Analysis 4 Problem Set 7

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1 Exercise 1

1.1 Exercise 1.1

Using Proposition 4.1 in the lecture notes, we know that

$$\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n \quad \text{for } |x| < 1. \tag{1}$$

From (1), we obtain

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \quad \text{for } |x| < 1. \tag{2}$$

1.2 Exercise 1.2

Using Proposition 3.1 in the lecture notes and (2), we have

$$\begin{aligned}\arctan(x) &= \arctan(0) + \int_0^x \frac{1}{1+s^2} ds \\ &= \int_0^x \sum_{n=0}^{+\infty} (-1)^n s^{2n} ds = \sum_{n=0}^{+\infty} (-1)^n \int_0^x s^{2n} ds = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.\end{aligned}$$

2 Exericse 2

2.1 Exercise 2.1

We assume

$$\frac{x^2+x-3}{(x-2)^2(2x-1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{2x-1}.$$

By direct computation, we have

$$\frac{x^2+x-3}{(x-2)^2(2x-1)} = \frac{(2A+C)x^2 + (-5A+2B-4C)x + (2A-B+4C)}{(x-2)^2(2x-1)}$$

It follows that $A = B = 1$ and $C = -1$. So we obtain

$$\frac{x^2+x-3}{(x-2)^2(2x-1)} = \frac{1}{x-2} + \frac{1}{(x-2)^2} - \frac{1}{2x-1}. \tag{3}$$

2.2 Exercise 2.2

Using Proposition 4.1 in the lecture notes, we know that

$$\frac{1}{x-2} = -\frac{1}{2}\frac{1}{1-\frac{x}{2}} = -\sum_{n=0}^{+\infty} \frac{x^n}{2^{n+1}} \text{ for } |x| < 2 \quad \text{and} \quad \frac{1}{2x-1} = -\frac{1}{1-2x} = -\sum_{n=0}^{+\infty} 2^n x^n \text{ for } |x| < \frac{1}{2}. \tag{4}$$

From Proposition 3.1 in the lecture notes and (4), we obtain

$$\frac{1}{(x-2)^2} = \sum_{n=0}^{+\infty} \frac{n+1}{2^{n+2}} x^n \quad \text{for } |x| < 2. \tag{5}$$

2.3 Exercise 2.3

Using (3), (4) and (5), we obtain

$$\frac{x^2+x-3}{(x-2)^2(2x-1)} = \sum_{n=0}^{\infty}\left(2^n - \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}}\right)x^n \quad \text{for } |x| < \frac{1}{2}.$$

3 Exercise 3

3.1 Exercise 3.1

We denote $\left(\sum \frac{z^{2n}}{(2n)!}\right)\left(\sum \frac{(-1)^n z^{2n}}{(2n)!}\right) = \sum c_n z^{2n}$. From Theorem 2.3, we obtain

$$c_n = \sum_{k+\ell=n} \frac{(-1)^\ell}{(2k)!(2\ell)!} = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{(2n-2\ell)!(2\ell)!} = \frac{1}{(2n)!}\sum_{\ell=0}^{n}(-1)^\ell C_{2n}^{2\ell}. \tag{6}$$

when $n = 2k+1$ with $k \in \mathbb{N}$, using (6)

$$c_n = \frac{1}{(2n)!}\left(\sum_{\ell \text{ even}} C_{2n}^{2\ell} - \sum_{\ell \text{ odd}} C_{2n}^{2\ell}\right) = \frac{1}{(2n)!}\left(\sum_{\ell \text{ even}} C_{2n}^{2\ell} - \sum_{\ell \text{ odd}} C_{2n}^{2(n-\ell)}\right) = 0.$$

when $n = 2k$ with $k \in \mathbb{N}$. We consider $\mathrm{Re}(1+i)^{4k}$, using Binomial theorem, we obtain

$$\mathrm{Re}(1+i)^{4k} = \mathrm{Re}\sum_{m=0}^{4k} i^m C_{4k}^{m} = \sum_{\ell=0}^{n}(-1)^\ell C_{2n}^{2\ell}. \tag{7}$$

On the other hand, by direct computation, we have

$$\mathrm{Re}(1+i)^{4k} = \mathrm{Re}\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{4k} = 4^k e^{ik\pi} = (-1)^k 4^k. \tag{8}$$

From (6), (7) and (8), we obtain

$$c_n = \frac{(-1)^k 4^k}{(4k)!} \quad \text{with } n = 2k.$$

3.2 Exercise 3.2

From Exercise 3.1 and Proposition 4.1, we obtain

$$\cos x \cosh x = \left(\sum \frac{x^{2n}}{(2n)!}\right)\left(\sum \frac{(-1)^n x^{2n}}{(2n)!}\right) = \sum_{n=0}^{+\infty} \frac{(-1)^n 4^n}{(4n)!} x^{4n}.$$

