Analysis 4 Problem Set 10

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1 Exercise 1

1.1 Exercise 1.1

We can calculate

$$\begin{aligned}
2\pi c_n(f) &= \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \\
&= \int_{-\pi}^{0} -e^{-inx}\,dx + \int_{0}^{\pi} e^{-inx}\,dx \\
&= -(\frac{1}{in}e^{in\pi} - 1 + \frac{1}{in}e^{-in\pi} - 1) \\
&= \frac{2}{in}(1-(-1)^n).
\end{aligned}$$

Thus,

$$\begin{aligned}
c_{2k+1}(f) &= \frac{2}{i\pi(2k+1)} \\
c_{2k}(f) &= 0.
\end{aligned}$$

1.2 Exercise 1.2

Because we know that f is 2π periodic and piecewise C^1, we know that the Fourier series of f converges for all x towards $\frac{f(x^-)+f(x^+)}{2}$. On the intervals $(-\pi,0)$, $(0,\pi)$, we know that $f(x) = \frac{f(x^-)+f(x^+)}{2}$, so we in particular have that $f(x)$ is equal to its Fourier series. On the other hand, at 0, π, we must have that $f(x) = 0$ in order for the Fourier series to converge.

1.3 Exercise 1.3

Plugging in $x = \frac{\pi}{2}$ into the Fourier series for f, we have that

$$\begin{aligned}
f\left(\frac{\pi}{2}\right) &= \sum_{k=-\infty}^{\infty} \frac{2}{i\pi(2k+1)} e^{i(2k+1)\frac{\pi}{2}} \\
&= \frac{2}{\pi}\sum_{k=-\infty}^{\infty} \frac{1}{2k+1}(-1)^k \\
\frac{\pi}{4} - 1 &= \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}.
\end{aligned}$$

We use Parseval's to see that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{4}{\pi^2(2k+1)^2} &= 1 \\
\sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} &= \frac{\pi^2}{4} \\
\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} &= \frac{\pi^2}{8}.
\end{aligned}$$

We have previously shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} &= -\sum_{n=1}^{\infty} \frac{1}{n^2} + 2\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \\ &= \frac{\pi^2}{12}\end{aligned}$$

2 Exercise 2

2.1 Exercise 2.1

First, we note that since f is both 2π periodic and C^k, by extending Lemma 4.3, we have that $c_n(f^{(k)}) = (in)^k c_n(f)$. Moreover, by Parseval's Theorem, we know that

$$\sum_{n\in\mathbb{Z}} |c_n(f^{(k)})|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f^{(k)}(x)|^2\,dx.$$

Then we know that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |c_n(f^{(k)})|^2 &= \sum_{n\in\mathbb{Z}} n^{2k}|c_n(f)|^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} |f^{(k)}(x)|^2\,dx < \infty.\end{aligned}$$

2.2 Exercise 2.2

Consider $x, x+h \in [0, 2\pi)$. Then,

$$\begin{aligned}|f(x) - f(y)| &= \left|\sum_{n\in\mathbb{Z}} c_n(f)e^{inx} - \sum_{n\in\mathbb{Z}} c_n(f)e^{in(x+h)}\right| \\ &\le \sum_{n\in\mathbb{Z}} |c_n(f)|\left|e^{inh} - 1\right| \\ &\le \sum_{n<M} |c_n(f)|\left|e^{inh} - 1\right| + \sum_{n\ge M} |c_n(f)|\left|e^{inh} - 1\right|\end{aligned}$$

To control the first term, we use the fact that $e^{inh} \to 1$ as $h \to 0$. To control the second term we use the fact that since $\sum_{n\in\mathbb{Z}} n^2|c_n(f)|^2 < \infty$ implies that $\sum_{n\in\mathbb{Z}} |c_n(f)| < \infty$, which implies that for sufficiently large M, we know that $\sum_{n\ge M} |c_n(f)| \le \epsilon$.

2.3 Exercise 2.3

We have previously calculated that the Fourier coefficients associated to f is $\frac{\pi}{2} + \sum_{n\ge 0} \frac{4\cos((2n+1)x)}{\pi(2n+1)^2}$.

Thus, it is clear that $\sum n^2|c_n(f)|^2 < \infty$, but $|x|$ is clearly not C^1, having a discontinuous derivative at the origin.

3 Exercise 3

3.1 Exercise 3.1

Consider

$$\begin{aligned}
\left|\int_{-\pi}^{\pi} D_n(t)f(x_0-t)\,dt - f(x_0)\right| &= \left|\sum_{k=-n}^{n}\int_{-\pi}^{\pi} e^{ikt}f(x_0-t)\,dt - f(x_0)\right| \\
&= \left|\sum_{k=-n}^{n}\int_{-\pi}^{\pi} e^{ik(x_0-t)}f(t)\,dt - f(x_0)\right| \\
&= \left|\sum_{k=-n}^{n} c_k(f)e^{ikx_0} - f(x_0)\right|.
\end{aligned}$$

Since we know that the Fourier series associated to f converges to f at x_0, we know that there exists N such that for all $n > N$, $\left|\sum_{k=-n}^{n} c_k(f)e^{ikx_0} - f(x_0)\right| \le \epsilon$ for any fixed positive ϵ, so we are done.

