
Learning Journal

Organized by Sessions with Topic Summaries

Instructions

- Use this document to track your learning sessions.
- Each session entry should include:
 1. Topics covered (briefly).
 2. Key insights or definitions.
 3. Problems attempted (with references or solutions if necessary).
 4. Questions or areas for follow-up.

Make Anki flashcards of whatever makes sense. Even topics if you'd like, just to leverage their spaced repetition algorithm.

- After completing a topic, write a summary essay, incorporating insights from session notes and solved examples.

Resources

Physics

- Eigenchris
- MIT OpenCourseware
- Richard Behiel
- Steve Brunton

Math

- VisualMath on YT
 - Lectures on quantum topology without topology seem really cool! His course notes are also free.
 - Many overview videos
 - Algebraic Topology
 - Algebraic Geometry
 - etc.
- The Bright Side of Mathematics

Learning Plan

*This is a first pass on some topics I find interesting. I don't need to understand everything the first time. I don't need to go through all the subtopics the first time. I need to understand the main idea, understand why and when it is useful, and try my hand at some basic problems to use it myself. I can always pick up these topics again since I keep a record here. Just remember to jot down your questions, the things you don't understand (and why) and what you'd like to investigate further in the future. You can **always** come back! The below learning plan is not set in stone. Be open for modifications and new ideas. Follow your curiosity.*

Calculus of Variations in Physics - 21th December

- ~~Understand the derivation of the Euler-Lagrange Equations~~
- Solve the QM problem in Hand and Finch (17th December)
- Answer questions below and synthesize and finalize notes on the topic (20th December)

Questions

- How is it used in modern physics today?
- Does one use different "actions" when working on separate problems. If so, could one find a problem "midway" between those problems and see what the action looks like there? Maybe smoothly interpolate an action between these two problems to gain a deeper understanding of how and why they need different descriptions. Maybe find a general description which they are both special cases of.

Special Relativity, Classical Field Theory and Tensors - 28th December

- Susskind Book (26rd December)
- Learn some tensor notation from Tong Notes
- Solving some basic problems with 4-vectors (27th December)
- Synthesize and finalise (28th December)

Floquet Theory - 31st December

Schuller's Course and self-defined problems - 31st January

Linear Algebra Refresher - 7th February

Complex Analysis Basics - 14th February

MIT Quantum Mechanics Course - 7th February

Misc. things

- Green's Function solution of Poisson's equation to get the electric potential from Griffiths

Session 1: Session Date: 14th December, 2024

Main Topic: Set Theory

Resource: Geometrical Anatomy of Physics

Topics Covered

- Space
- Maps
- Domain
- Codomain
- Image
- Preimage
- Bijection
- Inverse
- Equivalence Relation
- Equivalence Class
- Quotient Space
- What amplitudes are
- What a Positive Grassmannian is

Key Insights

Definitions from Schuller's lecture series

Definition. A **space** is a set with some underlying structure. We often study structure-preserving maps between such spaces.

Definition. A **map** is a relation between two sets. More formally, we can write that given two sets A and B , a map $\phi : A \rightarrow B$ is a relation such that for each element $a \in A$ there exists exactly one element $b \in B$ such that $\phi(a, b)$. We write that $a \mapsto \phi(a)$.

Definition. Here, A is the **domain** and B is the **codomain**. The codomain is also called the target.

Definition. The **image** of a set $C \subseteq A$ under a map ϕ is the set one gets, which will be a subset of (or equal to) the codomain, by collecting everything that ϕ maps to when applied to C . We write $\phi(C) \equiv \text{im}_\phi(C) := \{\phi(c) \mid c \in C\}$.

Definition. The **preimage** is the set, which will be a subset of (or equal to) the domain, one gets by considering which elements in the domain one has to apply ϕ to to get certain

elements in the codomain. Let for example $V \subseteq B$, then $\text{preim}_\phi(V) := \{a \in A \mid \phi(a) \in V\}$

Remark. The inverse is only defined for bijections, but the preimage is defined for all maps, and we will often meet it in topology! I was confused at first as to why we know that $\text{preim}_\phi(B) = A$ without requiring surjectiveness, but this is because when we write $\phi : A \rightarrow B$ we are already stating that ϕ is applied to, or at least makes sense to apply to, all of A . Now the image might not be all of B (it just "lives" in B), but the preimage of B is the set of all of the values in the domain which under the map ϕ ends up in B - but that is of course all of A , since from the definition, applying ϕ to any element in A it will end up B .

