CAGD - Homework 5

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Task 1

In this task we convert between barycentric and homogeneous coordinates.
Consider the points

$$p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

and let

$$q_1 = \begin{pmatrix} 0.25 \\ 0.25 \\ 0.5 \end{pmatrix}$$

in barycentric coordinates with respect to p_0, p_1, p_2. We want to express q_1 in homogeneous coordinates.
First, we express q_1 in Cartesian coordinates

$$q_1 = 0.25p_0 + 0.25p_1 + 0.5p_2 = \begin{pmatrix} 0.25 + 1.5 + 0.25 \\ 0.25 + 1.5 + 0.5 \\ 0.25 + 1.5 + 0.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2.25 \\ 2.25 \end{pmatrix}.$$

For any $\omega \in \mathbb{R}$, the homogeneous coordinates of q_1 are

$$q_1 = \begin{pmatrix} 2\omega \\ 2.25\omega \\ 2.25\omega \\ \omega \end{pmatrix},$$

which is what we wanted to determine.
Now let

$$q_2 = \begin{pmatrix} 5 \\ 4 \\ 4 \\ 3 \end{pmatrix}$$

in homogeneous coordinates. We wish to express q_2 in barycentric coordinates with respect to p_0, p_1, p_2.
First, we express q_2 in Cartesian coordinates

$$q_2 = \frac{1}{3}\begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}.$$

We now want to determine the coefficients a_0, a_1, a_2 such that

$$\sum_{i=0}^{2} a_i p_i$$

and

$$\sum_{i=0}^{2} a_i = 1.$$

This can be done by solving the linear equation system

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 \\ 4 \\ 4 \\ 3 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

The solution is given by

$$a_0 = 1, \quad a_1 = \frac{1}{3}, \quad a_2 = -\frac{1}{3},$$

so in barycentric coordinates with respect to p_0, p_1, p_2,

$$q_2 = \frac{1}{3}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix},$$

which is what we wanted to determine.

Task 2

In this task we interpolate the points $(0, 0)$, $(6, 10)$, $(7, 10.2)$ and $(9, 8)$ using B-Spline curves of degree 2 and 3. The steps we need to take are 1. Set the parameter values $t_0, ..., t_s$ using one of the three methods Uniform parametrization, Chord length parametrization or Centripetal parametrization. 2. Calculate the knots using either the Uniformly spaced method or the de Boor averaging method. 3. Get the control points by solving the system

$$\begin{pmatrix} N_0^p(t_0) & N_1^p(t_0) & \cdots & N_s^p(t_0) \\ N_0^p(t_1) & N_1^p(t_1) & \cdots & N_s^p(t_1) \\ \vdots & \vdots & & \vdots \\ N_0^p(t_s) & N_1^p(t_s) & \cdots & N_s^p(t_s) \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_s \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_s \end{pmatrix}.$$

where N_j^p is the j:th basis function of degree p, b_j the control points and d_j the interpolation points.

Task 3

In this task we construct a circle using second degree NURBS and the de Boor algorithm. The code can be seen in Appendix I.
The following nine control points are used

$$\{(-1,0),(-1,1),(0,1),(1,1),(1,0),(1,-1),(0,-1),(-1,-1),(-1,0)\},$$

consisting of the vertices and midpoints of the unit square centered at $(0,0)$ and its sides, respectively. As we want the curve to pass through the midpoints of the vertices and as these, apart from the starting point, will be situated at $\frac{1}{4}$, $\frac{2}{4}$ and $\frac{3}{4}$ of the whole curve, we need to include these points in the knot sequence with multiplicity two.
As we also want the curve to be clamped, this results in the knot sequence

