Xerebêlerebébis
UFMG
Theoretical Guide

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1 Identities

1.1 Series

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1$$

$$\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1$$

1.2 Binomial Identities

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \qquad \binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{k}\binom{n}{k}$$

$$\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h} \qquad \binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$$

$$\textstyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad \sum_{k=0}^{n} k^2\binom{n}{k} = (n+n^2)2^{n-2}$$

$$\textstyle\sum_{j=0}^{k} \binom{m}{j}\binom{n-m}{k-j} = \binom{n}{k} \qquad \sum_{j=0}^{m} \binom{m}{j}^2 = \binom{2m}{m}$$

$$\textstyle\sum_{m=0}^{n} \binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1} \qquad \sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$$

$$\textstyle\sum_{r=0}^{m} \binom{n+r}{r} = \binom{n+m+1}{m} \qquad \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{n} = \mathrm{Fib}(n+1)$$

$$\textstyle(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \qquad (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

1.3 Bits Manipulation

x & (x - 1) : Turn off the rightmost 1-bit in a word, producing 0 if none (e.g., 01011000 → 01010000). This can be used to determine if an unsigned integer is a power of 2 or is 0: apply the formula followed by a 0-test on the result.

x | (x + 1) : Turn on the rightmost 0-bit in a word, producing all 1's if none (e.g., 10100111 → 10101111)

x & (x + 1) : Turn off the trailing 1's in a word, producing **x** if none (e.g., 10100111 → 10100000)

x | (x - 1) : Turn on the trailing 0's in a word, producing x if none (e.g., 10101000 → 10101111)

∼x & (x + 1) : Create a word with a single 1-bit at the position of the rightmost 0-bit in **x**, producing 0 if none (e.g., 10100111 → 00001000)

∼x | (x - 1) : Create a word with a single 0-bit at the position of the rightmost 1-bit in **x**, producing all 1's if none (e.g., 10101000 → 11110111)

∼x & (x - 1) or ∼(x | -x) : Create a word with 1's at the positions of the trailing 0's in **x**, and 0's elsewhere, producing 0 if none (e.g., 01011000 → 00000111)

∼x | (x + 1) : Create a word with 0's at the positions of the trailing 1's in **x**, and 1's elsewhere, producing all 1's if none (e.g., 10100111 → 11111000)

x & (-x) : Isolate the rightmost 1-bit, producing 0 if none (e.g., 01011000 → 00001000)

x ⊕ (x - 1) : Create a word with 1's at the positions of the rightmost 1-bit and the trailing 0's in **x**, producing all 1's if no 1-bit, and integer 1 if no trailing 0's (e.g., 01011000 → 00001111)

x ⊕ (x + 1) : Create a word with 1's at the positions of the rightmost 0-bit and the traling 1's in **x**, producing all 1's if no 0-bit, and the integer 1 if no trailing 1's (e.g., 01010111 → 00001111)

((x & (-x)) + x) & x : Turn off the rightmost contiguous string of 1's (e.g., 01011100 → 01000000)

Index of MSB(x): `__builtin_clz(1) - __builtin_clz(x)`

Index of LSB(x): `__builtin_ctz(x)`

2 Number Theory

2.1 Maximal Prime Gaps:

For numbers until 10^9 the maximal gap is 400.
For numbers until 10^{18} the maximal gap is 1500.

2.2 Prime counting function - $\pi(x)$

The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

2.3 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033
$10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$

2.4 Number of Divisors

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

n	367567200	6983776800	13967553600	321253732800
$d(n)$	1152	2304	2688	5376

$18401055938125660800 \approx 2\text{e}18$ is highly composite with 184320 divisors.
For numbers up to 10^{88}, $d(n) < 3.6\sqrt[3]{n}$.

2.5 Lucas's Theorem

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod p$$

For p prime. n_i and m_i are the coefficients of the representations of n and m in base p.

2.6 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \ (mod \ p)$$

$$a^{p-1} \equiv 1 \ (mod \ p)$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a + b)^p \equiv a^p + b^p \ (mod \ p)$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \ (mod \ p)$$

3 Geometry

3.1 Pythagorean Triples

For all natural a, b, c satisfying $a^2 + b^2 = c^2$ there exist $m, n \in \mathbb{N}$ and $m > n$ such that (reverse is also true):

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

3.2 Heron's Formula

The area of a triangle can be written as $A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$, where a, b, c are the lengths of its sides and $s = \frac{a+b+c}{2}$.
This can be generalized to compute the area A of a quadrilateral with sides a, b, c, d, with $s = \frac{a+b+c+d}{2}$ and α, γ any two opposite angles:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\left(\cos^2\left(\frac{\alpha+\gamma}{2}\right)\right)}$$

3.3 Colinear Points

Three points are colinear on $\mathbb{R}^2$ iff:

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 0$$

The absolute value of this determinant is twice the area of the triangle ABC.

