Theoretical Guide
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UFMG

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1 Identities

1.1 Series

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$$

$$g_k(n) = \sum_{i=1}^{n} i^k = \frac{1}{k+1}\left(n^{k+1} + \sum_{j=1}^{k}\binom{k+1}{j+1}(-1)^{j+1} g_{k-j}(n)\right)$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

1.2 Binomial Identities

$$\tbinom{n}{k} = \tfrac{n}{k}\tbinom{n-1}{k-1} \qquad \tbinom{n-1}{k} - \tbinom{n-1}{k-1} = \tfrac{n-2k}{k}\tbinom{n}{k}$$

$$\tbinom{n}{h}\tbinom{n-h}{k} = \tbinom{n}{k}\tbinom{n-k}{h} \qquad \tbinom{n}{k} = \tfrac{n+1-k}{k}\tbinom{n}{k-1}$$

$$\textstyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad \sum_{k=0}^{n} k^2\binom{n}{k} = (n+n^2)2^{n-2}$$

$$\textstyle\sum_{j=0}^{k}\binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k} \qquad \sum_{j=0}^{m}\binom{m}{j}^2 = \binom{2m}{m}$$

$$\textstyle\sum_{m=0}^{n}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1} \qquad \sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}$$

$$\textstyle\sum_{r=0}^{m}\binom{n+r}{r} = \binom{n+m+1}{m} \qquad \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k} = \mathrm{Fib}(n+1)$$

$$\textstyle(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \qquad (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$\textstyle 2\sum_{i=L}^{R}\binom{n}{i} - \binom{n}{L} - \binom{n}{R} = \sum_{i=L+1}^{R}\binom{n+1}{i}$$

2 Number Theory

2.1 Identities

$$\sum_{d|n}\varphi(d) = n$$

$$\sum_{\substack{i<n \\ \gcd(i,n)=1}} i = n \cdot \frac{\varphi(n)}{2}$$

$$|\{(x,y) :\, 1 \le x, y \le n,\, \gcd(x,y) = 1\}| = \sum_{d=1}^{n}\mu(d)\left\lfloor\frac{n}{d}\right\rfloor^2$$

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\gcd(x,y) = \sum_{k=1}^{n} k \sum_{k|l}\left\lfloor\frac{n}{l}\right\rfloor^2 \mu\left(\frac{l}{k}\right)$$

$$\sum_{x=1}^{n}\sum_{y=x}^{n}\gcd(x,y) = \sum_{g=1}^{n}\sum_{g|d}^{n} g\cdot\varphi\left(\frac{d}{g}\right)$$

$$\sum_{x=1}^{n}\sum_{y=1}^{n}\mathrm{lcm}(x,y) = \sum_{d=1}^{n} d\,\mu(d)\sum_{d|l}^{n} l\binom{\lfloor\frac{n}{l}\rfloor+1}{2}^2$$

$$\sum_{x=1}^{n}\sum_{y=x+1}^{n}\mathrm{lcm}(x,y) = \sum_{g=1}^{n}\sum_{g|d}^{n} d\cdot\varphi\left(\frac{d}{g}\right)\cdot\frac{d}{g}\cdot\frac{1}{2}$$

$$\sum_{x\in A}\sum_{y\in A}\gcd(x,y) = \sum_{t=1}^{n}\left(\sum_{l|t}\frac{t}{l}\mu(l)\right)\left(\sum_{t|a}\mathrm{freq}[a]\right)^2$$

$$\sum_{x\in A}\sum_{y\in A}\mathrm{lcm}(x,y) = \sum_{t=1}^{n}\left(\sum_{l|t}\frac{l}{t}\mu(l)\right)\left(\sum_{a\in A,\, t|a} a\right)^2$$

2.2 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

2.3 Prime counting function - $\pi(x)$

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

2.4 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033
$10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$

2.5 Number of Divisors

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

n	367567200	6983776800	13967553600	321253732800
$d(n)$	1152	2304	2688	5376

$18401055938125660800 \approx 2e18$ is highly composite with 184320 divisors.
For numbers up to 10^{88}, $d(n) < 3.6\sqrt[3]{n}$.

