromagnetically levitated spherical bodies in axisymmetric magnetic fields generated by a set of coaxial circular current loops.[13] In this study, we extend our previous

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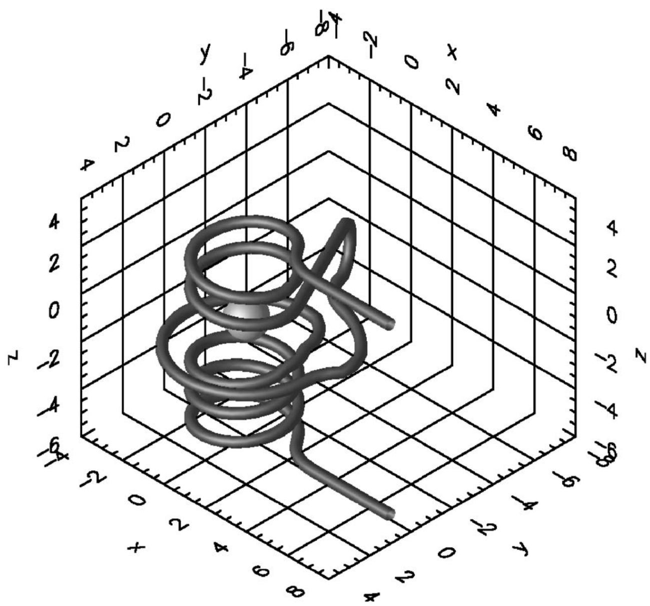

FIG. 1. Diagonal view of a model coil with sphere.

analysis to general three-dimensional (3D) ac magnetic fields generated by coils made of thin windings. In addition, we work out a theoretical approach to analyze the spin-up instability of a sphere in such 3D magnetic fields. The electromagnetic problem is solved by the combination of numerical and analytical methods. First, we find numerically the expansion of the vector potential of the external magnetic field generated by the coil in terms of spherical harmonics, and then we use analytical techniques similar to that of Lohoefer.[14,15] We employ the previously introduced gauge transformation that allows us to obtain a relatively simple 3D analytical solution for a sphere in terms of the vector potential only.[13]

The paper is organized as follows. The problem is formulated in Sec. II. The numerical expansion of the magnetic field of the coil is described in Sec. III. Section IV presents analytical solutions for both the base state and a small displacement which are used to further analyze the stability of small-amplitude oscillations. Spin-up instability is considered in Sec. V, where the governing equations and an analytical solution for a slowly rotating sphere are presented. Numerical results for a model coil are given in Sec. VI. The paper is concluded with a summary in Sec. VII.

II. PROBLEM FORMULATION

Consider a sphere of radius R and conductivity σ moving at velocity $\boldsymbol{v}$ in the magnetic field $\boldsymbol{B}$ created by a coil made of a thin wire forming a closed contour L and supplied by an ac current with circular frequency ω and amplitude I_0, as shown in Fig. 1. The induced electric field follows from the first Maxwell equation as $\boldsymbol{E}=-\boldsymbol{\nabla}\Phi-\partial_t\boldsymbol{A}$, where Φ is the scalar electric potential and $\boldsymbol{A}$ is the magnetic vector potential. The density of the electric current induced in a moving medium is given by Ohm's law,

$$\boldsymbol{j}=\sigma(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})=\sigma(-\boldsymbol{\nabla}\Phi-\partial_t\boldsymbol{A}+\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{A}).$$

The electric field induced by either translational or rotational solid body motion can be represented as

$$\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{A}=\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A})-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}-\boldsymbol{A}\times\boldsymbol{\Omega},$$

where $\boldsymbol{v}$ is a spatially invariant vector for translational motion or $\boldsymbol{v}=\boldsymbol{\Omega}\times\boldsymbol{r}$ for a solid-body rotation with angular velocity $\boldsymbol{\Omega}$ which is zero in the first case. Assuming that the frequency of the ac magnetic field is sufficiently low to neglect the displacement current, the second Maxwell equation leads to

$$\partial_t\boldsymbol{A}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}+\boldsymbol{A}\times\boldsymbol{\Omega}=\frac{1}{\mu_0\sigma}\boldsymbol{\nabla}^2\boldsymbol{A}, \tag{1}$$

where the gauge invariance of $\boldsymbol{A}$ has been employed to define the scalar potential as

$$\Phi=\boldsymbol{v}\cdot\boldsymbol{A}-\frac{1}{\mu_0\sigma}\boldsymbol{\nabla}\cdot\boldsymbol{A}, \tag{2}$$

which reduces to the Coulomb gauge $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$ in free space with $\sigma=0$. Boundary conditions at the surface follow from the continuity of the magnetic field,

$$[\boldsymbol{A}]_S=[\partial_n\boldsymbol{A}]_S=0, \tag{3}$$

where $[f]_S$ denotes a jump of the quantity f across the boundary S; $\partial_n\equiv(\boldsymbol{n}\cdot\boldsymbol{\nabla})$ is the derivative normal to the boundary.

In order to simplify the following analysis by reducing the number of independent parameters, we nondimensionalize all quantities by using the magnetic diffusion time $\tau_m=\mu_0\sigma R^2$ and the radius of the sphere R as the time and length scales. The current density and the vector potential are scaled by I_0/R^2 and $\mu_0 I_0$, respectively. For the sake of brevity, we keep in the following for the dimensionless quantities the same notation as for the dimensional ones.

III. EXPANSION OF THE EXTERNAL MAGNETIC FIELD IN TERMS OF SPHERICAL HARMONICS

The electromagnetic problem formulated above can be solved analytically provided that the external magnetic field is given in terms of spherical harmonics. Although such a representation can be obtained analytically for simple inductors, such as a set of circular loops,[13] in general, a numerical approach is called for. For the coil made of a thin current filament specified by a contour L, the nondimensionalized vector potential of the external magnetic field follows from the Biot-Savart law,

$$\boldsymbol{A}_0^e(\boldsymbol{R})=\frac{1}{4\pi}\int_L\frac{d\boldsymbol{R}'}{|\boldsymbol{R}-\boldsymbol{R}'|}. \tag{4}$$

Note that for the vector potential to satisfy the Coulomb gauge, L has to be closed. This solution can be expanded in spherical harmonics about the center of the sphere located at $\boldsymbol{R}_0$ as

$$\boldsymbol{A}_0^e(\boldsymbol{r}) = \frac{1}{4\pi}\int_L d\boldsymbol{R}'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{|\boldsymbol{R}_0-\boldsymbol{R}'|^{l+1}} \times Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\boldsymbol{I}_l^m\bar{X}_l^m(\boldsymbol{r}), \quad (5)$$

where $r=|\boldsymbol{r}|=|\boldsymbol{R}-\boldsymbol{R}_0|$ is the spherical radius and $Y_{lm}(\theta,\phi) = \sqrt{[(2l+1)/4\pi][(l-m)!/(l+m)!]}P_l^m(\cos\theta)e^{im\phi}$ is spherical harmonic[16] defined using the associated Legendre functions of the first kind $P_l^m(x)$;[17] the asterisk denotes the complex conjugate. Note that this expansion describes the external magnetic field in the spherical volume which is not crossed by the coil. Although the expansion coefficients are formally defined by the contour integrals

$$\boldsymbol{I}_l^m = \frac{1}{2l+1}\int_L X_l^{m*}(\boldsymbol{R}_0-\boldsymbol{R}')d\boldsymbol{R}',$$

where $X_l^m(\boldsymbol{r})=r^{-l-1}Y_{lm}(\theta,\phi)$ and $\bar{X}_l^m(\boldsymbol{r})=r^lY_{lm}(\theta,\phi)$ are the outer and the inner solutions of the Laplace equation associated to the spherical harmonic $Y_{lm}(\theta,\phi)$;[13] they can be calculated more efficiently in the following way. To find the expansion coefficients up to the degree N, we first approximate the coil by linear filaments and then use the well-known analytical formula for the vector potential of each such element to evaluate numerically (4) on the surface of the sphere at the grid points with $N+1$ Gaussian quadrature nodes in $x_j^N=\cos\theta_j^N$, where x_j^N is the jth root of the Legendre polynomial $P_N^0(x)=0$,[17] and $2N+2$ equispaced points in ϕ,

$$(\theta_j,\phi_k) = \left[\arccos(x_j^{N+1}),\ \frac{\pi k}{N+1};\quad j=0,\ldots,N;\ k=0,\ldots,2N+1\right].$$

