Quiz 1 2020

Allen Fang and Xu Yuan

1 Exercise 1

Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = n^2(x^{n-1} - x^n) \quad \text{for } n \in \mathbb{N}^+.$$

1. Show that for any $x \in [0,1]$
$$\lim_{n\to\infty} f_n(x) = 0.$$
2. Calculate $\int_0^1 f_n(x)\,\mathrm{d}x$ for $n \in \mathbb{N}^+$.
3. Calculate
$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x.$$
4. Conclude that the sequence $(f_n)_{n\in\mathbb{N}^+}$ does not converge uniformly to 0 on $[0,1]$.

2 Exercise 2

Suppose f is a C^2 function on $[-\delta,\delta]$ with $f(0) = 0$, $0 < f'(0) < 1$. Set

$$f_0(x) = f(x) \quad \text{and} \quad f_n(x) = f(f_{n-1}(x)) \quad \text{for } n \in \mathbb{N}^+.$$

1. Show that for any $0 < \delta_1 < \delta$,
$$|f(x)| \le \left(f'(0) + \frac{1}{2}M\delta_1\right)|x| \quad \text{for } x \in (-\delta_1, \delta_1),$$
where
$$M = \max_{x\in[-\delta,\delta]} |f''(x)|.$$
Hint: Taylor formula.
2. Show that there exists $0 < \delta_2 < \delta$ such that
$$|f(x)| < q|x| \quad \text{for } x \in (-\delta_2, \delta_2),$$
where $0 < q < 1$.
3. Show that for all $n \in \mathbb{N}^+$
$$|f_n(x)| < q^n|x| \quad \text{for } x \in (-\delta_2, \delta_2).$$
Conclude that the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on $(-\delta_2, \delta_2)$.

3 Exercise 3

Find the radius of convergence R for $\sum a_n x^n$ with the given coefficients a_n.

1. Consider
$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(2x)^n}{n}.$$
2. Consider
$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{nx^n}{e^n}.$$
3. Consider
$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{\pi^n x^n}{n^\pi}.$$
4. Consider
$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!}.$$
5. Consider
$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{x^n}{C_{2n}^n}, \quad \text{where } C_m^k = \frac{m!}{k!(m-k)!} \quad \text{for all } m, k \in \mathbb{N}^+.$$

4 Exercise 4

Let $I = [0,1]$ and $f_n : I \to \mathbb{R}$, defined by
$$f_n(x) = \left(1 + \frac{x}{n}\right)^n, \quad \text{for } n \in \mathbb{N}^+.$$
show that the sequence $(f_n)_{n\in\mathbb{N}^+}$ converges uniformly to $f(x) = e^x$ on I.

5 Exercise 5

1. Write the partial fraction decomposition of $\frac{5}{(x^2+4)(x^2-1)}$.
2. Write $\frac{1}{1+x^2}$ and $\frac{1}{1-x^2}$ as a power series.
3. Write $\frac{5}{(x^2+4)(x^2-1)}$ as a power series, and compute the radius of convergence.