2.6 Lucas's Theorem

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}$$

For p prime. n_i and m_i are the coefficients of the representations of n and m in base p.

2.7 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a+b)^p \equiv a^p + b^p \pmod{p}$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod{p}$$

2.8 Taking modulo at the exponent

If a and m are coprime, then

$$a^n \equiv a^{n \bmod \varphi(m)} \pmod{m}$$

Generally, if $n \geq \log_2 m$, then

$$a^n \equiv a^{\varphi(m) + [n \bmod \varphi(m)]} \pmod{m}$$

2.9 Mobius invertion

If $g(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} g(d)\mu(\frac{n}{d})$.
A more useful definition is: $\sum_{d|n} \mu(d) = [n = 1]$
Example, sum of LCM:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{lcm}(i,j) &= \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}[\gcd(i,j)=k]\frac{ij}{k} \\
&= \sum_{k=1}^{n}\sum_{a=1}^{\lfloor \frac{n}{k}\rfloor}\sum_{b=1}^{\lfloor \frac{n}{k}\rfloor}[\gcd(a,b)=1]abk \\
&= \sum_{k=1}^{n} k\sum_{a=1}^{\lfloor \frac{n}{k}\rfloor} a\sum_{b=1}^{\lfloor \frac{n}{k}\rfloor} b\sum_{d=1}^{\lfloor \frac{n}{k}\rfloor}[d|a]\,[d|b]\,\mu(d) \\
&= \sum_{k=1}^{n} k\sum_{d=1}^{\lfloor \frac{n}{k}\rfloor}\mu(d)\left(\sum_{a=1}^{\lfloor \frac{n}{k}\rfloor}[d|a]\,a\right)\left(\sum_{b=1}^{\lfloor \frac{n}{k}\rfloor}[d|b]\,b\right) \\
&= \sum_{k=1}^{n} k\sum_{d=1}^{\lfloor \frac{n}{k}\rfloor}\mu(d)\left(\sum_{p=1}^{\lfloor \frac{n}{kd}\rfloor} p\right)\left(\sum_{q=1}^{\lfloor \frac{n}{kd}\rfloor} q\right)
\end{aligned}$$

2.10 Chicken McNugget Theorem

Given two **coprime** numbers n, m, the largest number that cannot be written as a linear combination of them is $nm - n - m$.

- There are $\frac{(n-1)(m-1)}{2}$ non-negative integers that cannot be written as a linear combination of n and m;
- For each pair $(k, nm - n - m - k)$, for $k \geq 0$, exactly one can be written.

3 Geometry

3.1 Pythagorean Triples

For all natural a, b, c satisfying $a^2 + b^2 = c^2$ there exist $m, n \in \mathbb{N}$ and $m > n$ such that (reverse is also true):

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

3.2 Heron's Formula

The area of a triangle can be written as $A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$, where a, b, c are the lengths of its sides and $s = \frac{a+b+c}{2}$.
This can be generalized to compute the area A of a quadrilateral with sides a, b, c, d, with $s = \frac{a+b+c+d}{2}$ and α, γ any two opposite angles:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\left(\cos^2\left(\frac{\alpha+\gamma}{2}\right)\right)}$$

3.3 Colinear Points

Three points are colinear on $\mathbb{R}^2$ iff:

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 0$$

The absolute value of this determinant is twice the area of the triangle ABC.