Definition. A map is **surjective** if $\phi(A) \equiv \text{im}_\phi(A) = B$ - that is, if all of B is "hit" by applying ϕ to all of A . A map is **injective** if for $a_1, a_2 \in A$ we have that $\phi(a_1) = \phi(a_2) \Rightarrow a_1 = a_2$.

The most important notion: A map is called **bijective** if it is both surjective and injective.

Definition. When a map is bijective, then a unique **inverse** exists. This is the map such that $\phi^{-1} \circ \phi = \text{id}_A$ while $\phi \circ \phi^{-1} = \text{id}_B$. In other words, it "undoes" a mapping. Reading \circ as "after" helps to learn the order of application.

Remark. Generically, if there exists one bijection between sets, then there exists many. A bijection is just a "pairing up" of elements - if you can come up with one way, then you can certainly come up with many (unless you try to design a counterexample I guess).

Definition. If there exists any bijection between two sets A and B then we say that they are (set-theoretically) isomorphic ("of the same shape"). We write $A \cong_{\text{set}} B$.

Definition. An **equivalence relation** is any relation between elements in a set which is both *reflexive*, *symmetric* and *transative*. Letting \sim denote the relation, we write these as

$$\begin{aligned} a &\sim a && \text{(reflexive)} \\ a &\sim b \Leftrightarrow b \sim a && \text{(symmetric)} \\ a &\sim b \wedge b \sim c \Rightarrow a \sim c && \text{(transative)} \end{aligned}$$

We denote all of the elements of A which are equivalent to some $m \in A$ under the given equivalence relation as $[m] := \{n \in A \mid n \sim m\}$.

Remark. I am very proud since I was able to prove the following:

- i) $a \in [m] \Rightarrow [a] = [m]$
- ii) Either $[a] = [m]$ or $[a] \cap [m] = \emptyset$

The first of these results imply that any member of an equivalence class equally well represents the whole class. The second one implies that an equivalence relation completely splits the set A into disjoint equivalence classes - we say that it "partitions" the set.

Definition. The set of all equivalence classes formed by applying the equivalence relation \sim to the set A is called the **quotient space** and is written as $A \setminus \sim$. Intuitively, the quotient space is what you get when you sort your large set A into smaller sets by using some rules defined by the given equivalence relation. Examples of an equivalence relation is modulo division by some prime number. One can then take say the quotient space $\mathbb{Z} \setminus \text{mod } p$.

Takeaways

A map $\phi : A \rightarrow B$ applies, by definition, to **all** elements in the domain A . This crucially does not necessarily mean that all elements in B has a corresponding element in A under this map; this property is exactly *surjectivity*.

The set obtained from applying the map to the entire domain is the *image* of the map. If the codomain and the image are equal, then the map is surjective. Thus we can always redefine the codomain to be the image and then the map becomes surjective. But this is often not very interesting.

But the fact that the map is understood to apply to all the elements of the domain is needed to understand why for the map $\phi : A \rightarrow B$ we find that

$$\text{preim}_{\phi}(B) = A$$

where for some $V \subseteq B$ we define the preimage as

$$\text{preim}_{\phi}(V) = \{a \in A \mid \phi(a) \in V\}$$

Amplitudes and the positive Grassmannian

Studying amplitudes is about studying what we expect to happen when fundamental particles interact at very high energies, like when being smashed into each other at the LHC. Physicists then calculate the probabilities related with the particle scattering in different directions with different energies and momenta. These probabilities are precisely the "amplitudes" in "scattering" and "scattering amplitudes". The reason why this is interesting is because if we want to know if our theory is right - or if we want to know precisely when and where it is not - then we need to have very precise expectations from experiments such that we know when they deviate.

