Learning Journal

Organized by Sessions with Topic Summaries

Instructions

- Use this document to track your learning sessions.
- Each session entry should include:
  1. Topics covered (briefly).
  2. Key insights or definitions.
  3. Problems attempted (with references or solutions if necessary).
  4. Questions or areas for follow-up.

  Make Anki flashcards of whatever makes sense. Even topics if you'd like, just to leverage their spaced repitition algorithm.
- After completing a topic, write a summary essay, incorporating insights from session notes and solved examples.

Resources

Physics

- Eigenchris
- MIT OpenCourseware
- Richard Behiel
- Steve Brunton

Math

- VisualMath on YT
  - Lectures on quantum topology without topology seem really cool! His course notes are also free.
  - Many overview videos
  - Algebraic Topology
  - Algebraic Geometry
  - etc.
- The Bright Side of Mathematics

Learning Plan

This is a first pass on some topics I find interesting. I don't need to understand everything the first time. I don't need to go through all the subtopics the first time. I need to understand the main idea, understand why and when it is useful, and try my hand at some basic problems to use it myself. I can always pick up these topics again since I keep a record here. Just remember to jot down your questions, the things you don't understand (and why) and what you'd like to investigate further in the future. You can ***always*** *come back! The below learning plan is not set in stone. Be open for modifications and new ideas. Follow your curiosity.*

Special Relativity, Classical Field Theory and Tensors

- Special Relativity Basics
- 4-vectors
- Lorentz Transformations ("Boosts")
- Covariant and Contravariant
- Invariant Intervals
- Adding Velocities
- Momentum-Energy Relation
- Transformation of forces under boosts (and Minkowski Force)
- Tensors
- Lorentz Group
- Poincaré Group
- Learn some tensor notation from Tong Notes
- Susskind Book
- Synthesize and finalise

Schuller's Course and self-defined problems - 31st Janurary

Linear Algebra Refresher - 7th February

MIT Quantum Mechanics Course - 7th February

Complex Analysis Basics - 14th February

Misc. things

- Green's Function solution of Poisson's equation to get the electric potential from Griffiths

Session 1: Session Date: 3rd January, 2025

Main Topic: Electromagnetism

Topics Covered: Ch. 10 Griffiths

- Solution to the four-dimensional wave-equation
- Liénard-Wiechert potentials
- Fields from a moving point charge

Notes

Idea: Potential Formulation of Electrodynamics

The goal of Chapter 10 in Griffiths is to find a general solution to Maxwell's equations in terms of potentials (we are working in the Lorenz Gauge). The very central thing to keep in mind is that information travels at the speed of light, such that the potential at some distance $\imath$ isn't given by the source distributions at some time t, but rather at the retarded time (dependent on the distance to the source)

$$\boxed{t_r = t - \frac{\imath}{c}} \tag{10.25}$$

By finding the solution to Maxwell's equations in the potential formulation

$$\Box^2 V(\mathbf{r}, t) = -\frac{1}{\epsilon_0}\rho(\mathbf{r}, t)$$
$$\Box^2 \mathbf{A}(\mathbf{r}, t) = -\mu_0 \mathbf{J}(\mathbf{r}, t)$$

we can subsequently calculate the fields. But since the relevant *time* now depends on position (space-time), differentiating sources is non-trivial. And this even applies to point charges. This is because classical electrodynamics formulates sources in terms of densities, and as such we have to define the densities related to point charges as the density of a charge with finite extension as the size goes to zero. It seems weird that there can be different retarded times from the same field point to a point charge, but that's apparently necessary with classical electrodynamics.

The Generalized Potentials

We have already found solutions for electro*statics* (with the Coloumb Gauge)

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\imath} d\tau'$$
$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\imath} d\tau'$$

The naïve idea is to just make the transition

$$\rho(\mathbf{r}') \to \rho(\mathbf{r}', t)$$
$$\mathbf{J}(\mathbf{r}') \to \mathbf{J}(\mathbf{r}', t)$$

BUT, since the fields can maximally communicate at the speed of light, the sources should not be evaluated at some global time t, but rather we should evaluate them at the *retarded* time:

$$t_r = t - \frac{\imath}{c}$$

which means that any field point "sees" the charge distribution at a different time depending on the radial distance from the source (and as such there are in fact infinitely many retarded times, dependent on the relative distance). It turns out that the generalisation of Maxwell's equations in terms of retarded potentials in fact is just the naïve potentials, but with the retarded times instead

$$\boxed{V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\imath} d\tau', \qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{\imath} d\tau'} \tag{10.26}$$

Notice how we on the left have find the potential at the point $\mathbf{r}$ at the time t, whereas we on the right have the source point $\mathbf{r}'$ and the retarded time t_r. But since this is integrated away, we end up with the correct dependencies. Since every field point sees the source point at a distinct retarded time (depending on radial distance) it does not seem so out of touch to think about $(\mathbf{r}', t_r)$ as a coordinate in 4 dimensions without making the time coordinate too special. We can already see how a "native relativistic" formulation of these equations could be quite elegant.