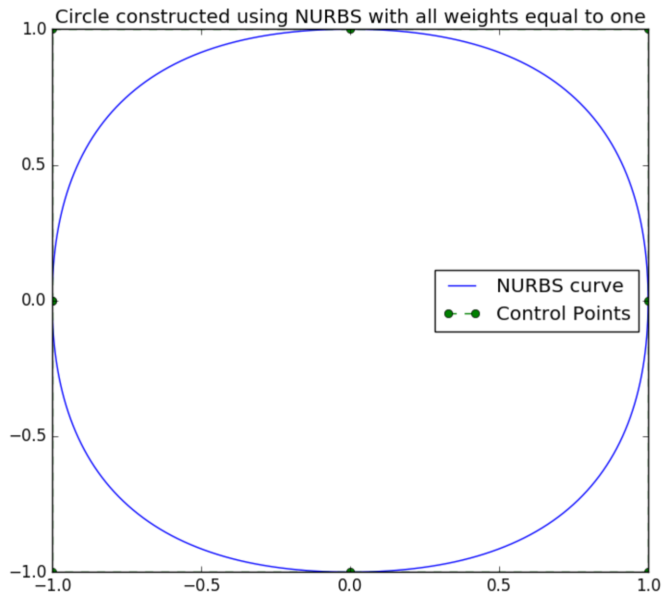
$$\{0,0,0,0.25,0.25,0.5,0.5,0.75,0.75,1,1,1\}.$$

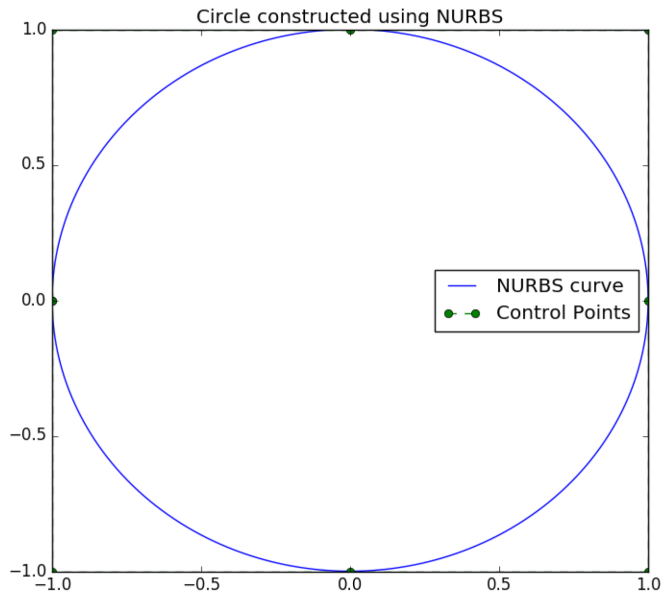
If all weights are set to one, we end up with a B-spline curve with the following appearance

In order to obtain an actual circle, we change the weights corresponding to the control points located in the vertices of the square. By trial and error, we find the correct weights ω to be

$$\omega = \{1,0.707,1,0.707,1,0.707,1,0.707,1\},$$

so the weights corresponding to the vertices of the square are approximately $\frac{\sqrt{2}}{2}$. The circle can be seen below.


Circle constructed using NURBS with all weights equal to one
NURBS curve
Control Points
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

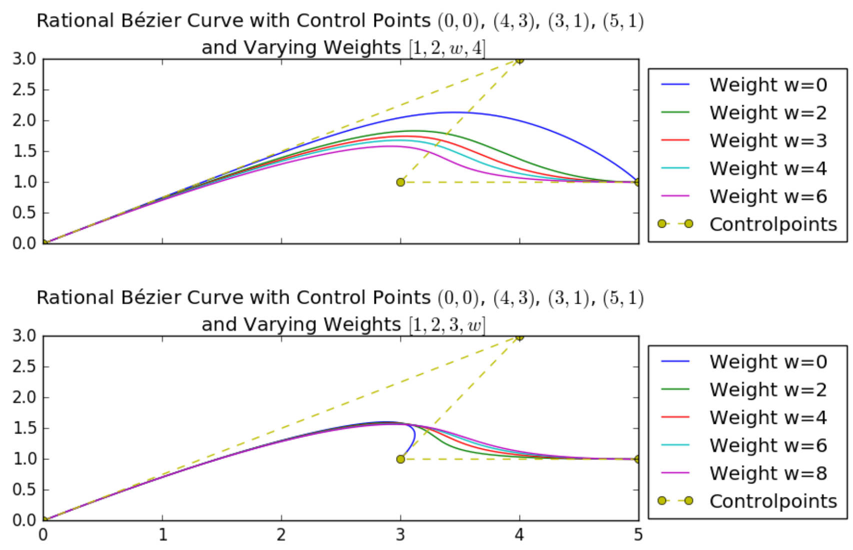

Circle constructed using NURBS
NURBS curve
Control Points
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

Task 4

In this task we contructed a rational Bzier curve, with the Rational de Casteljau Algorithm, with varying weights. The control points that we used were $(0,0)$, $(4,3)$, $(3,1)$ and $(5,1)$ and the weights were originally $[1,2,3,4]$.

