
Learning Journal

Organized by Sessions with Topic Summaries

Instructions

- Use this document to track your learning sessions.
- Each session entry should include:
 1. Topics covered (briefly).
 2. Key insights or definitions.
 3. Problems attempted (with references or solutions if necessary).
 4. Questions or areas for follow-up.

Make Anki flashcards of whatever makes sense. Even topics if you'd like, just to leverage their spaced repetition algorithm.

- After completing a topic, write a summary essay, incorporating insights from session notes and solved examples.

Resources

Physics

- Eigenchris
- MIT OpenCourseware
- Richard Behiel
- Steve Brunton

Math

- VisualMath on YT
 - Lectures on quantum topology without topology seem really cool! His course notes are also free.
 - Many overview videos
 - Algebraic Topology
 - Algebraic Geometry
 - etc.
- The Bright Side of Mathematics

Learning Plan

*This is a first pass on some topics I find interesting. I don't need to understand everything the first time. I don't need to go through all the subtopics the first time. I need to understand the main idea, understand why and when it is useful, and try my hand at some basic problems to use it myself. I can always pick up these topics again since I keep a record here. Just remember to jot down your questions, the things you don't understand (and why) and what you'd like to investigate further in the future. You can **always** come back! The below learning plan is not set in stone. Be open for modifications and new ideas. Follow your curiosity.*

~~Calculus of Variations in Physics~~

- ~~• Understand the derivation of the Euler-Lagrange Equations~~
- ~~• Solve the QM problem in Hand and Finch (17th December)~~
- ~~• Answer questions below and synthesize and finalize notes on the topic (20th December)~~

~~Questions~~

- ~~• How is it used in modern physics today?~~
- ~~• Does one use different "actions" when working on separate problems. If so, could one find a problem "midway" between those problems and see what the action looks like there? Maybe smoothly interpolate an action between these two problems to gain a deeper understanding of how and why they need different descriptions. Maybe find a general description which they are both special cases of.~~

~~Special Relativity, Classical Field Theory and Tensors~~

- ~~• Susskind Book~~
- ~~• Learn some tensor notation from Tong Notes~~
- ~~• Solving some basic problems with 4-vectors~~
- ~~• Synthesize and finalise~~

Schuller's Course and self-defined problems - 31st January

Linear Algebra Refresher - 7th February

Complex Analysis Basics - 14th February

MIT Quantum Mechanics Course - 7th February

Misc. things

- Green's Function solution of Poisson's equation to get the electric potential from Griffiths

Session 1: Session Date: 3rd January, 2025

Main Topic: Electromagnetism

Topics Covered: Ch. 10 Griffiths

- Solution to the four-dimensional wave-equation
- Liénard-Wiechert potentials
- Fields from a moving point charge

Notes

Idea: Potential Formulation of Electrodynamics

The goal of Chapter 10 in Griffiths is to find a general solution to Maxwell's equations in terms of potentials (we are working in the Lorenz Gauge). The very central thing to keep in mind is that information travels at the speed of light, such that the potential at some distance r isn't given by the source distributions at some time t , but rather at the retarded time (dependent on the distance to the source)

$$\boxed{t_r = t - \frac{r}{c}} \quad (10.25)$$

By finding the solution to Maxwell's equations in the potential formulation

$$\begin{aligned} \square^2 V(\mathbf{r}, t) &= -\frac{1}{\epsilon_0} \rho(\mathbf{r}, t) \\ \square^2 \mathbf{A}(\mathbf{r}, t) &= -\mu_0 \mathbf{J}(\mathbf{r}, t) \end{aligned}$$

we can subsequently calculate the fields. But since the relevant *time* now depends on position (space-time), differentiating sources is non-trivial. And this even applies to point charges. This is because classical electrodynamics formulates sources in terms of densities, and as such we have to define the densities related to point charges as the density of a charge with finite extension as the size goes to zero. It seems weird that there can be different retarded times from the same field point to a point charge, but that's apparently necessary with classical electrodynamics.