This can directly be showed to solve Maxwell's equations

$$\Box^2 V(\mathbf{r},t) = -\frac{1}{\epsilon_0}\rho(\mathbf{r},t)$$
$$\Box^2 \mathbf{A}(\mathbf{r},t) = -\mu_0 \mathbf{J}(\mathbf{r},t)$$

just by taking the gradient and then the divergence.

Mathematical Takeaways

Remember the chain rule! Some takeaways from that derivation is

$$\begin{aligned}\nabla\left(\frac{\rho(\mathbf{r}',t_r)}{\imath}\right) &= \partial_i\left(\frac{\rho(\mathbf{r},t_r)}{\imath}\right) \\ &= \frac{1}{\imath}\partial_i\rho + \rho\partial_i\left(\frac{1}{\imath}\right) \\ &= \frac{1}{\imath}\frac{\partial\rho}{\partial t}\frac{\partial t}{\partial t_r}\frac{\partial t_r}{\partial x_i} - \rho\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\end{aligned}$$

and since

$$\partial_{t_r} t = 1$$

we get that

$$\begin{aligned}\nabla\left(\frac{\rho(\mathbf{r}',t_r)}{\imath}\right) &= -\frac{1}{c\imath}\dot{\rho}\partial_i\imath - \rho\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \\ &= -\frac{1}{c}\dot{\rho}\frac{\hat{\boldsymbol{\imath}}}{\imath} - \rho\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\end{aligned}$$

Component Form and Free Indicies are your Best Friends Use "dimensional analysis" with free indicies. A gradient is a vector for example. Thus when we look at the above derivation, we see that we have a vector on the left hand side. Therefore every term on the right hand side *needs* to have one - and only one - free index. When taking the divergence we get a scalar because we have an index contraction, and thus every term should be of the form $[(\cdot)_i (\cdot)_i]$ - thus *zero* free indicies. The advantage of writing every thing out in components allows us to use the usual product and Leibniz rules for differentation and we just have to respect indicies.

In showing that the retarded potentials given here are indeed correct, one uses that

$$\begin{aligned}
\nabla\left(\frac{1}{\imath}\right) &= -\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \\
\nabla\imath &= \hat{\boldsymbol{\imath}} \\
\nabla\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\imath}\right) &= \frac{1}{\imath^2} \\
\nabla\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) &= 4\pi\delta^3(\hat{\boldsymbol{\imath}})
\end{aligned}$$

Coordinate Free Expressions It is often the case that we analyse some geometrical problem in the plane or in 3D using a specific coordinate system and specific angles. After having found expressions for certain quantities, see if you can write them in a coordinate free form - this means writing it in terms of dot products or cross products between vectors. This is a very powerful (and natural) way to generalize formulas, since there is nothing special about the coordinate systems we use other than the fact that axes are orthonormal. A simple example is the drawing from page 458 in Griffiths, where we use the unit vector $\hat{\boldsymbol{\imath}}$ to write the cosine of the angle in a coordinate free form:

$$v\cos\theta = \hat{\boldsymbol{\imath}}\cdot\mathbf{v}$$

Regarding this, notice the cool coordinate free from using the above ideas ($\mathbf{A}$ and $\mathbf{B}$ are vector fields dependent on position):

$$[\nabla(\mathbf{A}\cdot\mathbf{B})]_i = \partial_i\left(A_jB_j\right) = B_j\partial_iA_j + A_j\partial_iB_j = [\mathbf{J}_\mathbf{A}(\mathbf{r})\mathbf{B} + \mathbf{J}_\mathbf{B}(\mathbf{r})\mathbf{A}]_i$$

where $\mathbf{J}_\mathbf{A}(\mathbf{r})$ denotes the Jacobian of the vector field $\mathbf{A}$ with respect to the cartesian axes. The Jacobian is a matrix (2 free indicies) and the i'th element is thus

$$[\mathbf{J}_\mathbf{A}(\mathbf{r})\mathbf{B}]_i = [\mathbf{J}_\mathbf{A}(\mathbf{r})]_{ij}\,B_j$$

where Einstein summation is still implied.