We apply subsequently the fast Fourier transform (FFT) along the azimuthal angle,

$$\boldsymbol{I}^m(\theta_j) = \int_0^{2\pi}\boldsymbol{A}_0^e(\theta_j,\phi)e^{-im\phi}d\phi = \sum_{k=0}^{2N+1}\boldsymbol{A}_0^e(\theta_j,\phi_k)e^{-im\phi_k}, \quad m=0,\ldots,2N+1,$$

and the Legendre transform by using Gaussian quadrature along the poloidal angle that results in

$$\boldsymbol{I}_l^m = (-1)^m\sqrt{2\pi}\int_0^{\pi}\boldsymbol{I}^m(\theta)\bar{P}_l^m[\cos(\theta)]\sin(\theta)d\theta = (-1)^m\sqrt{2\pi}\sum_{j=0}^{N}\boldsymbol{I}^m(\theta_j)\bar{P}_l^m[\cos(\theta_j)]\sin(\theta_j)w_j^{N+1},$$

where $\bar{P}_l^m(x)$ are the normalized Legendre functions related to the spherical harmonics as $Y_{lm}(\theta,\phi)=(-1)^m/\sqrt{2\pi}\bar{P}_l^m(\cos\theta)e^{im\phi}$; w_j^{N+1} are the weights of Gaussian quadrature.[17] Due to the properties of the spherical harmonics we have $\boldsymbol{I}_l^{-m}=(-1)^m\boldsymbol{I}_l^{m*}$. Note that this expansion procedure has to be repeated whenever the position of the sphere changes.

IV. ANALYTICAL SOLUTION

A. Base state

For a sphere at rest, the vector potential is sought as $\boldsymbol{A}(\boldsymbol{r},t)=\Re[\boldsymbol{A}_0(\boldsymbol{r})e^{i\bar{\omega}t}]$, where $\boldsymbol{A}_0$ is a complex amplitude and $\bar{\omega}=\omega\mu_0\sigma R^2$ is the dimensionless ac frequency which will be the main parameter of this study. For the interior of the sphere, Eq. (1) reads in the dimensionless form as

$$\nabla^2\boldsymbol{A}_0 = i\bar{\omega}\boldsymbol{A}_0, \quad (6)$$

while for the exterior excluding the coil we have

$$\nabla^2\boldsymbol{A}_0 = 0. \quad (7)$$

Note that this equation does not contain the current density in the coil as a source term because we consider only the exterior region which is not crossed by the coil. This is sufficient for our purposes because the magnetic field generated by the coil in this region is already given by Eq. (5). For the interior, the solution is

$$\boldsymbol{A}_0(\boldsymbol{r}) = \sum_{l=0}^{\infty}\bar{g}_l j_l(r\sqrt{\bar{\omega}/i})\sum_{m=-l}^{l}\boldsymbol{I}_l^mY_{lm}(\theta,\phi),$$

where $\bar{g}_l$ is an unknown coefficient to be determined from the boundary conditions (3); $j_l(x)$ is the spherical Bessel function of order l.[17] For the exterior, the solution is represented as a superposition of external and induced fields $\boldsymbol{A}_0(\boldsymbol{r})=\boldsymbol{A}_0^e(\boldsymbol{r})+\boldsymbol{A}_0^i(\boldsymbol{r})$, where the latter is sought as

$$\boldsymbol{A}_0^i(\boldsymbol{r}) = \sum_{l=1}^{\infty}g_l\sum_{m=-l}^{l}\boldsymbol{I}_l^mX_l^m(\boldsymbol{r}). \quad (8)$$

From the boundary condition (3) applied at $r=1$, we find

$$\bar{g}_l = \frac{2l+1}{\sqrt{\bar{\omega}/i}}\frac{1}{j_{l-1}(\sqrt{\bar{\omega}/i})}, \quad g_l = \frac{j_{l+1}(\sqrt{\bar{\omega}/i})}{j_{l-1}(\sqrt{\bar{\omega}/i})}. \quad (9)$$

Note that for a nonaxisymmetric external magnetic field, the solution obtained in this way does not in general satisfy the Coulomb gauge for the induced field outside the sphere. This can be corrected by adding a gradient of a harmonic gauge potential $\Lambda_0^e(\boldsymbol{r})$ defined as

$$\Lambda_0^e(\boldsymbol{r}) = \sum_{l=1}^{\infty}\frac{2l-1}{2l+1}\sum_{m=-l}^{l}\lambda_l^mX_l^m(\boldsymbol{r})$$

to the vector potential of the external magnetic field.[13] Since this gauge transformation results only in the replacement of the original expansion coefficients $\boldsymbol{I}_l^m$ by

$$\mathcal{I}_l^m = \boldsymbol{I}_l^m + \left(\frac{\boldsymbol{e}_\eta^*}{\sqrt{2}}N_{l+1}^{m-1}\lambda_{l+1}^{m-1} - \frac{\boldsymbol{e}_\eta}{\sqrt{2}}N_{l+1}^{m+1}\lambda_{l+1}^{m+1} + \boldsymbol{e}_zN_{l+1}^m\lambda_{l+1}^m\right)\Big/ N_l^m, \quad (10)$$

where $\boldsymbol{e}_\eta=(\boldsymbol{e}_x+i\boldsymbol{e}_y)/\sqrt{2}$ is a complex unity vector and $N_l^m = \sqrt{(l-m)!(l+m)!/(2l+1)}$; the transformed solution keeps satisfying the boundary conditions (3) which are ensured by Eq. (9) independent of this transformation. Then the Coulomb gauge applied to the induced field outside the sphere leads to

$$\frac{\boldsymbol{e}_\eta^* \cdot \boldsymbol{\mathcal{I}}_l^{m-1}}{\sqrt{2}N_l^{m-1}} - \frac{\boldsymbol{e}_\eta \cdot \boldsymbol{\mathcal{I}}_l^{m+1}}{\sqrt{2}N_l^{m+1}} - \frac{\boldsymbol{e}_z \cdot \boldsymbol{\mathcal{I}}_l^m}{N_l^m} = 0,$$

which defines the coefficients of the gauge potential as

$$\lambda_{l+1}^m = \frac{(l+1)^2 - m^2}{(l+1)(2l+1)} \frac{N_l^m}{N_{l+1}^m} \times \left(\frac{N_l^m}{N_l^{m-1}} \frac{\boldsymbol{e}_\eta^* \cdot \boldsymbol{I}_l^{m-1}}{\sqrt{2}} - \frac{N_l^m}{N_l^{m+1}} \frac{\boldsymbol{e}_\eta \cdot \boldsymbol{I}_l^{m+1}}{\sqrt{2}} - \boldsymbol{e}_z \cdot \boldsymbol{I}_l^m \right). \quad (11)$$

The transformed vector potential $\boldsymbol{\mathcal{A}}_0^e$ can be associated with a current distribution on the spherical surface passing between the sphere and the coil at some radius R_0,

$$\boldsymbol{j}_0^e(\boldsymbol{r}) = \delta(r - R_0) \sum_{l=0}^{\infty} (2l+1) R_0^{l-1} \sum_{m=-l}^{l} \boldsymbol{\mathcal{I}}_l^m Y_{lm}(\theta,\phi) = -\boldsymbol{\nabla}^2 \boldsymbol{\mathcal{A}}_0^e, \quad (12)$$

where $\delta(x)$ is the Dirac delta function, and the solution for $\boldsymbol{\mathcal{A}}_0^e$ at $r > R_0$ is analogous to (8). Such a surface current, in turn, is associated with a jump of the induction $[\boldsymbol{B}_0^e]_S$ across the surface S as $\boldsymbol{j}_0^e(\boldsymbol{r}) = \boldsymbol{n} \times [\boldsymbol{B}_0^e]_S$, where $\boldsymbol{n}$ is the surface normal. Since $\boldsymbol{n} = \boldsymbol{e}_r$ for a spherical surface, we have $(\boldsymbol{e}_r \cdot \boldsymbol{j}_0^e) = 0$. This condition is ensured by the gauge transformation (10) which, thus, replaces the original 3D current distribution by an equivalent spherical surface current which both generate the same magnetic field in the region occupied by the sphere.[13]