Studying amplitudes is therefore very much at the heart of our most fundamental understanding of nature, and it actually sounds really exciting.

The Grassmannian is a way to group and classify subspaces embedded in larger spaces. For example, $\text{Gr}(k, d)$ is the collection of all k -dimensional subspaces going through the origin in the larger d -dimensional space. I think. The positive Grassmannian is the subspace of the Grassmannian which only has positive minors along (**all or certain**) axes. I think minors are subdeterminants or something? I'm not sure. But the intuition is that if we are considering the space of all lines in 3D going through the origin $\text{Gr}(1, 3)$, then the positive Grassmannian would be only the lines with positive slope. The Grassmannian kind of "keeps track" of all these distinct geometric objects (lines with different slopes and directions) by only having them as points. The full Grassmannian just discussed would uniquely identify each point on the upper hemisphere with a line (except for lines

going through the "equator" in the (x, y) -plane, if the hemisphere is formed by cutting a sphere in two in the (x, y) -plane).

Apparently, great advances were made in calculating scattering amplitudes by using the positive Grassmannian. And this was just around 10 years ago - so it is still relatively new!. Calculating these amplitudes was (and probably still is) usually done by adding hundreds of Feynman diagrams for even the simplest calculations - and many thousands for a bit more interesting interactions. And most of these terms sum to zero or something very concise, which is very difficult to understand from the size of the sum. In other words, a lot of redundancies are inherent in the Feynman diagram way of calculating scattering amplitudes, and people are working on more direct ways of doing it since there must be a reason for why many answers come out so beautiful and concise even though the actual calculation is the most messy thing ever.

Problems Attempted

1. Proving the statements regarding equivalence classes (huge victory!)

Follow-Up Questions

- Do a short write up of the proofs above. Remind yourself of the general proof-technique to use if one wants to show a "either - or" statement. Reminder: show $p \Rightarrow \neg q$ because then $\neg(\neg q) \Rightarrow \neg p$ (through contrapositive) which is the same as $q \Rightarrow \neg p$.

Session 2: Session Date: 15th December, 2024

Main Topic: Electromagnetism

Topics Covered

- A small window into amplitudes (Cheung Lecture)
- Fields from electric monopoles, magnetic dipoles and the classical electron radius

Key Insights

Write down definitions, theorems, or takeaways. Use this space for concise notes.

Definitions

Theorems

Takeaways

The Classical Electron Radius: What it is and how to calculate it. **What assumption in the calculation makes it so wrong?**

When doing the assignment we also found that the closer you integrate both the electric field and the magnetic field to the electric charge and the magnetic dipole respectively, the more energy you find - and this goes towards infinity as the integration approaches all space (by making the exempt sphere around the mono/dipole smaller). Take the electric point charge. This infinite energy makes sense since we can imagine that we take the total charge of the point charge and split it up into smaller, less charged point particles. Now, gathering those point particles in the same point to get the total charge would require us to work an infinite amount against the fields, since the strength of the field becomes infinite as the point charges go to sit on top of each other. The same principle applies with the magnetic dipole. Given a perfect magnetic dipole $\mathbf{m} = I \int d\mathbf{a}$, we see that we need an infinite amount of current to get a finite m since $\int d\mathbf{a}$ is zero for a perfect dipole. But running an infinite current also requires an infinite amount of energy. So that divergence makes sense from Maxwell's theories as well.

Problems Attempted

1. Calculating the "classical electron radius"
2. **Outline the method here**
3. **Remind yourself why it makes sense that the energy stored in the fields explode as you integrate closer and closer to either the electric charge or the magnetic dipole (something about putting together the total charge of one charge)**

Follow-Up Questions

- Why do we expect Hamilton's principle to work for scalar fields in arbitrary dimensions? We know that it works well in 3 dimensions because we can do experiments - but is it a leap of faith to do it in higher dimensions, or do we have some clue to its validity even in higher dimensions? What if the look of the least action principle is a

special case in 3D, and there is a more general mathematical principle - a geometry maybe - which underlies the whole thing in arbitrary D -dimensional space?