The Generalized Potentials

We have already found solutions for *electrostatics* (with the Coloumb Gauge)

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{r} d\tau' \\ \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{r} d\tau' \end{aligned}$$

The naïve idea is to just make the transition

$$\begin{aligned} \rho(\mathbf{r}') &\rightarrow \rho(\mathbf{r}', t) \\ \mathbf{J}(\mathbf{r}') &\rightarrow \mathbf{J}(\mathbf{r}', t) \end{aligned}$$

BUT, since the fields can maximally communicate at the speed of light, the sources should not be evaluated at some global time t , but rather we should evaluate them at the *retarded* time:

$$t_r = t - \frac{r}{c}$$

which means that any field point "sees" the charge distribution at a different time depending on the radial distance from the source (and as such there are in fact infinitely many retarded times, dependent on the relative distance). It turns out that the generalisation of Maxwell's equations in terms of retarded potentials in fact is just the naïve potentials, but with the retarded times instead

$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{r} d\tau', \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{r} d\tau'} \quad (10.26)$$

Notice how we on the left have find the potential at the point \mathbf{r} at the time t , whereas we on the right have the source point \mathbf{r}' and the retarded time t_r . But since this is integrated away, we end up with the correct dependencies. Since every field point sees the source point at a distinct retarded time (depending on radial distance) it does not seem so out of touch to think about (\mathbf{r}', t_r) as a coordinate in 4 dimensions without making the time coordinate too special. We can already see how a "native relativistic" formulation of these equations could be quite elegant.

This can directly be showed to solve Maxwell's equations

$$\begin{aligned} \square^2 V(\mathbf{r}, t) &= -\frac{1}{\epsilon_0} \rho(\mathbf{r}, t) \\ \square^2 \mathbf{A}(\mathbf{r}, t) &= -\mu_0 \mathbf{J}(\mathbf{r}, t) \end{aligned}$$

just by taking the gradient and then the divergence.

Mathematical Takeaways

Remember the chain rule! Some takeaways from that derivation is

$$\begin{aligned} \nabla \left(\frac{\rho(\mathbf{r}', t_r)}{r} \right) &= \partial_i \left(\frac{\rho(\mathbf{r}, t_r)}{r} \right) \\ &= \frac{1}{r} \partial_i \rho + \rho \partial_i \left(\frac{1}{r} \right) \\ &= \frac{1}{r} \frac{\partial \rho}{\partial t} \frac{\partial t}{\partial t_r} \frac{\partial t_r}{\partial x_i} - \rho \frac{\hat{\mathbf{r}}}{r^2} \end{aligned}$$

and since

$$\partial_{t_r} t = 1$$

we get that

$$\begin{aligned} \nabla \left(\frac{\rho(\mathbf{r}', t_r)}{r} \right) &= -\frac{1}{c r} \dot{\rho} \partial_i r - \rho \frac{\hat{\mathbf{r}}}{r^2} \\ &= -\frac{1}{c} \dot{\rho} \frac{\hat{\mathbf{r}}}{r} - \rho \frac{\hat{\mathbf{r}}}{r^2} \end{aligned}$$

Component Form and Free Indices are your Best Friends Use "dimensional analysis" with free indices. A gradient is a vector for example. Thus when we look at the above derivation, we see that we have a vector on the left hand side. Therefore every term on the right hand side *needs* to have one - and only one - free index. When taking the divergence we get a scalar because we have an index contraction, and thus every term should be of the form $[(\cdot)_i (\cdot)_i]$ - thus *zero* free indices. The advantage of writing every thing out in components allows us to use the usual product and Leibniz rules for differentiation and we just have to respect indices.

In showing that the retarded potentials given here are indeed correct, one uses that

$$\begin{aligned}\nabla \left(\frac{1}{r} \right) &= -\frac{\hat{\mathbf{r}}}{r^2} \\ \nabla r &= \hat{\mathbf{r}} \\ \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r} \right) &= \frac{1}{r^2} \\ \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) &= 4\pi\delta^3(\hat{\mathbf{r}})\end{aligned}$$

