Rábalabaxúrias
UFMG
Theoretical Guide

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1 Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Geometric Series:

$$a_n = a_1 * c^{n-1} \qquad \sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2}-(n+1)c^{n+1}+c}{(c-1)^2}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

2 Identities

2.1 Combinatorics

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$	$\sum_{k=0}^{n} \binom{n}{k} = 2^n$
$\binom{n}{k} = \binom{n}{n-k}$	$\binom{n}{k} = \frac{n}{k}\binom{n-k}{m-k}$
$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$	$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$
$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}$	$\sum_{k=0}^{n} \binom{k}{k} = \binom{n+1}{m+1}$
$\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$	

2.2 Newton's Binomial

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

2.3 Bits Manipulation

x & (x - 1) : Turn off the rightmost 1-bit in a word, producing 0 if none (e.g., 01011000 → 01010000). This can be used to determine if an unsigned integer is a power of 2 or is 0: apply the formula followed by a 0-test on the result.

x | (x + 1) : Turn on the rightmost 0-bit in a word, producing all 1's if none (e.g., 10100111 → 10101111)

x & (x + 1) : Turn off the trailing 1's in a word, producing **x** if none (e.g., 10100111 → 10100000)

x | (x - 1) : Turn on the trailing 0's in a word, producing x if none (e.g., 10101000 → 10101111)

∼x & (x + 1) : Create a word with a single 1-bit at the position of the rightmost 0-bit in **x**, producing 0 if none (e.g., 10100111 → 00001000)

∼x | (x - 1) : Create a word with a single 0-bit at the position of the rightmost 1-bit in **x**, producing all 1's if none (e.g., 10101000 → 11110111)

∼x & (x - 1) or ∼(x | -x) : Create a word with 1's at the positions of the trailing 0's in **x**, and 0's elsewhere, producing 0 if none (e.g., 01011000 → 00000111)

∼x | (x + 1) : Create a word with 0's at the positions of the trailing 1's in **x**, and 1's elsewhere, producing all 1's if none (e.g., 10100111 → 11111000)

x & (-x) : Isolate the rightmost 1-bit, producing 0 if none (e.g., 01011000 → 00001000)

x ⊕ (x - 1) : Create a word with 1's at the positions of the rightmost 1-bit and the trailing 0's in **x**, producing all 1's if no 1-bit, and integer 1 if no trailing 0's (e.g., 01011000 → 00001111)

x ⊕ (x + 1) : Create a word with 1's at the positions of the rightmost 0-bit and the traling 1's in **x**, producing all 1's if no 0-bit, and the integer 1 if no trailing 1's (e.g., 01010111 → 00001111)

((x & (-x)) + x) & x : Turn off the rightmost contiguous string of 1's (e.g., 01011100 → 01000000)

Index of MSB(x): `__builtin_clz(0) - __builtin_clz(x) - 1`

Index of LSB(x): `__builtin_ctz(x)`

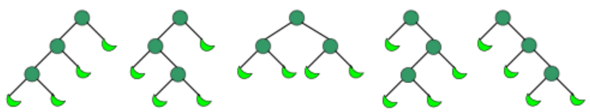
2.4 Catalan Numbers

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360, 1002242216651368.

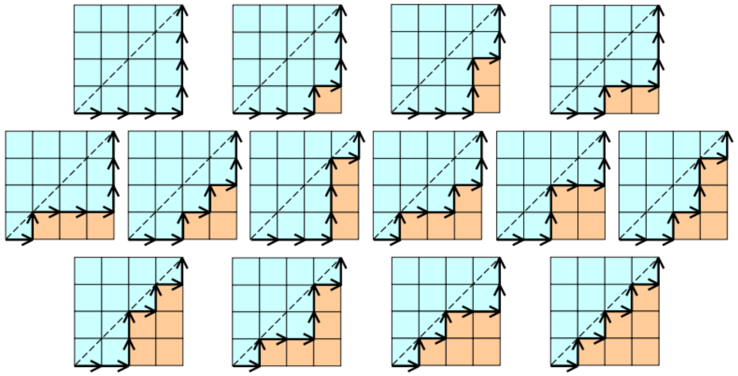
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n} \frac{n+k}{k}, \quad n \geq 0$$

Applications:

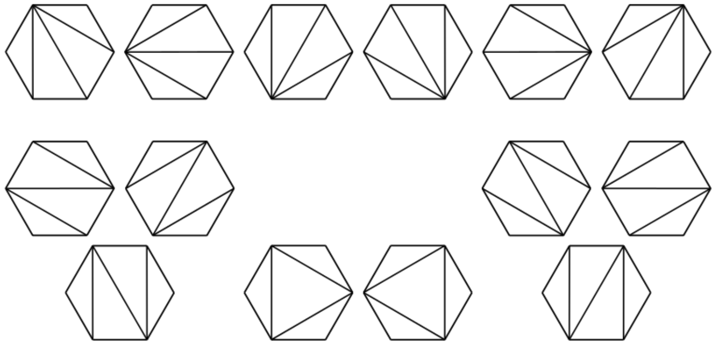
- C_n is the number of Dyck words of length 2n. A Dyck word is a string consisting of n Xs and n Ys such that no initial segment of the string has more Ys than Xs. For example, the following are the Dyck words of length 6.
  XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY
- Re-interpreting the symbol X as an open parenthesis and Y as a close parenthesis, Cn counts the number of expressions containing n pairs of parentheses which are correctly matched.
  ((())) ()(()) ()()() (())() ...
- Successive applications of a binary operator can be represented in terms of a full binary tree. (A rooted binary tree is full if every vertex has either two children or no children.) It follows that C_n is the number of full binary trees with n + 1 leaves:



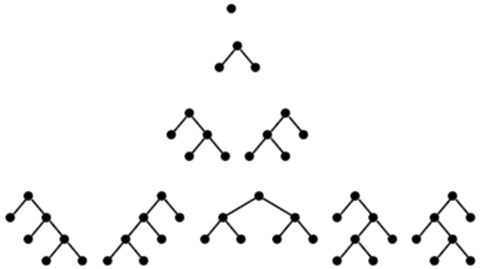
- C_n is the number of monotonic lattice paths along the edges of a grid with n x n square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards. Counting such paths is equivalent to counting Dyck words: X stands for “move right" and Y stands for “move up". The following diagrams show the case n = 4:



- C_n is the number of different ways a convex polygon with n + 2 sides can be cut into triangles by connecting vertices with straight lines (a form of Polygon triangulation). The following hexagons illustrate the case n = 4:



- C_n is the number of rooted binary trees with n internal nodes (n + 1 leaves or external nodes). Illustrated in following Figure are the trees corresponding to n = 0,1,2 and 3. There are 1, 1, 2, and 5 respectively. Here, we consider as binary trees those in which each node has zero or two children, and the internal nodes are those that have children.



Binary Trees

- C_n is the number of different ways n + 1 factors can be completely parenthesized (or the number of ways of associating n applications of a binary operator). For n = 3, for example, we have the following five different parenthesizations of four factors:
  ((ab)c)d   (a(bc))d   (ab)(cd)   a((bc)d)   a(b(cd))

3 Number Theory

Maximal Prime Gaps:
For numbers until 10^9 the maximal gap is 400.
For numbers until 10^{18} the maximal gap is 1500.

Number of prime numbers in intervals:
There is aprox. $8 * 10^4$ primes between 1 e 10^6.
There is aprox. $6 * 10^5$ primes between 1 e 10^7.
There is aprox. $5 * 10^6$ primes between 1 e 10^8.
There is aprox. $5 * 10^7$ primes between 1 e 10^9.

Primes less than 1000:
2 3 5 7 11 13 17 19 23 29 31 37
41 43 47 53 59 61 67 71 73 79 83 89
97 101 103 107 109 113 127 131 137 139 149 151
157 163 167 173 179 181 191 193 197 199 211 223
227 229 233 239 241 251 257 263 269 271 277 281
283 293 307 311 313 317 331 337 347 349 353 359
367 373 379 383 389 397 401 409 419 421 431 433

439	443	449	457	461	463	467	479	487	491	499	503
509	521	523	541	547	557	563	569	571	577	587	593
599	601	607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733	739	743
751	757	761	769	773	787	797	809	811	821	823	827
829	839	853	857	859	863	877	881	883	887	907	911
919	929	937	941	947	953	967	971	977	983	991	997

Other primes:
The largest prime smaller than 10 is 7.
The largest prime smaller than 10^2 is 97.
The largest prime smaller than 10^3 is 997.
The largest prime smaller than 10^4 is 9973.
The largest prime smaller than 10^5 is 99991.
The largest prime smaller than 10^6 is 999983.
The largest prime smaller than 10^7 is 9999991.
The largest prime smaller than 10^8 is 99999989.
The largest prime smaller than 10^9 is 999999937.
The largest prime smaller than 10^{10} is 9999999967.
The largest prime smaller than 10^{11} is 99999999977.
The largest prime smaller than 10^{12} is 999999999989.
The largest prime smaller than 10^{13} is 9999999999971.
The largest prime smaller than 10^{14} is 99999999999973.
The largest prime smaller than 10^{15} is 999999999999989.
The largest prime smaller than 10^{16} is 9999999999999937.
The largest prime smaller than 10^{17} is 99999999999999997.
The largest prime smaller than 10^{18} is 999999999999999989.