Note especially that this is how a Jacobian is denoted in component form: ∂_iA_j.

Liénard-Wichart Potentials: Geometrical Derivation

Liénard-Wichart Potentials: Dirac-Delta Proof

Problems Attempted

Problem 10.12. A piece of wire bent into a loop, as shown in Fig. 10.5, carries a current that increases linearly with time:

$$I(t) = kt \quad (-\infty < t < \infty).$$

Calculate the retarded vector potential $\mathbf{A}$ at the center. Find the electric field at the center. Why does this (neutral) wire produce an *electric* field? (Why can't you determine the *magnetic* field from this expression for $\mathbf{A}$?)

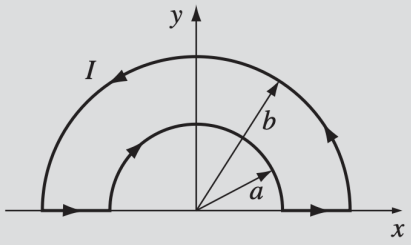


Fig. 10.5

Problem 10.12 in Griffiths In general, the relevant one-dimensional integral is along the wire:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}(\mathbf{r}',t_r)}{\imath}dl'$$

But the current only depends on time and not on position. Along the top wire we get

$$\mathbf{I}(t_r) = k(t-\frac{b}{c})\hat{\boldsymbol{\phi}} = (t-\frac{b}{c})\left[-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}\right]$$

while along the bottom we have

$$\mathbf{I}(t_r) = k(t-\frac{a}{c})(-\hat{\boldsymbol{\phi}}) = (t-\frac{a}{c})\left[\sin\theta\hat{\mathbf{x}}-\cos\theta\hat{\mathbf{y}}\right]$$

such that from those contributions we get

$$\begin{aligned}\mathbf{A}_{\text{top and bottom}}(\mathbf{r},t) = \frac{k\mu_0}{4\pi}\Bigg[\hat{\mathbf{x}}\int_0^\pi d\theta\sin\theta\left(a\left(t-\frac{a}{c}\right)+b\left(t-\frac{b}{c}\right)\right)\\ \hat{\mathbf{y}}\int_0^\pi d\theta\cos\theta\left(b\left(t-\frac{b}{c}\right)+a\left(t-\frac{a}{c}\right)\right)\Bigg]\end{aligned}$$

plenty of things are independent of θ so we get

$$\mathbf{A}_{\text{top and bottom}}(\mathbf{r},t) = \frac{k\mu_0}{2\pi}\left(t(a+b)-\frac{1}{c}\left(a^2+b^2\right)\right)$$

whereas from the horizontal segments we get (I think they will be equal, but the "doppler" effect would actually play in here with a point charge. This is why we have terms involving the velocity vector in that case. But here I think they should be the same in the sense that it is symmetrical w.r.t. time):

$$\begin{aligned}\mathbf{A}_{\text{left horizontal}}(\mathbf{r},t) &= \frac{k\mu_0}{4\pi}\hat{\mathbf{x}}\int_b^a \frac{t-\frac{x'}{c}}{x'}dx'\\ &= \frac{k\mu_0}{4\pi}\hat{\mathbf{x}}\left(t\ln\left(\frac{a}{b}\right)-\frac{a-b}{c}\right)\end{aligned}$$

but of course with the right horizontal, we are integrating the exact same thing, but from $a \to b$. Thus they cancel. This seems wierd to me right now, since with the right hand rule from the current, I would think the field would have to contribute something. But I do see that the center is on the line of the two horizontal segments. But if we only had one side, that should prevent the field, right? I got a contribution from the left side alone. It was just cancelled by the right side it seems.

Follow-Up Questions/Ideas/ToDos

- It seems weird that there can be different retarded times from the same field point to a point charge, but that's apparently necessary with classical electrodynamics. What's the full story?
- In problem 10.12 above, it confuses me that flipping the limits of integration is the same as flipping the direction of the current in some sense. When we are running along the inner half circle (the one at radius a), I have the urge to say that it is running in the direction $-\hat{\boldsymbol{\phi}}$ while the angle being sweeped is from $\pi \to 0$. But this is "a double flip" and leaves the integral unchanged. At the same time, I feel like the two horizontal wire segments shouldn't cancel. They are running in the same direction one would think. But with respect to the center point, they *are* running in oppositse directions of course. Don't think so much about things where you can say: Of course it has to be this way. Of course you don't want to double flip, since you see that it leaves everything unchanged. Don't worry so much!