The associated external current distribution (12) can be used to find conveniently the time-averaged total force on the sphere,

$$\boldsymbol{F}_0 = \int_V \langle \boldsymbol{j} \times \boldsymbol{B} \rangle dV = \frac{1}{2} \int_V \Re[\boldsymbol{j}_0 \times \boldsymbol{B}_0^*] dV,$$

where the integral is taken over the volume of the sphere V. Taking into account that $\boldsymbol{j}_0 = \boldsymbol{\nabla} \times \boldsymbol{B}_0^i$ and $\boldsymbol{B}_0 = \boldsymbol{B}_0^e + \boldsymbol{B}_0^i$, where $\boldsymbol{B}_0^e$ is a purely real quantity, we obtain $\boldsymbol{F}_0 = \frac{1}{2}\int_V \Re[\boldsymbol{j}_0] \times \boldsymbol{B}_0^e dV$. Eventually, the momentum conservation law results in

$$\boldsymbol{F}_0 = -\frac{1}{2} \int_{\bar{V}} \boldsymbol{j}_0^e \times \Re[\boldsymbol{B}_0^i] dV,$$

where the integral is taken over the space outside the sphere $\bar{V}$. The last integral can be evaluated straightforwardly because (12) is defined in terms of the Dirac δ function. Taking into account that the induced magnetic field outside the sphere is

$$\boldsymbol{B}_0^i = \sum_{l=1}^{\infty} g_{l-1} \sum_{m=-l}^{l} \left(\frac{\boldsymbol{e}_\eta^* \times \boldsymbol{\mathcal{I}}_l^{m-1}}{\sqrt{2}N_{l-1}^{m-1}} - \frac{\boldsymbol{e}_\eta \times \boldsymbol{\mathcal{I}}_l^{m+1}}{\sqrt{2}N_{l-1}^{m+1}} - \frac{\boldsymbol{e}_z \times \boldsymbol{\mathcal{I}}_l^m}{N_{l-1}^m} \right) \times N_l^m X_l^m,$$

after some algebra we obtain

$$\boldsymbol{F}_0 = \sum_{l=1}^{\infty} \sqrt{4l^2-1} \Re[g_{l-1}] \sum_{m=0}^{l} \Re[\boldsymbol{e}_\eta (\boldsymbol{\mathcal{I}}_l^{m*} \cdot \boldsymbol{\mathcal{I}}_{l-1}^{m+1}) \times \sqrt{(l-m-1)(l-m)/2} - \boldsymbol{e}_\eta^* (\boldsymbol{\mathcal{I}}_l^{m*} \cdot \boldsymbol{\mathcal{I}}_{l-1}^{m-1}) \times \sqrt{(l+m-1)(l+m)/2} + \boldsymbol{e}_z (\boldsymbol{\mathcal{I}}_l^{m*} \cdot \boldsymbol{\mathcal{I}}_{l-1}^{m}) \sqrt{l^2 - m^2}]/c_m, \quad (13)$$

where $c_m = 1$ and $c_m = 2$ for $m \neq 0$ and $m = 0$, respectively. This force is used to find the equilibrium position of the sphere, the stability of which is analyzed in the following.

B. Perturbation due to a small displacement

Consider a small displacement of the sphere by $\boldsymbol{x}$ that corresponds to a change of its center from the position $\boldsymbol{r}$ to $\boldsymbol{r}+\boldsymbol{x}$. In the frame of reference related to the sphere, this is equivalent to a perturbation of the external magnetic field,

$$\boldsymbol{A}_0^e(\boldsymbol{r}+\boldsymbol{x}) \approx \boldsymbol{A}_0^e(\boldsymbol{r}) + (\boldsymbol{x} \cdot \boldsymbol{\nabla}) \boldsymbol{A}_0^e(\boldsymbol{r}) = \boldsymbol{A}_0^e(\boldsymbol{r}) + \boldsymbol{A}_1^e(\boldsymbol{r}),$$

where

$$\boldsymbol{A}_1^e(\boldsymbol{r}) = (\boldsymbol{x} \cdot \boldsymbol{\nabla}) \boldsymbol{A}_0^e(\boldsymbol{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \boldsymbol{J}_l^m \bar{X}_l^m(\boldsymbol{r}),$$

with the expansion coefficients defined as $\boldsymbol{J}_l^m = \mathbf{M}_l^m \cdot \boldsymbol{x}$ by using the matrix

$$\mathbf{M}_l^m = \frac{2l+3}{2l+1} \frac{1}{N_l^m} \left(\frac{\boldsymbol{\mathcal{I}}_{l+1}^{m-1} \boldsymbol{\cdot} \boldsymbol{e}_\eta^*}{\sqrt{2}} N_{l+1}^{m-1} - \frac{\boldsymbol{\mathcal{I}}_{l+1}^{m+1} \boldsymbol{\cdot} \boldsymbol{e}_\eta}{\sqrt{2}} N_{l+1}^{m+1} + \boldsymbol{\mathcal{I}}_{l+1}^m \boldsymbol{\cdot} \boldsymbol{e}_z N_{l+1}^m \right),$$

where $[\boldsymbol{a} \boldsymbol{\cdot} \boldsymbol{b}]_{ij} = a_i b_j$ denotes the dyadic product forming a matrix from two vectors. To satisfy the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A}_1^i = 0$, we again add to the external vector potential $\boldsymbol{A}_1^e(\boldsymbol{r})$ a gradient of a harmonic gauge potential,

$$\Lambda_1^e(\boldsymbol{r}) = \sum_{l=1}^{\infty} \frac{2l-1}{2l+1} \sum_{m=-l}^{l} \tilde{\lambda}_l^m X_l^m(\boldsymbol{r}).$$

This results in the replacement of the coefficients $\boldsymbol{J}_l^m$ by

$$\boldsymbol{\mathcal{J}}_l^m = \boldsymbol{J}_l^m + \frac{1}{N_l^m} \left(\frac{\boldsymbol{e}_\eta^*}{\sqrt{2}} N_{l+1}^{m-1} \tilde{\lambda}_{l+1}^{m-1} - \frac{\boldsymbol{e}_\eta}{\sqrt{2}} N_{l+1}^{m+1} \tilde{\lambda}_{l+1}^{m+1} + \boldsymbol{e}_z N_{l+1}^m \tilde{\lambda}_{l+1}^m \right).$$

Similarly, the corresponding induced field is obtained as

$$\boldsymbol{\mathcal{A}}_1^i(\boldsymbol{r}) = \sum_{l=1}^{\infty} g_l \sum_{m=-l}^{l} \boldsymbol{\mathcal{J}}_l^m X_l^m(\boldsymbol{r}).$$

Applying the Coulomb gauge, we find $\tilde{\lambda}_{l+1}^m = \boldsymbol{Q}_{l+1}^m \cdot \boldsymbol{x}$, where

$$\boldsymbol{Q}_{l+1}^m = \frac{2l+3}{(2l+1)^2} \frac{N_{l+1}^m}{(l+1)} \times \left(\frac{\boldsymbol{e}_\eta^* \cdot \mathbf{M}_l^{m-1}}{\sqrt{2}N_l^{m-1}} - \frac{\boldsymbol{e}_\eta \cdot \mathbf{M}_l^{m+1}}{\sqrt{2}N_l^{m+1}} - \frac{\boldsymbol{e}_z \cdot \mathbf{M}_l^m}{N_l^m} \right).$$