Random Primes

1000000009	1000000021	1000000033	1000000087	1000000093
1000000097	1000000103	1000000123	1000000181	1000000207
1000000223	1000000241	1000000271	1000000289	1000000297
1000000321	1000000349	1000000363	1000000403	1000000409
2000003273	2000003281	2000003293	2000003303	2000003333
2000003351	2000003353	2000003359		

Metodo de Newton

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- O numero de casas decimais corretas, em media, dobra a cada iteracao. (Em caso de duvida, default=30).

Metodo dos Quadrados Minimos
Queremos estimar valores de determinada variável y. Para isso, consideramos os valores de outra variavel x que acreditamos ter poder de explicação sobre y conforme a fórmula:

$$y = \alpha + \beta x + \epsilon$$

- α: Parametro do modelo chamado de constante (porque não depende de x).
- β: Parametro do modelo chamado de coeficiente da variavel x.
- ϵ: Erro - representa a variação de y que nao eh explicada pelo modelo.

Temos tambem uma base de dados com n valores observados de y e x. Ao fazer a estimativa de α, β, e ϵ, mudamos a notação de algumas variáveis:
$\alpha \rightarrow$ a.
$\beta \rightarrow$ b.
$\epsilon \rightarrow$ e.
Desse modo, ao estimar o modelo usando a base de dados, estamos estimando, na verdade:

$$y_i = a + bx_i + e_i$$

onde i indica cada uma das n observacoes da base de dados e e passa a ser chamado de residuo, ao inves de erro. O metodo dos minimos quadrados minimiza a soma dos quadrado dos residuos, ou seja, minimiza $\sum_{i=1}^{n} e_i^2$.
Minimizando S(a, b) $= \sum_{i=1}^{n}(y_i - a - bx_i)^2$ temos que:

$$a = \bar{y} - b\bar{x}$$

$$b = \frac{\sum_{i=1}^{n} x_i(y_i - \bar{y})}{\sum_{i=1}^{n} x_i(x_i - \bar{x})} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

onde $\bar{y}$ eh a media amostral de y e $\bar{x}$ eh a media amostral de x.

Funcao de Ackermann-Peter

$$A(m,n) = \begin{cases} n+1 & se \quad m = 0 \\ A(m-1, 1) & se \quad m > 0 \quad e \quad n = 0. \\ A(m-1, A(m, n-1)) & se \quad m > 0 \quad e \quad n > 0. \end{cases}$$

- A(4, 2) tem 19729 digitos.

m\n	0	1	2
0	1	2	3
1	2	3	4
2	3	5	7
3	5	13	29
4	$13 = 2^{2^2} - 3$	$65533 =$ 2**3 - 3	$2^{65536} - 3 =$ 2**4 -3

m\n	3	4	n
0	4	5	$n+1$
1	5	6	$n+2 = 2+(n+3)-3$
2	9	11	$2n+3 = 2(n+3)-3$
3	61	125	$2^{n+3} - 3$
4	$2^{2^{2^{65536}}} - 3 =$ 2**5 - 3	2**6 - 3	2**(n+ 2) - 3

Obs.: 2**n $= 2^{2^{\cdots}}$ 2, elevado a 2, elevado a 2... n vezes.
Ex.: 2**1 $= 2^2$; 2**2 $= 2^{2^2}$

Numeros de Mersenne

$$\mathcal{M}(n) = 2^n - 1$$

onde n eh um inteiro nao-negativo. Os numeros da forma $\mathcal{M}(n)$ sao conhecidos como numeros de Mersenne.
Um numero eh chamado de perfeito se eh igual a metade da soma de todos os seus divisores positivos. Por exemplo, os divisores de 6 sao 1, 2, 3 e 6. Somando estes numeros obtemos:

$$1 + 2 + 3 + 6 = 12 = 2 * 6$$

Logo 6 eh perfeito. Em comparacao, nenhum primo eh perfeito. Todos os perfeitos pares sao da forma $2^{n-1}2(2^n - 1)$ com $2^n - 1$ primo. Assim, para achar os perfeitos pares basta achar os primos de Mersenne. Ninguem sabe se existem ou nao numeros perfeitos impares.

