An Intertemporal Price Equilibrium Model

Author(s): T. Takayama and G. G. Judge

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T. Takayama and G. G. Judge

IN THE early 1950's, Enke and Samuelson[1] formulated the problem concerning equilibrium among spatially separated markets and suggested ways of obtaining a solution. Building on this work, Takayama and Judge[2] in a recent paper have shown how a class of spatial equilibrium problems may be solved by quadratic programming techniques. Noting the fact that economic relations in time have many of the properties of economic relations in space, Samuelson[3] in a 1957 paper discussed the problem of intertemporal price equilibrium and suggested how the tools for analyzing spatial competitive relations may be applied to the more complex problems of equilibrium commodity prices over time. Within this context it is the modest purpose of this paper to show how the spatial formulation of Takayama and Judge may be *extended* to obtain the competitive price and product allocation when markets are separated by both space and time. In particular, it will be shown (1) how the concept of net social payoff[4] may be used as a basis for deducing the conditions of spatial and intertemporal equilibrium and (2) if linear dependencies between regional supply, demand and price for each time period $t(t=1, 2, \cdots, T)$ are postulated and, given transportation costs among regions and storage costs over time periods, how the problem can be converted into a quadratic programming problem that can be solved directly for the competitive solution. Introducing the time dimension into the model opens the way to handling many of the knotty problems concerned with price adjustment and allocation over time when products can be stored and the products between time periods are considered substitutes in consumption. In an effort to achieve brevity and clarity, this presentation is restricted to a single commodity specification. Extension to the multicommodity dimension is straightforward given the initial formulation.

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[1] S. Enke, "Equilibrium Among Spatially Separated Markets," *Econometrica*, Vol. 19, 1951, pp. 40–47; and P. S. Samuelson, "Spatial Price Equilibrium and Linear Programming," *Am. Econ. Review*, Vol. 42, 1952, pp. 283–303.

[2] T. Takayama and G. G. Judge, "Spatial Equilibrium and Quadratic Programming," *J. Farm Econ.*, Vol. 46, February 1964, pp. 67–93.

[3] P. S. Samuelson, "Intertemporal Price Equilibrium: A Prologue to the Theory of Speculation," *Weltwirtschaftliches Archiv*, Band 79, 1957, pp. 181–221.

[4] P. S. Samuelson, *op. cit.*, pp. 287–292.

T. Takayama *is visiting professor and* G. G. Judge *is professor of agricultural economics at the University of Illinois.*

Notation

To facilitate discussion, the definitions and notation to be employed are summarized as follows:

Let

i, j denote the regions, where $i, j = 1, 2, \cdots, n$.

t denote the discrete time periods, where $t = 1, 2, \cdots, T$.

$P_d = \{p_i(t)\}$ denote regional demand prices, $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$.

$D = \{d_i(t)\}$ denote the regional demand relations, $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$.

The $d_i(t)$ are linear functions of the prices $p_i(t)$ such that

$$d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t);\ \alpha_i(t) > 0,\ \beta_i(t) \geq 0. \tag{1}$$

$P^s = \{p^j(t)\}$ denote the regional supply prices, $j = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$.

$S = \{s_j(t)\}$ denote the regional supply relations.

The $s_j(t)$ are linear functions of the regional prices $p^j(t)$ such that

$$s_j(t) = \theta_j(t) + \gamma_j(t)p^j(t);\ \theta_j(t) \lesseqgtr 0,\ \gamma_j(t) \geq 0. \tag{2}$$

$X = \{x_{ij}(t, \tau)\}$ denote the commodity flow from region i to region j (for all i, j and t) from time period τ to t, where $\tau = 1, 2, \cdots, t$. To account for the asymmetrical nature of time on the supply side let $\{x_{ij}(\rho, t)\}$ denote the interregional flows from time period t to ρ, where $\rho = t, t+1, \cdots, T$.

