Analysis 4 Problem Set 5

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1 Exercise 1

1.1 Exercise 1.1

If $\limsup \left|\frac{a_{n+1}}{a_n}\right| = \ell < 1$ then there exists $N \geq 0$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq \frac{1+\ell}{2} \quad \text{for all } n \geq N.$$

Let $a = \frac{1+\ell}{2} < 1$. We have

$$|a_n| \leq a^{n-N}|a_N| \quad \text{for all } n \geq N.$$

Since the series $\sum_{n=0}^{+\infty} a^n$ is convergent, it follows from the comparison principle that $\sum_0^\infty a_n$ is also convergent.

1.2 Exercise 1.2

If $\liminf \left|\frac{a_{n+1}}{a_n}\right| = \ell > 1$ then there exists $N \geq 0$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \geq \frac{1+\ell}{2} \quad \text{for all } n \geq N.$$

Let $a = \frac{1+\ell}{2} > 1$. We have

$$|a_n| \geq a^{n-N}|a_N| \quad \text{for all } n \geq N,$$

then the sequence $(a_n)_{n\geq 0}$ does not converge to 0, and so the series $\sum_{n=0}^\infty a_n$ is divergent.

1.3 Exercise 1.3

First, we consider following sequence,

$$a_n = 1 \quad \text{for all } n \in \mathbb{N}.$$

It is not difficult to check $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$ and the series $\sum_{n=0}^\infty a_n$ is divergent.

Second, we consider following sequence,

$$a_n = (-1)^n \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

It is not difficult to check $\lim \left|\frac{a_{n+1}}{a_n}\right| = 1$ and the series $\sum_{n=0}^\infty a_n$ is convergent.

2 Exercise 2

2.1 Exercise 2.1

Note that

$$\left(n^3\right)^{\frac{1}{n}} = e^{3\frac{\ln n}{n}} \quad \text{and} \quad \lim_{n\to\infty} \frac{\ln n}{n} = 0. \tag{1}$$

Using (1) and the definition of the radius of convergence, we obtain

$$R = \frac{1}{\limsup_{n\to\infty} (n^3)^{\frac{1}{n}}} = 1.$$

2.2 Exercise 2.2

First, we claim following estimate

$$\lim_{n\to\infty} (n!)^{\frac{1}{n}} = +\infty. \tag{2}$$

In actually, fix $M > 0$, we have

$$(n!)^{\frac{1}{n}} \geq M^{(n-M)\frac{1}{n}} = M^{1-\frac{M}{n}} \quad \text{for all } n \geq M.$$

Let $n \to +\infty$, we obtain

$$\liminf_{n\to+\infty} (n!)^{\frac{1}{n}} \geq M.$$

Let $M \to +\infty$, we obtain (2). Using (2) and the definition of the radius of convergence, we obtain

$$R = \frac{1}{\limsup_{n\to\infty} \left(\frac{2^n}{n!}\right)^{\frac{1}{n}}} = +\infty.$$

2.3 Exercise 2.3

Using the similar argument in Exercise 2.1, we have

$$\lim_{n\to+\infty} \left(\frac{2^n}{n^2}\right)^{\frac{1}{n}} = \lim_{n\to+\infty} \frac{2}{(n^2)^{\frac{1}{n}}} = 2.$$

Using the definition of the radius of convergence, we obtain

$$R = \frac{1}{\limsup_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{\frac{1}{n}}} = \frac{1}{2}.$$

2.4 Exercise 2.4

Using the similar argument in Exercise 2.1, we have

$$\lim_{n\to+\infty} \left(\frac{n^3}{3^n}\right)^{\frac{1}{n}} = \lim_{n\to+\infty} \frac{(n^3)^{\frac{1}{n}}}{3} = \frac{1}{3}.$$

Using the definition of the radius of convergence, we obtain

$$R = \frac{1}{\limsup_{n\to\infty} \left(\frac{n^3}{3^n}\right)^{\frac{1}{n}}} = 3.$$

3 Exercise 3

3.1 Exercise 3.1

From (2), we know that

$$R = \frac{1}{\limsup (n!)^{\frac{1}{n}}} = +\infty.$$

Using Theorem 1.5 in Lecture notes, we obtain the series $\sum \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.

3.2 Exercise 3.2

From Proposition 3.1 in Lecture notes, we obtain $\exp(x) \in C^1(\mathbb{R})$ and

$$\exp'(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} \left(\frac{x^{n-1}}{(n-1)!}\right) = \exp(x).$$

3.3 Exercise 3.3

From the definition of $\exp(x)$, we have

$$\exp(-x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R}.$$

Using the definition of $\cosh(x)$ and $\sinh(x)$, we obtain

$$\cosh(x) = \frac{\exp(x)+\exp(-x)}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}+(-1)^n\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

and

$$\sinh(x) = \frac{\exp(x)-\exp(-x)}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}-(-1)^n\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}.$$

4 Exercise 4

4.1 Exercise 4.1

If $\sum a_n$ is convergent, then there exists $N \geq 0$ such that

$$|a_n| \leq \frac{1}{2} \quad \text{for all } n \geq N.$$

It follows that

$$\limsup |a_n|^{\frac{1}{n}} \leq \limsup (\frac{1}{2})^{\frac{1}{n}} \leq 1.$$

Using the definition of radius of convergence, we obtain the power series $\sum a_n z^n$ has a radius of convergence greater or equal to 1.