Session 2: Session Date: 4th of January, 2025

Main Topic: Electromagnetism

Topics Covered

- Liénard-Wiechert Potentials Continued

Key Insights

Write down definitions, theorems, or takeaways. Use this space for concise notes.

Problems Attempted

Derivation of the Liénard-Wiechert Potentials

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\imath(t_r)}d\tau'$$

Since

$$t_r = t - \frac{\imath(t_r)}{c} = t - \frac{|\mathbf{r}-\mathbf{w}(t_r)|}{c}$$

we have that

$$\imath(t_r) = |\mathbf{r}-\mathbf{w}(t_r)| = c(t-t_r)$$

where $\mathbf{w}(t_r)$ is the trajectory of the particle, evaluated at the retarded time (since this is the reality that any different point in space-time sees!).

We also have that

$$\rho(\mathbf{r}',t_r) = q\delta^3(\mathbf{r}'-\mathbf{w}(t_r))$$

This can be read as "Any observer will see a point charge q not at the position r' $\mathbf{w}(t)$ at the current time t, but instead at the position $\mathbf{w}(t_r)$ which is where the particle was a while ago, at the time t_r". Remember that we are integrating over the primed coordinates.

In fact this is our regular expression for a point charge just in terms of retarded times since

$$\mathbf{r}' - \mathbf{w}(t_r) = \boldsymbol{\imath}$$

Thus I would think that our expression would be

$$V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int \frac{\delta^3(\mathbf{r}'-\mathbf{w}(t_r))}{|\mathbf{r}-\mathbf{w}(t_r)|}d\tau'$$

But my lecture notes does as following: Introduces another integration step, changes integration variable AND has a different denominator (which functional dependence I am not sure of) without a different numerator:

$$V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int d\mathbf{r}'\int dt\frac{\delta^3(\mathbf{r}'-\mathbf{w}(t))}{|\mathbf{r}-\mathbf{r}'|}\delta(t-t_r)$$

I can see how performing the $d\mathbf{r}'$ integral would give the same denominator, but now the integral is suddenly very different. It is over time and the numerator is a delta function involing t and t_r. Could you explain everything that is happening here for me? I would really like to master these types of integral transforms and *wield* them so I am more comfortable with doing something like this in the future by myself.

Now, here comes an interesting step, which I don't quite understand. We introduce another integration while integrating over $\mathbf{r}'$ instead of $d\tau'$:

$$V(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int d\mathbf{r}' \int dt \frac{\delta^3(\mathbf{r}' - \mathbf{w}(t))}{|\mathbf{r} - \mathbf{w}(t)|}\delta(t - t_r)$$

Follow-Up Questions/Ideas/ToDos

- *Write down any gaps in understanding or questions to revisit.*

Session 3: Session Date: 5th of January, 2025

Main Topic: Electromagnetism

Topics Covered

- Liénard-Wiechert Potentials

Key Insights

Write down definitions, theorems, or takeaways. Use this space for concise notes.

Clarification of integration techniques When we write the integral

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d\tau'$$

we have to be very clear about what we are calculating. First of all, We are integrating over a fixed, stationary *volume* in space. This is why no time dependence is implied in the denominator. Every point in the volume we are integrating over will have a static distance to the field point $|\mathbf{r}-\mathbf{r}'|$. Secondly, writing $d\tau'$ is the same as writing d^3r' or even $d\mathbf{r}'$. So the integration variable didn't change in the above derivation. It was just another notation. Third, even though the integration *region* is fixed and independent of time, the *density* of charge at that point is not. And *this* is what we wish to evaluate at the retarded time, t_r. But the retarded time depends on the distance from the source point to the field point

$$t_r = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

And here comes the integral trick being used above: Instead of having to remember that we should evaluate our charge distribution at the retarded time (an implicit dependence $\rho(\mathbf{r},t_r)$), which depends on the primed coordinates which we are integrating over, making the whole thing quite convoluted, we can instead integrate over *all* time, and just use a delta function to pick out the retarded time only and thus make the dependence excplicit mathematically ($\delta(t'-t_r)$). This separates time and space as integration variables formally, so to say. And if our functions are nice enough (which they often are in physics), this subsequently allows us to carry out first the (independent) space integral and *then* the time integral (we are really using the Green's functions solution to the wave equation here), while the dependencies are being "managed" explicitly by the delta functions:

$$\begin{aligned} V(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}d^3r' \\ &= \frac{1}{4\pi\epsilon_0}\int d^3r' \int_{-\infty}^{\infty} dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t_r) \\ &= \frac{1}{4\pi\epsilon_0}\int d^3r' \int_{-\infty}^{\infty} dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-\left(t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)) \end{aligned}$$

