Función Price

$f(x) = (2x_0^3 x_1 - x_1^3)^2 + (6x_0 - x_1^2 + x_1)^2$

n=2

$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$

$\boldsymbol{z} = \begin{bmatrix} 2x_0^3 x_1 - x_1^3 \\ 6x_0 - x_1^2 + x_1 \end{bmatrix}$

$f(\boldsymbol{x}) = \boldsymbol{z}'\boldsymbol{z}$

$J_x \boldsymbol{z} = \begin{bmatrix} \frac{\partial z_0}{\partial x_0} & \frac{\partial z_0}{\partial x_1} \\ \frac{\partial z_1}{\partial x_0} & \frac{\partial z_1}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 6x_0^2 x_1 & 2x_0^3 - 3x_1^2 \\ 6 & -2x_1 + 1 \end{bmatrix}$

$\nabla f(\boldsymbol{x}) = 2\,(\boldsymbol{J}_x'\boldsymbol{z})\,\boldsymbol{z}$

Función de Rosenbrock

$f(x) = \sum_{i=0}^{n/2-1} 100(x_{2i+1} - x_{2i}^2)^2 + (1 - x_{2i})^2$

n=2

$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$

$f(x) = 100(x_1 - x_0^2)^2 + (1 - x_0)^2$

$f(x) = [10(x_1 - x_0^2)]^2 + [1 - x_0]^2$

$\boldsymbol{z} = \begin{bmatrix} 10(x_1 - x_0^2) \\ 1 - x_0 \end{bmatrix}$

$f(\boldsymbol{x}) = \boldsymbol{z}'\boldsymbol{z}$

$$J_x\boldsymbol{z} = \begin{bmatrix} \frac{\partial z_0}{\partial x_0} & \frac{\partial z_0}{\partial x_1} \\ \frac{\partial z_1}{\partial x_0} & \frac{\partial z_1}{\partial x_1} \end{bmatrix} = \begin{bmatrix} -20x_0 & 10 \\ -1 & 0 \end{bmatrix}$$

$$\nabla f(\boldsymbol{x}) = 2\,(\boldsymbol{J}_x'\boldsymbol{z})\,\boldsymbol{z}$$

n = 4

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$f(x) = 100(x_1 - x_0^2)^2 + (1 - x_0)^2 + 100(x_3 - x_2^2)^2 + (1 - x_2)^2$$

$$f(x) = [10(x_1 - x_0^2)]^2 + [1 - x_0]^2 + [10(x_3 - x_2^2)]^2 + [1 - x_2]^2$$

$$\boldsymbol{z} = \begin{bmatrix} 10(x_1 - x_0^2) \\ 1 - x_0 \\ 10(x_3 - x_2^2) \\ 1 - x_2 \end{bmatrix}$$

$$f(\boldsymbol{x}) = \boldsymbol{z}'\boldsymbol{z}$$

$$J_x\boldsymbol{z} = \begin{bmatrix} \frac{\partial z_0}{\partial x_0} & \frac{\partial z_0}{\partial x_1} & \frac{\partial z_0}{\partial x_2} & \frac{\partial z_0}{\partial x_3} \\ \frac{\partial z_1}{\partial x_0} & \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \\ \frac{\partial z_2}{\partial x_0} & \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} \\ \frac{\partial z_3}{\partial x_0} & \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -20x_0 & 10 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -20x_2 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\nabla f(\boldsymbol{x}) = 2\,(\boldsymbol{J}_x'\boldsymbol{z})\,\boldsymbol{z}$$

n = 6

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{aligned} f(x) = {} & 100(x_1 - x_0^2)^2 + (1 - x_0)^2 + \\ & 100(x_3 - x_2^2)^2 + (1 - x_2)^2 + \\ & 100(x_5 - x_4^2)^2 + (1 - x_4)^2 \end{aligned}$$

$$\begin{aligned} f(x) = {} & [10(x_1 - x_0^2)]^2 + [1 - x_0]^2 + \\ & [10(x_3 - x_2^2)]^2 + [1 - x_2]^2 + \\ & [10(x_5 - x_4^2)]^2 + [1 - x_4]^2 \end{aligned}$$

$$\boldsymbol{z} = \begin{bmatrix} 10(x_1 - x_0^2) \\ 1 - x_0 \\ 10(x_3 - x_2^2) \\ 1 - x_2 \\ 10(x_5 - x_4^2) \\ 1 - x_4 \end{bmatrix}$$