3.4 Coplanar Points

Four points are coplanar in $\mathbb{R}^3$ iff:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0$$

3.5 Trigonometry

3.5.1 Angle Sum

$$\sin(a \pm b) = \sin a \cos b \pm \cos\ \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

3.5.2 Sum-to-Product Transformation

$$\sin a \pm \sin b = 2\sin\frac{a\pm b}{2}\cos\frac{a\mp b}{2}$$

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$$

$$\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$$

$$\tan a \pm \tan b = \frac{\sin(a\pm b)}{\cos a\cos b}$$

3.6 Centroid of a polygon

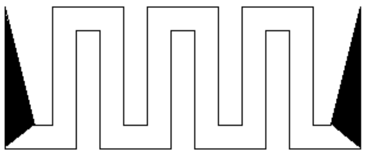
The coordites of the centroid of a non-self-intersecting closed polygon is:

$$\frac{1}{3A}\left(\sum_{i=0}^{n-1}(x_i+x_{i+1})(x_iy_{i+1}-x_{i+1}y_i), \sum_{i=0}^{n-1}(y_i+y_{i+1})(x_iy_{i+1}-x_{i+1}y_i)\right),$$

where A is twice the signed area of the polygon.

3.7 Two Ears Theorem

Every simple polygon with more than 3 vertices has at least two non-overlapping ears (a ear is a vertex whose diagonal induced by its neighbors which lies strictly inside the polygon). Equivalently, every simple polygon can be triangulated. Example of a simple polygon with exactly two ears:



4 Probability

4.1 Moment Generating Functions

Let X be a random variable. Define $M_X(t) = E[e^{tX}]$.

when X is Discrete

$$M_X(t) = \sum_{i=1}^{\infty} e^{tx_i}p_i$$

when X is Continuous

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}f(x)dx$$

Then we have:

$$M_X(0) = 0 \qquad M_X^{'}(0) = E[x] \qquad \frac{d^kM_X(0)}{dt^k} = E[x^k]$$

4.2 Distributions

4.2.1 Binomial

- X is the number of successes in a sequence of n independent experiments.

$$P(X=k) = \binom{n}{k}p^k(1-p)^{n-k} \qquad E[X] = np \qquad Var(X) = np(1-p)$$

4.2.2 Geometric

- X is the number of failures in a sequence of independent experiment of Bernoulli until the first success.

$$P(X=k) = (1-p)^kp \qquad E[X] = \frac{1}{p} \qquad Var(X) = \frac{1-p}{p^2}$$

5 Graphs

5.1 Planar Graphs

1. If G has k connected components, then $n - m + f = k + 1$.
2. $m \leq 3n - 6$. If G has no triangles, $m \leq 2n - 4$.
3. The minimum degree is less or equal 5. And can be 6 colored in $\mathcal{O}(n+m)$.

5.2 Counting Minimum Spanning Trees - $\tau(G)$

- **Cayley's Formula**: $\tau(K_n) = n^{n-2}$.
- **Complete Bipartite Graphs**: $\tau(K_{p,q}) = p^{q-1}q^{p-1}$.
- **Kirchhoff's Theorem**: More generally, if we define the Laplacian matrix $\mathbf{L}(G) = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the diagonal matrix with entries equal to the degree of vertices and $\mathbf{A}$ is the adjacency matrix. For $\mathbf{L}(G)_{ab}$ equal to $\mathbf{L}(G)$ without row a and column b, we have $\tau(G) = \det \mathbf{L}(G)_{ab}$, for any row a and column b.

5.3 Prüfer's Sequence

The Prüfer sequence is a bijection between labeled trees with n vertices and sequences with $n-2$ numbers from 1 to n.
To get the sequence from the tree:

- While there are more than 2 vertices, remove the leaf with smallest label and append it's neighbour to the end of the sequence.

To get the tree from the sequence:

- The degree of each vertex is 1 more than the number of occurrences of that vertex in the sequence. Compute the degree d, then do the following: for every value x in the sequence (in order), find the vertex with smallest label y such that $d(y) = 1$ and add an edge between x and y, and also decrease their degrees by 1. At the end of this procedure, there will be two vertices left with degree 1; add an edge between them.

5.4 Erdős-Gallai Theorem

A sequence of non-negative integers $d_1 \geq ... \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + ... + d_n$ is even and

$$\sum_{i=1}^{k} d_i \ \leq \ k(k-1) + \sum_{i=k+1}^{n} min(d_i, k)$$

holds for every k in $1 \leq k \leq n$.