4.2 Exercise 4.2

We denote $S_{-1} = 0$ then

$$\sum_{n=0}^{m} a_n x^n = \sum_{n=0}^{m}(S_n - S_{n-1})\,x^n = (1-x)\sum_{n=0}^{m-1} S_n x^n + S_m x^m$$

We let $m \to +\infty$ and obtain

$$f(x) = (1-x)\sum_{n=0}^{+\infty} S_n x^n \quad \text{for all } |x| < 1. \tag{3}$$

4.3 Exercise 4.3

Suppose $S = \sum a_n$. Let ε be given, choose N such that

$$|S - S_n| \leq \varepsilon \quad \text{for all } n > N.$$

Then, since

$$(1-x)\sum_{n=0}^{\infty} x^n = 1 \quad \text{for all } |x| < 1.$$

We obtain from (3)

$$|f(x) - S| = \left|(1-x)\sum_{n=0}^{+\infty}(S_n - S)x^n\right| \leq (1-x)\sum_{n=0}^{N}|S_n - S|\,|x|^n + \varepsilon \leq 2\varepsilon$$

if $x > 1 - \delta$, for some suitably chosen $\delta > 0$.

5 Exercise 5

Note that $a_n \in \mathbb{Z}$ and there is an infinite numbers of $a_n \neq 0$, we obtain there is an infinite numbers of $|a_n| \geq 1$. It follows that

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \geq 1.$$

Using the definition of radius of convergence, we obtain $R \leq 1$.

6 Exercise 6

6.1 Exercise 6.1

Since

$$\left(\frac{1}{2n-1}\right)^{\frac{1}{4n-2}} \to 1 \quad \text{as } n \to +\infty$$

we obtain the radius of convergence $R = 1$.

6.2 Exercise 6.2

Using Proposition 3.1 in Lecture notes, we obtain $f \in C^1((-1,1))$ and

$$f'(x) = \sum_{n=1}^{+\infty} \frac{4n-2}{2n-1} x^{4n-3} = \frac{2}{x^3} \sum_{n=1}^{+\infty} x^{4n} = \frac{2x}{1-x^4} \quad \text{for all } |x| < 1.$$

Note that $f(0) = 0$, we have

$$f(x) = \int_0^x f'(s)ds = \int_0^x \frac{2s}{1-s^4} ds = \int_0^{x^2} \frac{1}{1-s^2} ds = \frac{1}{2} \int_0^{x^2} \left(\frac{1}{1+s} + \frac{1}{1-s}\right) ds = \frac{1}{2}\left(\ln(1+x^2) - \ln(1-x^2)\right).$$

7 Exercise 7

7.1 Exercise 7.1

Since

$$\lim_{n\to+\infty} (\log(n))^{\frac{1}{2n}} = 1,$$

we obtain the radius of convergence $R = 1$.

7.2 Exercise 7.2

Since

$$\lim_{n\to+\infty} (|1 + a^n|)^{\frac{1}{n}} = \max(1, |a|) \quad \text{for all } |a| \neq 1.$$

we obtain the radius of convergence $R = \frac{1}{\max(1,|a|)}$.

7.3 Exercise 7.2

Since

$$\lim_{n\to+\infty} \left(a^{\sqrt{n}}\right)^{\frac{1}{n}} = \lim_{n\to+\infty} e^{\frac{\ln a \sqrt{n}}{n}} = 1.$$

we obtain the radius of convergence $R = 1$.

7.4 Exercise 7.3

Since

$$\lim_{n\to+\infty} 1^{\frac{1}{n!}} = 1.$$

we obtain the radius of convergence $R = 1$.

8 Exercise 8

We denote the radius of convergence of the power series $\sum a_n b_n z^n$ is R''. Note that

$$\limsup_{n\to+\infty} |a_n b_n|^{\frac{1}{n}} = \limsup_{n\to+\infty} \left(|a_n|^{\frac{1}{n}} |b_n|^{\frac{1}{n}}\right) \le \limsup_{n\to+\infty} \left(|a_n|^{\frac{1}{n}}\right) \cdot \limsup_{n\to+\infty} \left(|b_n|^{\frac{1}{n}}\right) \le \frac{1}{RR'}$$

It follows that

$$R'' = \frac{1}{\limsup_{n\to+\infty} |a_n b_n|^{\frac{1}{n}}} \ge RR'. \tag{4}$$

The inequality (4) is optimal. Let

$$a_n = b_n = 1 \quad \text{for all } n \in \mathbb{N}.$$

We obtain $R = R' = R'' = 1$.