Week 10, April 18th: Fourier series

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1 Important exercises

Exercise 1. We consider the 2π periodic functions defined on $[-\pi, \pi[$ by

$$f(-\pi) = \alpha,\ f(x) = -1 \text{ for } x \in]-\pi, 0[,\ f(0) = \beta,\ f(x) = 1 \text{ for } x \in]0, \pi[.$$

1. Calculate the Fourier series associated to f.
2. Show that f is equal to its Fourier series on $]-\pi, 0[$ and $]0, \pi[$. What should be the values of α and β in order for f to be equal to its Fourier series everywhere ?
3. Deduce the values of

$$\sum_{k\geq 1} \frac{(-1)^k}{2k+1}, \quad \sum_{k\geq 1} \frac{1}{(2k+1)^1}, \quad \sum \frac{1}{n^2}, \quad \frac{(-1)^{n-1}}{n^2}.$$

Exercise 2.

1. Let f be a 2π periodic function, which is C^k. Show that

$$\sum_{n=-\infty}^{\infty} n^{2k} |c_n(f)|^2 < \infty.$$

2. Let f be a 2π periodic, piecewise continuous function such that $\sum n^2 |c_n(f)|^2 < \infty$. Show that f is continuous.
3. By considering the 2π periodic function defined on $[-\pi, \pi]$ by $|x|$, show that $\sum n^2 |c_n(f)|^2$ does not imply that f is C^1.

Exercise 3. Let f and g be two piecewise continuous, 2π periodic functions. We assume that there exists $x_0 \in]-\pi, \pi[$ and $\delta > 0$ such that $f(x) = g(x)$ for all $x \in]x_0 - \delta, x_0 + \delta[$ and such that the Fourier series associated to f converges to f at x_0.

1. Let $D_n(x) = \sum_{k=-n}^{n} e^{ikx}$. Show that for all $\varepsilon > 0$ there exists N such that for all $n \geq N$

$$\left| \int_{-\pi}^{\pi} D_n(t) f(x_0 - t) dt - f(x_0) \right| \leq \varepsilon.$$

2. Show that

$$\begin{aligned}
&\int_{-\pi}^{\pi} D_n(t)g(x_0-t)dt - g(x_0)\\
&= \int_{-\pi}^{\pi} D_n(t)f(x_0-t)dt - f(x_0) + \int_{-\pi}^{-\delta} D_n(t)(g(x_0-t)-g(x_0))dt + \int_{\delta}^{\pi} D_n(t)(g(x_0-t)-g(x_0))dt\\
&\quad - \int_{-\pi}^{-\delta} D_n(t)(f(x_0-t)-f(x_0))dt + \int_{\delta}^{\pi} D_n(t)(f(x_0-t)-f(x_0))dt.
\end{aligned}$$

3. Thanks to Riemann-Lebesgue lemma show that

$$\int_{-\pi}^{-\delta} D_n(t)(g(x_0-t)-g(x_0))dt \to 0$$
$$\int_{\delta}^{\pi} D_n(t)(g(x_0-t)-g(x_0))dt \to 0$$

as $n \to \infty$ and the same holds for f.

4. Conclude that the Fourier series associated to g converges to g at x_0. In other words, the properties of convergence of the Fourier series is a "local" property.

2 More involved exercises

Exercise 4. Let $\alpha \in \mathbb{R}\setminus\mathbb{Z}$. We define the 2π periodic function $f_\alpha(t) = \cos(\alpha t)$ for $t \in [-\pi, \pi]$.

1. Calculate the Fourier series associated to f_α. Deduce that for all $t \in \mathbb{R}\setminus\pi\mathbb{Z}$ we have

$$\frac{1}{\tan(t)} = \frac{1}{t} + 2t\sum_{n=1}^{\infty}\frac{1}{t^2 - n^2\pi^2}. \tag{1}$$

2. Let $x \in]0,\pi[$ and $g : [0,x] \to \mathbb{R}$ defined by $g(0) = 0$ and $g(t) = \frac{1}{\tan(t)} - \frac{1}{t}$ for $t \in]0,x[$. Calculate $\int_0^x g(t)dt$.

3. Show that for all $t \in]-\pi,\pi[$ we have

$$\sin(t) = t\prod_{n=1}^{\infty}\left(1 - \frac{t^2}{n^2\pi^2}\right).$$

4. By differentiating (1) show that for all $t \in]-\pi,\pi[\setminus\{0\}$

$$\frac{1}{\sin^2(t)} = \sum_{n=-\infty}^{\infty}\frac{1}{(t-n\pi)^2}.$$