Hand In I

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Part 1.

1.1 Let

$$H = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2} \tag{1}$$

and let

$$\psi_m(\phi) = Ae^{im\phi}$$

Since

$$H\psi_m = -\frac{\hbar^2}{2I}\frac{\partial^2\psi_m}{\partial\phi^2} = \frac{\hbar^2 m^2}{2I}\psi_m = E_m\psi_m,$$

The set $\{\psi_m(\phi)\}$ clearly constitutes a collection of eigenfunctions for the Hamiltonian operator as defined in (1), since $\hbar^2 m^2/2I$ is a number.

1.2 Since the particle is on a ring, we have the physical constraint that

$$\psi_m(\phi + 2\pi) = \psi_m(\phi)$$

which means that we require

$$\begin{aligned} e^{im\phi} &= e^{im(\phi+2\pi m)} \\ &\Rightarrow e^{i2\pi m} = 1 \end{aligned}$$

For this equation to hold, we thus get the requirement that $m \in \mathbb{Z}$, or

$$m = \ldots, -2, -1, 0, 1, 2, \ldots$$

1.3 We see that

$$\langle \psi_m \mid \psi_m \rangle = |A|^2 \int_{-\pi}^{\pi} d\phi\; e^{i(m-m)\phi} = 2\pi|A|^2$$

For the state to be normalized, or equivalently $|\psi_m|^2 = \langle \psi_m \mid \psi_m \rangle = 1$, we thus choose

$$A = \frac{1}{\sqrt{2\pi}}$$

1.4 The state

$$\Psi(x, t = 0) = \frac{1}{\sqrt{\pi}}\sin(2\phi)$$

can also be written as a superposition of the (complete set of) eigenfunctions of the Hamiltonian operator as defined in (1):

$$\begin{aligned} \Psi(x, t = 0) = \frac{1}{\sqrt{\pi}}\sin(2\phi) &= \frac{1}{\sqrt{\pi}}\left(\frac{e^{i2\phi} - e^{-i2\phi}}{2i}\right) \\ &= -\frac{i\sqrt{2\pi}}{2\sqrt{\pi}}\left(\frac{1}{\sqrt{2\pi}}e^{i2\phi} - \frac{1}{\sqrt{2\pi}}e^{-i2\phi}\right) \\ &= -\frac{i}{\sqrt{2}}\left(\psi_2(\phi) - \psi_{-2}(\phi)\right) \end{aligned}$$

The state is clearly already normalized (as it should be if it were realizable in the first place!) and we know just tack on the "wiggle factors" to obtain $\Psi(x,t)$:

$$\Psi(x,t) = \frac{1}{i\sqrt{2}} \left(\psi_2(\phi) e^{-iE_{(2)}t/\hbar} - \psi_{(-2)}(\phi) e^{-iE_{-2}t/\hbar} \right)$$

As is seen from (1), we have that

$$E_m = \frac{\hbar^2 m^2}{2I}$$

such that we may explicitly expand

$$\begin{aligned} \Psi(x,t) &= \frac{1}{i\sqrt{2}} \left(\frac{1}{\sqrt{2\pi}} e^{i2\left(\phi - \frac{\hbar}{I}t\right)} - \frac{1}{\sqrt{2\pi}} e^{-i2\left(\phi - \frac{\hbar}{I}t\right)} \right) \\ &= \frac{1}{\sqrt{\pi}} \sin\left(2 \left[\phi - \frac{\hbar}{I} t \right] \right) \end{aligned}$$

for $t > 0$.