Week 13, May 23th: Differential equations

Instructor: Cécile Huneau (cecile.huneau@polytechnique.edu)
Tutorial Assistants:
- Allen Fang (groups ?, allen.fang@sorbonne-universite.fr)
- Yuan Xu (groups ?, xu.yuan@polytechnique.edu)

1 Important exercises

Exercise 1. We consider the system of differential equations

$$\begin{cases} x' = 4y + 2t \\ y' = -\frac{x}{t^2} + \frac{4y}{t} + e^t \end{cases} \tag{1}$$

1. Show that for all $t_0 > 0$ and $(x_0, y_0) \in \mathbb{R}^2$ there exists a unique solution $\psi :]0, +\infty[\to \mathbb{R}$.
2. Show that $\psi_1(t) = (t^4, t^3)$ and $\psi_2(t) = (4t, 1)$ are solutions to the homogeneous equation.
3. Give the space of solutions to (1).

Exercise 2. Let $(y_0, z_0) \in \mathbb{R}^2$. We consider the Cauchy problem

$$\begin{cases} y' = \sqrt{1 + z^2} - \cos(t)z \\ z' = \arctan(yz) + 3y \end{cases}$$

with $(y(0), z(0)) = (x_0, y_0)$. Show that there exists a unique solution $\phi : \mathbb{R} \to \mathbb{R}^2$.

Exercise 3. We consider the differential equation $t^2 y'' - 2y = 3t^2$.

1. We consider first this equation on the domain $]0, +\infty[\times\mathbb{R}$. Can you find two independent solutions to the homogeneous equation $t^2 y'' - 2y = 0$? (Otherwise ask your teaching assistant).
2. Use the method of variation of the constant to find a particular solution, then give the space of solutions.
3. Solve the equation on the domain $]-\infty, 0[\times\mathbb{R}$.
4. Do there exists C^2 solutions to the equation $t^2 y'' - 2y = 3t^2$ which are defined on $\mathbb{R}$?

2 More involved exercises

Exercise 4. We consider the differential system on $\mathbb{C}^n$: $Y' = A(t)Y$, where $A : \mathbb{R} \to M_n(\mathbb{C})$ is continuous and T periodic.

1. Show that this system admits a non trivial solution V such that there exists $\lambda \in \mathbb{C}$ such that for all $t \in \mathbb{R}$ $V(t+T) = \lambda V(t)$.

2. We consider n linearly independent solutions $V_1, .., V_n$. We note $M(t)$ the matrix whose column vectors are given by $V_1(t)..V_n(t)$. Show that there exists an invertible matrix C such that for all $t \in \mathbb{R}$ we have $M(t+T) = M(t)C$.

Exercise 5. Let $E = \mathbb{R}^d$, and $A : \mathbb{R} \to M_d(\mathbb{R})$ be a continuous function. For $(t_0, y_0) \in \mathbb{R} \times E$ we call $v(t, t_0, y_0)$ the solution to the Cauchy problem $y' = A(t)y$ with $y(t_0) = y_0$. Prove the following statements

1. For fixed $t, s \in \mathbb{R}$, the map $E \to E$, $y \mapsto v(t, s, y)$ is an invertible linear map, that we note $C(t, s)$.

2. For fixed $s \in \mathbb{R}$, the map $t \mapsto C(t, s)$ is a solution to the Cauchy problem
$$w'(t) = A(t) \circ w(t), \; w(s) = \mathrm{id}.$$

3. For $s, t, u \in \mathbb{R}$ we have
$$C(t, s) = C(t, u)C(u, s), \; C(t, s) = C(s, t)^{-1}.$$

4. The map $(t, s) \mapsto C(t, s)$ is continuous.

5. Let $\psi : \mathbb{R} \to E$ be a continuous function. Show that the solution f to the Cauchy problem $y' = A(t)y + \psi$, $y(t_0) = y_0$ is given by
$$f(t) = C(t, t_0)y_0 + \int_{t_0}^{t} C(t, s)\psi(s)ds.$$