Week 2, February 14th: Series of functions

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1 Important exercises

Exercise 1. Let $I = [a, b]$ be a segment of $\mathbb{R}$ and $f : I \to \mathbb{R}$ a C^1 function. For $n \geq 0$ we define $f_n : I \to \mathbb{R}$ by $f_n(x) = \int_a^x f(t)\cos(nt)dt$. With an integration by parts, show that the sequence of fucntions f_n converges uniformly to zero.

Exercise 2. We say that a sequence of functions $f_n : \Omega \to \mathbb{R}$ is uniformly bounded if there exists $M > 0$ such that for all $n \in \mathbb{N}$ and $x \in \Omega$, we have $|f_n(x)| \leq M$.

1. Let (f_n) be a sequence of functions which is uniformly bounded and converges uniformly to f. Show that f is bounded.
2. Let (f_n) and (g_n) be two sequences of functions which are uniformly convergent. Show that $(f_n + g_n)$ is uniformly convergent.
3. Let (f_n) and (g_n) be two sequences of functions which are uniformly bounded and uniformly convergent. Show that $(f_n g_n)$ is uniformly convergent.
4. Construct two sequences of function (f_n) and (g_n) uniformly convergent such that the sequence $(f_n g_n)$ converges pointwise but not uniformly.

Exercise 3. Let (a_n) be a sequence of real positive numbers such that a_n is decreasing and tends to zero. We consider the series $\sum(-1)^n a_n$.

1. We note $S_n = \sum_0^n (-1)^n a_n$. Show that the sequences $(S_{2p})_{p\in\mathbb{N}}$ and (S_{2p+1}) converge toward the same finite limit S.
2. Show that for all n, $|S - s_n| \leq a_{n+1}$ and that the series $\sum(-1)^n a_n$ tends to S.

Exercise 4. For $n \geq 1$, define $f_n : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{\cos(nx)}{n^2}$. Show that the series $\sum f_n$ is normally convergent and that the function $\sum_{n=1}^{\infty} f_n$ is continuous.

Exercise 5. For all $n \geq 1$ we define $f_n : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{x}{n^2+x^2}$.

1. Show that the series $\sum f_n$ converges pointwise.
2. Show that for all p, there exists $x \in \mathbb{R}$ such that $\sum_{n=p+1}^{2p} \frac{x}{n^2+x^2} \geq \frac{1}{5}$. Deduce that $\sum f_n$ is not uniformly convergent on $\mathbb{R}$.

3. Show that the limit function $f = \sum_{n=1}^{\infty} f_n$ is continuous.

4. Show that the sequence of functions $\sum(-1)^n f_n$ is uniformly convergent on $\mathbb{R}$ but not normally convergent.

2 More involved exercises

Exercise 6. We define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = 0 \text{ for } x < \frac{1}{n+1}, \quad f_n(x) = \sin^2(\frac{\pi}{x}) \text{ for } \frac{1}{n+1} \leq x \leq \frac{1}{n}, \quad f_n(x) = 0 \text{ for } \frac{1}{n} < x.$$

1. Show that the sequence of functions f_n converges pointwise, but not uniformly to zero.

2. Show that the series $\sum |f_n|$ converges pointwise, but that the series $\sum f_n$ does not converge uniformly.

Exercise 7. Let f_n and g_n be two sequences of functions $\Omega \to \mathbb{K}$ such that

- The partial sums $\sum_{n=1}^{N} f_n$ are uniformly bounded,
- The sequence (g_n) converges uniformly to zero on Ω,
- For all $x \in \Omega$, the sequence $g_n(x)$ is decreasing.

Show that the series $\sum f_n g_n$ converges uniformly on Ω.

Tip : You may calculate first, for two numerical sequences a_n, b_n

$$\sum_{n=1}^{q} (\sum_{k=1}^{n} a_n)(b_n - b_{n+1}).$$

Exercise 8. The aim of this exercise is to show that the sequence of functions $f_n : [0, +\infty[\to \mathbb{R}$ defined by $f_n(x) = \left(1 - \frac{x}{n}\right)^n$ for $x \leq n$ and $f_n(x) = 0$ for $x \geq n$ converges uniformly to $f(x) = e^{-x}$.

1. Let $n \in \mathbb{N}$. Consider the function $\phi : [0, n] \to \mathbb{R}$, $x \mapsto e^{-x} - f_n(x)$. Show that there exists $\alpha \in [0, n]$ such that $0 \leq \phi(x) \leq \phi(\alpha)$ for all $x \in [0, n]$ and $\phi'(\alpha) = 0$.

2. Show that $\phi(\alpha) = \frac{\alpha}{n} e^{-\alpha}$ and deduce that $\phi(\alpha) \leq \frac{1}{en}$.

3. Conclude.