Week 5, March 14th: Power series

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1 Important exercises

Exercise 1. Recall the proof of d'Alembert rule : let $\sum a_n$ be a numerical series

1. If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, show that the series $\sum a_n$ is convergent.
2. If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, show that the series $\sum a_n$ is divergent.
3. Find a convergent series and a divergent series with $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Exercise 2. Determine the radius of convergence of the following power series

1. $\sum n^3 z^n$
2. $\sum \frac{2^n}{n!} z^n$
3. $\sum \frac{2^n}{n^2} z^n$
4. $\sum \frac{n^3}{3^n} z^n$

Exercise 3.

1. Show that the series $\sum \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$. We note $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
2. Calculate $\exp'(x)$.
3. We note $\cosh(x) = \frac{\exp(x)+\exp(-x)}{2}$ and $\sinh(x) = \frac{\exp(x)-\exp(-x)}{2}$. Show that for all $x \in \mathbb{R}$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Exercise 4. **Abel theorem**

Let a_n be a numerical sequence such that $\sum a_n$ is convergente.

1. Show that the power series $\sum a_n z^n$ has a radius of convergence greater or equal to 1.
2. Let $f :]-1,1[\to \mathbb{C}$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Let $S_n = \sum_{k=0}^{n} a_k$. Show that for all $|x| < 1$

$$f(x) = (1-x)\sum_{n=0}^{\infty} S_n x^n.$$

3. Show that $f(x)$ tend to $\sum_{n=0}^{\infty} a_n$ as x tend to 1 with $x < 1$.

Exercise 5. Let $\sum a_n z^n$ be a power series with $a_n \in \mathbb{Z}$. Assume that there is an infinite numbers of $a_n \neq 0$. Show that the radius of convergence is less than 1.

Exercise 6.

1. What is the radius of convergence of the power series $\sum \frac{z^{4n-2}}{2n-1}$?
2. Let $f :]-1,1[\to \mathbb{R}$ be defined by $f(x) = \sum \frac{x^{4n-2}}{2n-1}$. Calculate $f'(x)$ and then $f(x)$.

2 More involved exercises

Exercise 7. Determine the radius of convergence of the following power series

1. $\sum \log(n) z^{2n}$,
2. $\sum (1 + a^n) z^n$ where $a \in \mathbb{C}$ is such that $|a| \neq 1$,
3. $\sum a^{\sqrt{n}} z^n$ where $a > 0$,
4. $\sum z^{n!}$.

Exercise 8. Let $\sum a_n z^n$ and $\sum b_n z^n$ be two power series with radius of convergence R and R'. What can you say about the radius of convergence of the power series $\sum a_n b_n z^n$?