C. Stability of small-amplitude oscillations

According to our previous theory,[12] small-amplitude oscillations of the sphere with $\boldsymbol{x}(t)=\boldsymbol{x}_0\cos(\Omega t)$ give rise to a perturbation of the integral force on the sphere,

$$\boldsymbol{F}_1=\Re[(-\mathbf{K}+i\Omega\boldsymbol{\Gamma})\cdot\boldsymbol{x}_0e^{i\Omega t}]=-\mathbf{K}\cdot\boldsymbol{x}+\boldsymbol{\Gamma}\cdot\frac{d\boldsymbol{x}}{dt},$$

where $\mathbf{K}$ and $-\boldsymbol{\Gamma}$ are the effective electromagnetic stiffness and damping matrices, respectively, whose projections on vector $\boldsymbol{\epsilon}_x=\boldsymbol{x}/|\boldsymbol{x}|$ are[13]

$$\mathbf{K}\cdot\boldsymbol{\epsilon}_x=\frac{1}{2}\int_{\bar{V}}[\boldsymbol{j}_0^e\times\Re[\boldsymbol{B}_1^i]+\boldsymbol{j}_1^e\times\Re[\boldsymbol{B}_0^i]]dV,$$

$$\boldsymbol{\Gamma}\cdot\boldsymbol{\epsilon}_x=-\frac{1}{2}\partial_{\bar{\omega}}\int_{\bar{V}}\boldsymbol{j}_0^e\times\Im[\boldsymbol{B}_1^i]dV.$$

After some algebra, we obtain

$$\boldsymbol{\Gamma}=\sum_{l=0}^{\infty}(2l+1)\Im[\partial_{\bar{\omega}}g_l]\left\{\sum_{m=0}^{l}\Re[\mathbf{M}_l^{m\dagger}\cdot\mathbf{M}_l^m]/c_m-\frac{(l+1)(2l+1)^2}{2l+3}\sum_{m=0}^{l+1}\Re[\boldsymbol{Q}_{l+1}^{m*}\cdot\boldsymbol{Q}_{l+1}^m]/c_m\right\},\tag{14}$$

where † denotes the Hermitian conjugate matrix. Similarly, we find

$$\mathbf{K}=-\sum_{l=0}^{\infty}(2l+1)\Re[g_l]\left\{\sum_{m=0}^{l}\Re[\mathbf{M}_l^{m\dagger}\cdot\mathbf{M}_l^m]-\frac{(l+1)(2l+1)^2}{2l+3}\sum_{m=0}^{l+1}\Re[\boldsymbol{Q}_{l+1}^{m*}\cdot\boldsymbol{Q}_{l+1}^m]\right\}\Big/c_m+\sum_{l=0}^{\infty}(2l+1)\Re[g_{l-1}]\sum_{m=0}^{l}\Re[\mathbf{M}_l^{m\dagger}\cdot\mathbf{L}_l^m]/c_m,\tag{15}$$

where

$$\mathbf{L}_l^m=N_l^m\left[\frac{\boldsymbol{\mathcal{I}}_{l-1}^{m-1}\cdot\boldsymbol{e}_\eta^*}{\sqrt{2}N_{l-1}^{m-1}}-\frac{\boldsymbol{\mathcal{I}}_{l-1}^{m+1}\cdot\boldsymbol{e}_\eta}{\sqrt{2}N_{l-1}^{m+1}}+\frac{\boldsymbol{\mathcal{I}}_{l-1}^{m}\cdot\boldsymbol{e}_z}{N_{l-1}^{m}}\right].$$

Note that both the damping (14) and stiffness (15) matrices are symmetric.

The complex dimensionless frequency Ω of small-amplitude oscillations of the sphere about the equilibrium position is defined by the characteristic equation[12]

$$\Omega^2\boldsymbol{x}_0=N(\mathbf{K}-i\Omega\boldsymbol{\Gamma})\cdot\boldsymbol{x}_0,\tag{16}$$

where $N=(\mu_0R)^3(\sigma I_0)^2/m$ is a dimensionless interaction parameter characterizing the ratio of the electromagnetic and inertial force with m as the mass of the sphere. To solve this equation, we employ the classical perturbation theory for an eigenvalue problem by considering the damping to be a small perturbation relative to the stiffness.[18] This is true for sufficiently high ac frequencies ($\bar{\omega}>10$) where the relative magnitude of the damping decreases as $\sim\bar{\omega}^{-3/2}$.[12,13] Neglecting the damping, the leading-order solution for the frequency is $\Omega_i^{(0)}=\sqrt{N\lambda_i^{(0)}}$, where $\lambda_i^{(0)}$ $(i=1,2,3)$ are three eigenvalues of the stiffness matrix $\mathbf{K}$ further referred to as the principal stiffness coefficients. These frequencies correspond to the three principal directions of the normal mode oscillations that are defined by the corresponding eigenvectors $\boldsymbol{x}_i^{(0)}$. Note that due to the symmetry of $\mathbf{K}$, the eigenvectors are mutually orthogonal and the eigenvalues are real. For the given equilibrium position to be statically stable, all three eigenfrequencies have to be real which in turn requires the corresponding principal stiffness coefficients to be positive, i.e., $\lambda_i^{(0)}>0$.

The first-order correction due to the damping follows from

$$\lambda_i^{(1)}\boldsymbol{x}_i^{(0)}+\lambda_i^{(0)}\boldsymbol{x}_i^{(1)}=\mathbf{K}\cdot\boldsymbol{x}_i^{(1)}-i\Omega_i^{(0)}\boldsymbol{\Gamma}\cdot\boldsymbol{x}_i^{(0)}.$$

Since the leading-order eigenvectors are mutually orthogonal, we can use them as basis to represent the perturbation as $\boldsymbol{x}_i^{(1)}=\Sigma_{j=1}^3C_{ij}^{(0)}\boldsymbol{x}_j^{(0)}$. Substituting this expansion into the above equation, we obtain

$$\sum_{j=1}^{3}(\lambda_i^{(0)}-\lambda_j^{(0)})C_{ij}^{(0)}\boldsymbol{x}_j^{(0)}=-\lambda_i^{(1)}\boldsymbol{x}_i^{(0)}-i\Omega_i^{(0)}\boldsymbol{\Gamma}\cdot\boldsymbol{x}_i^{(0)}.$$

Multiplying the equation above scalarly by $\boldsymbol{x}_i^{(0)}$ and taking into account the orthogonality condition, we find $\lambda_i^{(1)}=-i\Omega_i^{(0)}\Gamma_i$, where the growth rate is defined as

$$\Gamma_i=\frac{(\boldsymbol{x}_i^{(0)}\cdot\boldsymbol{\Gamma}\cdot\boldsymbol{x}_i^{(0)})}{(\boldsymbol{x}_i^{(0)}\cdot\boldsymbol{x}_i^{(0)})}.$$

Then the first-order correction to the frequency is $\Omega_i^{(1)}=\frac{1}{2}\lambda_i^{(1)}/\sqrt{\lambda_i^{(0)}/N}=-\frac{1}{2}iN\Gamma_i$. Thus, the development in time of the oscillation amplitude is determined by the sign of the growth rate Γ_i. For $\Gamma_i>0$, oscillations will be dynamically unstable with an amplitude growing in time.