To evaluate a rational Bzier curve in $\mathbb{R}^2$ we must first project the points to $\mathbb{R}^3$ using homogeneous coordinates. Thereafter the algorithm proceeds as usual. Before we plot, we traverse the coordinates to be cartesian again to be able to plot the curve on the plane.

The first plot demonstrate how the curve changes when the third weight varies between 0 and 6. The second plot demonstrates the changes when the last weight varies between 0 and 8. What is clear is that if the weight is 0 we neglect the relevant control point.



Do-over Homework 2 Task 5

<u>**Claim 1:**</u> The Lagrange form is invariant under all affine maps.

<u>**Proof:**</u> Let $\varphi : \mathbb{E}^n \to \mathbb{E}^n$ be any affine map. Then φ can be expressed as

$$\varphi(u) = Au + v,$$

with $A \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$. Let p be a polynomial of degree n. Then p can be expressed as a linear combination of Lagrange basis polynomials L_i^n of degree n, i.e.

$$p(x) = \sum_{i=0}^{n} L_i^n(x) c_i,$$

with $c_i \in \mathbb{R}^n$ for $i = 0, 1, \ldots, n$. Recall that the Lagrange basis fulfills the partition of unity, i.e.

$$\sum_{i=0}^{n} L_i^n(x) = 1.$$

We have that

$$\begin{aligned}
\varphi(p(x)) &= \varphi\left(\sum_{i=0}^{n} L_i^n(x)c_i\right) \\
&= A\sum_{i=0}^{n} L_i^n(x)c_i + v \\
&= \sum_{i=0}^{n} L_i^n(x)Ac_i + v \\
&= \sum_{i=0}^{n} L_i^n(x)Ac_i + 1 \cdot v \\
&= \sum_{i=0}^{n} L_i^n(x)Ac_i + \sum_{i=0}^{n} L_i^n(x)v \\
&= \sum_{i=0}^{n} L_i^n(x)(Ac_i + v) \\
&= \sum_{i=0}^{n} L_i^n(x)\varphi(c_i)
\end{aligned}$$

which is what we wanted to show. □

Corollary: The Lagrange form in invariant under all affine domain transformations.
Proof: As all affine domain transformations are affine maps, the result follows directly from Claim 1. □

Claim 2: The monomial form is **not** invariant under all affine domain transformations.

Proof: Let $\varphi : [c, d] \to [0, 1]$ be the affine domain transformation defined by

$$\varphi(u) = \frac{u - c}{d - c}$$

and set

$$A = \frac{1}{d - c}, \quad v = \frac{c}{d - c},$$

so that

$$\varphi(u) = Au + v.$$

Now let p be a polynomial of degree 1 on such that $\operatorname{Im}(p|_{[a,d]}) = [c, d]$. Then we can express p as

$$\begin{aligned} p(x) &= \sum_{i=0}^{1} x^i c_i \\ &= c_0 + xc_1, \quad x \in [a, b]. \end{aligned}$$

Then, for $x \in [a, b]$,

$$\begin{aligned} \varphi(p(x)) &= \varphi(c_0 + xc_1) \\ &= A(c_0 + xc_1) + v \end{aligned}$$

Furthermore, for $x \in [a, b]$,

$$\begin{aligned} \sum_{i=0}^{1} x^i \varphi(c_i) &= \varphi(c_0) + x\varphi(c_1) \\ &= Ac_0 + v + x(Ac_1 + v) \end{aligned}$$

Thus, for $x \in [a, b]$, $\varphi(p(x)) = \sum_{i=0}^{1} x^i \varphi(c_i)$ iff $xv = 0$ which is not true in general. Hence, the monomial for is not invariant under all affine domain transformations. $\square$