Coordinate Free Expressions It is often the case that we analyse some geometrical problem in the plane or in 3D using a specific coordinate system and specific angles. After having found expressions for certain quantities, see if you can write them in a coordinate free form - this means writing it in terms of dot products or cross products between vectors. This is a very powerful (and natural) way to generalize formulas, since there is nothing special about the coordinate systems we use other than the fact that axes are orthonormal. A simple example is the drawing from page 458 in Griffiths, where we use the unit vector $\hat{\mathbf{r}}$ to write the cosine of the angle in a coordinate free form:

$$v \cos \theta = \hat{\mathbf{r}} \cdot \mathbf{v}$$

Regarding this, notice the cool coordinate free from using the above ideas (\mathbf{A} and \mathbf{B} are vector fields dependent on position):

$$[\nabla(\mathbf{A} \cdot \mathbf{B})]_i = \partial_i (A_j B_j) = B_j \partial_i A_j + A_j \partial_i B_j = [\mathbf{J}_\mathbf{A}(\mathbf{r})\mathbf{B} + \mathbf{J}_\mathbf{B}(\mathbf{r})\mathbf{A}]_i$$

where $\mathbf{J}_\mathbf{A}(\mathbf{r})$ denotes the Jacobian of the vector field \mathbf{A} with respect to the cartesian axes. The Jacobian is a matrix (2 free indices) and the i 'th element is thus

$$[\mathbf{J}_\mathbf{A}(\mathbf{r})\mathbf{B}]_i = [\mathbf{J}_\mathbf{A}(\mathbf{r})]_{ij} B_j$$

where Einstein summation is still implied.

Note especially that this is how a Jacobian is denoted in component form: $\partial_i A_j$.

Liénard-Wichart Potentials: Geometrical Derivation

Liénard-Wichart Potentials: Dirac-Delta Proof

Problems Attempted

Problem 10.12. A piece of wire bent into a loop, as shown in Fig. 10.5, carries a current that increases linearly with time:

$$I(t) = kt \quad (-\infty < t < \infty).$$

Calculate the retarded vector potential \mathbf{A} at the center. Find the electric field at the center. Why does this (neutral) wire produce an *electric* field? (Why can't you determine the *magnetic* field from this expression for \mathbf{A} ?)

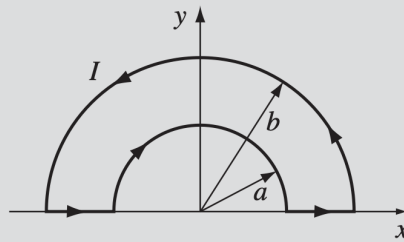


Fig. 10.5

Problem 10.12 in Griffiths In general, the relevant one-dimensional integral is along the wire:

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}', t_r)}{r} dl'$$

But the current only depends on time and not on position. Along the top wire we get

$$\mathbf{I}(t_r) = k\left(t - \frac{b}{c}\right)\hat{\phi} = \left(t - \frac{b}{c}\right)[- \sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{y}}]$$

while along the bottom we have

$$\mathbf{I}(t_r) = k\left(t - \frac{a}{c}\right)(-\hat{\phi}) = \left(t - \frac{a}{c}\right)[\sin \theta \hat{\mathbf{x}} - \cos \theta \hat{\mathbf{y}}]$$

such that from those contributions we get

$$\begin{aligned} \mathbf{A}_{\text{top and bottom}}(\mathbf{r}, t) = \frac{k\mu_0}{4\pi} & \left[\hat{\mathbf{x}} \int_0^\pi d\theta \sin \theta \left(a \left(t - \frac{a}{c} \right) + b \left(t - \frac{b}{c} \right) \right) \right. \\ & \left. \hat{\mathbf{y}} \int_0^\pi d\theta \cos \theta \left(b \left(t - \frac{b}{c} \right) + a \left(t - \frac{a}{c} \right) \right) \right] \end{aligned}$$

plenty of things are independent of θ so we get

$$\mathbf{A}_{\text{top and bottom}}(\mathbf{r}, t) = \frac{k\mu_0}{2\pi} \left(t(a + b) - \frac{1}{c}(a^2 + b^2) \right)$$

whereas from the horizontal segments we get (I think they will be equal, but the "doppler" effect would actually play in here with a point charge. This is why we have terms involving the velocity vector in that case. But here I think they should be the same in the sense that it is symmetrical w.r.t. time):

$$\begin{aligned} \mathbf{A}_{\text{left horizontal}}(\mathbf{r}, t) &= \frac{k\mu_0}{4\pi} \hat{\mathbf{x}} \int_b^a \frac{t - \frac{x'}{c}}{x'} dx' \\ &= \frac{k\mu_0}{4\pi} \hat{\mathbf{x}} \left(t \ln \left(\frac{a}{b} \right) - \frac{a - b}{c} \right) \end{aligned}$$

but of course with the right horizontal, we are integrating the exact same thing, but from $a \rightarrow b$. Thus they cancel. This seems wierd to me right now, since with the right hand rule from the current, I would think the field would have to contribute something. But I do see that the center is on the line of the two horizontal segments. But if we only had one side, that should prevent the field, right? I got a contribution from the left side alone. It was just cancelled by the right side it seems.