Above is the general idea. Now in the case of a point particle, we can convinently write the distribution as follows:

$$\rho(\mathbf{r},t') = q\delta^3(\mathbf{r}'-\mathbf{w}(t'))$$

such that when we carry out the space integration, we get

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \int_{-\infty}^{\infty} dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t' - \left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right))$$
$$= \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty} dt' \frac{\delta(t' - \left(t - \frac{|\mathbf{r}-\mathbf{w}(t')|}{c}\right))}{|\mathbf{r}-\mathbf{w}(t')|}$$

at which point we use the identity:

$$\delta(h(t)) = \sum_i \frac{\delta(t-t_i)}{|h'(t_i)|}$$

where $h(t_i) = 0$. This identity is proven by doing a change of variables in the integral

$$\int_{-\infty}^{\infty} f(t)\delta(h(t))dt$$

Two delta functions are considered equal if they produce the same result when convoluted with the same function over the same integral. You might think that $u = h(t)$ would force $h^{-1}(u) = t$ to be defined everywhere, but that is not the case since you only need $h(t)$ to be one-to-one (so there is an inverse) locally around the zeroes of $h(t)$. This is perfectly consistent with the identity which sums up all these contributions.

After using the identity we also come across the differentation

$$\frac{d}{dt}|\mathbf{r}-\mathbf{w}(t)| = \frac{d}{dt}\sqrt{(\mathbf{r}-\mathbf{w}(t))\cdot(\mathbf{r}-\mathbf{w}(t))}$$
$$= \frac{\mathbf{r}-\mathbf{w}(t)}{|\mathbf{r}-\mathbf{w}(t)|}\cdot\left(-\frac{d\mathbf{w}}{dt}\right)$$
$$= -\hat{\boldsymbol{\imath}}\cdot\mathbf{v}$$

And those where the important steps in the derivation. The reason why the separation vector in the retarded time suddenly became $\mathbf{w}(t)$ still eludes my previous explanation that the separation vector is from a fixed source point to a fixed field point. Is that because we are also changing

$$t_r = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \to t - \frac{|\mathbf{r}-\mathbf{w}(t)|}{c}$$

when doing the spatial integral? And if so, then I guess we haven't competely separated time and space, have we? *I have restated the language above a little better: The delta function doesn't completely separate time and space - they* are *dependent. But it keeps track of this depencence explicitly and mathematically, instead of us having to remember the implicit dependence through* t_r.

Session 4: Session Date: 6th of January, 2025

Prestudy from yesterday

To Read

- Chapter 11 in Griffiths
- Fourier Transform Chapter in Arfken probably.

Problems

- Exam and a problem from ch. 11 Griffiths.
- Fourier Transform of a square wave
- Fourier Transform of a Gaussian.

Main Topic: *What you are working on/torwards. What is the context?*

Didn't get around to any of the things because I spent it on assignment and exams. But I'll get around to it! I can alwas get back to the ideas above.

One really cool argument that Jörg told me about to day in lab was how we saw that the wave pulse *shape* had changed when the wave got reflected. And as such, we could immediately conclude that the amount of dampening of the reflected wave has to be frequency dependent. Why? Because in frequency space (after af Fourier transform), we see that the shape of the wave pulse depends on the amplitudes of the frequencies used to build it. This means that if all frequencies are damped equally, we can pull out in front, Fourier transform back to the space domain and then have a scaled version of our originally shaped waveform. But since the *shape* is different, that means that some frequencies were damped more than others - in other words, the dampening is frequency dependent!

Session 5: Session Date: 8th of January, 2025

Main Topic: Determinants and Jacobian Determinant, Electromagnetism and Special Relativity

Topics Covered

- Permutations, $sgn(\sigma)$ and the symmetric group
- Leibniz' formula for the determinant
- Contraction of $\epsilon_{ijk}T^{ij}$ where T^{ij} is symmetric
- Coordinate transformations in integrals and the Jacobian
- Intuition about line integral formula

Key Insights

Permutations and the Symmetric group A permutation σ is a bijective (one-to-one) map on a set onto *itself* (which means it just reorders the elements of the set). The collection of all possible permutations of a set is called *the symmetric group* and is denoted with S_n. A common notation is cyclic notation. It is probably best illustrated with an example. Denoting the original set ordering on top and the permuted set on the bottom, let's write the following permutation in cyclic notation

$$\begin{gathered}\{1,2,3,4,5,6,7\}\\ \downarrow\\ \{2,3,6,5,4,1,7\}\end{gathered}$$

