Lecture 12 - Integrals on $\mathbb{R}^2$

4.2 Integrating continuous maps on compact sets of $\mathbb{R}^2$

4.2.1 On rectangles

- **Prop 4.2.1:** I and J two compact intervals of $\mathbb{R}$. $f : I \times J \longrightarrow \mathbb{C}$ confinuous. The maps $y \mapsto \int_I f(x,y)dx$ and $x \mapsto \int_I f(x,y)dy$ are continuous on J and I. Furthermore,

$$\int_I (\int_I f(x,y)dy)dx = \int_I (\int_I f(x,y)dx)dy$$

- **Def 4.2.1:** I and J two compact intervals of $\mathbb{R}$. f continuous on $I \times J$. We define

$$\int_{I\times J} f(x,y)dxdy = \int_I dx \int_J dy f(x,y)$$

4.2.2 On a domain defined by 2 continuous maps

- **Prop 4.2.2:** I compact interval of $\mathbb{R}$. f and g two continuous maps on I s.t $f \leq g$ and let

$$\Omega = \{(x,y) | x \in I \text{ and } f(x) \leq y \leq g(x)\}$$

  Let φ a continuous function on Ω. The map

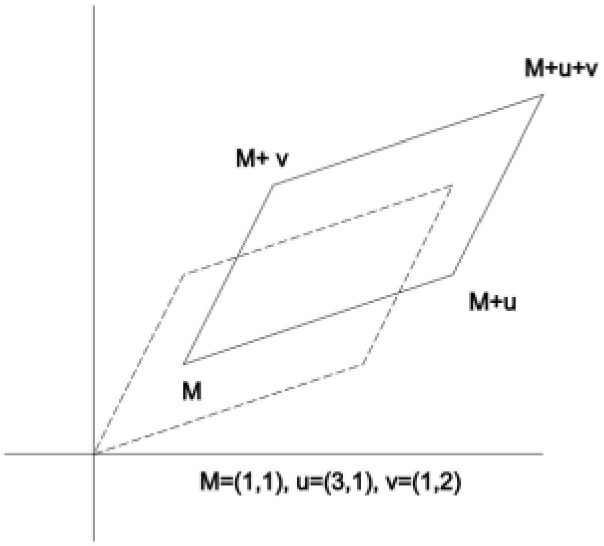
$$x \mapsto \int_{f(x)}^{g(x)} \varphi(x,y)dy$$

  is continuous on I.

4.2.3 On parallelograms

- Let Ω be a parallelogram. It is defined by a point $M = (a,b)$ and $u, v \in \mathbb{R}^2$. We then have that

$$M_1 = M + u$$
$$M_2 = M + u + v$$
$$M_3 = M + v$$



- **Def 4.2.3:** Let Ω be a parallelogram and L an affine map s.t. $\Omega = L([0,1]^2)$. Lef f a continuous map on Ω. We define

$$\int_\Omega f(x,y)dxdy = \int_{[0,1]^2} |\det L| f(L(t,s))dtdst$$

where det L is the determinant of the linear map underlying L.

> An affine map is a function of the form $f(x) = Ax + b$, where A is a matrix and b is a vector. L maps the unit square $[0,1]^2$ onto the parallelogram Ω.

4.2.4 On domains that are C^1 diffeomorphic to a rectangle

- **Def 4.2.4 - diffeomophism:** Let Ω open set of $\mathbb{R}^2$. Let $\varphi : \Omega \mapsto \mathbb{R}^2$. We say φ is a C^1 diffeomorphism if:
  1. φ is C^1
  2. φ is bijective
  3. φ has a C^1 inverse

- **Remark 4.2.3:** The differential of φ is always invertible

- **Def 4.2.5 - Jacobian of φ:** The absolute value of the determinant of the Jacobian matrix of φ. We call it $jac(\varphi)$

- **Def 4.2.6 - diffeomophic to a rectangle:** We say $\Omega \subseteq \mathbb{R}^2$ is C^1 diffeomorphic to a rectangle if $\exists$ a C^1 diffeomorphism φ and a rectangle $I \times J$ s.t $\Omega = \varphi(I \times J)$

- **Def 4.2.7:** We set

$$\int_{\Omega} f(x,y)dxdy = \int_{I\times J} f \circ \varphi(t,s) jac(\varphi)(t,s) dtds$$

- **Def 4.2.8:** Let $(\Omega_k)_k$ a finite family of domains C^1 diffeomorphic to a rectangle. Assume that $\forall j \neq k$,

$$|\Omega_k \cap \Omega_j| = 0$$

  Let f a continuous map on $\Omega = \cup \Omega_k$. We define

$$\int_{\Omega} f(x,y)dxdy = \sum_k \int_{\Omega_k} f(x,y)dxdy$$