$H = \{h_{ij}(\tau)\}$ denote the unit transportation cost for shipping the commodity from i to j for all i, j and τ.

$B = \{b_i(t, \tau)\}$ denote the storage cost, insurance, interest on capital, etc., for carrying the product from time period τ to t for all i and t.

$R = H + B = \{h_{ij}(\tau) + b_i(t, \tau)\}$ denote the combined storage and transportation costs.

$C = \{\alpha_i(t) - \theta_i(t)\}$ for all i and t. $\theta_i(1)$ and $\alpha_i(T)$ contain the initial stocks and final carry-over quantities.

$W(1) = \{\delta_i\}$ denote the inventory (stock) at time period 1, where $\delta_i \geq 0$.

$W(T) = \{\mu_i\}$ denote the carry-over stock at the end of time period T, where $\mu_i \geq 0$.

The Formulation

By using the Samuelson concept of net social payoff (NSP), the problem for the n region, T time period case may be written in mathematical form as:

To maximize

$$(3)\quad \text{NSP} = f(X) = \sum_t \sum_i \int_0^{\sum_\tau \sum_j x_{ji}(t,\tau)} p_i(t)d\xi_i(t)$$
$$- \sum_t \sum_i \int_0^{\sum_\rho \sum_j x_{ji}(\rho,t)} p^i(t)d\xi^i(t) - \sum_t \sum_\tau \sum_i \sum_j h_{ij}(\tau)x_{ij}(t,\tau)$$
$$- \sum_t \sum_\tau \sum_i \sum_j b_i(t,\tau)x_{ij}(t,\tau) - \sum_t \sum_i a_i(t)$$

subject to

$$(4)\qquad x_{ij}(t,\tau),\ x_{ij}(\rho,t) \geq 0$$

where $a_i(t)$ is the sum of producers' and consumers' surplus in the ith region at time period t under pretrade equilibrium.

The necessary conditions for the maximum NSP over time defined above when converted to the Lagrangean form, $\phi(X, \lambda) = f(X) + \lambda'X$, are as follows:[5]

$$(5)\qquad \frac{\partial\phi(X,\lambda)}{\partial x_{ij}(t,\tau)} = p_j(t) - p^i(\tau) - h_{ij}(\tau) - b_i(t,\tau) + \lambda_{ij}(t,\tau) \leq 0$$

and

$$(6)\qquad \frac{\partial\phi(X,\lambda)}{\partial x_{ij}(t,\tau)} \cdot x_{ij}(t,\tau) = 0$$

$$(7)\qquad \frac{\partial\phi(X,\lambda)}{\partial \lambda_{ij}(t,\tau)} = x_{ij}(t,\tau) \geq 0$$

and

$$(8)\qquad \frac{\partial\phi(X,\lambda)}{\partial \lambda_{ij}(t,\tau)} \cdot \lambda_{ij}(t,\tau) = x_{ij}(t,\tau)\lambda_{ij}(t,\tau) = 0.$$

$\lambda_{ij}(t,\tau)$ is a Lagrangean for the constraint $x_{ij}(t,\tau) \geq 0$ and $\lambda_{ij}(t,\tau) \geq 0$ for all i, j, t, and τ. Another set of implicit constraints for the above problem is $p_i(t), p^i(t) \geq 0$ for all i and t.

An economic interpretation of these equilibrium conditions is that prices between any two regions in time period t can differ at most by the cost of transportation and in equilibrium for those regions for which flows take place $p_j(t) - p^i(t) = h_{ij}(t)$. When $p_j(t) - p^i(t) < h_{ij}(t)$, no flows take place. The difference in price between any two regions and any two time periods

[5] The following inequality system corresponds to the Kuhn-Tucker optimality conditions for nonlinear programming. H. W. Kuhn and A. W. Tucker, "Non-Linear Programming," in *Proceedings of the Second Symposium on Mathematical Statistics and Probability*, J. Neyman (ed.), University of California Press, 1951, pp. 481–492.

can differ at most by the cost of transportation plus storage and in equilibrium for those regions for which intertemporal flows take place $p_j(t) - p^i(\tau) = h_{ij}(\tau) + b_i(t, \tau)$. When no flows take place, the inequality holds. These conditions are consistent with dynamic competitive behavior.