5.5 Maximum Matching in Complete Multipartite graphs

The size of the maximum matching in a complete multipartite graph with n vertices and k vertices in its largest partition is (reference):

$$|M| = \min\left(\left\lfloor \frac{n}{2} \right\rfloor, n-k\right)$$

5.6 Dilworth's Theorem

5.6.1 Node-disjoint Path Cover

The node disjoint path cover in a `DAG` is equal to $|V| - |M|$, where M is the maximum matching in the bipartite flow network.

5.6.2 General Path Cover

The general path cover in a `DAG` is equal to $|V| - |M|$, where M is the maximum matching in the bipartite flow network of the transitive closure graph.

5.6.3 Dilworth's Theorem

The size of the maximum **antichain** in a `DAG`, that is, the maximum size of a set S of vertices such that no vertex in S can reach another vertex in S, is equal to size of the minimum **general** path cover.

5.7 Sum of Subtrees of a Tree

For a rooted tree T with n vertices, let $sz(v)$ be the size of the subtree of v. Then the following holds:

$$\sum_{v \in V} \left[\mathrm{sz}(v) + \sum_{u \text{ child of } v} \mathrm{sz}(u)(\mathrm{sz}(v) - \mathrm{sz}(u)) \right] = n^2$$

5.8 Number of eulerian subgraphs

Let G be a graph with n vertices, m edges and c connected components. We consider an eulerian subgraph of G to be any subgraph H with the same vertex set as G and where every vertex has even degree. The number of such subgraphs is:

$$2^{m-n+c}$$

6 Counting Problems

6.1 Stirling numbers of the first kind

These are the number of permutations of $[n]$ with exactly k disjoint cycles. They obey the recurrence:

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0$$

- The sum products of the $\binom{n}{k}$ subsets of size k of $\{0, 1, \ldots, n-1\}$ is $\left[{n \atop n-k}\right]$.
- $\sum_{k=0}^{n} \left[{n \atop k}\right] = n!$
- $\sum_{k=0}^{n} \left[{n \atop k}\right] x^k = x(x-1)(x-2)...(x-n+1)$

6.2 Stirling numbers of the second kind

These are the number of ways to partition $[n]$ into exactly k non-empty sets. They obey the recurrence:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1, \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ n \end{Bmatrix} = 0$$

A "closed" formula for it is:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

6.3 How many functions $f\colon [n] \to [k]$ are there?

$[n]$	$[k]$	Any f	Injective	Surjective
dist	dist	k^n	$\frac{k!}{(n-k)!}$	$k!\left\{{n \atop k}\right\}$
indist	dist	$\binom{k+n-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{n-k}$
dist	indist	$\sum_{i=1}^{k}\left\{{n \atop i}\right\}$	$[n \leq k]$	$\left\{{n \atop k}\right\}$
indist	indist	$\sum_{i=1}^{k} p_i(n)$	$[n \leq k]$	$p_k(n)$

Where $p_k(n)$ is the number of ways to partition n into k terms.

6.4 Derangement

A derangement is a permutation that has no fixed points. Let d_n be the number of ways of derangement of a sequence of the sequence $1 \ldots n$. We have the recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$. Moreover, d_n is the closest integer to $\frac{n!}{e}$.

$$d_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

6.5 Bell numbers

These count the number of ways to partition $[n]$ into subsets. They obey the recurrence:

$$\mathcal{B}_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k$$

x	5	6	7	8	9	10	11	12
$\mathcal{B}_x$	52	203	877	4.140	21.147	115.975	678.570	4.213.597

6.6 Eulerian numbers

The Eulerian number $T(n,k)$ is the number of permutations of the numbers from 1 to n in which exactly k elements are greater than the previous element (permutations with k "ascents").

$$T(n,k) = \sum_{j=0}^{k}(-1)^j(k-j)^{(n+1)}\binom{n+1}{j}$$

6.7 Burside's Lemma

Let G be a group that acts on a set X. The Burnside Lemma states that the number of distinct orbits is equal to the average number of points fixed by an element of G.