- Se n eh um inteiro composto, entao $\mathcal{M}(n)$ eh composto.

Teoremas de Fermat

Seja P um numero primo e a um numero inteiro, entao:

$$a^p \equiv a \ (mod\ p)$$

$$a^{p-1} \equiv 1 \ (mod\ p)$$

Lema: Seja p um numero primo e a e b inteiros. Entao:

$$(a+b)^p = a^p + b^p \ (mod\ p)$$

Lema: Seja p um numero primo e a inteiro. Entao, O inverso de a mod p eh a^{p-2}:

$$inv(a) \equiv a^{p-2} \ (mod\ p)$$

Highly Composite Numbers
Highly Composite Numbers menores que 2*10^9

n	d(v)	n	d(v)	n	d(v)
1	1	2	2	4	3
6	4	12	6	24	8
36	9	48	10	60	12
120	16	180	18	240	20
360	24	720	30	840	32
1260	36	1680	40	2520	48
5040	60	7560	64	10080	72
15120	80	20160	84	25200	90
27720	96	45360	100	50400	108
55440	120	83160	128	110880	144
166320	160	221760	168	277200	180
332640	192	498960	200	554400	216
665280	224	720720	240	1081080	256
1441440	288	2162160	320	2882880	336
3603600	360	4324320	384	6486480	400
7207200	432	8648640	448	10810800	480
14414400	504	17297280	512	21621600	576
32432400	600	36756720	640	43243200	672
61261200	720	73513440	768	110270160	800
122522400	864	147026880	896	183783600	960
245044800	1008	294053760	1024	367567200	1152
551350800	1200	698377680	1280	735134400	1344
1102701600	1440	1396755360	1536		

Maximum number of divisors

r \ c	1	2	3	4	5	6	7	8	9
0	1	2	2	3	3	4	4	4	4
1	4	6	8	9	10	12	12	12	12
2	12	18	20	24	24	24	24	30	32
3	32	40	48	48	48	60	60	64	64
4	64	80	96	96	100	120	120	120	128
5	128	160	180	192	200	216	224	240	240
6	240	288	336	360	384	384	400	432	448
7	448	512	576	640	672	672	720	768	768
8	768	960	1024	1152	1152	1200	1280	1344	1344
9	1344	1536	1792	1920	2016	2048	2304	2304	2304
10	2304	2688	3072	3072	3456	3456	3584	3600	3840
11	4032	4608	5040	5376	5760	5760	6144	6144	6144
12	6720	7680	8064	8640	9216	9216	10080	10080	10368
13	10752	12288	13440	13824	14400	15360	16128	16128	16128
14	17280	20160	21504	23040	23040	24576	24576	25920	26880
15	26880	30720	32768	34560	36864	36864	40320	40320	41472
16	41472	48384	51840	55296	57600	57600	61440	64512	64512
17	64512	73728	82944	86016	92160	92160	96768	98304	103680
18	103680	115200	124416	129024	138240	138240	147456	147456	153600

Upper bound on the number of divisors of the integers in $[1, c10^r]$.

4 Geometry

Conversao - angulos:

De radiano para grau:

grau = radiano*(180/PI)

De grau para radiano:

radiano = grau*(PI/180)

Triplas Pitagóricas

Uma tripla pitagórica consiste de três inteiros, a, b e c tal que $a^2 + b^2 = c^2$.
A fórmula de Euclides pode ser usada para gerar uma tripla pitagórica dado um par arbitrário de inteiros positivos m e n onde $m > n$:
$a = m^2 - n^2$
$b = 2mn$
$c = m^2 + n^2$

Teorema de Herao

A area de um triângulo pode ser dada por: A = $\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$ em que s representa o semi-perimetro do triângulo e a, b e c são os comprimentos de seus tres lados.

Central Angle

Let ϕ_1, λ_1 and ϕ_2, λ_2 be the geographical latitude and longitude of two points 1 and 2, and $\Delta\phi$, $\Delta\lambda$ their absolute differences; then $\Delta\sigma$, the central angle between them, is given by the spherical law of cosines:
$\Delta\sigma = arccos(sin\phi_1\, sin\phi_2 + cos\phi_1\, cos\phi_2 cos\Delta\lambda)$. The distance d, i.e. the arc-length, for a sphere of radius r and $\Delta\sigma$ given in $d = r\Delta\sigma$.