We can think very algorithmically about this permutation as follows:

- 1 is sent to 6
- 6 is sent to 3
- 3 is sent to 2
- 2 is sent to 1

This is now a closed cycle. In other words, if we continue we just repeat the same cycle. We write the above cycle as

$$\kappa_1 = (1632)$$

where the cycle ends before we get back to where we started, such that the string only contains unique numbers. But just because 4 and 5 and 7 do not appear in the above cycle it doesn't mean that they weren't permutated, just that their permutation cycle is *disjoint* from the first one. We see that we also have the disjoint cycle

$$\kappa_2 = (45)$$

and the trivial 'cycle' (which by convention is omitted) where 7 is left in its starting position. The complete permutation is thus written as the multiplication of the two sub-permutations. The group multiplication of this group is function composition, that is,

first applying one permutation and then applying the next permutation on the result of that first permutation

$$\tau \star \sigma(x_i) = \tau(\sigma(x_i)) = \tau \circ \sigma(x_i)$$

where the star is used to show a multiplication as a group operation and the $\circ$ shows that in this case it is our classical notion of function composition. The advantage of using cycle notation where we identify all disjoint sub-permutation cycles is that since the order of two disjoint cycles obviously doesn't matter, we find that

$$\tau \circ \sigma(x_i) = \sigma \circ \tau(x_i) \qquad \text{if } \tau \text{ and } \sigma \text{ are disjoint}$$

We can thus write the full permutation as

$$\sigma = \kappa_1\kappa_2 = (1632)(45) = (45)(1632) = \kappa_2\kappa_1$$

where the products are implicit between parantheses.

Any permutation can obviously be represented by succesively interchanging two elements at a time. We call such a switch a *transposition* and often denote it as τ. We can thus decompose any permutation as

$$\sigma = \tau_1\tau_2 \dots \tau_r$$

The number of transposition used in decomposing a permutation is *not* unique, but it is unique on the equivalence class r mod 2. We can thus define a unique characteristic/map(?)

$$sgn(\sigma) = (-1)^r = \begin{cases} +1 & \text{if } r \text{ is even} \\ -1 & \text{if } r \text{ is odd} \end{cases}$$

The reason for introducing all this is the following topic.

Leibniz' determinant formula The determinant of a 3×3 matrix is given as follows

$$det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - cfg$$

We can notice a pattern based on permutations: In each row i of the matrix, we pick a single column j. This gives us three elements and we multiply these together. Let's say we pick (in order of increasing row index) $1, 3, 2$. This would mean that we form the product

$$a_{11}a_{23}a_{32} = afh$$

And now, we count the transpositions needed to permute the column indicies of this set into an ordered (increasing) set (which in the case of an $n \times n$ matrix is just $\{1, \dots, n\}$). Here, we just need a single transposition

$$\sigma = (32) = \tau_1$$

But if we had instead started with $2, 3, 1$ we would have to do

$$\sigma = \tau_1\tau_2 = (32)(12)$$

Note that permutations in general do not commute unless the are disjoint. Decomposing a permutation into transpositions will in general not only have distinct transpositions of course, since this would imply that the starting permutation already only consisted of disjoint transpositions (or trivial permutations).

If we sum all the possible triplets that can be constructed and let the sign with which they contribute in the sum be determined by whether the number of transpositions needed to permute the triplet into "identity" triplet $\{1, 2, 3\}$, we in fact get the determinant! And this sign is **exactly** what the sign function (funnily enough) gives us! Just go back and check that the determinant for a 3×3 works like this. For example, the term bdi has a negative sign, since b is in column 2, d is in column 1 and i is in column. We thus need a single transposition to do

$$\{2, 1, 3\} \overset{\sigma=(12)}{\longrightarrow} \{1, 2, 3\}$$

and for this map

$$sgn(\sigma) = (-1)^1 = -1$$

The incredible fact is, that this way of computing the determinant works in N dimensions! The proof of this is quite technical and relies on understanding multilinear maps (which are tensors in fact). The proof shows that the only possible map that can satisfy all the abstract properties of the determinant is precisely given by the Leibniz' formula (which is the generalisation of the above idea to N dimensions). It could be fun to revisit when you understand tensors!

