Analysis 4 Problem Set 4

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1 Exercise 1

1.1 Exercise 1.1

We know that $e^{-x} < \frac{1}{x}$ for all $x > 0$. Then, in particular, for $x \in [a, +\infty)$, we have that

$$f_n(x) < \frac{1}{an^2}.$$

$\sum \frac{1}{an^2}$ clear converges, so we have that $\sum f_n$ is normally convergent on $[a, +\infty)$. To see that $\sum f_n(x)$ is pointwise convergent on $(0, +\infty)$, consider that

$$f_n(x) \leq \frac{1}{xn^2} \quad \text{for all } n \in \mathbb{N}.$$

Thus for any fixed $x \in (0, +\infty)$, since we know that $\sum \frac{1}{n^2}$ converges, we have that $\sum f_n(x)$ converges pointwise.

1.2 Exercise 1.2

We first see that since each $f_n(x)$ is decreasing, it follows immediately that f is also decreasing. To show that $f \in C^0$, we first realize that because $\sum f_n$ is normally convergent on $[a, \infty)$, it is also uniformly convergent on $[a, \infty)$, for any $a > 0$. Then, since for each x there exists an interval $[a, \infty)$, such that $x \in [a, \infty)$, $a > 0$, then we have that $f(x)$ is continuous for all $x \in (0, +\infty)$.

We argue that $\lim_{x \to \infty} f(x) = 0$. This is clear by using the bound

$$\sum f_n(x) \leq \sum_{n=1}^{\infty} \frac{1}{xn^2} \leq \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Taking the limit as $x \to \infty$ and using the fact that $\sum \frac{1}{n^2}$ converges to a finite limit shows $\lim_{x \to \infty} f(x) = 0$.

1.3 Exercise 1.3

We show that f is unbounded by showing that $\lim_{x \to 0} f(x) = \infty$. Using the fact that $f_n(x)$ is a decreasing function, we know that

$$\int_0^{n+1} e^{-xt^2} \, dt \leq \sum_{k=0}^{n} f_n(x)$$

by using the fact that the left Riemann sum of a monotonically decreasing function bounds the integral of the same function from above.

Then, using u-substitution (as done below in Exercise 4),

$$\frac{1}{\sqrt{x}} \int_0^{\infty} e^{-t^2} dt \leq f(x).$$

Then, since $\int_0^{\infty} e^{-t^2} dt < \infty$, we have that as $x \to 0$ from above, $f(x)$ diverges.

1.4 Exercise 1.4

To show that $\int_0^\infty e^{-xt^2}\,dt < \infty$, let $s = \sqrt{x}t$. Then using u-substitution,

$$\int_0^\infty e^{-xt^2}\,dt = \frac{1}{\sqrt{x}}\int_0^\infty e^{-s^2}\,ds.$$

Then, since we know that the integral on the right hand side in the above equality is finite, $\int_0^\infty e^{-xt^2}\,dt < \infty$.

1.5 Exercise 1.5

Since we have that e^{-xt^2} is a decreasing function in t, we know that using the left Riemann sum bounds the integral from above, and using the right Riemann sum bounds the integral from below. That is, we know that we have the following inequality

$$\sum_{k=1}^{n} e^{-xk^2} \le \int_0^n e^{-xt^2}\,dt \le \sum_{k=0}^{n-1} e^{-xk^2}$$

Then, the inequality on the left yields

$$\sum_{k=0}^{n} e^{-xk^2} \le 1 + \int_0^n e^{-xt^2}\,dt$$

while the inequality on the right yields

$$e^{-xn^2} + \int_0^n e^{-xt^2}\,dt \le \sum_{k=0}^{n} e^{-xk^2}.$$

Then using Exercise 1.4 and taking the limit $n \to \infty$,

$$\sqrt{x}e^{-xn^2} + \int_0^\infty e^{-t^2}\,dx \le \sqrt{x}f(x) \le \sqrt{x} + \int_0^\infty e^{-t^2}\,dt.$$

Taking the limit superior and limit inferior as $x \to 0$ yields the conclusion.

2 Exercise 2

2.1 Exercise 2.1

Let α denote the limit of $\sum \alpha_n$. Then,

$$\sum_{k=n}^{m} \alpha_k H(x - x_k) \le \sum_{k=n}^{m} \alpha_k.$$

But we know that $\sum \alpha_n$ is an absolutely convergent numerical series. Thus, for any $\epsilon > 0$, there exists some N such that for all $m > n > N$, $\sum_{k=n}^{m} \alpha_k < \epsilon$. By the same token,

$$\sum_{k=n}^{m} \alpha_k H(x - x_k) < \epsilon \quad \text{for all } m > n > N, \text{ independent of } x. \tag{1}$$

Thus, we have that the series converges uniformly on (a, b).