3.4 Coplanar Points

Four points are coplanar in $\mathbb{R}^3$ iff:

$$\begin{vmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \\ x_D & y_D & z_D & 1 \end{vmatrix} = 0$$

3.5 Trigonometry

3.5.1 Angle Sum

$$\sin(a \pm b) = \sin a \cos b \pm \cos\ \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

3.5.2 Sum-to-Product Transformation

$$\sin a \pm \sin b = 2 \sin \frac{a \pm b}{2} \cos \frac{a \mp b}{2}$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\tan a \pm \tan b = \frac{\sin(a \pm b)}{\cos a \cos b}$$

3.6 Centroid of a polygon

The coordites of the centroid of a non-self-intersecting closed polygon is:

$$\frac{1}{3A}\left(\sum_{i=0}^{n-1}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \sum_{i=0}^{n-1}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)\right),$$

where A is twice the signed area of the polygon.

4 Probability

4.1 Moment Generating Functions

Let X be a random variable. Define $M_X(t) = E[e^{tX}]$.

when X is Discrete

$$M_X(t) = \sum_{i=1}^{\infty} e^{tx_i} p_i$$

when X is Continuous

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Then we have:

$$M_X(0) = 0 \qquad M_X'(0) = E[x] \qquad \frac{d^k M_X(0)}{dt^k} = E[x^k]$$

4.2 Distributions

4.2.1 Binomial

- X is the number of successes in a sequence of n independent experiments.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad E[X] = np \qquad Var(X) = np(1-p)$$

4.2.2 Geometric

- X is the number of failures in a sequence of independent experiment of Bernoulli until the first success.

$$P(X = k) = (1-p)^k p \qquad E[X] = \frac{1}{p} \qquad Var(X) = \frac{1-p}{p^2}$$

5 Graphs

5.1 Planar Graphs

1. If G has k connected components, then $n - m + f = k + 1$.
2. $m \leq 3n - 6$. If G has no triangles, $m \leq 2n - 4$.
3. The minimum degree is less or equal 5. And can be 6 colored in $\mathcal{O}(n + m)$.

5.2 Counting Minimum Spanning Trees - $\tau(G)$

- **Cayley's Formula**: $\tau(K_n) = n^{n-2}$.
- **Complete Bipartite Graphs**: $\tau(K_{p,q}) = p^{q-1}q^{p-1}$.
- **Kirchhoff's Theorem**: More generally, if we define the Laplacian matrix $\mathbf{L}(G) = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the diagonal matrix with entries equal to the degree of vertices and $\mathbf{A}$ is the adjacency matrix. For $\mathbf{L}(G)_{ab}$ equal to $\mathbf{L}(G)$ without row a and column b, we have $\tau(G) = \det \mathbf{L}(G)_{ab}$, for any row a and column b.

5.3 Prüfer's Sequence

The Prüfer sequence is a bijection between labeled trees with n vertices and sequences with $n-2$ numbers from 1 to n.
To get the sequence from the tree:

- While there are more than 2 vertices, remove the leaf with smallest label and append it's neighbour to the end of the sequence.

To get the tree from the sequence:

- The degree of each vertex is 1 more than the number of occurrences of that vertex in the sequence. Compute the degree d, then do the following: for every value x in the sequence (in order), find the vertex with smallest label y such that $d(y) = 1$ and add an edge between x and y, and also decrease their degrees by 1. At the end of this procedure, there will be two vertices left with degree 1; add an edge between them.

5.4 Erdős-Gallai Theorem

A sequence of non-negative integers $d_1 \geq ... \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + ... + d_n$ is even and

$$\sum_{i=1}^{k} d_i \ \leq \ k(k-1) + \sum_{i=k+1}^{n} min(d_i, k)$$

holds for every k in $1 \leq k \leq n$.