- When is the Fourier Transform a smart thing to use? I remember Brian said something about that an instinctive response when seeing a function $f(r_i - r_j)$ should be to Fourier Transform. How come? Are there other "forms" of functions where it immediately simplifies the problem (most of the time). What is k -space, and why does the Fourier Transform take us there?

Also, look at the picture you took of the blackboard last week when Jens Paaske wrote something about Fourier Transforming a single wave packet. Try it out for yourself with the normal distribution as the wave. Here you of course have to figure out how to write it "as a wave" (probably just replace x with $x - vt$) as well as how to integrate it properly. Exciting!

Session 3: Session Date: 16th December, 2024**Main Topic: Electromagnetism****Topics Covered**

- Total momentum in fields from momentum density
- Angular momentum in the fields from momentum density
- Complex current from complex impedances

Key Insights**Takeaways**

If one has a field whose magnitude only depends on r but whose direction is given by $\hat{\phi}$, then any integral one full revolution in the ϕ -direction will of course give $\mathbf{0}$. This makes great sense both when you think about it geometrically (you will add up as many components in one direction as in any other, hence summing up to zero) or if you write it out in cartesian components you will integrate $\sin \phi$ and $\cos \phi$ around a full period which gives zero.

Follow-Up Questions

- How does one derive rest energy? I don't remember where it comes from.

Session 4: Session Date: 17th December, 2024

Main Topic: Electromagnetism

Topics Covered

- Maxwell's equations in terms of potentials only
- Gauge Transformations
- 4-vector notation
- d'Alembert operator
- Retarded potentials

Key Insights

Derivation Recap

$$\text{i) } \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\text{ii) } \nabla \cdot \mathbf{B} = 0$$

$$\text{iii) } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\text{iv) } \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

ii) allows us to write

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

since the divergence of any curl is zero. This allows us to write

$$\begin{aligned} \nabla \times \mathbf{E} &= -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \\ \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) &= 0 \end{aligned}$$

And since the curl of any gradient is zero too, we know that we can write the above as the (negative) gradient of a potential too:

$$-\nabla V = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$$

or

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

such that Gauss' law becomes

$$\begin{aligned} \nabla \cdot \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) &= -\nabla^2 V - \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = \frac{\rho}{\epsilon_0} \\ \Rightarrow \nabla^2 V + \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} &= -\rho/\epsilon_0 \end{aligned}$$

In the static case ($\partial_t \mathbf{A} = 0$) this reduces to Laplace's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

We can also rewrite the Ampère-Maxwell law, keeping the source on the right and moving anything else (the fields or potentials) to the left:

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

which in terms of the potentials gives

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c^2} \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) &= \mu_0 \mathbf{J} \\ = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{1}{c^2} \nabla \left(\frac{\partial V}{\partial t} \right) + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \\ = \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) - \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} \end{aligned}$$

Defining the d'Alembertian as

$$\square^2 \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

and letting

$$L \equiv \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t}$$

we can succinctly write the Ampère-Maxwell law as:

$$\boxed{\square^2 \mathbf{A} - \nabla L = -\mu_0 \mathbf{J}}$$

Notice how we can use this in Gauss' law:

$$\begin{aligned} \nabla^2 V + \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} &= -\rho/\epsilon_0 \\ \Rightarrow \nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} + \frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) &= -\rho/\epsilon_0 \\ = \boxed{\square^2 V + \frac{\partial L}{\partial t} = -\rho/\epsilon_0} \end{aligned}$$

Such that Maxwell's equation in terms of potentials become

Follow-Up Questions

- How come the retarded potential formulation is not equivalent to putting a heaviside inside the integration.
- Why will you get all modes below the cutoff frequency inside wave guides?

- Show what Gauge invariance means
- Do an example from the blackboard
- Rederive boundary conditions for the fields (both free and in matter)
- Learn more about the 4-vector formulation and how $g_{\mu\nu}$ can "raise or lower indices"
- Lorentz transformations as rotations in 4-space (Mogen's notes)
- Coulomb Gauge ($\nabla \cdot \mathbf{A} = 0$)
- Lorenz Gauge ($L = 0$)
- Walk through the full retarded potential derivation.