CAGD - Homework 1

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Task 1

UNSURE ABOUT $\mathbb{R}^2$ AND $\mathbb{E}^2$.
In this task we want to find the affine map $\Phi : \mathbb{E}^2 \to \mathbb{E}^2$ such that

$$\begin{aligned}(0,0) &\overset{\Phi}{\mapsto} (1,0)\\ (1,0) &\overset{\Phi}{\mapsto} (1,-1)\\ (1,1) &\overset{\Phi}{\mapsto} (3,-3)\\ (0,1) &\overset{\Phi}{\mapsto} (3,-2)\end{aligned}$$

We look for Φ on the form $\Phi(x) = Ax + v$, $A \in \mathbb{R}^{2\times 2}$, $v \in \mathbb{R}^2$. To determine A and v we solve a series of equations.
Set

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

We now solve a series of equations to determine $a_{11}, a_{12}, a_{21}, a_{22}, v_1, v_2$.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow v_1 = 1 \tag{1}$$

Insert $v_1 = 1$ and continue

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} a_{11} = 0 \\ a_{21} = -1 - v_2 \end{cases} \tag{2}$$

Insert $a_{11} = 0$, $a_{21} = -1 - v_2$ and continue

$$\begin{bmatrix} 0 & a_{12} \\ -1 - v_2 & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \Rightarrow \begin{cases} a_{12} = 2 \\ a_{22} = -2 \end{cases} \tag{3}$$

Insert $a_{12} = 2$, $a_{22} = -2$ and continue

$$\begin{bmatrix} 0 & 2 \\ -1 - v_2 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow v_2 = 0 \tag{4}$$

Thus,

$$A = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and we are done. $\square$

Task 3

We want to show that linear interpolation $\Phi : \mathbb{R} \to \mathbb{E}^2$ is an affine map.
Proof: Let p, q be arbitrary points in $\mathbb{E}^2$. Then

$$\Phi(t) = (1-t)p + tq, \quad t \in \mathbb{R}$$

is the linear interpolation of p and q. Let $x \in \mathbb{R}$ be a barycentric combination, i.e. $x = \sum_{i=1}^{n} \alpha_i x_i$ with $x_i, \alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \alpha_i = 1$. Then

$$\begin{aligned}
\Phi(x) &= \Phi(\sum_{i=1}^{n} \alpha_i x_i) \\
&= (1 - \sum_{i=1}^{n} \alpha_i x_i)p + (\sum_{i=1}^{n} \alpha_i x_i)p \\
&\overset{(i)}{=} (\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \alpha_i x_i)p + \sum_{i=1}^{n} \alpha_i x_i p \\
&= \sum_{i=1}^{n} \alpha_i [(1-x_i)p + x_i q] \\
&= \sum_{i=1}^{n} \alpha_i \Phi(x_i)
\end{aligned}$$

where (i) follows from $\sum_{i=1}^{n} \alpha_i = 1$. This is what we wanted to show. $\square$