Trigonalization

Let K be a field

- **Def - Upper Triangular:** A matrix A is upper triangular if $a_{i,j} = 0$ for all $i > j$
- **Def - Triagonizable:** $A \in M_n(K)$ is *triagonizable* when A is similar to a upper triangular matrix. That is, there is $P \in GL_n(K)$ s.t PAP^{-1} is upper triangular

- **Lemma 1:** A triagonalizable matrix always admits an eigenvalue

- **Theorem:** Let $A \in M_n(K)$. The following are equivalent:
  1. A is triagonalizable
  2. the characteristic polynomial χ_A of A splits over K
  3. rhe minimal polynomial P_A of A splits over K

> In this context, "split over" means that the polynomial can be written as the product of its linear factors. Which means can be factorized.

- **Corollary 1:** Any matrix over C is trigonalizable
- **Corollary 2:** Any nilpotent matrix is trigonalizable. More precisely, it is similar to an upper triangular matrix with only 0's on the diagonal

> A nilpotent matrix is a square matrix A such that $A^k = 0$ for some positive integer k.

- **Proposition 3:** For any $A \in M_n(K)$ trigonalizable, we have:
  - $trace(A) = \sum_{\lambda \in A} \lambda$
  - $det(A) = \prod_{\lambda \in A} \lambda$

Reduction through topology

- **Lemma 1:** Let $|||.|||$ be a submultiplicative norm on $M_n(K)$. Then for any matrix $A \in M_n(K)$ with $|||A||| < 1$ the matrix $Id - A$ is invertible.

> Suppose we have a vector space V and a norm ||·|| on V. Then the **submultiplicative inequality** states that:
> $||x + y|| \leq ||x|| * ||y||$

- **Theo 1:** The subset $GL_n(K)$ is open in $M_n(K)$
- **Theo 2:** $GL_n(K)$ is dense in $M_n(K)$
- **Theo 3:** The set of diagonalizable matrices of $M_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$

- **Corollary 1 - Cayley Hamilton:** $\forall A \in M_n(\mathbb{C}), \chi_A(A) = 0$
- **Lemma 2:** Let $A \in M_n(\mathbb{C})$. For any eigenvalue λ of A, we have

$$|\lambda| \leq |||A|||$$

- **Theo 4:** For any $A \in M_n(\mathbb{C})$, we have

$$\log \rho(A) = \lim_n \frac{\log |||A|||}{n}$$