2.2 Exercise 2.2

Consider $x \notin (x_n)$. Then, since x_n is a sequence of distinct points, there must be some δ such that there are no points in $(x_n)_{n=1}^N$ within $I := (x - \delta, x + \delta)$. Then, let $y \in I$, from the definition of H,

$$H(x - x_n) = H(y - x_n) \quad \text{for all } n = 1, \cdots, N. \tag{2}$$

Using (1) and (2),

$$\begin{aligned}
|f(x)-f(y)| &= \left|\sum_{n=1}^{\infty}\left(\alpha_n H(x-x_n)-\alpha_n H(x-x_n)\right)\right| \\
&\le \left|\sum_{n=1}^{N}\left(\alpha_n H(x-x_n)-\alpha_n H(x-x_n)\right)\right| + \left|\sum_{n=N+1}^{\infty}\left(\alpha_n H(x-x_n)-\alpha_n H(x-x_n)\right)\right| \\
&\le \left|\sum_{n=N+1}^{\infty}\alpha_n H(x-x_n)\right| + \left|\sum_{n=N+1}^{\infty}\alpha_n H(y-x_n)\right| \le 2\epsilon.
\end{aligned}$$

In particular, this shows continuity.

3 Exercise 3

Using Weierstrass Theorem, there exists a sequence of polynomials P_n which converges uniformly to f on $[0,1]$. Then, let $P_n(x) = \sum_{k=0}^{k_n} a_{n,k}x^k$. Using $\int_0^1 f(x)x^n dx = 0$ for all $n \in \mathbb{N}$, we have

$$\int_0^1 f(x)P_n(x) = \sum_{k=0}^{k_n} a_{n,k}\int_0^1 f(x)x^k dx = 0. \tag{3}$$

Using (3) and P_n converges uniformly to f, we obtain

$$\begin{aligned}
\int_0^1 f^2(x)dx &= \int_0^1 f(x)\left(f(x)-P_n(x)\right)dx + \int_0^1 f(x)P_n(x)dx \\
&\le \int_0^1 |f(x)||\left(f(x)-P_n(x)\right)|dx \\
&\le \left(\max_{x\in[0,1]}|f(x)|\right)\left(\max_{x\in[0,1]}|f(x)-P_n(x)|\right) \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

It follows that $f(x) = 0$ for any $x \in [0,1]$.

4 Exercise 4

4.1 Exercise 4.1

We can directly calculate

$$\begin{aligned}
B_n(1) &= \sum_{k=0}^{n} C_k^n x^k(1-x)^{n-k} \\
&= 1. \\
B_n(x) &= \sum_{k=0}^{n} C_k^n \frac{k}{n}x^k(1-x)^{n-k} \\
&= x \\
B_n(x^2) &= \sum_{k=0}^{n} C_k^n \frac{k^2}{n^2}x^k(1-x)^{n-k} \\
&= \frac{(n-1)x^2}{n} + \frac{x}{n}.
\end{aligned}$$

The first identity follows directly from the binomial theorem.

Then, consider

$$\begin{aligned}
\partial_u\left(\sum_{k=0}^{n} C_k^n u^k v^{n-k}\right) &= \partial_u(u+v)^n \\
&= n(u+v)^{n-1}.
\end{aligned}$$

It follows then that

$$\sum_{k=0}^{n} C_k^n \frac{k}{n} u^k v^{n-k} = (u+v)^{n-1} u. \tag{4}$$

Letting $u = x$, $v = 1 - x$ yields the second identity.

From (4), we obtain

$$\sum_{k=0}^{n} C_k^n \frac{k^2}{n} u^{k-1} v^{n-k} = \partial_u \left(\sum_{k=0}^{n} C_k^n \frac{k}{n} u^k v^{n-k} \right) = (n-1)(u+v)^{n-2} u + (u+v)^{n-1}.$$

Multiply by $\frac{u}{n}$, then we get the following

$$\sum_{k=0}^{n} C_k^n \frac{k^2}{n^2} u^k v^{n-k} = \frac{(n-1)(u+v)^{n-2}}{n} u^2 + \frac{(u+v)^{n-1}}{n} u.$$

Then again, letting $u = x$, $v = 1 - x$ yields the third identity.

4.2 Exercise 4.2

By direct computation,

$$\left(\frac{k}{n} - x \right)^2 = \frac{k^2}{n^2} - 2x \frac{k}{n} + x^2.$$

Useing our previous calculations

$$\begin{aligned}
\sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^2 b_n^k &= B_n(x^2) - 2x B_n(x) + x^2 B_n(1) \\
&= \frac{(n-1)x^2}{n} + \frac{x}{n} - x^2 \\
&= \frac{x - x^2}{n}.
\end{aligned}$$

Then, notice that for any $\eta > 0$

$$\begin{aligned}
\eta^2 \sum_{k, \left| \frac{k}{n} - x \right| \geq \eta} C_k^n x^k (1-x)^{n-k} &\leq \sum_{k, \left| \frac{k}{n} - x \right| \geq \eta} C_k^n \left(\frac{k}{n} - x \right)^2 x^k (1-x)^{n-k} \\
&= \frac{x - x^2}{n} \\
&\leq \frac{1}{n}
\end{aligned}$$

since $x \in (0, 1)$. Diving both sides of the above inequality by η^2 yields the conclusion.