Follow-Up Questions/Ideas/Todos

- It seems weird that there can be different retarded times from the same field point to a point charge, but that's apparently necessary with classical electrodynamics. **What's the full story?**
- In problem 10.12 above, it confuses me that flipping the limits of integration is the same as flipping the direction of the current in some sense. When we are running along the inner half circle (the one at radius a), I have the urge to say that it is running in the direction $-\hat{\phi}$ while the angle being swept is from $\pi \rightarrow 0$. But this is "a double flip" and leaves the integral unchanged. At the same time, I feel like the two horizontal wire segments shouldn't cancel. They are running in the same direction one would think. But with respect to the center point, they *are* running in oppositse directions of course. **Don't think so much about things where you can say: Of course it has to be this way. Of course you don't want to double flip, since you see that it leaves everything unchanged. Don't worry so much!**

Session 2: Session Date: 4th of January, 2025

Main Topic: Electromagnetism

Topics Covered

- Liénard-Wiechert Potentials Continued

Key Insights

Write down definitions, theorems, or takeaways. Use this space for concise notes.

Problems Attempted

Derivation of the Liénard-Wiechert Potentials

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{r(t_r)} d\tau'$$

Since

$$t_r = t - \frac{r(t_r)}{c} = t - \frac{|\mathbf{r} - \mathbf{w}(t_r)|}{c}$$

we have that

$$r(t_r) = |\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r)$$

where $\mathbf{w}(t_r)$ is the trajectory of the particle, evaluated at the retarded time (since this is the reality that any different point in space-time sees!).

We also have that

$$\rho(\mathbf{r}', t_r) = q\delta^3(\mathbf{r}' - \mathbf{w}(t_r))$$

This can be read as "Any observer will see a point charge q not at the position \mathbf{r}' $\mathbf{w}(t)$ at the current time t , but instead at the position $\mathbf{w}(t_r)$ which is where the particle was a while ago, at the time t_r ". Remember that we are integrating over the primed coordinates.

In fact this is our regular expression for a point charge just in terms of retarded times since

$$\mathbf{r}' - \mathbf{w}(t_r) = \mathbf{r}$$

Thus I would think that our expression would be

$$V(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta^3(\mathbf{r}' - \mathbf{w}(t_r))}{|\mathbf{r} - \mathbf{w}(t_r)|} d\tau'$$

But my lecture notes does as following: Introduces another integration step, changes integration variable AND has a different denominator (which functional dependence I am not sure of) without a different numerator:

$$V(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int d\mathbf{r}' \int dt \frac{\delta^3(\mathbf{r}' - \mathbf{w}(t))}{|\mathbf{r} - \mathbf{r}'|} \delta(t - t_r)$$

I can see how performing the $d\mathbf{r}'$ integral would give the same denominator, but now the integral is suddenly very different. It is over time and the numerator is a delta function involving t and t_r . Could you explain everything that is happening here for me? I would really like to master these types of integral transforms and *wield* them so I am more comfortable with doing something like this in the future by myself.

Now, here comes an interesting step, which I don't quite understand. We introduce another integration while integrating over \mathbf{r}' instead of $d\tau'$:

$$V(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int d\mathbf{r}' \int dt \frac{\delta^3(\mathbf{r}' - \mathbf{w}(t))}{|\mathbf{r} - \mathbf{w}(t)|} \delta(t - t_r)$$

Follow-Up Questions/Ideas/ToDoS

- *Write down any gaps in understanding or questions to revisit.*

Session 3: Session Date: 5th of January, 2025

Main Topic: Electromagnetism

Topics Covered

- Liénard-Wiechert Potentials

Key Insights

Write down definitions, theorems, or takeaways. Use this space for concise notes.

