Analysis 4 Problem Set 12

Allen Fang and Xu Yuan

1 Exercise 1

1.1 Exercise 1.1

This is a separable equation, so we can solve it by direct integration:

$$\begin{aligned}
\int_{y(t_0)}^{y(t)} e^{-y}\,dy &= \int_{t_0}^{t} s\,ds \\
-e^{y(t)} + e^{y(t_0)} &= \frac{1}{2}(t^2 - t_0^2) \\
e^{y(t)} &= \frac{1}{2}(t_0^2 - t^2) + e^{y(t_0)} \\
y(t) &= y(t_0)\ln\frac{1}{2}(t_0^2 - t^2).
\end{aligned}$$

1.2 Exercise 1.2

This equation can be solved by substitution. Consider $z(t) = y'(t)$. Then, the original ODE is reduced by

$$z' + 3z^2 = 0.$$

Now this is a linear ODE that we can solve directly as follows

$$\begin{aligned}
\int_{z(t_0)}^{z(t)} -\frac{1}{z^2} &= \int_{t_0}^{t} 3 \\
\frac{1}{z(t)} - \frac{1}{z(t_0)} &= 3t - 3t_0 \\
z(t) &= \frac{1}{3t - 3t_0 + \frac{1}{z_0}}.
\end{aligned}$$

Then, we can use the fact that $z(t) = y'(t)$ to see that

$$y(t) - y(t_0) = \ln\left(3t - 3t_0 + \frac{1}{y'(t_0)}\right)$$

1.3 Exercise 1.3

Once again, this equation is separable, so we can directly integrate to solve

$$\begin{aligned}
\int_{y(t_0)}^{y(t)} y^{-\frac{1}{2}}\,dy &= \int_{t_0}^{t} s^2\,ds \\
2\left(\sqrt{y(t)} - \sqrt{y(t_0)}\right) &= \frac{1}{3}\left(t^3 - t_0^3\right) \\
y(t) &= \left(\frac{1}{6}\left(t^3 - t_0^3\right) + \sqrt{y(t_0)}\right)^2
\end{aligned}$$

2 Exercise 2

2.1 Exercise 2.1

In this case, we have $F(t,y) = ty^2 - y$. Then, it is clear that F is continuous. F is also locally Lipschitz in the second variable, as

$$\begin{aligned} |F(t,y) - F(t,x)| &= \left|ty^2 - tx^2 - y + x\right| \\ &\leq (t|y+x| + 1)|y-x|, \end{aligned}$$

and on any given compact interval, $t|y+x|+1$ is bounded.

Then, Corollary 2.8 in the lecture notes directly yields that there exists a unique maximal solution to the Cauchy problem.

2.2 Exercise 2.2

We first realize that 0 solves the given differential equation, and that moreover, it is a maximal solution. Then, since we have shown in Exercise 2.1 that maximal solutions are unique, we must have precisely that $\phi_0 = 0$.

By direct computation, we verify that $\phi_1(t) = \frac{1}{1+t}$ solves the Cauchy problem.

2.3 Exercise 2.3

Writing $z = \phi_r^{-1}$, we see that z must solve the equation

$$z' = z - t.$$

Then, we can solve this explicitly by variating constants, which yields that

$$ze^{-t} - z_0 e^{-t_0} = (t+1)e^{-t} - (t_0+1)e^{-t_0}$$

Thus, since we are prescribing initial data $(t_0, \phi_r(0)) = (0, r)$, we have that

$$y = \frac{1}{(t+1) + (\frac{1}{r} - 1)e^t}$$

Thus, we see that on the interval (a,b), $0 \leq \phi_r(t) \leq \frac{1}{1+t}$.

2.4 Exercise 2.4

Since we know that $0 \leq \phi_r(t) \leq \frac{1}{1+t}$ for $t \in (a,b)$, we can use Proposition 2.10 to see that the solution can be extended to a strictly bigger interval.

2.5 Exercise 2.5

Using the explicit solution from Exercise 2.3, it's clear that the the solution is defined on the interval $(-\frac{1}{r}, \infty)$. For $0 < r < 1$, the intersection of these domains of existence is clearly $(-1, \infty)$.

3 Exercise 3

3.1 Exercise 3.1

If $\alpha < 0$, then the domain of the equation is restricted to when $y \neq 0$, while there is no such restriction if $\alpha > 0$.

3.2 Exercise 3.2

Assume for the sake of contradiction that there exists a C^1 solution ϕ, and $\phi(0) > 0$, but $\phi(t) \le 0$, then $\phi(s) = 0$ for some $0 < s \le t$.