4 Exercise 4

If y is a function defined by a power series $\sum_{n=0}^{+\infty} a_n x^n$ with radius of convergence $R > 0$, then we have on $(-R, R)$

$$xy'' = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{+\infty}(n+1)n a_{n+1} x^n,$$

$$y' = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty}(n+1)a_{n+1}x^n \quad \text{and} \quad xy = \sum_{n=0}^{+\infty} a_n x^{n+1}.$$

It follows that

$$xy'' + y' + xy = a_1 + \sum_{n=1}^{+\infty}\left((n+1)^2 a_{n+1} + a_{n-1}\right)x^n = 0 \tag{9}$$

From (9), we obtain

$$a_1 = 0 \quad \text{and} \quad \frac{a_{n+2}}{a_n} = -\frac{1}{(n+2)^2} \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

Using (10), we have

$$a_{2n} = \frac{(-1)^n}{\Pi_{k=0}^n (2k)^2} a_0 \quad \text{and} \quad a_{2n+1} = 0 \quad \text{for all } n \in \mathbb{N}.$$

So that we obtain

$$y(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{\Pi_{k=0}^n (2k)^2} a_0 x^{2n} \quad \text{for } x \in \mathbb{R}.$$

5 Exercise 5

5.1 Exercise 5.1

Note that using the same method from previous weeks

$$\lim_{n\to+\infty} \left| \frac{(-1)^n}{n(2n+1)} \right|^{\frac{1}{n}} = 1,$$

we obtain $R = 1$.

5.2 Exercise 5.2

From Proposition 3.1 and Proposition 4.1 in the lecture notes, we obtain

$$f'(x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^{2n} = \ln(1 + x^2) \quad \text{for } |x| < 1. \tag{11}$$

Note that $f(0) = 0$ and using (11), we obtain

$$f(x) = f(0) + \int_0^x f'(s)ds = \int_0^x \ln(1 + s^2)ds = x\ln(1 + x^2) - 2 + 2\arctan x. \tag{12}$$

5.3 Exercise 5.3

Note that $\sum \frac{(-1)^n}{n(2n+1)}$ is convergent and using (12), we have

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n(2n+1)} = \lim_{x\to1} f(x) = \ln 2 - 2 + \frac{\pi}{2}.$$

6 Exercise 6

6.1 Exercise 6.1

Note that as

$$\left(\frac{1}{n^2}\right)^{\frac{1}{n}} \to 1 \quad \text{as } n \to +\infty,$$

we obtain $R = 1$. Because $\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$, we have for all $|z| = 1$ the series $\sum_{n=1}^{+\infty} \frac{z^n}{n^2}$ is convergent. So then for $|z| \le 1$ the series $\sum_{n=1}^{+\infty} \frac{z^n}{n^2}$ is convergent.

6.2 Exercise 6.2

Because $\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$, we obtain that the series $\sum_{n=1}^{+\infty} \frac{x^n}{n^2}$ is normally convergent on $[-1,1]$. Using Theorem 2.3 in Chapter 1, we obtain $f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2}$ is continuous on $[-1,1]$.

From $R = 1$, Proposition 3.1 and Proposition 4.1 in Chapter 2, we obtain

$$f'(x) = \sum_{n=1}^{+\infty} \frac{x^{n-1}}{n} = -\frac{\ln(1-x)}{x}. \tag{13}$$

We denote

$$F(x) = f(x) + f(1-x) + \ln x \ln(1-x) \quad \text{for } 0 < x < 1.$$

6.3 Exercise 6.3

Using (13), we have

$$F'(x) = f'(x) - f'(1-x) + \frac{\ln(1-x)}{x} - \frac{\ln x}{1-x} = 0$$

We then obtain that there exists $c \in \mathbb{R}$ such that for all

$$F(x) = f(x) + f(1-x) + \ln x \ln(1-x) = c \tag{14}$$

6.4 Exercise 6.4

Note that

$$\ln x \ln(1-x) = x \ln x \cdot \frac{\ln(1-x)}{x} \to 0 \quad \text{as } x \to 0. \tag{15}$$

Let $x \to 0$ in (14) and using (15), we obtain

$$c = f(0) + f(1) = \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Let $x = \frac{1}{2}$ in (14), we obtain

$$\sum_{n=1}^{+\infty} \frac{1}{2^n n^2} = \frac{c}{2} - \frac{\ln^2(\frac{1}{2})}{2} = \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2}.$$

7 Exercise 7

7.1 Exericise 7.1

Step 1. We first show that if $R > 0$, then $|a_n| \le q^n$ for all $n \in \mathbb{N}$.

If we have $R > 0$. From the definition of the radius of convergence, we have there exists $M > 0$

$$\limsup_{n\to+\infty} |a_n|^{\frac{1}{n}} = M < +\infty. \tag{16}$$

From (16), we obtain there exists $N \in \mathbb{N}$ such that

$$|a_n|^{\frac{1}{n}} < M + 1 \quad \text{for all } n > N. \tag{17}$$

We denote

$$q = \max\left(|a_1|, |a_2|^{\frac{1}{2}}, \cdots, |a_N|^{\frac{1}{N}}, M+1\right).$$

From the definition of q and (17), we obtain $|a_n| \le q^n$ for all $n \in \mathbb{N}$.