5.5 Maximum Matching in Complete Multipartite graphs

The size of the maximum matching in a complete multipartite graph with n vertices and k vertices in its largest partition is (reference):

$$|M| = \min\left(\left\lfloor \frac{n}{2} \right\rfloor, n-k\right)$$

5.6 Dilworth's Theorem

5.6.1 Node-disjoint Path Cover

The node disjoint path cover in a `DAG` is equal to $|V| - |M|$, where M is the maximum matching in the bipartite flow network.

5.6.2 General Path Cover

The general path cover in a `DAG` is equal to $|V| - |M|$, where M is the maximum matching in the bipartite flow network of the transitive closure graph.

5.6.3 Dilworth's Theorem

The size of the maximum **antichain** in a `DAG`, that is, the maximum size of a set S of vertices such that no vertex in S can reach another vertiex in S, is equal to size of the minimum **general** path cover.

5.7 Sum of Subtrees of a Tree

For a rooted tree T with n vertices, let $sz(v)$ be the size of the subtree of v. Then the following holds:

$$\sum_{v \in V(T)} \sum_{u \text{ child of } v} \mathrm{sz}(u) \cdot (\mathrm{sz}(v) - \mathrm{sz}(u)) = 2 \cdot n^2$$

6 Counting Problems

6.1 Stirling numbers of the first kind

These are the number of permutations of $[n]$ with exactly k disjoint cycles. They obey the recurrence:

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0$$

- The sum products of the $\binom{n}{k}$ subsets of size k of $\{0, 1, \ldots, n-1\}$ is $\left[{n \atop n-k}\right]$.
- $\sum_{k=0}^{n} \left[{n \atop k}\right] = n!$
- $\sum_{k=0}^{n} \left[{n \atop k}\right] x^k = x(x-1)(x-2)...(x-n+1)$

6.2 Stirling numbers of the second kind

These are the number of ways to partition $[n]$ into exactly k non-empty sets. They obey the recurrence:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1, \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ n \end{Bmatrix} = 0$$

A “closed” formula for it is:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

6.3 How many functions $f\colon [n] \to [k]$ are there?

$[n]$	$[k]$	Any f	Injective	Surjective
dist	dist	k^n	$\frac{k!}{(n-k)!}$	$k!\left\{{n \atop k}\right\}$
indist	dist	$\binom{k+n-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{n-k}$
dist	indist	$\sum_{i=1}^{k}\left\{{n \atop i}\right\}$	$[n \leq k]$	$\left\{{n \atop k}\right\}$
indist	indist	$\sum_{i=1}^{k} p_i(n)$	$[n \leq k]$	$p_k(n)$

Where $p_k(n)$ is the number of ways to partition n into k terms.

6.4 Derangement

A derangement is a permutation that has no fixed points. Let d_n be the number of ways of derangement of a sequence of the sequence $1 \ldots n$. We have the recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$. Moreover, d_n is the closest integer to $\frac{n!}{e}$.

$$d_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}$$

6.5 Bell numbers

These count the number of ways to partition $[n]$ into subsets. They obey the recurrence:

$$\mathcal{B}_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k$$

x	5	6	7	8	9	10	11	12
$\mathcal{B}_x$	52	203	877	4.140	21.147	115.975	678.570	4.213.597

6.6 Eulerian numbers

The Eulerian number $T(n, k)$ is the number of permutations of the numbers from 1 to n in which exactly k elements are greater than the previous element (permutations with k “ascents”).

$$T(n,k) = \sum_{j=0}^{k}(-1)^j(k-j)^{(n+1)}\binom{n+1}{j}$$

6.7 Burside’s Lemma

Let G be a group that acts on a set X. The Burnside Lemma states that the number of distinct orbits is equal to the average number of points fixed by an element of G.

$$T = \frac{1}{|G|}\sum_{g \in G}|\texttt{fix}(g)|$$

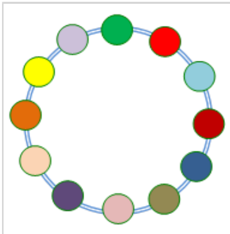
Where a orbit $\texttt{orb}(x)$ is defined as

$$\texttt{orb}(x) = \{y \in X : \exists g \in G\ gx = y\}$$

and $\texttt{fix}(g)$ is the set of elements in X fixed by g

$$\texttt{fix}(g) = \{x \in X : gx = x\}$$

Example: With k distinct types of beads how many distinct necklaces of size n can be made? Considering that two necklaces are equal if the rotation of one gives the other.



$$T = \frac{1}{n+1}\sum_{i=0}^{n} k^{gcd(i,n)}$$

6.8 Catalan Numbers

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360, 1002242216651368.