V. SPIN-UP INSTABILITY

A. Governing equations

Here we consider a solid sphere, as in Sec. II, which rotates slowly with angular velocity $\boldsymbol{\Omega}$ in a 3D ac magnetic field. The induced magnetic field can be found more easily by proceeding to the frame of reference rotating together with the sphere. In this frame of reference, the sphere is at rest while the source of the magnetic field rotates with angular velocity $-\boldsymbol{\Omega}$. Thus, on one hand, Eq. (1) reduces to (6), as for a body at rest with $\boldsymbol{\Omega}=\boldsymbol{v}=0$. On the other hand, the temporal variation of the external magnetic field becomes more complicated because of the apparent rotation of the coil. In the laboratory frame of reference, the magnetic field of the coil is specified by two vector fields, the radius vector $\boldsymbol{r}$ and the magnetic vector potential $\boldsymbol{A}_0^e$, which both appear to rotate with angular velocity $-\boldsymbol{\Omega}$ in the rotating frame of reference. Thus, the vector potential of the external magnetic field may be represented in the rotating frame of reference as

$$\boldsymbol{A}_0^{e\prime}(\boldsymbol{r}',t)=\mathbf{G}(-\alpha)\cdot\boldsymbol{A}_0^e(\mathbf{G}(-\boldsymbol{\alpha})\cdot\boldsymbol{r},t),\tag{17}$$

where $\mathbf{G}(\boldsymbol{\alpha})$ is the matrix of rotation by the angle $\boldsymbol{\alpha}=t\boldsymbol{\Omega}$. Further, we focus on arbitrary slow rotations with infinitesimally small $\boldsymbol{\Omega}$ for which the rotation angle $\boldsymbol{\alpha}$ remains small for any limited time t. For small rotation angles, we have

$\mathbf{G}(\boldsymbol{\alpha})\cdot\boldsymbol{x} \approx \boldsymbol{x}+\boldsymbol{\alpha}\times\boldsymbol{x}$ and, consequently, Eq. (17) may be approximated by the power series expansion in $\boldsymbol{\alpha}$,

$$\begin{aligned}\boldsymbol{A}_0^{e\prime}(\boldsymbol{r}',t) &\approx [\boldsymbol{A}_0^e(\boldsymbol{r}) - (t\boldsymbol{\Omega}\times\boldsymbol{r}\cdot\boldsymbol{\nabla})\boldsymbol{A}_0^e(\boldsymbol{r}) - t\boldsymbol{\Omega}\times\boldsymbol{A}_0^e(\boldsymbol{r})]e^{i\bar{\omega}t}\\ &= \boldsymbol{A}_0^e(\boldsymbol{r})e^{i\bar{\omega}t} + i\partial_{\bar{\omega}}[\boldsymbol{A}_1^e(\boldsymbol{r})e^{i\bar{\omega}t}],\end{aligned}$$

where $\boldsymbol{A}_1^e(\boldsymbol{r}) = (\boldsymbol{\Omega}\times\boldsymbol{r}\cdot\boldsymbol{\nabla})\mathbf{A}_0^e(\mathbf{r}) + \boldsymbol{\Omega}\times\boldsymbol{A}_0^e(\boldsymbol{r})$ is the amplitude of the external field perturbation due to a small rotation, whereas its time dependence is expressed by means of a parametric frequency derivative as $te^{i\bar{\omega}t} = -i\partial_{\bar{\omega}}e^{i\bar{\omega}t}$. The latter transformation leads to a simple solution for the perturbation of the induced field. Because of the linearity of the electromagnetic problem, we can first find the harmonically alternating solution of the induced field corresponding to the external field $\boldsymbol{A}_1^e$ defined above and then take the frequency derivative from this intermediate solution to obtain the final solution for the perturbation of the induced field. To find the intermediate solution note that, since the perturbation of the external field $\boldsymbol{A}_1^e$ represents a small rotation of the external base field, the perturbation of the induced field can also be obtained by the rotation of the induced base field,

$$\begin{aligned}\boldsymbol{A}_1^{i\prime}(\boldsymbol{r}',t) &= i\partial_{\bar{\omega}}\{[(\boldsymbol{\Omega}\times\boldsymbol{r}\cdot\boldsymbol{\nabla})\boldsymbol{\mathcal{A}}_0^i(\boldsymbol{r}) + \boldsymbol{\Omega}\times\boldsymbol{\mathcal{A}}_0^i(\boldsymbol{r})]e^{i\bar{\omega}t}\}\\ &= i\partial_{\bar{\omega}}[(\boldsymbol{\Omega}\times\boldsymbol{r}\cdot\boldsymbol{\nabla})\boldsymbol{\mathcal{A}}_0^i(\boldsymbol{r}) + \boldsymbol{\Omega}\times\boldsymbol{\mathcal{A}}_0^i(\boldsymbol{r})]e^{i\bar{\omega}t}\\ &\quad - [(t\boldsymbol{\Omega}\times\boldsymbol{r}\cdot\boldsymbol{\nabla})\boldsymbol{\mathcal{A}}_0^i(\boldsymbol{r}) + t\boldsymbol{\Omega}\times\boldsymbol{\mathcal{A}}_0^i(\boldsymbol{r})]e^{i\bar{\omega}t},\end{aligned}$$

where $\boldsymbol{\mathcal{A}}_0^i$ is the amplitude of the induced base field (8) transformed according to (10). Note that $\boldsymbol{\mathcal{A}}_0^i$ is frequency dependent because of containing the coefficients g_l defined by (9). Eventually, returning back to the laboratory frame of reference, we obtain a simple solution for the perturbation of the induced field $\boldsymbol{A}_1^i(\boldsymbol{r},t) = i\partial_{\bar{\omega}}\boldsymbol{A}_1^i(\boldsymbol{r})\mathrm{e}^{i\bar{\omega}t}$, where

$$\boldsymbol{A}_1^i(\boldsymbol{r}) = (\boldsymbol{\Omega}\times\boldsymbol{r}\cdot\boldsymbol{\nabla})\boldsymbol{\mathcal{A}}_0^i(\boldsymbol{r}) + \boldsymbol{\Omega}\times\boldsymbol{\mathcal{A}}_0^i(\boldsymbol{r}).$$

The time-averaged total torque is

$$\begin{aligned}\boldsymbol{M} &= \int_V \boldsymbol{r}\times\langle\boldsymbol{j}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)\rangle dV\\ &= \frac{1}{2}\int_V \boldsymbol{r}\times\Re[\boldsymbol{j}(\boldsymbol{r})\times\boldsymbol{B}^*(\boldsymbol{r})]dV.\end{aligned}$$

Taking into account that $\boldsymbol{j} = \boldsymbol{\nabla}\times\boldsymbol{B}^i$ and $\boldsymbol{B} = \boldsymbol{B}^e + \boldsymbol{B}^i$, where $\boldsymbol{B}^e$ is purely real, we obtain $\boldsymbol{M} = \frac{1}{2}\int_V \boldsymbol{r}\times\Re[\boldsymbol{j}]\times\boldsymbol{B}^e dV$. Conservation of the angular momentum leads to $\boldsymbol{M} = -\frac{1}{2}\int_{\bar{V}}\boldsymbol{r}\times\boldsymbol{j}^e\times\Re[\boldsymbol{B}^i]dV$, where the last integral is taken over the volume $\bar{V}$ outside the sphere. Since there is no torque created by a harmonically alternating field on a sphere at rest[14,19] and $(\boldsymbol{r}\cdot\boldsymbol{j}_0^e) = 0$, as discussed after Eq. (12), the last integral can be represented as $\boldsymbol{M} = \frac{1}{2}\int_{\bar{V}}\boldsymbol{j}_0^e\Im[\boldsymbol{r}\cdot\partial_{\bar{\omega}}\boldsymbol{B}_1^i]dV$. After some algebra, we eventually obtain the torque in terms of the angular velocity of the sphere and the induced base field as

$$\boldsymbol{M} = \frac{1}{2}\int_{\bar{V}}(\boldsymbol{\Omega}\cdot\boldsymbol{j}_0^e)\boldsymbol{r}\times\boldsymbol{\nabla}(\boldsymbol{r}\cdot\boldsymbol{\nabla}\times\Im[\partial_{\bar{\omega}}\boldsymbol{\mathcal{A}}_0^i])dV. \quad (18)$$

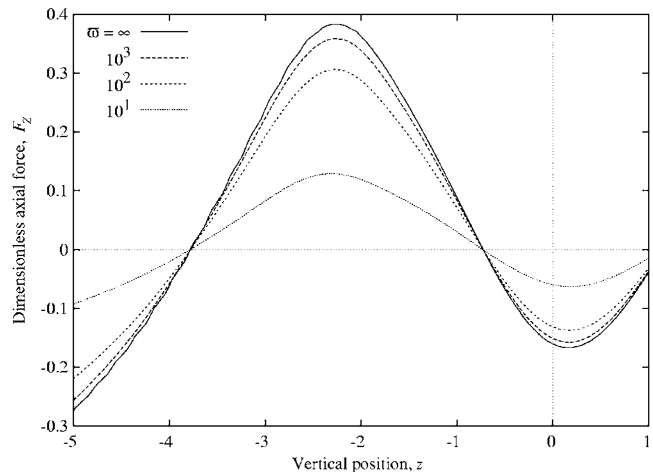

FIG. 2. Dimensionless axial electromagnetic force on the sphere depending on its vertical position at various dimensionless ac frequencies.

