CORRECTION WEEK 1

Solution of exercise 1.

1. Note that
$$\sum_{n=1}^{\infty}|f_n| \le \sum_{n=1}^{\infty}\frac{|x|}{x^2+n^2+1} \le |x|\sum_{n=1}^{\infty}\frac{1}{n^2+1} < +\infty.$$
It follows that the series of $\sum f_n$ pointwise convergent. Because f_n is odd, so that the limit $f(x)=\sum_n f_n$ is also odd.

2. Note that (Here we use $\sin x \le |x|$)
$$\sum_{n=1}^{\infty}|f_n| \le \sum_{n=1}^{\infty}\left(\frac{|x|}{n}\right)^2 \le |x|^2\sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty.$$
It follows that the series of $\sum f_n$ pointwise convergent. Because f_n is even, so that the limit $f(x)=\sum_n f_n$ is also even.

3. Note that (Here we use $\ln(1+x)\le x$ for all $x\ge 0$.)
$$\sum_{n=1}^{\infty}|f_n| \le \sum_{n=1}^{\infty}\left(\frac{x}{n^2}\right) \le x\sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty.$$
It follows that the series of $\sum f_n$ pointwise convergent. Because f_n is increasing, so that the limit $f(x)=\sum_n f_n$ is also increasing.

4. The series of functions $\sum f_n$ does not pointwise convergent. First, when $z=1$, we have, $\sum_{k=1}^{n} f_k = n$. It is clear that $\sum_{n=1}^{\infty} f_n$ does not convergent. Second, when $z\ne 1$, we denote $z=e^{i\theta}$, $\theta\in(0,2\pi)$. From the formula of the sum of the geometric series, we have
$$\sum_{k=1}^{n} f_k = \sum_{k=1}^{n} e^{ik\theta} = \frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}}.$$
It is clear that $\sum_{n=1}^{\infty} f_n$ does not convergent.

Solution of exercise 2.

It suffer to prove that, for all ε, there exists $N\in\mathbf{N}$ such that, for all $n\ge N$, $m\ge N$, $\left|\sum_{k=n}^{m}\frac{\sqrt{f_k}}{k}\right| < \varepsilon$. By the series $\sum f_n$ is pointwise convergent and $\sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty$, we have for all ε, there exists $N\in\mathbf{N}$ such that, for all $n\ge N$, $m\ge N$,
$$\sum_{k=n}^{m} f_k < \varepsilon \quad \text{and} \quad \sum_{k=n}^{m}\frac{1}{k^2} < \varepsilon. \tag{0.1}$$
Using Cauchy–Schwarz inequality and (0.1), we obtain
$$\left|\sum_{k=n}^{m}\frac{\sqrt{f_k}}{k}\right| < \frac{1}{2}\left(\sum_{k=n}^{m} f_k + \sum_{k=n}^{m}\frac{1}{k^2}\right) < \varepsilon.$$

Solution of exercise 3.

It suffer to prove that, for all ε, there exists $N \in \mathbf{N}$ such that, for all $n \geq N$, $m \geq N$, $|\sum_{k=n}^{m} f_k g_k| < \varepsilon$. From $\sum f_n$ is a pointwise convergent series and f_n positive, we have ,for all ε, there exists $N \in \mathbf{N}$ such that, for all $n \geq N$, $m \geq N$, $|\sum_{k=n}^{m} f_k| < \varepsilon$. Note that, $(g_n(x)$ is bounded, we denote $M_x = \sup_n |g_n(x)|$. It follows that

$$|\sum_{k=n}^{m} f_k g_k| < \sum_{k=n}^{m} |f_k||g_k| < M_x \sum_{k=n}^{m} |f_k| < M_x \varepsilon.$$

Solution of exercise 4.

For $x = 0$, it is clear that $f_n(0) = 0$. For $x \neq 0$, by Cauchy–Schwarz inequality, we obtain

$$|f_n(x)| = \frac{1}{|x^{-1} + n^2 x|} \leq \frac{1}{2n}.$$

It follows that (f_n) uniform converging on $\mathbb{R}$.

Solution of exercise 5.

1. By direct computation,

$$f_n(x) = \frac{nx^2}{1+nx^2} = 1 - \frac{1}{1+nx^2}.$$

We denote $f(x) = \lim_{n\to\infty} f_n(x)$. Let $n \to \infty$, we have

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Note that $f(x)$ does not continuous at 0 and $f_n(x)$ is continuous function series. From Theorem 2.3, we know f_n does note uniform convergence.

