Analysis 4 Problem Set 3

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1 Exercise 1

1.1 Exercise 1.1

Fix x, then because we have that$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we know that there exists some N be such that

$$\sum_{n=p}^{q} \frac{1}{n^2} \leq \frac{\varepsilon}{|x|+1} \quad \text{for all } q > p > N.$$

Then, we have

$$\left| \sum_{n=p}^{q} \frac{x}{n(n+x)} \right| < |x| \left| \sum_{n=p}^{q} \frac{1}{n^2} \right|$$
$$\leq \varepsilon,$$

for $n > N$. This shows pointwise convergence using the Cauchy criterion.

1.2 Exercise 1.2

First, we notice that for all $x \in [0, \infty)$

$$f_n(x) = \frac{x}{n(n+x)} = \frac{1}{n} - \frac{1}{n+x}. \tag{1}$$

This shows that $f_n(x) \geq 0$ on $[0, \infty)$ and $f_n(x)$ is an increasing function. It follows that

$$\max_{x \in [0,a]} |f_n(x)| = |f_n(a)| = \frac{a}{n(n+a)}.$$

Note that, for all $a > 0$,

$$\sum_{n=1}^{\infty} \frac{a}{n(n+a)} \leq a \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus, we have that $\sum f_n$ is normally convergent on $[0, a]$. Using Corollary 2.4, f is continuous on $[0, a]$, for all $a > 0$. Let $a \to +\infty$, we obtain f is continuous on $[0, \infty)$.

1.3 Exercise 1.3

Using $f_n(x)$ is an increasing function, we obtain for all $0 \leq x < y < \infty$,

$$\sum_{n=1}^{N} f_n(x) < \sum_{n=1}^{N} f_n(y) \quad \text{for all } N \in \mathbb{N}.$$

Let $N \to +\infty$, we obtain f is an increasing function.

1.4 Exercise 1.4

Using (1), we obtain for all integers $1 \le p \le N$,

$$\sum_{n=1}^{N} \frac{p}{n(n+p)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+p} \right) = \sum_{n=1}^{p} \frac{1}{n} + \sum_{n=p}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n+p}$$
$$= \sum_{n=1}^{p} \frac{1}{n} + \sum_{n=p}^{N} \frac{1}{n} - \sum_{n=p+1}^{p+N} \frac{1}{n} = \sum_{n=1}^{p} \frac{1}{n} - \sum_{n=N+1}^{N+p} \frac{1}{n}. \tag{2}$$

Let $N \to +\infty$ in (2), we obtain for each integer $p \ge 1$,

$$f(p) = \sum_{n=1}^{\infty} \frac{p}{n(n+p)} = \sum_{n=1}^{p} \frac{1}{n}. \tag{3}$$

1.5 Exercise 1.5

From $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and (3), we have for any $M > 0$, there exists $p \in \mathbb{N}$ such that $f(p) > M$. Recall that f is an increasing function, we have

$$f(x) > M \quad \text{for all } x \in [p, \infty).$$

This shows $\lim_{x \to +\infty} f(x) = +\infty$.

1.6 Exercise 1.6

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ and

$$\left| \frac{f_n}{x} \right| \le \frac{1}{n(n+x)} \le \frac{1}{n^2} \quad \text{for all } x \in [0, \infty).$$

This shows the series $\sum \frac{f_n}{x}$ is normally convergent on $[0, \infty)$. From the series $\sum \frac{f_n}{x}$ is normally convergent on $[0, \infty)$, we have for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$,

$$\sup_{x \in [0,\infty)} \left| \frac{f(x)}{x} - \sum_{n=1}^{N} \frac{f_n(x)}{x} \right| \le \varepsilon. \tag{4}$$

Using the definition of $f_n(x)$, we denote $M = \left(\sum_{n=1}^{N} \frac{1}{n} \right)^{-1} \varepsilon$,

$$\sup_{x > M} \left| \sum_{n=1}^{N} \frac{f_n(x)}{x} \right| \le \sup_{x > M} \left(\sum_{n=1}^{N} \frac{1}{nx} \right) \le M \left(\sum_{n=1}^{N} \frac{1}{n} \right) \le \varepsilon. \tag{5}$$