B. Analytical solution

Substituting the associated external current (12) and the solution for the induced base field (8) into (18), we find $\boldsymbol{M} = -\mathbf{M}\cdot\boldsymbol{\Omega}$, where the effective angular damping matrix is defined as

$$\mathbf{M} = \sum_{l=0}^{\infty}(2l+1)\Im[\partial_{\bar{\omega}}g_l]\sum_{m=0}^{l}\Re[\boldsymbol{\mathcal{I}}_l^{m*}\cdot\boldsymbol{L}_l^m]/c_m, \quad (19)$$

by making use of

$$\boldsymbol{L}_l^m = N_l^m\left[\frac{\boldsymbol{e}_\eta^*(l+1-m)}{\sqrt{2}N_l^{m-1}}s_l^{m-1} - \frac{\boldsymbol{e}_\eta(l+1-m)}{\sqrt{2}N_l^{m+1}}s_l^{m+1} + \frac{\boldsymbol{e}_z m}{N_l^m}s_l^m\right]$$

and

$$s_l^m = (l+1)N_l^m\left[\frac{\boldsymbol{e}_\eta^*\cdot\boldsymbol{\mathcal{I}}_l^{m-1}}{\sqrt{2}N_l^{m-1}} + \frac{\boldsymbol{e}_\eta\cdot\boldsymbol{\mathcal{I}}_l^{m+1}}{\sqrt{2}N_l^{m+1}}\right].$$

Note that $\mathbf{M}$ is symmetric as the linear damping (14) and the stiffness (15) matrices in the case of oscillatory stability obtained in Sec. IV C.

The evolution of small rotational perturbations of the sphere is governed by the following dimensionless equation; $d\boldsymbol{\Omega}/dt = -\frac{5}{2}N\mathbf{M}\cdot\boldsymbol{\Omega}$, where N is the interaction parameter introduced in (16). This equation has three particular solutions of the form $\boldsymbol{\Omega} = \boldsymbol{\Omega}_i e^{(5/2)N\Gamma_i t}$, where the angular growth rates Γ_i are the eigenvalues of $-\mathbf{M}$ and the three principal axes of rotation are defined by the corresponding eigenvectors.

VI. NUMERICAL RESULTS

The solutions obtained above have been validated by comparing with previously obtained analytical solutions for axisymmetric coils made of circular loops.[13] Here we will illustrate the application of our theory to the coil shown in Fig. 1. Note that the size of the coil is given in the dimensionless form by using the radius of the sphere as the length scale. The electromagnetic force is scaled by $F_0 = \mu_0 I_0^2$. The corresponding axial dimensionless electromagnetic force (13) acting on the sphere at various vertical positions in the coil is plotted in Fig. 2 for various dimensionless ac frequencies. This calculation is done by fixing the vertical position of the sphere and searching for the equilibrium position in the horizontal plane that corresponds to a sphere suspended on a long rope. This allows us to evaluate the suspension characteristics of the coil and its axial symmetry. As is seen in Fig. 2, an increase of the ac frequency results in a saturation of the force at a value which depends on the position of the sphere. The limit of high frequency (formally $\bar{\omega}=\infty$) gives the maximal force that can be attained at the given position. In addition, this limit provides a good approximation for the force when the dimensionless frequency is high enough ($\bar{\omega}>10^2$). The electromagnetic force tries to expel the sphere out of the coil when it is inserted at the lower end of the coil. Only when the sphere is inserted deep enough, there appears the electromagnetic force trying to suspend the sphere. The electromagnetic suspension force increases with the vertical position of the sphere until some distance from the bottom winding of the coil is reached where the force attains a maximum. With further increase of the vertical position, the electromagnetic suspension force decreases down to zero at some height where the direction of the force changes because of the upper counter winding. The electromagnetic force can balance gravity only for those positions where the resulting electromagnetic force is directed against gravity, i.e., where the force is positive. However, not all of these positions are stable. All positions below the force maximum are unstable with respect to small axial displacements of the sample because the electromagnetic force increases as the sphere is moved upwards and decreases as the sphere is moved downwards. Thus, a force perturbation will appear moving the sphere upwards or downwards away from the presumed equilibrium position. Consequently, a stable suspension is possible only in the range of vertical positions above the force maximum till the upper neutral point where the vertical direction of the electromagnetic force reverses.

The horizontal equilibrium positions of the sphere are shown in Fig. 3 versus the vertical position at various dimensionless ac frequencies. There are noticeable deviations of the sphere from the coil axis in the gap between the lower and the upper parts of the coil where the sphere is strongly pushed aside by the uncompensated connection between the base coil and the counter winding.

Small amplitude oscillations of the sphere are characterized by three stiffness coefficients and the associated oscillation directions specified by the spherical angles, defined after Eq. (16), which are plotted in Fig. 4 versus the vertical position of the sphere for $\bar{\omega}=\infty$. The azimuthal and poloidal angles are defined as usual with respect to the x and z axes shown in Fig. 1. There is one principal stiffness coefficient which becomes negative for the mode $i=1$ when the sphere is located too low or too high in the coil. Note that a negative stiffness coefficient implies the position to be statically unstable with respect to a small-amplitude displacement in the direction specified by the corresponding spherical angles. As is seen in Fig. 4(a), in the unstable regions at both ends of the coil, the mode $i=1$ is nearly axial as already suggested by the distribution of the axial force discussed above. The two other principal stiffness coefficients, which are positive, correspond to stable oscillations in the horizontal plane. The principal stiffness coefficient for the mode $i=1$ turns positive in the range of vertical positions between the extrema of the axial electromagnetic force where a statically stable suspension is thus possible. Note that in this range the mode $i=1$ changes its orientation from axial to horizontal while the mode $i=3$ does the opposite.