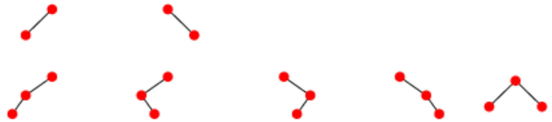
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} = \prod_{k=2}^{n} \frac{n+k}{k}, \quad n \geq 0$$

Applications:

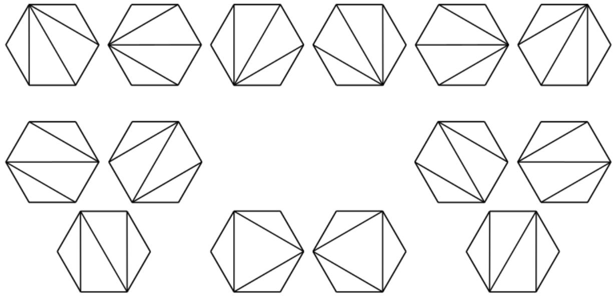
- C_n counts the number of expressions containing n pairs of parentheses which are correctly matched.

$$((())) \qquad ()(()) \qquad ()()() \qquad (())() \qquad \ldots$$

- Successive applications of a binary operator can be represented in terms of a full binary tree. (A rooted binary tree is full if every vertex has either two children or no children.) It follows that C_n is the number of full binary trees with n + 1 leaves:



- C_n is the number of different ways a convex polygon with n + 2 sides can be cut into triangles by connecting vertices with straight lines (a form of Polygon triangulation). The following hexagons illustrate the case n = 4:



6.9 Central Binomial Coefficient

To number of of subsets T of $S = \{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{-1, -1, \ldots, -1}_{n}\}$ that sum to 0 is

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} = \frac{2n!}{(n!)^2} \approx \frac{2^{2n}}{\sqrt{n \cdot \pi}}$$

- The number of factors of 2 in $\binom{2n}{n}$ is equal to the number of 1's in the binary representation if n.
- $\binom{2n}{n}$ is never squarefree for $n > 4$.

7 Dynamic Programming Optimizations

7.1 Divide and Conquer

DP to compute the minimum cost to divide an array into k subarrays; the cost of a solution is equal to the sum of the costs of each subarray. The cost of a subarray $A[i..j]$ is $c(i, j)$.

$$dp[i][k] = \min_{j \geq i} (dp[j+1][k-1] + c(i,j))$$

- Define A to be the functions satisfying

$$dp[i][k] = dp[A(i,k)+1][k-1] + c(i, A(i,k)).$$

  If A also satisfy $A(i,k) \leq A(i+1,k)$, then the dp is optimizable.

- Another sufficient condition is, for every $a < b < c < d$:

$$c(a,b) + c(b,c) \geq c(a,c) + c(b,d)$$

8 Other

8.1 Branching factors

The recurrence $T(n) = T(n-i) + T(n-j)$ is $\mathcal{O}(\tau(i,j)^n)$. Also, the recurrence $T(n) = T(n-i) + T(n-j) + f(n)$ is $\mathcal{O}(\tau(i,j)^n \cdot f(n))$.

i \ j	1	2	3	4	5
1	2.0000	1.6181	1.4656	1.3803	1.3248
2	1.6181	1.4143	1.3248	1.2721	1.2366
3	1.4656	1.3248	1.2560	1.2208	1.1939
4	1.3803	1.2721	1.2208	1.1893	1.1674
5	1.3248	1.2366	1.1939	1.1674	1.1487

Branching factors of binary branching vectors $\tau(i,j)$, rounded up.

8.2 Lagrange

Given a set of $k+1$ points

$$(x_0, y_0), \ldots, (x_j, y_j), \ldots, (x_k, y_k)$$

where no two x_j are the same, the interpolation polynomial in the Lagrange form is a linear combination

$$L(x) := \sum_{j=0}^{k} y_j l_j(x)$$

of Lagrange basis polynomials

$$l_j(x) := \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m} = \frac{(x-x_0)}{(x_j-x_0)} \cdots \frac{(x-x_{j-1})}{(x_j-x_{j-1})} \frac{(x-x_{j+1})}{(x_j-x_{j+1})} \cdots \frac{(x-x_k)}{(x_j-x_k)}$$