$$T = \frac{1}{|G|} \sum_{g \in G} |\texttt{fix}(g)|$$

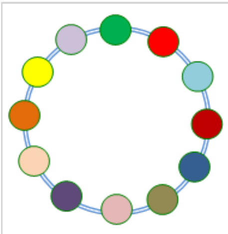
Where a orbit $\texttt{orb}(x)$ is defined as

$$\texttt{orb}(x) = \{y \in X : \exists g \in G \; gx = y\}$$

and $\texttt{fix}(g)$ is the set of elements in X fixed by g

$$\texttt{fix}(g) = \{x \in X : gx = x\}$$

Example: With k distinct types of beads how many distinct necklaces of size n can be made? Considering that two necklaces are equal if the rotation of one gives the other.



$$T = \frac{1}{n+1} \sum_{i=0}^{n} k^{gcd(i,n)}$$

6.8 Catalan Numbers

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360, 1002242216651368.

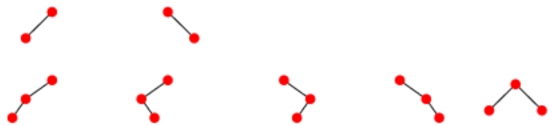
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n} \frac{n+k}{k}, \quad n \geq 0$$

Applications:

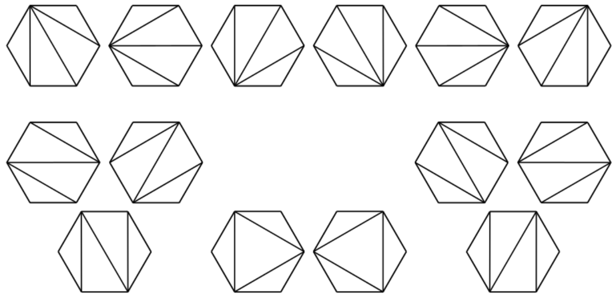
- C_n counts the number of expressions containing n pairs of parentheses which are correctly matched.

$$((())) \qquad ()(()) \qquad ()()() \qquad (())() \qquad \ldots$$

- Successive applications of a binary operator can be represented in terms of a full binary tree. (A rooted binary tree is full if every vertex has either two children or no children.) It follows that C_n is the number of full binary trees with n + 1 leaves:



- C_n is the number of different ways a convex polygon with n + 2 sides can be cut into triangles by connecting vertices with straight lines (a form of Polygon triangulation). The following hexagons illustrate the case n = 4:



6.9 Central Binomial Coefficient

To number of of subsets T of $S = \{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{-1, -1, \ldots, -1}_{n}\}$ that sum to 0 is

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} = \frac{2n!}{(n!)^2} \approx \frac{2^{2n}}{\sqrt{n \cdot \pi}}$$

- The number of factors of 2 in $\binom{2n}{n}$ is equal to the number of 1's in the binary representation if n.
- $\binom{2n}{n}$ is never squarefree for $n > 4$.

7 Dynamic Programming Optimizations

7.1 Divide and Conquer

DP to compute the minimum cost to divide an array into k subarrays; the cost of a solution is equal to the sum of the costs of each subarray. The cost of a subarray $A[i..j]$ is $c(i,j)$.

$$dp[i][k] = \min_{j \geq i} \left(dp[j+1][k-1] + c(i,j)\right)$$

- Define A to be the functions satisfying

$$dp[i][k] = dp[A(i,k)+1][k-1] + c(i, A(i,k)).$$

  If A also satisfy $A(i,k) \leq A(i+1,k)$, then the dp is optimizable.

- Another sufficient condition is, for every $a < b < c < d$:

$$c(a,d) + c(b,c) \geq c(a,c) + c(b,d)$$

8 Other

8.1 Branching factors

The recurrence $T(n) = T(n-i) + T(n-j)$ is $\mathcal{O}(\tau(i,j)^n)$. Also, the recurrence $T(n) = T(n-i) + T(n-j) + f(n)$ is $\mathcal{O}(\tau(i,j)^n \cdot f(n))$.