Given (3) and the necessary conditions for maximizing net social payoff over space and time, an equivalent programming formulation of the problem in the domain of price is as follows:

To maximize

$$(9) \qquad f(P) = \sum_i \sum_t \int_0^{p_i(t)} d_i(t) dp_i(t) - \int_0^{p^i(t)} s_i(t) dp^i(t)$$

subject to

$$(10) \qquad p_j(t) - p^i(\tau) - h_{ij}(\tau) - b_i(t, \tau) \leq 0$$

and

$$(11) \qquad p_j(t),\ p^i(t) \geq 0 \text{ for all i, j and t.}$$

The above problem can be rewritten as:

To maximize

$$(12) \quad f(P) = \sum_t \sum_i \alpha_i(t) p_i(t) - \frac{1}{2} \sum_t \sum_i \beta_i(t) (p_i(t))^2 - \sum_i \sum_t \theta_i(t) p^i(t) - \frac{1}{2} \sum_i \sum_t \gamma_i(t) (p^i(t))^2$$

or

$$f(P) = C'P - \tfrac{1}{2} P'QP$$

subject to

$$(13) \quad G'(n, T)P \leq R^{6} \text{ and } \qquad (14) \quad P \geq 0.$$

By using (1) a theorem[7] on the reducibility of a nonlinear programming problem, subject to linear constraints, to a linear programming problem with the same constraints and (2) the duality theorem of linear programming, we arrive at the following "primal-dual" programming formulation:

To maximize

$$(15) \qquad g(P, X) = [C - QP]'P - R'X \leq 0$$

[6] The structure of P, Q, G′(n, T) will be made clear in the tableau for the example to follow.
[7] T. Takayama and G. G. Judge, *op. cit.*

subject to

(13) $$G'(n, T)P \leq R,$$

(16) $$G(n, T)X + QP \geq C,$$

(17) $$P, X \geq 0.$$

Since (1) and (2) appear as equations, (16) can be converted to an equality system thus insuring that total supplies equal total demands. To convert (13) into an equality system a nonnegative slack vector V (the counterpart of X) is introduced. This results in the following programming problem:
To maximize (15) or

(18) $$g(X, V) = -V'X \leq 0$$

subject to

(19) $$G'(n, T)P + V = R,$$

(20) $$G(n, T)X + QP = C, \text{ and}$$

(21) $$P, X, V \geq 0.$$

To show the equivalence between the programming problem above and the initial problem with the maximizing vector X which is characterized by (5), (6), (7) and (8), it is sufficient to compare the system (18) through (21) with the other system (1), (2), (5), (6), (7), and (8). Constraints (19), (20) and (21) correspond to (1), (2), (5), (7) and (10), $\lambda_{ij}(t, \tau)$ to V and (6) and (8) to (18) at its maximum. Therefore, if the problem (18) through (21) is solved (i.e., if the joint constraints (18), (19) and (20), are feasible), then the solution exists and satisfies the maximum conditions for NSP and thus it is the solution for the initial problem (1), (2), (3), (4) and (10).

In regard to the existence of a solution, the following assumptions were made in developing the primal-dual programming formulation (18) through (22):

(a) f(P), (9), is continuous and strictly concave with respect to $p_i(t)$ when $p_i(t) = p^i(t)$ for all i, and t.

(b) A set P of all $p_i(t)$ for all i, and t satisfying (4), (10), (11), and (20) is nonvacuous. The set P is convex, closed and bounded, and thus convex and compact.

Due to the Generalized Weierstrass Theorem on the limit of the continuous function on a finite dimensional Cartesian product space and assumptions (a) and (b) above there exists a maximum on the convex compact set P. Thus the optimum solution $\overline{P}$ exists.