3.2 Exercise 3.2

First realize that $\int_{-\pi}^{\pi} D_n(t) = 2\pi$. Then, consider

$$\begin{aligned}
\int_{-\pi}^{\pi} D_n(t)g(x_0-t)\,dt - g(x_0) &= \int_{-\pi}^{-\delta} D_n(t)(g(x_0-t)-g(x_0))\,dt + \int_{\delta}^{\pi} D_n(t)(g(x_0-t)-g(x_0))\,dt \\
&\quad + \int_{-\delta}^{\delta} D_n(t)(g(x_0-t)-g(x_0))\,dt - g(x_0) + 2\pi f(x_0) \\
&= \int_{-\pi}^{-\delta} D_n(t)(g(x_0-t)-g(x_0))\,dt + \int_{\delta}^{\pi} D_n(t)(g(x_0-t)-g(x_0))\,dt \\
&\quad - \int_{-\pi}^{-\delta} D_n(t)(f(x_0-t)-f(x_0))\,dt - \int_{\delta}^{\pi} D_n(t)(f(x_0-t)-f(x_0))\,dt \\
&\quad + \int_{-\pi}^{\pi} D_n(t)f(x_0-t)\,dt - f(x_0)
\end{aligned}$$

3.3 Exercise 3.3

Using the expression $\sum_{k=-n}^{n} e^{ikx} = \frac{\sin((n+\frac{1}{2})x}{\sin(\frac{x}{2})}$, this is a direct application of the Riemann-Lebesgue lemma, as

$$\int_{-\pi}^{-\delta} D_n(t)(g(x_0-t)-g(x_0))\,dt = \int_{-\pi}^{-\delta} \sin((n+\frac{1}{2})t)\frac{D_n(t)(g(x_0-t)-g(x_0))}{\sin(\frac{t}{2})}\,dt$$

and that $\frac{D_n(t)(g(x_0-t)-g(x_0))}{\sin(\frac{t}{2}}$ is piecewise continuous and integrable on the domain of integration. A similar argument works for the other integrals in the problem. [Note: To see that $\frac{D_n(t)(g(x_0-t)-g(x_0))}{\sin(\frac{t}{2}}$ is piecewise continuous and integrable on the domain of integration, it was important that our domain of integration avoided a small neighborhood of 0).

3.4 Exercise 3.4

Combining Exercise 3.3 with Exercise 3.2 immediately yields that there exists some N such that for all $n \ge N$,

$$\left|\int_{-\pi}^{\pi} D_n(t)g(x_0-t)\,dt - g(x_0)\right| \le \epsilon.$$

which is exactly the result that we want.

4 Exercise 4

4.1 Exercise 4.1

Because $\cos x$ is even, we see that $\cos ax$ is even as well. Then, we know that

$$\begin{aligned} a_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \cos kx \, dx \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} (\cos(k-a)x + \cos(k+a)x) \, dx \end{aligned}$$

Which implies that

$$\begin{aligned} a_k &= \frac{1}{2\pi}\left(\frac{\sin(k-a)\pi}{k-a} + \frac{\sin(k+a)\pi}{k+a}\right) \\ &= \frac{(-1)^{k+1}}{\pi}\frac{a\sin\pi}{k^2-a^2}. \end{aligned}$$

Thus, we have that

$$\cos ax = \frac{\sin \pi a}{\pi a}\left(1 + 2a^2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\cos kx}{k^2-a^2}\right).$$

Then, we can divide both sides and plug in $x = \pi$ to get that

$$\frac{1}{\tan a\pi} = \frac{1}{\pi a}\left(1 + 2a^2 \sum_{k=1}^{\infty} \frac{1}{k^2-a^2}.\right)$$

Substituting $t = a\pi$ yields the desired result.

4.2 Exercise 4.2

Using Exercise 4.1, and direct integration, we get that

$$\ln\left(\frac{\sin x}{x}\right) = \int_0^x g(t)\,dt = \sum_{n=1}^{\infty}\int_0^x \frac{-2t}{n^2-t^2}\,dt = \sum_{n=1}^{\infty}\ln\left(1-\frac{x^2}{n^2}\right).$$

4.3 Exercise 4.3

This follows directly from the integration performed in Exercise 4.2.

4.4 Exercise 4.4

Consider

$$\begin{aligned} -\frac{1}{\sin^2(t)} &= -\frac{1}{t^2} - 2\sum_{n=1}^{\infty}\frac{t^2+n^2\pi^2}{(t^2-n^2\pi^2)^2} \\ \frac{1}{\sin^2(t)} &= \frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{1}{(n\pi+t)^2} + \frac{1}{(n\pi-t)^2} \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{(t-n\pi)^2}. \end{aligned}$$