Now here comes the formula. Let A be an $n \times n$ matrix. Just reading of the description we can write then write the determinant out like this:

$$det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\mathrm{a}_{1,\sigma(1)}\mathrm{a}_{2,\sigma(2)} \ldots \mathrm{a}_{\mathrm{n},\sigma(\mathrm{n})}$$

How is this the same? Well, let's try working it out in 3 dimensions again. First, we see that we are summing over all $\sigma \in S_n$. This means all possible permutations we could consider of the set $\{1, \ldots, n\}$, where as before we will interpret the entry index as the row number and the value at that index as the column number. How do we write that more formally? It is just

$$\sigma(i) = j$$

to denote that in row i we pick column enty j of course!! Note that using only permutations automatically forces us to pick a new column for each row, since otherwise we do not have a bijection (which permutations are defined to be). This is quickly seen if we choose the elements aeg which has the column indicies $\{1, 2, 1\}$. This can never be mapped to $\{1, 2, 3\}$ by reordering the elements; equivalently, any attempt at constructing such a map *cannot* be injective, since it forces us to map 1 onto two different elements (1 and 3).

Lastly, the $\text{sgn}(\sigma)$ makes sure that we get the sign correct for the current permutation we are using.

Using product notation we get the usual expression for the Leibniz formula:

$$\boxed{\det(A) = \sum_{\sigma \in S_n} \left(\text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right)}$$

Simply beautiful!

And it get's better than that! Right now we are using permutations and the sign function to only allow unique column indicies as well as keep track of what sign each term should have. But if we notice that "only allowing unique column indicies" is equivalent to saying that any terms with a repeating column index is just set to 0, we see that the exact same result is given by

$$\det(A) = \sum_{i_1, i_2, \ldots, i_n} \epsilon_{i_1, i_2, \ldots, i_n} a_{1,i_1} a_{2,i_2} \ldots a_{n,i_n}$$

where the sigma is often omitted due to Einstein notation. In other words, the Levi-Civita symbol counts if the number of transpositions needed to get from a permutation to whatever we define as the "identity" permutation is odd or even *while* only allowing actual permutations (by setting things that aren't permutations equal to zero).

Now that we understand this formula, we are ready to use it! But first, we need to note another cool, general identity:

Contraction of $\epsilon_{ijk} T^{ij}$ where T^{ij} is symmetric This is quite obvious in fact:

$$\begin{aligned}
\epsilon_{ijk} T^{ij} &= \frac{1}{2} \left(\epsilon_{ijk} T^{ij} + \epsilon_{jik} T^{ji} \right) \\
&= \frac{1}{2} \left(\epsilon_{ijk} T^{ij} - \epsilon_{ijk} T^{ji} \right) \\
&= \frac{1}{2} \left(\epsilon_{ijk} T^{ij} - \epsilon_{ijk} T^{ij} \right) \\
&= 0
\end{aligned}$$

In other words **contracting ϵ_{ijk} with a symmetric matrix always gives zero.**

Coordinate transformations in integrals and the Jacobian Consider that we want to integrate some function $f(x, y)$ which represents some density over a region $g(D)$. That would usually be written as

$$I = \iint_{g(D)} f(x, y) dx dy$$

But what does this integral look like in the coordinate system where the region we integrate over is just D? Say we label the coordinate axes in that system as (u, v). That means that a small region in the (u, v) system is sent to the region $(x, y) = g(u, v)$ and the value of the density function goes from $f(g(u, v)) \to f(x, y)$. In other words, the values