Gathering estimates (4) and (5),

$$\sup_{x > M} \left| \frac{f(x)}{x} \right| = \sup_{x > M} \left| \frac{f(x)}{x} - \sum_{n=1}^{N} \frac{f_n(x)}{x} + \sum_{n=1}^{N} \frac{f_n(x)}{x} \right| \le \sup_{x \in [0,\infty)} \left| \frac{f(x)}{x} - \sum_{n=1}^{N} \frac{f_n(x)}{x} \right| + \sup_{x > M} \left| \sum_{n=1}^{N} \frac{f_n(x)}{x} \right| \le 2\varepsilon.$$

This shows $\lim_{x \to +\infty} \frac{f(x)}{x} = 0$.

2 Exercise 2

2.1 Exercise 2.1

Fix $x \in \mathbb{R}$, note that,

$$\sum_{n=1}^{\infty} |f_n(x)| \le |x|^2 \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

This shows the series $\sum \frac{f_n}{x}$ is pointwise convergent.

2.2 Exercise 2.2

Fix $M > 0$, we have

$$\sup_{x\in[-M,M]} |f_n(x)| = \sup_{x\in[-M,M]} \left|\ln(1+\frac{x^2}{n^2})\right| \le \frac{M^2}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{M^2}{n^2} < \infty.$$

This shows $f = \sum f_n$ is normally convergent on $[-M, M]$. Using Corollary 2.4 and Theorem 2.8, we have f is continuous on $[-M, M]$. Let $M \to +\infty$, we obtain f is continuous on $\mathbb{R}$.

2.3 Exercise 2.3

From f_n is even function for all $n \in \mathbb{N}$, we have for all $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} f_n(-x) \quad \text{for all} \quad N \in \mathbb{N}. \tag{6}$$

Let $N \to +\infty$ in (6), we obtain $f(x) = f(-x)$ for all $x \in \mathbb{R}$. This shows f is even.

Note that, $f_n(x)$ is increasing on $[0, \infty)$ for all $n \in \mathbb{N}$, we obtain

$$\sum_{n=1}^{N} f_n(x) < \sum_{n=1}^{N} f_n(y) \quad \text{for all } 0 \le x < y < \infty \text{ and } n \in \mathbb{N}. \tag{7}$$

Let $N \to \infty$ in (7), we obtain $f(x) \le f(y)$ for all $0 \le x < y < \infty$. This shows f is increasing on $[0, \infty)$.

Note that

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \ge f_1(x) = \ln(1 + x^2) \to +\infty \quad \text{as } x \to \infty.$$

This shows $\lim_{x\to\infty} f(x) = +\infty$.

2.4 Exercise 2.4

Fix $I = [a, b]$ be a segment on $\mathbb{R}$. By direct computation,

$$f_n'(x) = \frac{\frac{2x}{n^2}}{1 + \frac{x^2}{n^2}} = \frac{2x}{n^2 + x^2} \quad \text{on } I.$$

Note that,

$$|f_n'(x)| = \left|\frac{2x}{n^2 + x^2}\right| \le \frac{|a| + |b|}{n^2} \quad \text{on } I \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|a| + |b|}{n^2} < \infty.$$

This shows the series of functions $\sum f_n'$ is normally convergent on I and using Exercise 2.1 and Theorem 3.2, we obtain f is derivable.

3 Exercise 3

3.1 Exercise 3.1

By direct computation,

$$\begin{aligned} f_n(x) + f_{n+1}(x) &= \frac{(-1)^n}{n(1+nx)} + \frac{(-1)^{n+1}}{(n+1)(1+(n+1)x)} \\ &= (-1)^n \left(\frac{1}{n(1+nx)} - \frac{1}{(n+1)(1+(n+1)x)} \right) \\ &= (-1)^n \frac{1 + (2n+1)x}{n(n+1)(1+nx)(1+(n+1)x)} \end{aligned} \tag{8}$$

Note that $1 + (2n+1)x \le 2(1 + (n+1)x)$ for all $x \in [0,\infty)$ and $n \in \mathbb{N}$, using (8) we obtain

$$|f_n(x) + f_{n+1}(x)| \le \frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1} \quad \text{for all} \quad x \in [0,\infty). \tag{9}$$