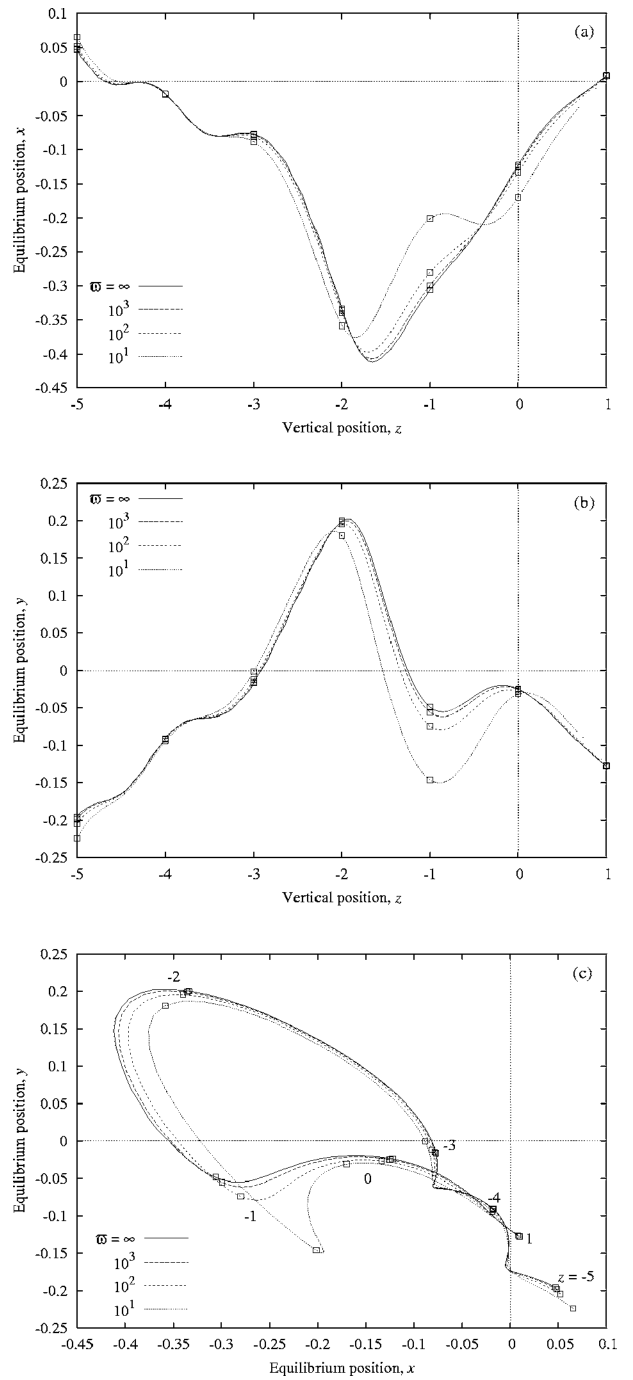

FIG. 3. Horizontal equilibrium positions of the center of the sphere along x (a) and y axes (b) depending on the vertical position z in the coil shown in Fig. 1 at various dimensionless ac frequencies; the resulting equilibrium position in the x-y plane at various vertical positions z when viewed from the top of the coil (c).

Besides the principal stiffness coefficients, which determine whether the levitated body can perform small-amplitude oscillations about the given equilibrium state and the frequency of these oscillations, we have the electromag-

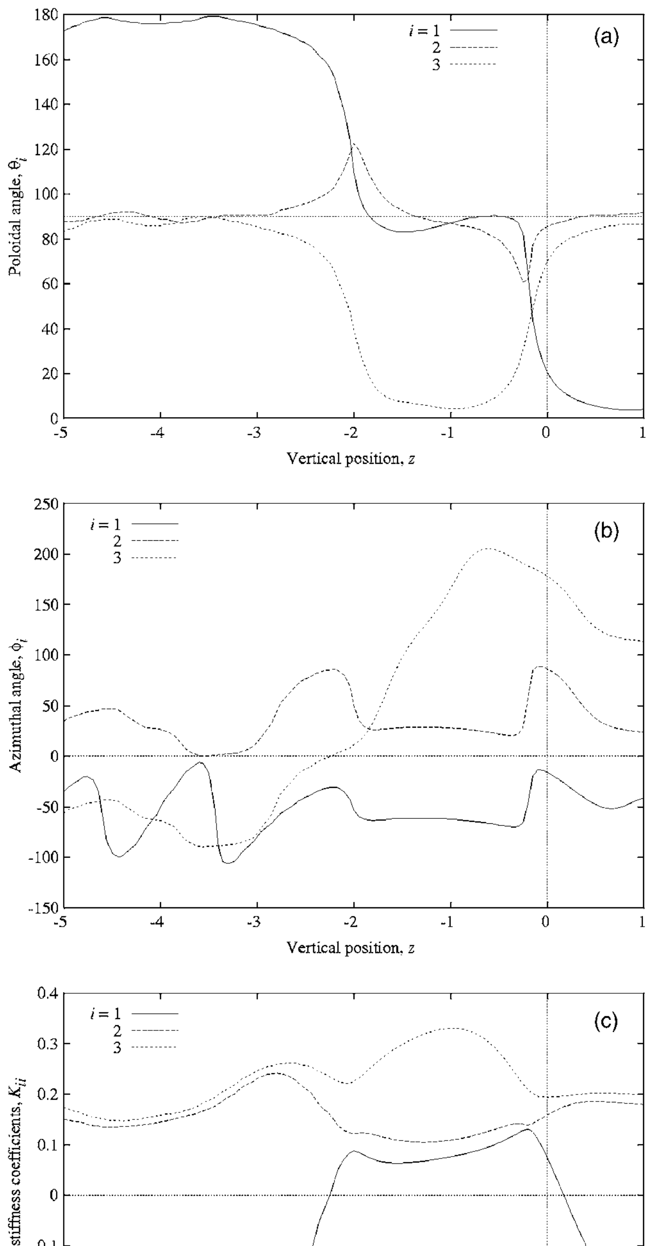

FIG. 4. Poloidal (a) and azimuthal (b) angles of the principal oscillation directions, and the corresponding stiffness coefficients (c) depending on the vertical position for $\bar{\omega}=\infty$ and a fixed current in the coil. All principal stiffness coefficients are required to be positive for the position to be stable.

netic damping coefficients of the oscillations which determine whether the oscillation amplitude is decaying or growing in time. Similarly, slow rotations of the sphere can be retarded or accelerated by the ac magnetic field depending on the sign of the principal angular electromagnetic damping coefficients which are the eigenvalues of (19). Calculated oscillation growth rates and spin-up rates of the sphere located at the vertical position $z=-1$ are plotted in Fig. 5 versus the dimensionless ac frequency. As is seen, the growth

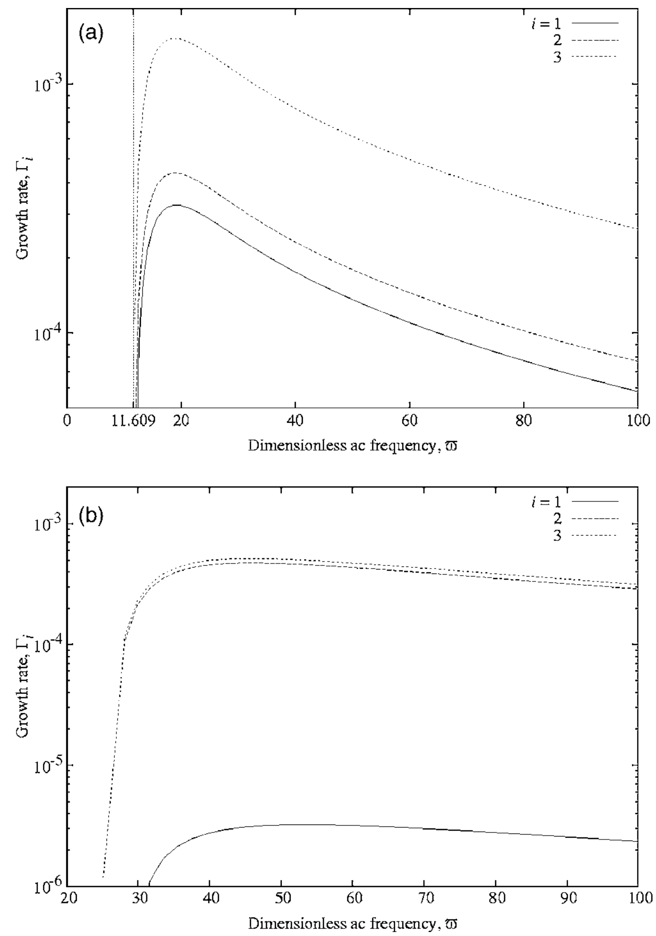

FIG. 5. Growth rates of principal oscillation (a) and rotation (b) modes depending on the dimensionless ac frequency for a sphere located at the vertical dimensionless position $z=-1$.

rates for all three principal oscillation modes become positive when the dimensionless ac frequency exceeds $\bar{\omega}_c \approx 11$ and attain maxima at $\bar{\omega}_d \approx 19$. Growth rates for both horizontal oscillation modes $i=1$ and 2 are about one-fourth of the growth rate of the axial oscillation mode $(i=3)$. These critical frequencies and the growth rate ratios are very close to the corresponding values in a simple spatially linear magnetic field.[12] This is because the sphere is located in the vicinity of a neutral point of the magnetic field, where the axial electromagnetic field turns to zero (see Fig. 2) and the radius of the coil is about twice as large as that of the sphere [see Fig. 1(c)]. Thus, the magnetic field generated by the coil in the vicinity of the sphere is obviously close to a spatially linear one.