2. Note that $e^{-(1+x^2)t^2}$ is positive, we have $f_{n+1}(x) > f_n(x)$. It follows that there exists $f(x)$, such that $f(x) = \lim_{n\to\infty} f_n(x)$. From the definition of $f_n(x)$ and consider change variable $s = \sqrt{1+x^2}t$,

$$f(x) = \int_0^\infty e^{-(1+x^2)t^2} dt = \frac{1}{\sqrt{1+x^2}} \int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2\sqrt{1+x^2}}.$$

We denote $a_n = \int_{n-1}^{n} e^{-t^2} dt$. It is not difficult to check $\sum_{n=1}^{\infty} a_n < \infty$ and $|f_n(x) - f_{n-1}(x)| \leq a_n$. From Theorem 2.8, we know f_n is uniform convergence.

3. By definition of $f_n(x)$, we have

$$f_n(x) = \begin{cases} x, & x < n \\ n, & x \geq 0 \end{cases}$$

It is clear that $f(x) = \lim_{n\to\infty} f_n(x) = x$ and f_n does not uniform convergence (because $f_{n+1}(n+1) - f_n(n+1) = 1$).

Solution of exercise 6.

From Ω is a compact set of E and $f : \Omega \to \mathbb{R}$ is a continuous functions, we have there exists $0 < M < +\infty$ such that $M = \max_{x\in\Omega} |f(x)|$. By the definition of f_n, we obtain

$$|f_n(x) - 0| \le |a_n||f(x)| \le M|a_n|. \tag{0.2}$$

Using $\lim_{n\to\infty} a_n = 0$ and (0.2), we obtain (f_n) converges uniformly to the null function on Ω.

Solution of exercise 7. From (f_n) be a sequence of continuous function $\Omega \to \mathbb{R}$ which converges uniformly to f, we have, for all ε, there exists $N \in \mathbf{N}$ such that, for all $n \ge N$,

$$\sup_{x\in\Omega} |f_n(x) - f(x)| < \varepsilon. \tag{0.3}$$

and $f(x)$ is a continuous functions. From (x_n) of Ω converging to x and $f(x)$ is a continuous functions, we have, for all ε, there exists $N \in \mathbf{N}$ such that, for all $n \ge N$,

$$|f(x_n) - f(x)| < \varepsilon. \tag{0.4}$$

Using (0.3) and (0.4), we obtain, for all ε, there exists $N \in \mathbf{N}$ such that, for all $n \ge N$,

$$|f_n(x_n)-f(x)| < |f_n(x_n)-f(x_n)|+|f(x_n)-f(x)| < \sup_{x\in\Omega} |f_n(x)-f(x)|+|f(x_n)-f(x)| < 2\varepsilon.$$

It follows that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

The converse is not true (See exercise 5.3). But if we assume Ω is compact and $f(x)$ is continuous function, the converse is also true.

Solution of exercise 8.

1. First, for all $\varepsilon \ll 1$, there exists $N \in \mathbf{N}$ such that, for all $n \ge N$, $|\cos^n(\varepsilon)| \le \varepsilon$. Second, note that, for all $\varepsilon \ll 1$, $|\sin(\varepsilon)| \le \varepsilon$. Gathering above estimates, we have, for all $n \ge N$,

$$\begin{aligned}
\sup_{x\in[0,\frac{\pi}{2}]} |f_n(x) - 0| &\le \sup_{x\in[0,\varepsilon]} |f_n(x)| + \sup_{x\in[\varepsilon,\frac{\pi}{2}]} |f_n(x)| \\
&\le \sup_{x\in[0,\varepsilon]} |\cos^n(x)\sin(x)| + \sup_{x\in[\varepsilon,\frac{\pi}{2}]} |\cos^n(x)\sin(x)| \\
&\le \sup_{x\in[0,\varepsilon]} |\sin(x)| + \sup_{x\in[\varepsilon,\frac{\pi}{2}]} |\cos^n(x)| \\
&\le |\sin(\varepsilon)| + |\cos^n(\varepsilon)| \le 2\varepsilon.
\end{aligned}$$

It follows that (f_n) converges uniformly to the null function on $[0, \frac{\pi}{2}]$.

2. Because $\cos(\delta) < 1$, we have, for all $\varepsilon \ll 1$, there exists $N \in \mathbf{N}$ such that, for all $n \ge N$, $(n+1)|\cos^n(\delta)| < \varepsilon$. It follows that

$$\begin{aligned}
\sup_{x\in[\delta,\frac{\pi}{2}]} |g_n(x) - 0| &\le \sup_{x\in[\delta,\frac{\pi}{2}]} |(n+1)\cos^n(x)| \cdot \sup_{x\in[\delta,\frac{\pi}{2}]} |\sin(x)| \\
&\le (n+1)|\cos^n(\delta)| \le \varepsilon.
\end{aligned}$$

So that, (g_n) converges uniformly to the null function on $[\delta, \frac{\pi}{2}]$.

3. By change variable $t = \cos x$, we have

$$(n+1)\int_0^{\frac{\pi}{2}} \cos^n(x)\sin(x)dx = (n+1)\int_0^1 t^n dt = 1$$

It follows that $\int_0^{\frac{\pi}{2}} g_n$ does not converge to zero as n tends to infinity.