Clarification of integration techniques

When we write the integral

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d\tau'$$

we have to be very clear about what we are calculating. First of all, We are integrating over a fixed, stationary *volume* in space. This is why no time dependence is implied in the denominator. Every point in the volume we are integrating over will have a static distance to the field point $|\mathbf{r} - \mathbf{r}'|$. Secondly, writing $d\tau'$ is the same as writing d^3r or even $d\mathbf{r}$. So the integration variable didn't change in the above derivation. It was just another notation. Third, even though the integration *region* is fixed and independent of time, the *density* of charge at that point is not. And *this* is what we wish to evaluate at the retarded time, t_r . But the retarded time depends on the distance from the source point to the field point

$$t_r = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}$$

And here comes the integral trick being used above: Instead of having to remember that we should evaluate our charge distribution at the retarded time (an implicit dependence $\rho(\mathbf{r}, t_r)$), which depends on the primed coordinates which we are integrating over, making the whole thing quite convoluted, we can instead integrate over *all* time, and just use a delta function to pick out the retarded time only and thus make the dependence explicit mathematically ($\delta(t' - t_r)$). This separates time and space as integration variables formally, so to say. And if our functions are nice enough (which they often are in physics), this subsequently allows us to carry out first the (independent) space integral and *then* the time integral (we are really using the Green's functions solution to the wave equation here), while the dependencies are being "managed" explicitly by the delta functions:

$$\begin{aligned} V(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \int_{-\infty}^{\infty} dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - t_r) \\ &= \frac{1}{4\pi\epsilon_0} \int d^3r' \int_{-\infty}^{\infty} dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right) \end{aligned}$$

Above is the general idea. Now in the case of a point particle, we can conveniently write the distribution as follows:

$$\rho(\mathbf{r}, t') = q\delta^3(\mathbf{r}' - \mathbf{w}(t'))$$

such that when we carry out the space integration, we get

$$\begin{aligned} V(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \int_{-\infty}^{\infty} dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right) \\ &= \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{\infty} dt' \frac{\delta\left(t' - \left(t - \frac{|\mathbf{r} - \mathbf{w}(t')|}{c}\right)\right)}{|\mathbf{r} - \mathbf{w}(t')|} \end{aligned}$$

at which point we use the identity:

$$\delta(h(t)) = \sum_i \frac{\delta(t - t_i)}{|h'(t_i)|}$$

where $h(t_i) = 0$. This identity is proven by doing a change of variables in the integral

$$\int_{-\infty}^{\infty} f(t) \delta(h(t)) dt$$

Two delta functions are considered equal if they produce the same result when convoluted with the same function over the same integral. You might think that $u = h(t)$ would force $h^{-1}(u) = t$ to be defined everywhere, but that is not the case since you only need $h(t)$ to be one-to-one (so there is an inverse) locally around the zeroes of $h(t)$. This is perfectly consistent with the identity which sums up all these contributions.

After using the identity we also come across the differentiation

$$\begin{aligned} \frac{d}{dt} |\mathbf{r} - \mathbf{w}(t)| &= \frac{d}{dt} \sqrt{(\mathbf{r} - \mathbf{w}(t)) \cdot (\mathbf{r} - \mathbf{w}(t))} \\ &= \frac{\mathbf{r} - \mathbf{w}(t)}{|\mathbf{r} - \mathbf{w}(t)|} \cdot \left(-\frac{d\mathbf{w}}{dt}\right) \\ &= -\hat{\mathbf{z}} \cdot \mathbf{v} \end{aligned}$$

And those were the important steps in the derivation. The reason why the separation vector in the retarded time suddenly became $\mathbf{w}(t)$ still eludes my previous explanation that the separation vector is from a fixed source point to a fixed field point. Is that because we are also changing

$$t_r = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \rightarrow t - \frac{|\mathbf{r} - \mathbf{w}(t)|}{c}$$

when doing the spatial integral? And if so, then I guess we haven't completely separated time and space, have we? *I have restated the language above a little better: The delta function doesn't completely separate time and space - they are dependent. But it keeps track of this dependence explicitly and mathematically, instead of us having to remember the implicit dependence through t_r .*

Problems Attempted

1. Problem statement or reference.
2. Solution (include partial work if needed).

Follow-Up Questions/Ideas/Todos

- Write down any gaps in understanding or questions to revisit.