For all $0 < \varepsilon \ll 1$, we denote $N \in \mathbb{N}$ and $N > \frac{1}{\varepsilon} + 1$. From (9), we have for all $m > n > N$,

$$\begin{aligned}
\left|\sum_{k=n}^{m} f_k(x)\right| &= \left|\frac{f_n(x)}{2} - \frac{f_{m+1}(x)}{2} + \sum_{k=n}^{m}\left(\frac{f_k(x) + f_{k+1}(x)}{2}\right)\right| \\
&\le \left|\frac{f_n(x)}{2}\right| + \left|\frac{f_{m+1}(x)}{2}\right| + \frac{1}{2}\sum_{k=n}^{m}\left(\frac{2}{n} - \frac{2}{n+1}\right) \\
&\le \frac{1}{2n} + \frac{1}{2(m+1)} + \frac{1}{n} - \frac{1}{m+1} \le 2\varepsilon.
\end{aligned}$$

This shows uniform convergence using the Uniform Cauchy criterion.

3.2 Exercise 3.2

Note that, $f_n(x) = \frac{(-1)^n}{n(1+nx)}$ is continuous on $[0,\infty)$ for all $n \in \mathbb{N}$. Using the series $\sum f_n$ converges uniformly to f and Corollary 2.4, we obtain f is continuous.

3.3 Exercise 3.3

First, we calculate $\lim_{x\to 0} f(x)$. Note that,

$$f(0) = \sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Because f is continuous on $[0,\infty)$, we obtain

$$\lim_{x\to 0} f(x) = f(0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Second, we calculate $\lim_{x\to+\infty} f(x)$. From the series $\sum f_n$ converges uniformly to f on $[0,\infty)$, for all $0 < \varepsilon \ll 1$, there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in[0,\infty)} \left|f(x) - \sum_{n=1}^{N} f_n(x)\right| \le \varepsilon. \tag{10}$$

Note that, $\lim_{x\to+\infty} f_n(x) = 0$ for all $n \in \mathbb{N}$. So that, for all $0 < \varepsilon \ll 1$, there exists $M > 0$ such that

$$\sup_{x>M} \left|\sum_{n=1}^{N} f_n(x)\right| \le \sum_{n=1}^{N}\left(\sup_{x>M} |f_n(x)|\right) \le \varepsilon. \tag{11}$$

Gathering estimates (10) and (11), we obtain

$$\sup_{x>M} |f(x)| = \sup_{x>M} \left|f(x) - \sum_{n=1}^{N} f_n(x) + \sum_{n=1}^{N} f_n(x)\right| \le \sup_{x\in[0,\infty)} \left|f(x) - \sum_{n=1}^{N} f_n(x)\right| + \sup_{x>M} \left|\sum_{n=1}^{N} f_n(x)\right| \le 2\varepsilon.$$

This shows $\lim_{x\to+\infty} f(x) = 0$.

4 Exercise 4

4.1 Exercise 4.1

From Theorem 2.3 in the notes, we know that since h_n are all continuous at any x_0, h is also continuous at any x_0. But then this holds for any x_0, so indeed, h is continuous on the entire interval.

4.2 Exercise 4.2

To see that H_n converge uniform to H, we consider

$$\begin{aligned}
|H_n(x) - H(x)| &= \left|\int_a^x h_n(t)\,dt - \int_a^x h(t)\,dt\right| \\
&= \left|\int_a^x h_n(t) - h(t)\,dt\right| \\
&\leq \int_a^x |h_n(t) - h(t)|\,dt
\end{aligned}$$

But then since we kow that h_n converges uniformly to h we know that there exists an N such that for all $n > N$, $|h_n(t) - h(t)| \leq \frac{\epsilon}{|I|}$, that is, that

$$\begin{aligned}
|H_n(x) - H(x)| &\leq \int_a^x \frac{\epsilon}{|I|}\,dt \\
&\leq \epsilon
\end{aligned}$$

which indeed shows that H_n converges uniformly to H.