Condicao de Existencia de Poligonos

- Generalizacao da desigualdade triangular
- Given a collection of positive numbers r_1,...,r_n, there exists a polygon in R^2 with the side-lengths r_1,...,r_n if and only if for every i, r_i $< 1/2$ * (r_1 + ... + r_n)

Lei dos Cossenos $a^2 = b^2 - c^2 - 2\,b\,c\,cosA$
Lei dos Senos $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

5 Probability

Expectation

$$E[X] = \sum_{i=1}^{\infty} x_i$$

Bayes Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

5.1 Distributions

5.1.1 Binomial

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad E[X] = n * p$$

- Calcula a probabilidade de ter K sucessos em n tentativas.

5.1.2 Geometric

- X eh o numero de fracassos em uma sequencia de ensaios independentes de Bernoulli ate que o primeiro sucesso seja observado.

$$P(X = k) = (1 - p)^k p$$

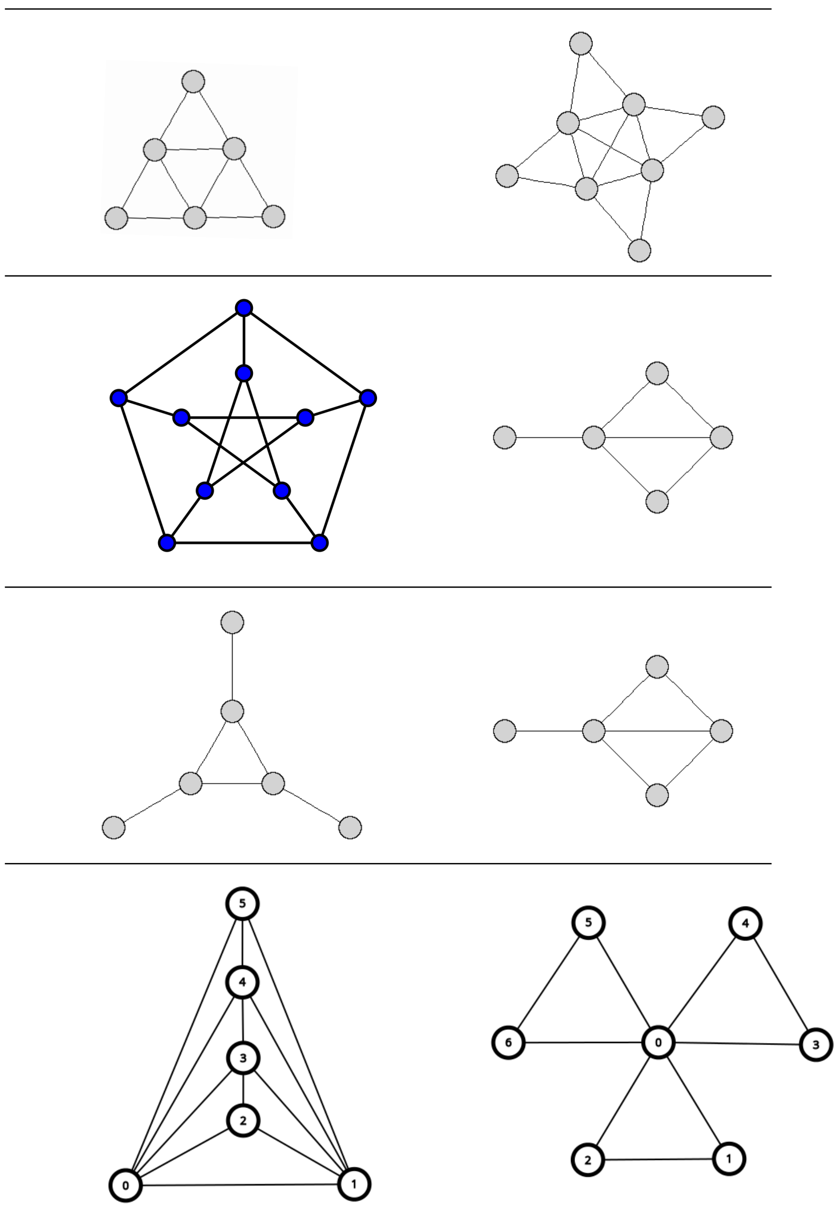
- Ou, se pensar que X eh o numero de tentativas ate o primeiro acerto...

$$P(X = k) = (1 - p)^{k-1} p$$

Expectation: $E[X] = 1/p$

6 Graphs

6.1 Random Graphs



6.2 Graph Classes

6.2.1 Planar Graphs

1. if G is a planar graph, $m \leq 3n - 6$.