If joint constraint sets (19), (20), and (21) are nonempty, then by the duality theorem of linear programming the optimum solution $\overline{X}$ exists satisfying (20) exactly.

Uniqueness of $\overline{P}$ follows from assumption (a), and uniqueness of $\overline{X}$ follows from the Kuhn-Tucker optimality conditions and the fact that the matrix $G(n, T)$ is linearly independent to the full maximal rank of $T(2n-1)$.

The formulation (18) through (21) is consistent with the Wolfe[8] and Barankin-Dorfman[9] algorithm for quadratic programming. The following tableau (Table 1) shows the characteristics of the algorithm for the two regions, two time periods, one product case. In the following tableau, the Z's are artificial variables already in the basis and M is any finite real number. In general the size of the tableau is $[2T(n^2+n)-n^2]\times 2T(n^2+2n)$. The tableau should make apparent the structure of P, C, G(n, T), X, V and Q which appear in (18) through (20).

Summary

In this paper we have attempted to show how equilibrium models involving space can be extended to also cover the time dimension. Under the assumption of perfect foresight concerning the future linear regional demand and supply relations and transportation and storage costs, the problem was converted to a quadratic programming specification which could be solved directly for the competitive time and space equilibrium quantities and prices. Since many commodities are storeable and demands of adjacent periods are substitutes for each other, it is important to have a technique for analyzing how stringency or abundance at one time is transmitted to adjacent time periods and thereby affects the level of demands, prices, and storage over time.

The model has been presented in a general form involving both the time and space dimensions. It could, of course, be reduced to consider only the problem of the time dimension. Also the case of either fixed regional demands or supplies over time could be handled by this specification. When regional demands over time are assumed predetermined, the model has many of the characteristics of the firm production smoothing model discussed by Modigliani and Hohn[10] and later simplified by Manne.[11]

The formulation and the method of analysis have proceeded under the ideal conditions of certainty. This is clearly unsatisfactory in a world where uncertainty is the rule rather than the exception. However, the hope is

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[8] P. Wolfe, "The Simplex Method for Quadratic Programming," *Econometrica*, Vol. 27, 1959, pp. 382–398.

[9] E. W. Barankin and R. Dorfman, *On Quadratic Programming*, University of California Publications in Statistics, 1958.

[10] F. Modigliani and F. E. Hohn, "Production Planning over Time," *Econometrica*, Vol. 23, 1955, pp. 46–66.

[11] A. S. Manne, "A Note on the Modigliani-Hohn Smoothing Model," *Management Science*, Vol. 3, 1957, pp. 371–379.

Table 1. Spatial and time equilibrium simplex tableau[a]