$i \backslash j$	1	2	3	4	5
1	2.0000	1.6181	1.4656	1.3803	1.3248
2	1.6181	1.4143	1.3248	1.2721	1.2366
3	1.4656	1.3248	1.2560	1.2208	1.1939
4	1.3803	1.2721	1.2208	1.1893	1.1674
5	1.3248	1.2366	1.1939	1.1674	1.1487

Branching factors of binary branching vectors $\tau(i,j)$, rounded up.

8.2 Lagrange

Given a set of $k+1$ points

$$(x_0, y_0), \ldots, (x_j, y_j), \ldots, (x_k, y_k)$$

where no two x_j are the same, the interpolation polynomial in the Lagrange form is a linear combination

$$L(x) := \sum_{j=0}^{k} y_j l_j(x)$$

of Lagrange basis polynomials

$$l_j(x) := \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m} = \frac{(x-x_0)}{(x_j-x_0)} \cdots \frac{(x-x_{j-1})}{(x_j-x_{j-1})} \frac{(x-x_{j+1})}{(x_j-x_{j+1})} \cdots \frac{(x-x_k)}{(x_j-x_k)}$$

8.3 Rational Root Theorem

The rational roots $\pm\frac{p}{q}$ of $P(x) = \sum_{i=0}^{n} a_i x^i$, with $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$ are such that $p \mid a_0$ and $q \mid a_n$. Note that we require that $a_0 \neq 0$; if this is not the case. take the least significant non-zero coefficient.

8.4 Slope Trick

- A function is slope-trick-able if it satisfies 3 conditions:
  1. It is continuous.
  2. It can be divided into multiple sections, where each section is a linear function with an integer slope.
  3. It is convex/concave.
- A slope-trick-able function can be represented by a linear function of the rightmost section and a multiset **S** containing all the slope changing points where the slope changes by 1.
- If $f(x)$ and $g(x)$ are slope-trick-able, then so is $f(x) + g(x)$ (merge the multiset and sum the linear functions).
- Example: given an array, each operation you are allowed to increase or decrease an element's value by 1. Find the minimum number of operations to make the array non decreasing.

- $f_i(x)$ = minimum number of operations to make the first i elements of the array non-decreasing, with the condition that $a_i \leq x$ is slope-trick-able.
- The dual of the problem is: given an array with the price of some stock by day, each day we can either buy or sell one unity of stock or do nothing. What is the largest possible profit?

8.5 Manhattan Distance Trick

This section is about $\mathbb{Z}^2$. Let $\mathcal{L}((x, y)) = (x + y, x - y)$. We have the distance functions:

- Manhattan distance: $M(p, q) = |p.x - q.x| + |p.y - q.y|$
- Chebyshev distance: $C(p, q) = \max(|p.x - q.x|, |p.y - q.y|)$

Then:

- $\mathcal{L}(\mathbb{Z}^2)$ scales $\mathbb{Z}^2$ by $\sqrt{2}$;
- $\mathcal{L}(\mathbb{Z}^2)$ rotates $\mathbb{Z}^2$ by $\frac{\pi}{4}$ clock-wise;
- For some $p \in \mathbb{Z}^2$, $\mathcal{L}(\{q : M(q, p) \leq d\})$ forms an axis-aligned square in $\mathcal{L}(\mathbb{Z}^2)$, with bottom-left corner $\mathcal{L}(p) - (d, d)$ and upper-right corner $\mathcal{L}(p) + (d, d)$;
- $M(p, q) = C(\mathcal{L}(p), \mathcal{L}(q))$, and $C(p, q) = M(\mathcal{L}^{-1}(p), \mathcal{L}^{-1}(q))$.

8.5.1 Higher Dimensions

On $\mathbb{Z}^d$, the last fact generalizes as a 2^{d-1} dimension transformation:

$$\mathcal{L}(p)_i = p_0 + \sum_{j=1}^{d} (-1)^{(i >> j \& 1)} p_j,$$

that is, the sign of the first coordinate is positive, and the others iterate through all 2^{d-1} possibilities.