4.3 Exercise 4.3

Exercise 4.3a We rewrite

$$g(x_2) - g(x_1) = (f_n'(x_1) - g(x_1) + g(x_2) - f_n'(x_2)) + (f_n'(x_2) - f_n'(x_1)).$$

Taking the absolute value of both sides and then applying the triangle inequality gets us the following inequality

$$\begin{aligned}
|g(x_2) - g(x_1) &\leq |f_n'(x_1) - g(x_1) + g(x_2) - f_n'(x_2)| + |f_n'(x_2) - f_n'(x_1)| \\
&\leq |f_n'(x_1) - g(x_1)| + |g(x_2) - f_n'(x_2)| + |f_n'(x_2) - f_n'(x_1)|.
\end{aligned}$$

Then, we see that by uniform convergence, there exists some N such that for all $n > N$, we have that $|f_n'(x_1) - g(x_1)| < \frac{\epsilon}{3}$ and that $|g(x_2) - f_n'(x_2)| \leq \frac{\epsilon}{3}$. Then, since we know that f_n' is continuous for each n, we also have that there exists a δ such that for $|x_1 - x_2| < \delta$, $|f_n'(x_1) - f_n'(x_2)| < \frac{\epsilon}{3}$.

Thus, for $n > N$, $|x_1 - x_2| < \delta$, we have that

$$\begin{aligned}
|g(x_2) - g(x_1)| &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\leq \epsilon.
\end{aligned}$$

Exercise 4.3b First, we see that $f_n(x) = f_n(a) + \int_a^x f_n'(t)\,dt$ by the Fundamental Theorem of Calculus. Then, by Exericse 3.2, $\int_a^x f_n'(t)\,dt$ converges uniformly to $\int_a^x g(t)\,dt$. Then, there is some N_1 such that for all $n > N_1$, $\int_a^x |f_n'(t) - g(t)|\,dt < \frac{\epsilon}{2}$. Furthermore, since we know that $f_n(a)$ converges to a limit $l \in \mathbb{R}$, then we have that there exists some N_2 such that for all $n > N_2$,

$$|f_n(a) - l| < \frac{\epsilon}{2}$$

Now, we use the triangle inequality to see that

$$\begin{aligned}
\left|f_n(x) - \left(l + \int_a^x g(t)\,dt\right)\right| &\leq |f_n(a) - l| + \int_a^x |f_n'(t) - g(t)|\,dt \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&\leq \epsilon
\end{aligned}$$

for $n > \max(N_1, N_2)$. This shows the sequence f_n converges uniformly to the functions f. From $f(x) = l + \int_a^x g(t)dt$ on I, we deduce that f is C^1 and that $g' = f$.

5 Exercise 5

From $\int_0^\infty |g(x)|dx < \infty$, we have for all $0 < \varepsilon \ll 1$, there exists $M > 0$ such that

$$\int_0^{\frac{1}{M}} |g(x)|dx + \int_M^\infty |g(x)| < \varepsilon. \tag{12}$$

Fix above M, using (f_n) converges uniformly on all compact set of $(0,\infty)$ to a function f and Proposition 3.1, we obtain there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\left| \int_{\frac{1}{M}}^M f_n(x)dx - \int_{\frac{1}{M}}^M f(x)dx \right| \le \varepsilon. \tag{13}$$

From $|f_n| \le |g|$ and (f_n) converges uniformly on all compact set of $(0,\infty)$ to a function f, we obtain

$$|f(x)| \le |g(x)| \quad \text{for all } x \in (0,\infty). \tag{14}$$

Gathering estimates (12), (13), (14) and $|f_n| \le |g|$, we obtain for all $n > N$,

$$\begin{aligned}
&\left| \int_0^\infty f_n(x)dx - \int_0^\infty f(x)dx \right| \\
&\le \left| \int_0^{\frac{1}{M}} f_n(x)dx - \int_0^{\frac{1}{M}} f(x)dx \right| + \left| \int_{\frac{1}{M}}^M f_n(x)dx - \int_{\frac{1}{M}}^M f(x)dx \right| + \left| \int_M^\infty f_n(x)dx - \int_M^\infty f(x)dx \right| \\
&\le \int_0^{\frac{1}{M}} (|f_n(x)| + |f(x)|)\,dx + \int_M^\infty (|f_n(x)| + |f(x)|)\,dx + \left| \int_{\frac{1}{M}}^M f_n(x)dx - \int_{\frac{1}{M}}^M f(x)dx \right| \\
&\le 2\int_0^{\frac{1}{M}} |g(x)|dx + 2\int_M^\infty |g(x)|dx + \varepsilon \le 3\varepsilon.
\end{aligned}$$

This shows $\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$.

