Analysis 4 Problem Set 14

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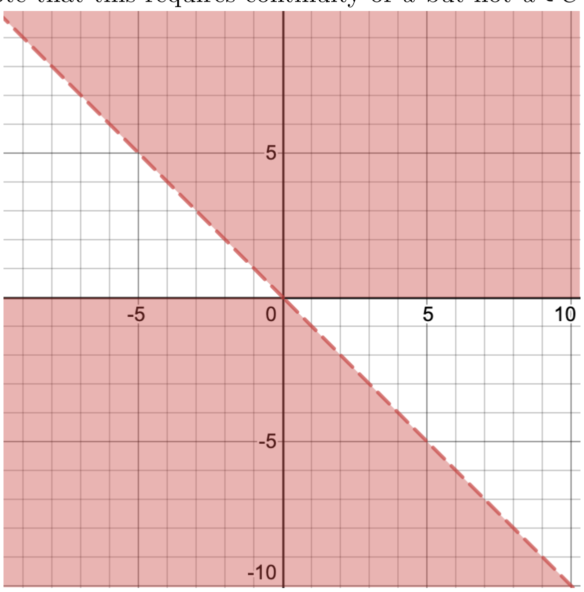
1 Exercise 1

Let us take the case where $u'(t) < f(t,u(t))$ and $\phi(t_0) \geq u(t_0)$ since the other proofs follow similarly. Then, assume for the sake of contradiction that actually, $u(s) > \phi(s)$ for some s. Let s_0 be the infimum of all such s. Then, by definition, we know that $u'(s_0) \geq \phi'(s_0)$. However, $u'(s_0) < f(s_0, u(s_0)) = f(s_0, \phi(s_0)) = \phi'(s_0)$. [Note that this requires continuity of u but not $u \in C^1$.



2 Exercise 2

2.1 Exercise 2.1

The solution is decreasing provided $y' < 0$. Using the ODE we see that this is equivalent to the condition that $y(y+t) > 0$, which holds in the area where $y > 0, y > -t$ and where $y < 0, y < -t$.

2.2 Exercise 2.2

Using Corollary 2.8, we see that since $F(t,y) = -y(y+t)$ is C^1 in y, it is in particular locally Lipschitz in y as well, and thus, the Cauchy problem has a unique maximal solution.

It is clear that the null function is a maximal solution to the Cauchy problem, and since we have just shown that maximal solutions are unique, $\phi_0 = 0$.

To see the last step, we use the same argument as used on Exercise Sheet 12. Namely, if there exists a solution that is nonpositive, then there must be a point, s at which $\phi(t_*) \leq 0$. Then, by continuity, there is a point at which $\phi(s) = 0$. But then considering the Cauchy problem

$$y' = -y(y+t) \quad (1)$$
$$y(s) = 0 \quad (2)$$

we see that 0 is a maximal solution to this Cauchy problem. Since we know maximal solutions are unique, this is a contradiction.

2.3 Exercise 2.3

Since one is positive, we know from Exercise 2.2 that $\phi_1(t) > 0$ for all $t \in [0, b)$. Then, since $t > 0$ as well, we have using the ODE directly that ϕ_1 must be decreasing in the region $0 \leq t < b$. Then, clearly the solution exists in the compact interval $[0, b] \times [b, 1]$, and since this is true for any finite b, using Proposition 2.10 shows that actually $b = \infty$. If $\phi_1(t)$ did not approach 0, then it must approach some other point α. Then, $\lim_{t\to\infty, y\to\alpha} F(t, y) = 0$, but this is clearly not the case. Note that this does not rely on the specific choice of $r = 1$. It suffices that $r > 0$.

2.4 Exercise 2.4

This follows directly from the fact that

$$u'(t) = -\frac{2}{(t+2)^2}$$

and that

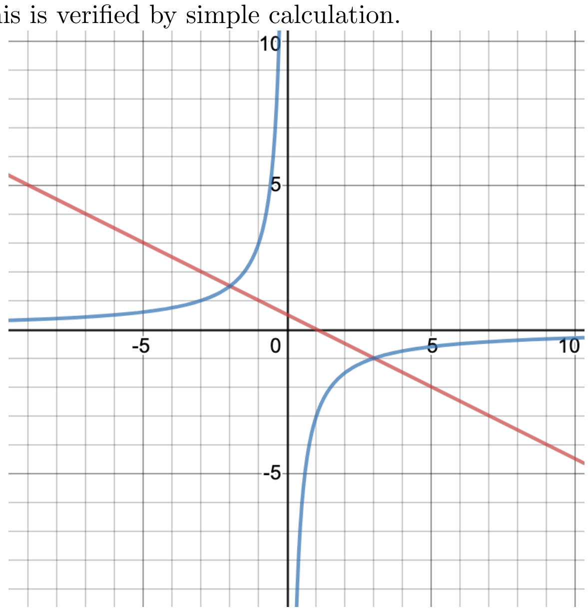
$$-u(t)(u(t) + 2) = -\frac{4}{(t+2)^2} - \frac{4}{t+2}$$