	C_j		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
C_i		Po	x_{11}^1	x_{12}^1	x_{21}^1	x_{22}^1	x_{11}^2	x_{12}^2	x_{21}^2	x_{22}^2	x_{11}^{12}	x_{12}^{12}	x_{21}^{12}	x_{22}^{12}	p_1^1	p_2^1	p^{11}	p^{12}	p_1^2	p_2^2	p^{21}	p^{22}	v_{11}^1	v_{12}^1	v_{21}^1	v_{22}^1	v_{11}^2	v_{12}^2	v_{21}^2	v_{22}^2	v_{11}^{12}	v_{12}^{12}	v_{21}^{12}	v_{22}^{12}
$-M$	Z_1	h_{11}^1													1		-1					**	1											
$-M$	Z_2	h_{12}^1														1	-1							1										
$-M$	Z_3	h_{21}^1													1			-1							1									
$-M$	Z_4	h_{22}^1														1		-1								1								
$-M$	Z_5	h_{11}^2																	1		-1						1							
$-M$	Z_6	h_{12}^2																		1	-1							1						
$-M$	Z_7	h_{21}^2																	1			-1							1					
$-M$	Z_8	h_{22}^2																		1		-1								1				
$-M$	Z_9	$h_{11}^1+b_1^{12}$															-1		1												1			
$-M$	Z_{10}	$h_{12}^1+b_2^{12}$															-1			1												1		
$-M$	Z_{11}	$h_{21}^1+b_1^{12}$																-1	1														1	
$-M$	Z_{12}	$h_{22}^1+b_2^{12}$																-1		1														1
$-M$	Z_{13}	α_1^1	1		1									*	β_1^1							***												
$-M$	Z_{14}	α_2^1		1		1										β_2^1																		
$-M$	Z_{15}	$-\theta_1^1-\delta_1$	-1	-1							-1	-1					γ_1^1																	
$-M$	Z_{16}	$-\theta_2^1-\delta_2$			-1	-1							-1	-1				γ_2^1																
$-M$	Z_{17}	$\alpha_1^2+\mu_1$					1		1		1		1						β_1^2															
$-M$	Z_{18}	$\alpha_2^2+\mu_2$						1		1		1		1						β_2^2														
$-M$	Z_{19}	$-\theta_1^2$					-1	-1													γ_1^2													
$-M$	Z_{20}	$-\theta_2^2$							-1	-1												γ_2^2												

[a] In the tableau $x_{ij}(t, \tau) = x_{ij}^{t\tau}$ for $t \neq \tau$ and x_{ij}^t for $t = \tau$; $v_{ij}(t, \tau) = v_{ij}^{t\tau}$ for $t \neq \tau$, and v_{ij}^t for $t = \tau$; $p_i(t) = p_i^t$; $p^j(t) = p^{jt}$; $h_{ij}(t) = h_{ij}^t$; $b_i(t, \tau) = b_i^{t\tau}$; $\alpha_i(t) = \alpha_i^t$; $\theta_i(t) = \theta_i^t$; $\beta_i(t) = \beta_i^t$; $\gamma_i(t) = \gamma_i^t$. The submatrices in the tableau denoted by *, **, and *** are G(2, 2), G′(2, 2), and Q.

that such an idealized model will help set the stage for fruitful analysis of more realistic markets and in its present form will be applicable for certain meaningful analyses. Also, the model presented is a finite horizon model. Since the future is infinitely long, we have maximized (3) over a finite time span by specifying a final level of inventory and obtaining a maximizing solution for such a level. To obtain the "right" final level of inventory, the question can only be answered by extending our time period further. This, however, leads us into an infinite regression, which means we must go beyond any finite time period. One practical way around this difficulty is to either obtain maximizing solutions for alternative levels of final inventories or to set up the objective function so as to minimize the deviations from the highest attainable level of payoff or satisfaction.[12]

The formulation is amenable to a variety of time ordered production patterns and thus should have many applications to practical problems of price adjustment and allocation over time and the algorithm suggested is efficient for obtaining a solution.

[12] P. S. Samuelson, *op. cit.*, pp. 214–215.

Discussion: An Intertemporal Price Equilibrium Model

R. A. SCHRIMPER

The extended model presented by Takayama and Judge greatly expands the scope of results obtained by programming solutions to spatial and temporal allocation problems. Many of the other models that have been developed in this area are now seen to be simply special cases of their more general formulation. For example, as noted in the paper, either the spatial or temporal dimension of the model can be collapsed and the problem under consideration could then be analyzed in terms of the remaining dimension. There are undoubtedly other refinements that could be added to the model to further enhance its generality, perhaps though, at the expense of some simplicity. This discussion is directed to possible refinements or alterations of the model in the following three areas: (1) discounting the objective function, (2) locating the storage activities, and (3) permitting storage costs to vary with the initial price of the commodity.

The Takayama and Judge model reemphasizes the fact that any allocation decision at the macro level or even the micro level is dated, and the

R. A. SCHRIMPER *is instructor in agricultural economics at North Carolina State College.*