6 Exercise 6

6.1 Exercise 6.1

We first note that, for all $q > p$,

$$\sum_{n=p}^q \left(\sum_{k=p}^n b_k \right) (a_n - a_{n+1}) + \left(\sum_{n=p}^q b_n \right) a_{q+1} = \sum_{n=p}^q a_n b_n \tag{15}$$

Using (15) and let $b_n = z^n$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{n=p}^q a_n z^n &= \sum_{n=p}^q \left(\sum_{k=p}^n z^k \right) (a_n - a_{n+1}) + \left(\sum_{n=p}^q z^n \right) a_{q+1} \\
&= \sum_{n=p}^q \left\{ \left(\frac{z^p(1 - z^{n-p+1})}{1-z} \right) (a_n - a_{n+1}) \right\} + \frac{z^p(1 - z^{q-p+1})}{1-z} a_{q+1}.
\end{aligned} \tag{16}$$

Now we calculate $|1 - z|$. Using $z(x) = \cos x + i \sin x$ and $\cos x = 1 - 2\sin^2 \frac{x}{2}$,

$$|1 - z| = \sqrt{(1 - \cos x)^2 + \sin^2 x} = \sqrt{2 - 2\cos x} = 2 \sin \frac{x}{2}. \tag{17}$$

Using (16) and (17), we obtain

$$\begin{aligned}
\left| \sum_{n=p}^q a_n z^n \right| &\le \sum_{n=p}^q \left\{ \left(\frac{|z^p(1 - z^{n-p+1})|}{|1-z|} \right) (|a_n - a_{n+1}|) \right\} + \frac{|z^p(1 - z^{q-p+1})|}{|1-z|} |a_{q+1}| \\
&\le \frac{2}{|1-z|} \left(\sum_{n=p}^q (|a_n - a_{n+1}|) + |a_{q+1}| \right).
\end{aligned} \tag{18}$$

Using (a_n) is a decreasing sequence in $\mathbb{R}$ converging to 0 and (18),

$$\left|\sum_{n=p}^{q} a_n z^n\right| \le \frac{2}{|1-z|}\left(\sum_{n=p}^{q}(|a_n - a_{n+1}|) + |a_{q+1}|\right) \le \frac{1}{\sin(\frac{x}{2})}\left(\sum_{n=p}^{q}(a_n - a_{n+1}) + a_{q+1}\right) \le \frac{a_p}{\sin(\frac{x}{2})}.$$

6.2 Exercise 6.2

Note that $z^n = \cos nx + i \sin nx$, we have for all $q > p$,

$$\left|\sum_{n=p}^{q} a_n \sin(nx)\right| = \left|\mathrm{Im}\left(\sum_{n=p}^{q} a_n z^n\right)\right| \le \frac{a_p}{\sin(\frac{x}{2})} \tag{19}$$

This shows pointwise convergence using the Cauchy criterion (Using $\lim_{n\to 0} a_n = 0$).

6.3 Exercise 6.3

Note that

$$\min_{x\in[u,2\pi-u]} |\sin(\frac{x}{2})| = \sin(\frac{u}{2}) \quad \text{for all } 0 < u < \pi. \tag{20}$$

Using (19) and (20), we have

$$\max_{x\in[u,2\pi-u]}\left|\sum_{n=p}^{q} a_n \sin(nx)\right| \le \max_{x\in[u,2\pi-u]}\left|\frac{a_p}{\sin(\frac{x}{2})}\right| \le \frac{a_p}{\sin(\frac{u}{2})}$$

This shows uniformly convergence on $[u, 2\pi - u]$ using the uniform Cauchy criterion (Using $\lim_{n\to 0} a_n = 0$). Note that $\sum_{n=1}^{N} a_n \sin(nx)$ is continuous on $[u, 2\pi - u]$. Then, since $f = \sum_{n=1}^{\infty} a_n \sin(nx)$ is the limit of a uniformly convergent series of continuous functions on $[u, 2\pi - u]$, using Corollary 2.4, it must also be continuous on $[u, 2\pi - u]$. Let $u \to 0$, we obtain f is continuous on $(0, 2\pi)$.