Growth rates of the principal rotational modes, which are plotted in Fig. 5(b), also become positive similar to the oscillation growth rates but at slightly higher dimensionless ac frequencies. Note that positive growth rates in this case imply that the electromagnetic torque, which arises as a result of a slow rotation of the sphere, tries to accelerate rather than to brake the rotation. This can result in a spin up of the sphere provided that there are no other factors such as, for example, an ambient gas or additional steady magnetic fields which might slow down the rotation. As in Fig. 5(b), two principal rotation modes $i=2$ and 3 have nearly equal growth

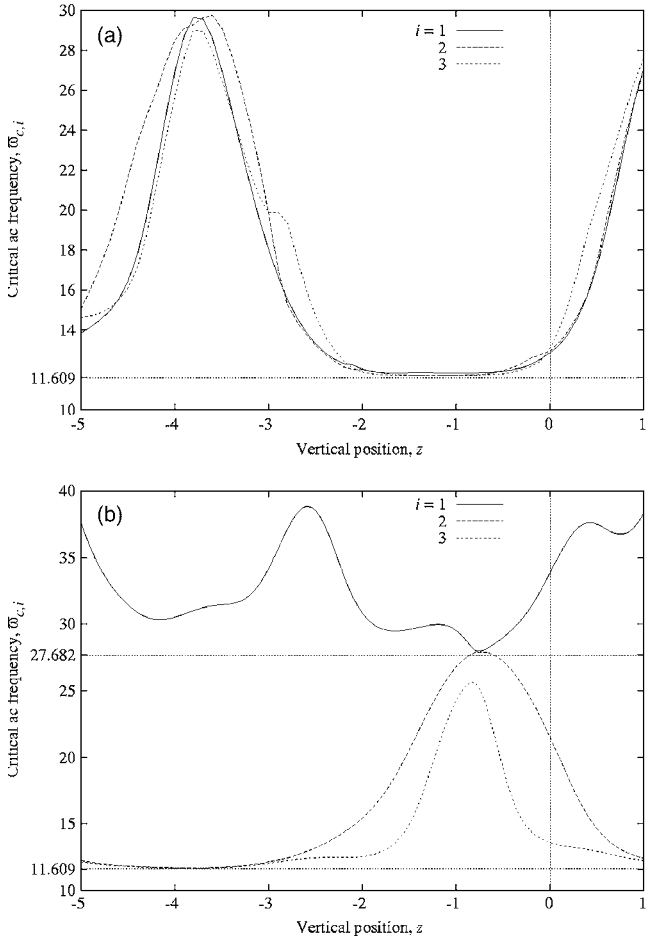

FIG. 6. Critical dimensionless ac frequencies for oscillatory (a) and rotational (b) modes depending on the vertical position of the sphere.

rates comparable in magnitude to those of the oscillation modes [see Fig. 5(a)], whereas the growth rate of the third mode $i=1$ is two orders of magnitude lower. Note that in axisymmetric magnetic fields, the sphere can spin up about any axis perpendicular to the symmetry axis while a rotation about the symmetry axis is not affected by the magnetic field at all which corresponds to a zero growth rate for this rotation.[11,13] This fact suggests that the first two modes correspond to a spin up about the horizontal axis while the third mode corresponds to the spin up about the vertical axis which is entirely due to the nonaxisymmetry of the magnetic field.

The critical dimensionless ac frequencies, by exceeding which positive oscillation or rotation growth rates appear, are plotted in Fig. 6 versus the vertical position of the sphere in the coil. When the sphere is located in the region between the lower and upper parts of the coil, the critical frequency for oscillatory modes is nearly constant and close to that for a spatially linear field. The critical frequency rises considerably, when the sphere is moved into the end parts of the coil where the electromagnetic force is determined by higher-order spherical harmonics of the magnetic field. Critical frequencies for the spin-up instability, plotted in Fig. 6(b), vary in the opposite way. First, in the bottom part of the coil, two critical frequencies for horizontal modes are almost constant and close to $\bar{\omega}_c=11.609$, corresponding to the critical frequency for spin up in a uniform ac magnetic field.[11] The third critical frequency for the spin up about the vertical axis, caused by the asymmetry of the magnetic field, is considerably higher. As the sphere approaches the neutral point between the lower and upper parts of the coil, the critical frequencies for horizontal modes rise and approach $\bar{\omega}_c=27.682$, which corresponds to the threshold for the spin-up instability in a spatially linear magnetic field.[11] As is seen, the critical frequency for the vertical mode decreases and also approaches this limit in the vicinity of the neutral point. Note that a stable suspension is possible only in some region between the lower and the upper parts of the coil.

VII. SUMMARY AND CONCLUSION

In this work, we have extended our previous stability theory of electromagnetically levitated spherical bodies to 3D magnetic fields. The analysis was carried out by a combination of numerical and analytical approaches. First, the coil generating the ac magnetic field was approximated by a number of linear elements and the corresponding contour integral was evaluated numerically to obtain the vector potential at grid points on the surface of the sphere. Then, FFT and Legendre transform using Gaussian quadrature were applied along the azimuthal and poloidal angles, respectively, to obtain the expansion of the vector potential in terms of spherical harmonics. Further, the 3D electromagnetic problem was solved analytically by using the gauge transformation developed in our previous work.[13] This allowed us to obtain the solution in terms of the vector potential only. Using this solution, we found the electromagnetic reaction force due to both a small displacement of the body from its equilibrium position and a slow motion. A theoretical approach based on the vector field transformation by small rotation in combination with a parametric frequency derivative was introduced to find the electromagnetic reaction torque due to a slow rotation of the sphere in a 3D ac magnetic field. This allowed us to simplify the analysis by avoiding the solution of the problem for an arbitrary rotation rate and then proceeding to the limit of slow rotation, as it was done previously for simple axisymmetric magnetic field configurations.[11,13] The electromagnetic reaction force and torque provided the basis for the stability analysis with respect to both small-amplitude oscillations and slow rotations. We considered both the static and the dynamic stability. The first is determined by the effective stiffness matrix, which characterizes the electromagnetic reaction force arising in response to a displacement of the levitated body from its equilibrium position. The position is statically stable if there is no component of the electromagnetic reaction force arising in the direction of the displacement. This requires all three principal stiffness coefficients, which are the eigenvalues of the symmetric stiffness matrix, to be positive. The principal stiffness coefficients, if positive, define the eigenfrequencies of the principal oscillation modes whereas the corresponding eigenvectors define the three mutually perpendicular spatial directions of these oscillations. The dynamic stability is determined by the linear electromagnetic damping matrix which characterizes the electromagnetic reaction force associated with a translational motion of the body. Oscillation growth rates were calculated by using the classical eigenvalue perturbation approach which was applicable because of the relative smallness of the damping effect. In this way, we found the critical ac frequencies by exceeding which the amplitude of oscillations can be amplified by the ac magnetic field provided that there are no additional stabilizing effects such as, for example, the viscosity of an ambient gas or dc magnetic fields. Similarly, the development of a slow rotation of the sphere was determined by the angular electromagnetic damping matrix. Eigenvalues of this symmetric matrix define the spin-up rates along three mutually perpendicular principal axes of rotation whose directions are given by the corresponding eigenvectors. The application of this theory was demonstrated for a 3D model coil.

Note that our theory assumes the magnetic field generated by the coil to be a single-phase field. It means that the magnetic field alternates along spatially fixed magnetic flux lines which are invariant in time, similar to the electromagnetic field in a linearly polarized wave. This obviously supposes the current in the coil to have a single phase. However, the ac magnetic field can induce eddy currents with phase shift in the coil itself similar to a body of finite conductivity placed in the coil. For a given current amplitude, the eddy currents result in a redistribution of the current over the cross section of the winding. This redistribution effect is obviously negligible when the cross section is small compared to the dimensions of the coil which was one of our basic assumptions in modeling the coil by linear current filaments. In addition, since eddy currents have to close over the cross section of the winding, their magnetic field is expected to be similar to that of a bifilar winding which decays much faster with distance than the magnetic field due to the net current. Nevertheless, when the cross section of the winding is large or the levitated body is placed close to the winding, the external magnetic field may be slightly rotating rather than purely alternating along the fixed directions which may be another mechanism responsible for rotation of levitated bodies observed in experiments.

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