2. if G is a planar graph, the minimum degree is less or equal 5. Hence, if G is planar, G has a vertex with degree 5 or less.

6.3 Theorems and Properties

6.3.1 Counting Minimum Spanning Trees

Cayley's Formula
Cayley's Formula tells us how many different trees we can construct on n vertices. For a K_n graph, there is n^{n-2} minimum spanning trees.

Complete Bipartite Graphs
If G is the complete bipartite graph $K_{p,q}$ then t(G) $= p^{q-1}q^{p-1}$.

Arbitrary Graphs
More generally, for any graph G, the number t(G) can be calculated in polynomial time as the determinant of a matrix derived from the graph, using Kirchhoff's matrix-tree theorem. Specifically, to compute t(G), one constructs a square matrix in which the rows and columns are both indexed by the vertices of G. The entry in row i and column j is one of three values:

- The degree of vertex i, if i = j,
- -1, if vertices i and j are adjacent, or
- 0, if vertices i and j are different from each other but not adjacent.

The resulting matrix is singular, so its determinant is zero. However, deleting the row and column for an arbitrarily chosen vertex leads to a smaller matrix whose determinant is exactly t(G).

6.3.2 Prufer's Enconding

The Prufer sequence of a labeled tree is a unique sequence associated with the tree. The sequence for a tree on n vertices has length n - 2, and can be generated by a iterative algorithm.

6.4 Erdos-Gallai Theorem

A sequence of non-negative integers $d_1 \geq ... \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + ... + d_n$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min(d_i, k)$$

holds for every k in $1 \leq k \leq n$.

7 Strings

Dec	Hex	Oct	Chr	Dec	Hex	Oct	HTML	Chr	Dec	Hex	Oct	HTML	Chr	Dec	Hex	Oct	HTML	Chr
0	0	000	NULL	32	20	040		Space	64	40	100	@	@	96	60	140	`	`
1	1	001	Start of Header	33	21	041	!	!	65	41	101	A	A	97	61	141	a	a
2	2	002	Start of Text	34	22	042	"	"	66	42	102	B	B	98	62	142	b	b
3	3	003	End of Text	35	23	043	#	#	67	43	103	C	C	99	63	143	c	c
4	4	004	End of Transmission	36	24	044	$	$	68	44	104	D	D	100	64	144	d	d
5	5	005	Enquiry	37	25	045	%	%	69	45	105	E	E	101	65	145	e	e
6	6	006	Acknowledgment	38	26	046	&	&	70	46	106	F	F	102	66	146	f	f
7	7	007	Bell	39	27	047	'	'	71	47	107	G	G	103	67	147	g	g
8	8	010	Backspace	40	28	050	(	(	72	48	110	H	H	104	68	150	h	h
9	9	011	Horizontal Tab	41	29	051	)	)	73	49	111	I	I	105	69	151	i	i
10	A	012	Line feed	42	2A	052	*	*	74	4A	112	J	J	106	6A	152	j	j
11	B	013	Vertical Tab	43	2B	053	+	+	75	4B	113	K	K	107	6B	153	k	k
12	C	014	Form feed	44	2C	054	,	,	76	4C	114	L	L	108	6C	154	l	l
13	D	015	Carriage return	45	2D	055	-	-	77	4D	115	M	M	109	6D	155	m	m
14	E	016	Shift Out	46	2E	056	.	.	78	4E	116	N	N	110	6E	156	n	n
15	F	017	Shift In	47	2F	057	/	/	79	4F	117	O	O	111	6F	157	o	o
16	10	020	Data Link Escape	48	30	060	0	0	80	50	120	P	P	112	70	160	p	p
17	11	021	Device Control 1	49	31	061	1	1	81	51	121	Q	Q	113	71	161	q	q
18	12	022	Device Control 2	50	32	062	2	2	82	52	122	R	R	114	72	162	r	r
19	13	023	Device Control 3	51	33	063	3	3	83	53	123	S	S	115	73	163	s	s
20	14	024	Device Control 4	52	34	064	4	4	84	54	124	T	T	116	74	164	t	t
21	15	025	Negative Ack.	53	35	065	5	5	85	55	125	U	U	117	75	165	u	u
22	16	026	Synchronous idle	54	36	066	6	6	86	56	126	V	V	118	76	166	v	v
23	17	027	End of Trans. Block	55	37	067	7	7	87	57	127	W	W	119	77	167	w	w
24	18	030	Cancel	56	38	070	8	8	88	58	130	X	X	120	78	170	x	x
25	19	031	End of Medium	57	39	071	9	9	89	59	131	Y	Y	121	79	171	y	y
26	1A	032	Substitute	58	3A	072	:	:	90	5A	132	Z	Z	122	7A	172	z	z
27	1B	033	Escape	59	3B	073	;	;	91	5B	133	[	[	123	7B	173	{	{
28	1C	034	File Separator	60	3C	074	<	<	92	5C	134	\	\	124	7C	174			|
29	1D	035	Group Separator	61	3D	075	=	=	93	5D	135	]	]	125	7D	175	}	}
30	1E	036	Record Separator	62	3E	076	>	>	94	5E	136	^	^	126	7E	176	~	~
31	1F	037	Unit Separator	63	3F	077	?	?	95	5F	137	_	_	127	7F	177	&#127;	Del