Then, consider the Cauchy problem with

$$\begin{aligned} y' &= a(t)y + b(t)y^\alpha \\ y(s) &= 0. \end{aligned}$$

By Cauchy-Lipschitz, this problem is well-posed, and it is obvious that the null solution is a solution to this ODE. But we, by assumption, have a non-null solution to this Cauchy problem, which is a contradiction.

3.3 Exercise 3.3

Using the definition of ψ, we can calculate that

$$\begin{aligned} \frac{\phi'}{\phi^\alpha} &= \frac{a(t)}{\phi^{\alpha-1}} + b(t) \\ \psi'(t) &= (1-\alpha)a(t)\psi(t) + (1-\alpha)b(t). \end{aligned}$$

3.4 Exercise 3.4

We can use the integrating factor method to solve for an explicit solution. Integrating from t_0 to t, we have that

$$\begin{aligned} \psi' e^{-\int_{t_0}^t a\,ds} - (1-\alpha)a(t)\psi e^{-(1-\alpha)\int_{t_0}^t a\,ds} &= (1-\alpha)b(t)e^{-(1-\alpha)\int_{t_0}^t a\,ds} \\ \psi(t)e^{-(1-\alpha)\int_{t_0}^t a\,ds} - \psi(t_0) &= \int_{t_0}^t (1-\alpha)b(t)e^{-(1-\alpha)\int_{t_0}^t (1-\alpha)a\,ds}\,dt \end{aligned}$$

3.5 Exercise 3.5

Consider $y = \phi_0 + z$. Then, substituting into our ODE, and using the fact that ϕ_0 is a solution to the Bernoulli equation,

$$z' = [2a(t)\phi_0(t) + b(t)]z + a(t)z^2.$$

But we see that this is a Bernoulli equation which can be solved explicitly as before.

4 Exercise 4

4.1 Exercise 4.1

By considering a constant function $y(t) = \alpha$, we see that the only constant solutions of the ODE are exactly when

$$\alpha = e^{\frac{2i\pi}{3}} := j \qquad \alpha = e^{\frac{4i\pi}{3}} = j^2.$$

4.2 Exercise 4.2

Define $z = y - j$. Then, rewriting the ODE in terms of z yields

$$z' + (2j+1)z + z^2 = 0.$$

If z vanishes at a point, then so does z', and we have that z is the null function. Otherwise, we can rewrite our equation by dividing through by z^2 as

$$\frac{z'}{z^2} + \frac{2j+1}{z} + 1 = 0.$$

or

$$-u' + i\sqrt{3}u + 1 = 0$$

where $u = \frac{1}{z}$, and we have used the definition of j. We can solve this equation explicitly by the family $v(t) = \lambda e^{i\sqrt{3}t}, \lambda \in \mathbb{C}$.

Substituting back then we see that our original ODE is solved by the family

$$y = \frac{1}{\lambda e^{i\sqrt{3}t} + \frac{i}{\sqrt{3}}} + j.$$

Since we are looking only for real solutions, we need to find $\lambda \in \mathbb{C}$ such that the imaginary part of the general solution disappears for all t. More precisely, we are looking for λ such that

$$-\frac{\sqrt{3}}{2} = -\frac{\Im(\lambda e^{i\sqrt{3}t} + \frac{i}{\sqrt{3}})}{|\lambda e^{i\sqrt{3}t} + \frac{i}{\sqrt{3}}|}^2.$$

Defining $I = \Im(\lambda e^{i\sqrt{3}t})$, we see that this is equivalent to looking for λ such that

$$\left(|\lambda|^2 + \frac{1}{3} + \frac{2I}{\sqrt{3}}\right)\frac{\sqrt{3}}{2} = I + \frac{1}{3}.$$

This is equivalent to $|\lambda|^2 = \frac{1}{3}$, so we see that we must restrict to the family of solutions where $\lambda = \frac{e^{i\theta}}{\sqrt{3}}$ to have real solutions.

5 Exercise 5

Since ϕ is a C^1 solution to $y' = f(y)$, and it is bounded, let us consider two different cases. If ϕ achieves a critical point on $\mathbb{R}$, then we are done. As $f(\phi(x_0)) = 0$ by definition.

If not, then we can assume that in particular $\phi(t)$ is (WLOG) monotonically increasing and bounded, and thus, in particular $\lim_{t\to\infty} \phi(t) = \phi_0$. Because f is a continuous function, and $\phi \in C^1$, we know that in particular $\phi'(\phi_0) = \lim_{t\to\infty} f(\phi(t))) = \lim_{t\to\infty} \phi'(t)$. is well-defined. Thus, it suffices to show that $\phi'(t) \to 0$ as $t \to \infty$.

Then, by an application of L'Hopital,

$$\begin{aligned}\lim_{t\to\infty} \frac{\phi(t) - \phi_0}{t} &= \lim_{t\to\infty} \phi'(t) \\ &= 0.\end{aligned}$$