Step 2. We now show the converse, that if $|a_n| \le q^n$ for all $n \in \mathbb{N}$, then $R > 0$.

If there exists $q > 0$ such that $|a_n| \le q^n$ for all $n \in \mathbb{N}$. We have

$$\limsup_{n\to+\infty} |a_n|^{\frac{1}{n}} \le q < +\infty.$$

From the definition of the radius of convergence, we obtain that the radius of convergence is strictly positive.

7.2 Exericise 7.2

If we suppose that $\sum_{n=0}^{+\infty} a_n z^n$ has an inverse $\sum_{n=0}^{+\infty} b_n z^n$, we have

$$\left(1 + \sum_{n=1}^{+\infty} a_n z^n\right)\left(\sum_{n=0}^{+\infty} b_n z^n\right) = 1. \tag{18}$$

We may compute the coefficients of the inverse series $\sum_{n=0}^{+\infty} b_n z^n$ via the explicit recursive formula

$$b_0 = 1 \quad \text{and} \quad b_n = -\sum_{k=1}^{n} a_k b_{n-k} \quad \text{for all } n \in \mathbb{N}. \tag{19}$$

From Exercise 7.1, we know that there exists $q > 0$ such that $|a_n| \leq q^n$ for all $n \in \mathbb{N}$. Now we prove that $|b_n| \leq (2q)^n$ for all $n \in \mathbb{N}$ by strong induction. Note that $|b_0| \leq (2q)^0$. We proves the statement $|b_{m+1}| \leq (2q)^{m+1}$ under the assumption that $|b_n| \leq (2q)^n$ holds for all natural n less than $m+1$. Using (19), we obtain

$$|b_{m+1}| \leq \sum_{k=1}^{m+1} |a_k| b_{m+1-k}| \leq \sum_{k=1}^{m+1} q^k (2q)^{m+1-k} \leq q^{m+1}\left(\sum_{k=1}^{m+1} 2^{m+1-k}\right) \leq (2q)^{m+1}.$$

So we obtain

$$|b_n| \leq (2q)^n \quad \text{for all } n \in \mathbb{N}.$$

From Exercise 7.1, we obtain the series $\sum_{n=0}^{+\infty} b_n z^n$ have a radius of convergence strictly positive. We conclude that $\frac{1}{f}$ can be written as $\sum_{n=0}^{+\infty} b_n z^n$ on a neighbour of 0.

8 Exercise 8

Fix $\varepsilon > 0$. Because $f(x) \to S$ as $x \to 1$ with $x < 1$, there exists $\delta > 0$ such that

$$\sup_{x \in (1-\delta,1)} \left|\sum_{n=0}^{+\infty} a_n x^n - S\right| < \varepsilon. \tag{20}$$

From $n|a_n| \to 0$ as $n \to +\infty$, there exists $N_1 > \frac{1}{\delta}$ such that

$$N|a_N| < \varepsilon \quad \text{and} \quad \frac{1}{N}\left(\sum_{n=0}^{N} n|a_n|\right) < \varepsilon \quad \text{for all } N > N_1. \tag{21}$$

Take $x = 1 - \frac{1}{N}$, so $x \in (1-\delta, 1)$ and $\frac{1}{1-x} = N$. Using (21), we obtain

$$\left|\sum_{n=N+1}^{+\infty} a_n x^n\right| \leq \sum_{n=N+1}^{+\infty} \left|n|a_n|\frac{x^n}{n}\right| \leq \frac{\varepsilon}{N+1} \sum_{n=N+1}^{+\infty} |x|^n \leq \frac{\varepsilon}{N+1} \cdot \frac{1}{1-x} \leq \frac{N\varepsilon}{N+1} \leq \varepsilon. \tag{22}$$

Note that

$$1 - x^n = (1-x)\left(1 + x + \cdots + x^{n-1}\right) \leq n(1-x). \tag{23}$$

Using (21) and (23), we obtain

$$\left|\sum_{n=0}^{N} a_n - \sum_{n=0}^{N} a_n x^n\right| \leq \sum_{n=0}^{N} |a_n|(1-x^n) \leq \sum_{n=0}^{N} n|a_n|(1-x) \leq \frac{1}{N}\left(\sum_{n=0}^{N} n|a_n|\right) \leq \varepsilon. \tag{24}$$

Using (20), (22) and (24), we have

$$\left|\sum_{n=0}^{N} a_n - S\right| \leq \left|\sum_{n=0}^{N} a_n - \sum_{n=0}^{N} a_n x^n\right| + \left|\sum_{n=0}^{+\infty} a_n x^n - S\right| + \left|\sum_{n=N+1}^{+\infty} a_n x^n\right| \leq 3\varepsilon \quad \text{for all } N > N_1.$$