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$i \backslash j$	1	2	3	4	5
1	2.0000	1.6181	1.4656	1.3803	1.3248
2	1.6181	1.4143	1.3248	1.2721	1.2366
3	1.4656	1.3248	1.2560	1.2208	1.1939
4	1.3803	1.2721	1.2208	1.1893	1.1674
5	1.3248	1.2366	1.1939	1.1674	1.1487

Branching factors of binary branching vectors $\tau(i, j)$, rounded up.

8 Other

8.1 Branching factors

8.2 Prime counting function ($\pi(x)$)

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1.229	9.592	78.498	664.579	5.761.455

8.3 Partition function

The partition function $p(x)$ counts show many ways there are to write the integer x as a sum of integers.

x	36	37	38	39	40	41	42
p(x)	17.977	21.637	26.015	31.185	37.338	44.583	53.174
x	43	44	45	46	47	100	
p(x)	63.261	75.175	89.134	105.558	125.754	190.569.292	

8.4 Catalan numbers

Catalan numbers are defined by the recurrence:

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

A closed formula for Catalan numbers is:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$$

8.5 Stirling numbers of the first kind

These are the number of permutations of I_n with exactly k disjoint cycles. They obey the recurrence:

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

8.6 Stirling numbers of the second kind

These are the number of ways to partition I_n into exactly k sets. They obey the recurrence:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}$$

A “closed” formula for it is:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n$$

8.7 Bell numbers

These count the number of ways to partition I_n into subsets. They obey the recurrence:

$$\mathcal{B}_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k$$

x	5	6	7	8	9	10	11	12
$\mathcal{B}_x$	52	203	877	4.140	21.147	115.975	678.570	4.213.597

8.8 Turán’s theorem

No graph with n vertices that is K_{r+1}-free can have more edges than the Turán graph: A k-partite complete graph with sets of size as equal as possible.

8.9 Generating functions

A list of generating functions for useful sequences:

$(1, 1, 1, 1, 1, 1, \ldots)$	$\frac{1}{1-z}$
$(1, -1, 1, -1, 1, -1, \ldots)$	$\frac{1}{1+z}$
$(1, 0, 1, 0, 1, 0, \ldots)$	$\frac{1}{1-z^2}$
$(1, 0, \ldots, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, \ldots)$	$\frac{1}{1-z^2}$
$(1, 2, 3, 4, 5, 6, \ldots)$	$\frac{1}{(1-z)^2}$
$(1, \binom{m+1}{m}, \binom{m+2}{m}, \binom{m+3}{m}, \ldots)$	$\frac{1}{(1-z)^{m+1}}$
$(1, c, \binom{c+1}{2}, \binom{c+2}{3}, \ldots)$	$\frac{1}{(1-z)^c}$
$(1, c, c^2, c^3, \ldots)$	$\frac{1}{1-cz}$
$(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$	$\ln \frac{1}{1-z}$

A neat manipulation trick is:

$$\frac{1}{1-z}G(z) = \sum_{n} \sum_{k \leq n} g_k z^n$$

8.10 Polyominoes

How many free (rotation, reflection), one-sided (rotation) and fixed n-ominoes are there?

n	3	4	5	6	7	8	9	10
free	2	5	12	35	108	369	1.285	4.655
one-sided	2	7	18	60	196	704	2.500	9.189
fixed	6	19	63	216	760	2.725	9.910	36.446

8.11 The twelvefold way (from Stanley)

How many functions $f\colon N \to X$ are there?