Appendix I

```
import scipy
from matplotlib import pyplot as plt


class NURBS:
    def __init__(self, grid, controlpoints, degree, weights):
        if len(controlpoints) != len(weights):
            raise ValueError('Not same number of controlpoints and
                weights')
        try:
            grid = scipy.array(grid)
            grid = grid.reshape((len(grid), 1))
        except ValueError:
            raise ValueError('Grid should be a one-dimensional list or
                array')
        self.grid = grid.reshape((len(grid), 1))
        self.controlpoints = controlpoints
        self.degree = degree
        self.weights = weights
        self.dim = scipy.shape(controlpoints)[1] + 1

    def __call__(self, u):
        index = self.get_index(u)
        r = self.get_mult(index, u)
        controlpoints, weights =
            self.get_controlpoints_and_weights(index,
                                                                 r,
                                                                 u)
        current_controlpoints = self.convert_to_homogeneous(
                                    controlpoints=controlpoints,
                                    weights=weights)
        num_controlpoints = len(current_controlpoints)
        d = scipy.zeros((num_controlpoints, num_controlpoints, self.dim))
        d[0] = current_controlpoints
        for s in range(1, num_controlpoints): # column
            for j in range(s, num_controlpoints): # rows
                left = index - self.degree + j
                right = index + j - s + 1
                a = (u - self.grid[left])\
                    / (self.grid[right] - self.grid[left])
                d[s, j] = (1 - a) * d[s-1, j-1] + a * d[s-1, j]
        return self.convert_to_cartesian(
            d[-1, -1].reshape((1, len(d[-1, -1])))).flatten()

    def get_mult(self, index, u):
        if u == self.grid[index]:
            return len([i for i in self.grid if i == self.grid[index]])
        elif u == self.grid[-1]:
            return len([i for i in self.grid if i == self.grid[-1]])
        else:
            return 0

    def convert_to_homogeneous(self, controlpoints, weights):
        homogeneous_controlpoints = scipy.zeros((controlpoints.shape[0],
                                                 controlpoints.shape[1] + 1))
        for i in range(len(homogeneous_controlpoints)):
            homogeneous_controlpoints[i] = weights[i] *\
                                           scipy.concatenate((controlpoints[i],
                                                              scipy.ones(1)))
        return homogeneous_controlpoints

    def convert_to_cartesian(self, homogeneous_controlpoints):
        cartesian_controlpoints = scipy.zeros(
            (homogeneous_controlpoints.shape[0],
             homogeneous_controlpoints.shape[1] - 1))

        for i in range(len(cartesian_controlpoints)):
            cartesian_controlpoints[i] = \
                         (1 / homogeneous_controlpoints[i][-1]) *\
                         homogeneous_controlpoints[i][:-1]
        return cartesian_controlpoints

    def get_controlpoints_and_weights(self, index, r, u):
        if r > self.degree:
            if u == self.grid[0]:
                current_controlpoints = self.controlpoints[:self.degree +
                    1]
                current_weights = self.weights[:self.degree + 1]
            else:
                current_controlpoints = self.controlpoints[-(self.degree
                    + 1):]
                current_weights = self.weights[- (self.degree + 1) :]
        else:
            current_controlpoints = \
                self.controlpoints[index - self.degree:index - r + 1]
            current_weights = self.weights[index - self.degree:index - r
                + 1]
        return current_controlpoints, current_weights

    def get_index(self, u):
        if u == self.grid[-1]:
            index = (self.grid < u).argmin() - 1
        else:
            index = (self.grid > u).argmax() - 1
        return index

    def plot(self, title, points=500, controlpoints=True):
        fig = plt.figure()
        ax = fig.add_subplot(111)
        ulist = scipy.linspace(self.grid[0], self.grid[-1], points)
        ax.plot(*zip(*[self(u) for u in ulist]), label='NURBS curve')
        if controlpoints:
            ax.plot(*zip(*self.controlpoints), 'o--', label='Control
                Points')

        lgd = ax.legend(loc='best')
        plt.title(title)
        return fig

if __name__ == '__main__':
    controlpoints = scipy.array([[-1, 0],
                                 [-1, 1],
                                 [0, 1],
                                 [1, 1],
                                 [1, 0],
                                 [1, -1],
                                 [0, -1],
                                 [-1, -1],
                                 [-1, 0]])

    grid = scipy.array([0, 0, 0, 0.25, 0.25, 0.5, 0.5, 0.75, 0.75, 1, 1,
        1])

    weights = scipy.ones(len(controlpoints))
    for i in [1, 3, 5, 7]:
        weights[i] = (2 ** 0.5) / 2

    nurbscurve = NURBS(controlpoints=controlpoints,
                     degree=2,
                     grid=grid,
                     weights=weights)
    fig = nurbscurve.plot('Circle constructed using NURBS')
    fig.show()
```