2.5 Exercise 2.5

Since $u(t) \geq F(t, y)$, and $u(0) < \phi_2(0)$, we can use Theorem 4.3 to conclude that $\phi_2(t) \geq u(t)$ on $(\max(a, -2), 0)$. We realize that this in particular implies that $-2 \leq a < 0$ since otherwise, $\phi_2(-2)$ would be unbounded, which means that it would not be a well-defined solution on (a, b). Then, since $u(0) = \phi_1(0)$, we can conclude using Theorem 4.3 that $\phi_1(t) \geq u(t)$ on $[a, 0]$, which in particular means, using the ODE, that $\phi_1'(t) < 0$ on $[a, 0]$.

2.6 Exercise 2.6

This is verified by simple calculation.



2.7 Exercise 2.7

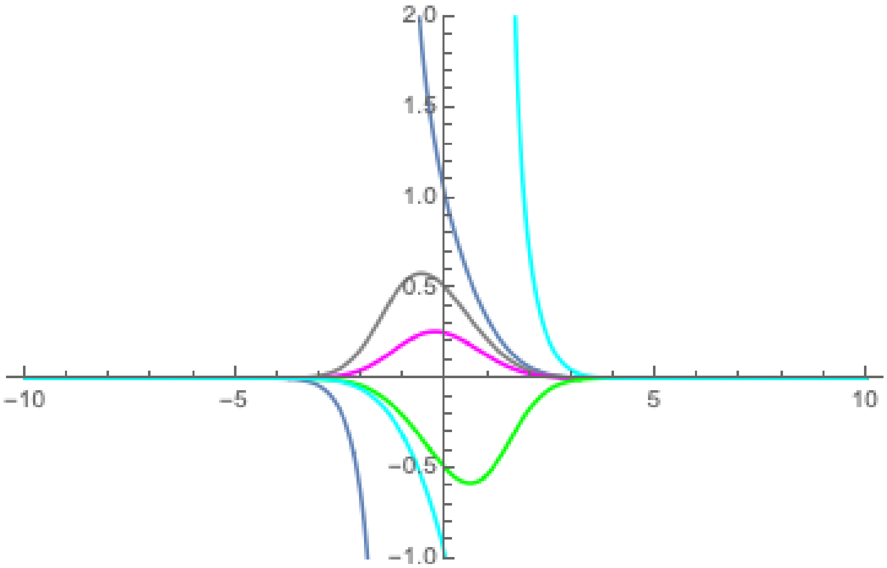
Consider the piecewise C^0 function defined by $z(t) = w(t)$ for $t \leq -2$, $z(t) = v(t)$ for $-2 < t \leq 0$. Then, $z'(t) \geq -z(t)(z(t) + t)$ for all $t < 0$, and since $z(0) \geq \phi_r(0)$ for $0 < r \leq \frac{1}{2}$, we have that $\phi_r(t) \leq z(t)$. But then recall that from before, we know that $\phi_r(t) > 0$. But then this directly implies that $\lim_{t\to-\infty}\phi_r(t) = 0$. (Notice that in particular, the comparison theorem allows us to extend the range of $\phi_r(t)$ to negative infinity). To see that $\lim_{t\to\infty}\phi_r(t) = 0$, it suffices to repeat the argument used for ϕ_1 above.

2.8 Exercise 2.8

Let $\phi' = -\phi(\phi + t)$. Then let $\psi(t) = -\phi(-t)$

$$\begin{aligned}\psi'(t) &= \phi'(-t)\\ &= -\phi(-t)(\phi(-t) - t)\\ &= -\psi(t)(\psi(t) + t).\end{aligned}$$

2.9 Exercise 2.9



We remark here that the graphs for ϕ_1, ϕ_{-1} contain singularities like we would expect and that the true maximal solution would just be one branch of the graphs shown here.

3 Exercise 3

3.1 Exercise 3.1

This is a separable first order ODE, and thus, we can solve it directly

$$\begin{aligned}y' &= -y^2\\ \int_{y(t_0}^{y(t)} \frac{1}{y^2}\,dy &= \int_{t_0}^{t} dt\\ -\left.\frac{1}{y}\right|_{y(t_0)}^{y(t)} &= t - t_0\\ -\frac{1}{y(t)} &= t - t_0 - \frac{1}{y_0}\\ y(t) &= -\frac{1}{t - t_0 + y_0^{-1}}.\end{aligned}$$

3.2 Exercise 3.2

From the explicit solution above, which can be extended to $t = \infty$, we see that it is clear that $\phi(t) \to 0$ as $t \to \infty$.