$$f(\boldsymbol{x}) = \boldsymbol{z}'\boldsymbol{z}$$

$$J_{\boldsymbol{x}}\boldsymbol{z} = \begin{bmatrix} \frac{\partial z_0}{\partial x_0} & \frac{\partial z_0}{\partial x_1} & \frac{\partial z_0}{\partial x_2} & \frac{\partial z_0}{\partial x_3} & \frac{\partial z_0}{\partial x_4} & \frac{\partial z_0}{\partial x_5} \\ \frac{\partial z_1}{\partial x_0} & \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} & \frac{\partial z_1}{\partial x_4} & \frac{\partial z_1}{\partial x_5} \\ \frac{\partial z_2}{\partial x_0} & \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} & \frac{\partial z_2}{\partial x_4} & \frac{\partial z_2}{\partial x_5} \\ \frac{\partial z_3}{\partial x_0} & \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} & \frac{\partial z_3}{\partial x_4} & \frac{\partial z_3}{\partial x_5} \\ \frac{\partial z_4}{\partial x_0} & \frac{\partial z_4}{\partial x_1} & \frac{\partial z_4}{\partial x_2} & \frac{\partial z_4}{\partial x_3} & \frac{\partial z_4}{\partial x_4} & \frac{\partial z_4}{\partial x_5} \\ \frac{\partial z_5}{\partial x_0} & \frac{\partial z_5}{\partial x_1} & \frac{\partial z_5}{\partial x_2} & \frac{\partial z_5}{\partial x_3} & \frac{\partial z_5}{\partial x_4} & \frac{\partial z_5}{\partial x_5} \end{bmatrix} = \begin{bmatrix} -20x_0 & 10 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -20x_2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20x_4 & 10 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\nabla f(\boldsymbol{x}) = 2\,(\boldsymbol{J}_{\boldsymbol{x}}'\boldsymbol{z})\,\boldsymbol{z}$$

Analizando los índices del vector $\boldsymbol{x}$ y $\boldsymbol{z}$. El vector $\boldsymbol{z}$ tiene elementos con la expresión $[10(x_{2i+1} - x_{2i}^2)]$ para índices pares (l_2) y elementos con la expresión $[1 - x_{2i}]$ para índices impares (l_1), entonces, los índices impares los definimos como $l_1 = 1,3,5, \dots$ y los índices pares como $l_2 = 0,2,4, \dots$; los elementos de $\boldsymbol{x}$ los agrupamos en elementos con índices impares y pares similar a los elementos de $\boldsymbol{z}$. Entonces el vector $\boldsymbol{z}$ se define como:

$$z_{l_2} \equiv 10\left(x_{l_1} - x_{l_2}^2\right)$$

$$z_{l_1} \equiv 1 - x_{l_2}$$

Entonces la función de rosenbrock para n par en forma vectorizada es:

$$f(\boldsymbol{x}) = \boldsymbol{z}'\boldsymbol{z}$$

Analizando los elementos de la matriz jacobiana $J_{\boldsymbol{x}}\boldsymbol{z}$ para n=2,4 y 6 ; se observa que existen tres elementos distintos en las posiciones de la matriz jacobiana, es decir.

Renglón par y columna par : $-20x_{2i}$
Renglon par y columna impar: $+10$
Renglon impar y columna par: -1

Entonces la matriz Jacobiana $J_{\boldsymbol{x}}\boldsymbol{z}$ para n par en forma vectorizada es:

$$J_{\boldsymbol{x}}\boldsymbol{z} = \boldsymbol{0}_{n\times n}$$

$$[J_{\boldsymbol{x}}\boldsymbol{z}]_{2i,2i} = -20x_{2i}$$

$$[J_{\boldsymbol{x}}\boldsymbol{z}]_{2i,2i+1} = +10$$

$$[J_{\boldsymbol{x}}\boldsymbol{z}]_{2i+1,2i} = -1$$

para i = 0,2,…,n/2

Entonces el vector gradiente $\nabla f(\boldsymbol{x})$ para n par en forma vectorizada es:

$$\nabla f(\boldsymbol{x}) = 2\,(\boldsymbol{J}_{\boldsymbol{x}}'\boldsymbol{z})\,\boldsymbol{z}$$

```python
# optimizer_adam_rosenbrock.py
# Adam optimization
"""
Dr. Juan R. Castro
FCQI, UABC Campus Tijuana
"""
# Importing Libraries
import numpy as np
```

```python
# objective Extended Rosenbrock function
def objfcn(x):
    # Minima -> f=0 at (1,.....,1)
    n = len(x) # n par
    z = np.zeros((n,1))
    l2 = np.array(range(0,n,2)) # indice par
    l1 = np.array(range(1,n,2)) # indice impar
    z[l2]=10.0*(x[l1]-(x[l2])**2.0)
    z[l1]=1.0-x[l2]
    f = z.T @ z
    return f[0,0]
```

```python
# Extended Rosenbrock gradient function
def objfcngrad(x):
    n = len(x) # n even
    Jz = np.zeros((n,n))
    z = np.zeros((n,1))
    l2 = np.array(range(0,n,2)) # indice par
    l1 = np.array(range(1,n,2)) # indice impar
    z[l2]=10.0*(x[l1]-(x[l2])**2.0)
    z[l1]=1.0-x[l2]

    for i in range(n//2):
        Jz[2*i,2*i]     = -20.0*x[2*i]
        Jz[2*i,2*i+1]   = 10.0
        Jz[2*i+1,2*i]   = -1.0

    gX = 2.0*Jz.T @ z
    return gX
```

```python
def objfcnjac(x):
    # Extended Rosenbrock Jacobian Function
    n = len(x) # n even
    Jz = np.zeros((n,n))
    z = np.zeros((n,1))
    l2 = np.array(range(0,n,2)) # indice par
    l1 = np.array(range(1,n,2)) # indice impar
    z[l2]=10.0*(x[l1]-(x[l2])**2.0)
    z[l1]=1.0-x[l2]

    for i in range(n//2):
        Jz[2*i,2*i]     = -20.0*x[2*i]
        Jz[2*i,2*i+1]   = 10.0
        Jz[2*i+1,2*i]   = -1.0

    gX = 2.0*Jz.T @ z
    normgX = np.linalg.norm(gX)
    return z, Jz, normgX
```