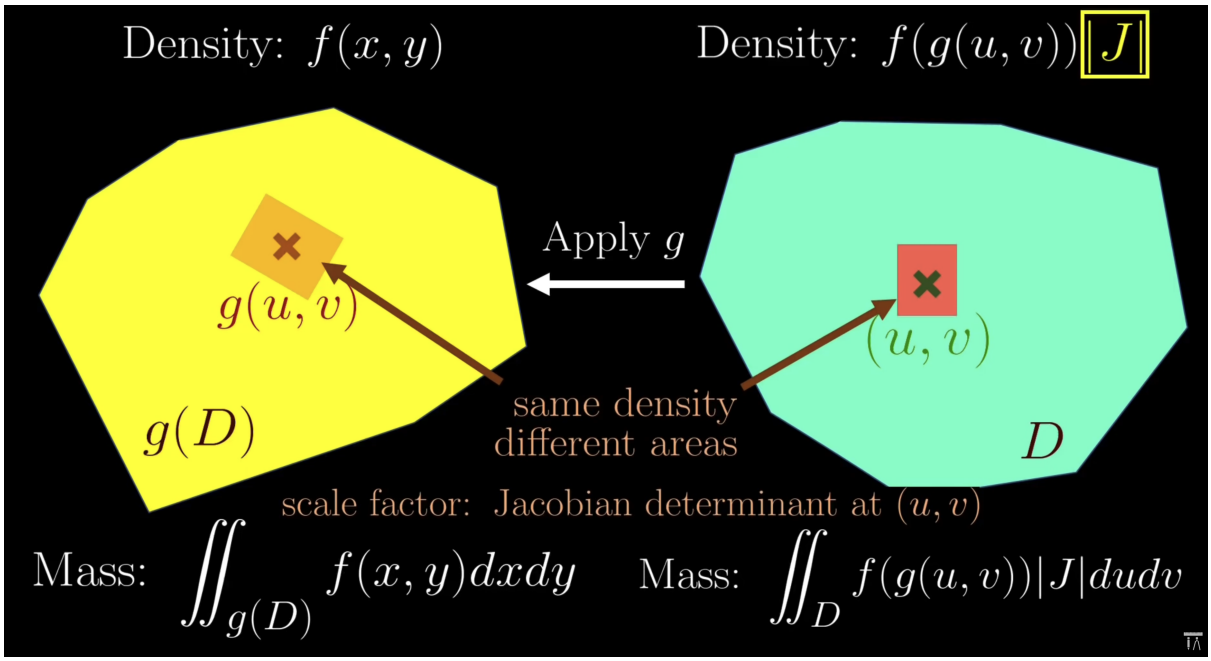
that f take on will not change, only the label which we give those points. But the are of out integration region *does* change. To make sure that the result we get from integrating using a different coordinate system, we need to make sure that we scale the area of the original disk D by an appropriate amount to make sure the integrals are equal. This appropiate amount can be different at each small region $dudv$ throughout D. It turns out to be given by the (absolute value of the) determinant of the Jacobian at the point

$$\begin{aligned}\text{abs}|\mathbf{J}|(u,v) &= \begin{vmatrix} \frac{\partial g_1}{\partial u}(u,v) & \frac{\partial g_1}{\partial v}(u,v) \\ \frac{\partial g_2}{\partial u}(u,v) & \frac{\partial g_2}{\partial v}(u,v) \end{vmatrix} \\ &= \left.\frac{\partial g_i}{\partial x_j}\right|_{(u,v)}\end{aligned}$$

to make sure that

$$\iint_{g(D)} f(x,y)dxdy = \iint_D f(g(u,v))\text{abs}|\mathbf{J}|dudv$$

It is probably best illustrated by the following image (credit to Mathemaniac on YouTube):



Liénard-Wiechert Potentials Derivation Finally we are able to use the above ideas to derive the Liénard-Wiechert potentials (again!) in a cool way. From Griffith's we end with the conclusion that for a moving point charge, any volume element (no matter the size) is transformed as

$$d\tau' \to \frac{d\tau}{1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c}$$

where the $d\tau'$ is the apparent volume (the one an observer looking at the volume element) sees, whereas $d\tau$ is the actual volume element if the particle were moving with the same uniform velocity as the observer.

Here comes the clever thinking: if this is the case, then that must mean that there should be some coordinate transformation which introduces the scaling factor as the absolute value of a Jacobian matrix! In other words, we should be able to find

$$|\det(\mathbf{J})| = \frac{1}{1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c}$$

for some map. An obvious candidate is the term in the delta function

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\delta^3(\mathbf{r}' - \mathbf{w}(t_r))}{}$$

Problems Attempted

1. *Problem statement or reference.*
2. *Solution (include partial work if needed).*

Follow-Up Questions/Ideas/ToDos

- Rewatch the Mathemaniac video to understand the Jacobian better as an approximation of a linear map

Session 6: Session Date: 9th of January, 2025

Prestudy from yesterday

To Study

- Liénard-Wiechert Potentials Derivation (again!)
- Chapter 11 in Griffiths
- Lorentz Transformations
- Contravariant and Covariant
- Metric
- Lorentz Transformations as a matrix-vector product (Λ^{ν}_{μ})
- Metric invariant under similarity transform with the Lorentz Matrix

Problems

- *List key problems here.*

Relevance

Main Topic: ***What you are working on/torwards. What is the context?***

Topics Covered

- *List key topics or subtopics here.*

Key Insights

Write down definitions, theorems, or takeaways. Use this space for concise notes.

Problems Attempted

1. *Problem statement or reference.*
2. *Solution (include partial work if needed).*

Follow-Up Questions/Ideas/ToDos

- *Write down any gaps in understanding or questions to revisit.*