N	X	Any f	Injective	Surjective
dist.	dist.	x^n	$(x)_n$	$x!\left\{ {n \atop x} \right\}$
indist.	dist.	$\binom{x+n-1}{n}$	$\binom{x}{n}$	$\binom{n-1}{n-x}$
dist.	indist.	$\left\{ {n \atop 1} \right\} + \ldots + \left\{ {n \atop x} \right\}$	$[n \leq x]$	$\left\{ {n \atop k} \right\}$
indist.	indist.	$p_1(n) + \ldots p_x(n)$	$[n \leq x]$	$p_x(n)$

Where $\binom{a}{b} = \frac{1}{b!}(a)_b$ and $p_x(n)$ is the number of ways to partition the integer n using x summands.

8.12 Common integral substitutions

And finally, a list of common substitutions:

$\int F(\sqrt{ax+b})dx$	$u = \sqrt{ax+b}$	$\frac{2}{a}\int uF(u)du$
$\int F(\sqrt{a^2-x^2})dx$	$x = a\sin u$	$a\int F(a\cos u)\cos u du$
$\int F(\sqrt{x^2+a^2})dx$	$x = a\tan u$	$a\int F(a\sec u)\sec^2 u du$
$\int F(\sqrt{x^2-a^2})dx$	$x = a\sec u$	$a\int F(a\tan u)\sec u\tan u du$
$\int F(e^{ax})dx$	$u = e^{ax}$	$\frac{1}{a}\int \frac{F(u)}{u}du$
$\int F(\ln x)dx$	$u = \ln x$	$\int F(u)e^u du$

8.13 Table of non-trigonometric integrals

Some useful integrals are:

$\int \frac{dx}{x^2+a^2}$	$\frac{1}{a}\arctan\frac{x}{a}$
$\int \frac{dx}{x^2-a^2}$	$\frac{1}{2a}\ln\frac{x-a}{x+a}$
$\int \frac{dx}{a^2-x^2}$	$\frac{1}{2a}\ln\frac{a+x}{a-x}$
$\int \frac{dx}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a}$
$\int \frac{dx}{\sqrt{x^2-a^2}}$	$\ln\left(u+\sqrt{x^2-a^2}\right)$
$\int \frac{dx}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\text{arcsec}\left\lvert\frac{u}{a}\right\rvert$
$\int \frac{dx}{x\sqrt{x^2+a^2}}$	$-\frac{1}{a}\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$
$\int \frac{dx}{x\sqrt{a^2+x^2}}$	$-\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)$

8.14 Table of trigonometric integrals

A list of common and not-so-common trigonometric integrals:

$\int \tan x dx$	$-\ln \lvert \cos x \rvert$
$\int \cot x dx$	$\ln \lvert \sin x \rvert$
$\int \sec x dx$	$\ln \lvert \sec x + \tan x \rvert$
$\int \csc x dx$	$\ln \lvert \csc x - \cot x \rvert$
$\int \sec^2 x dx$	$\tan x$
$\int \csc^2 x dx$	$\cot x$
$\int \sin^n x dx$	$\frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$
$\int \cos^n x dx$	$\frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$
$\int \arcsin x dx$	$x \arcsin x + \sqrt{1-x^2}$
$\int \arccos x dx$	$x \arccos x - \sqrt{1-x^2}$
$\int \arctan x dx$	$x \arctan x - \frac{1}{2} \ln \lvert 1 - x^2 \rvert$

8.15 Centroid of a polygon

The x coordinate of the centroid of a polygon is given by $\frac{1}{3A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$, where A is twice the signed area of the polygon.

8.16 Desaranjo

Seja d_n o número de maneiras de permutar uma sequência $1, \ldots, n$ de tal forma que nenhum número fique na sua posição inicial. Temos a recorrência $d_n = (n-1)(d_{n-1} + d_{n-2})$. Além disso, d_n é o inteiro mais próximo de $\frac{n!}{e}$.

8.17 Lagrange

Given a set of $k + 1$ data points

$$(x_0, y_0), \ldots, (x_j, y_j), \ldots, (x_k, y_k)$$

where no two x_j are the same, the **interpolation polynomial in the Lagrange form** is a linear combination

$$L(x) := \sum_{j=0}^{k} y_j \ell_j(x)$$

of Lagrange basis polynomials

$$\ell_j(x) := \prod_{\substack{0 \le m \le k \\ m \ne j}} \frac{x - x_m}{x_j - x_m} = \frac{(x - x_0)}{(x_j - x_0)} \cdots \frac{(x - x_{j-1})}{(x_j - x_{j-1})} \frac{(x - x_{j+1})}{(x_j - x_{j+1})} \cdots \frac{(x - x_k)}{(x_j - x_k)},$$