4.3 Exercise 4.3

To prove the Weierstrass theorem, we use the identity that

$$\sum_{k=0}^{n} C_k^n x^k (1-x)^{n-k} = 1$$

to see that

$$\begin{aligned}
|B_n(f)(x) - f(x)| &= \left| \sum_{k=0}^{n} C_k^n x^k (1-x)^{n-k} f\left(\frac{k}{n} \right) - f(x) \sum_{k=0}^{n} c_k^n x^k (1-x)^{n-k} \right| \\
&\leq \sum_{k=0}^{n} C_k^n x^k (1-x)^{n-k} \left| f\left(\frac{k}{n} \right) - f(x) \right|.
\end{aligned}$$

Now we split the last sum above into two sums, S_1, which is summed over those terms $\left|\frac{k}{n} - x\right| < \delta$, and S_2, which is summed over those terms $\left|\frac{k}{n} - x\right| \geq \delta$. δ is chosen such that for any x, $|f(y) - f(x)| < \frac{\epsilon}{2}$ if $|x - y| < \delta$, which is possible since $f \in C^0([0,1])$, and as $[0,1]$ is a compact interval, we moreover have that f is uniformly continuous on $[0,1]$.

Then we have that

$$\begin{aligned}
S_1 &\leq \sum C_k^n x^k (1-x)^{n-k} \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} \sum_{k=0}^{n} C_k^n x^k (1-x)^{n-k} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

On the other hand, we can control S_2 in the following manner. Since f is continuous on a bounded set, then we know that f in particular must also be bounded, say by M. Then,

$$\begin{aligned}
\delta^2 S_2 &\leq \sum C_k^n \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} \left| f\left(\frac{k}{n}\right) - f(x) \right| \\
&\leq \sum C_k^n \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} 2M \\
&= 2M \frac{x(1-x)}{n} \\
&\leq \frac{2M}{n}.
\end{aligned}$$

This yields that

$$S_2 \leq \frac{2M}{\delta^2 n}.$$

Then, we see that since we have fixed δ, we can choose N sufficiently large so that $\frac{2M}{\delta^2 n} \leq \frac{\epsilon}{2}$ if $n > N$.

Then, we have shown that for any x, given our choice of δ, there exists an N such that

$$|B_n(f)(x) - f(x)| \leq S_1 + S_2 < \epsilon$$

and we have proven uniform convergence.

5 Exercise 5

First consider $g_n(x) = f(x) - f_n(x)$. Since $f_n \to f$ pointwise, we also have that $g_n \to 0$ pointwise. Now let $\epsilon > 0$. Consider the family of sets $S_n = \{x : g_n(x) < \epsilon\}$.Then, g_n is continuous for all n, and S_n is open for all n, since $S_n = g_n^{-1}(-\infty, \epsilon)$.

Since $g_n(x) \to 0$ for all $x \in [a,b]$, there exists some N such that $x \in S_N$. Thus, we have that S_n forms an open cover for $[a,b]$. That is, that

$$[a,b] \subset \bigcup_n S_n.$$

But since $[a,b]$ is a compact interval, we have that it must be contained in a finite sub-cover, $S_{n_1}, \cdots, S_{n_m}$. That is, that

$$[a,b] \subset \bigcup_{i=0}^{m} S_{n_i}.$$

But this implies that for any $x \in [a,b]$, for any $p > n_m$, $g_n(x) < \epsilon$, so actually, the convergence is uniform.

6 Exercise 6

6.1 Exercise 6.1

Let $\epsilon > 0$. Then, since $f_k \to f$ pointwise, for each $x \in [a, b]$, there is an $N(x)$ such that for all $n \geq N(x)$, $|f_n(x) - f_m(x)| \leq \frac{\epsilon}{3}$ for $n, m > N(x)$. Then, consider $|x - y| \leq \frac{\epsilon}{3K}$, we have the following set of inequalities

$$\begin{aligned}|f_n(y) - f_m(y)| &\leq |f_n(y) - f_n(x)| + |f_n(x) - f_m(x)| + |f_m(y) - f_m(x)| \\ &\leq \epsilon.\end{aligned}$$

Notice that at this point we have *not* shown uniform convergence. Controlling the middle term in the above inequality relied on choosing an $N(x)$ that depends on x! The key to overcoming this difficulty is using the fact that $[a, b]$ is compact. Namely, we see that the collection of $\frac{\epsilon}{3K}$ open neighborhoods of $x \in [a, b]$ covers $[a, b]$. Thus, we actually have that $[a, b]$ is covered by a finite number of such neighborhoods, say M such neighborhoods, around $(x_i)_{i=1}^{M}$. Then, let $N = \max(N(x_i) : 1 \leq i \leq M)$. Then, this N is independent of x, and for $n, m > N$, we have that

$$|f_n(y) - f_m(y)| \leq \epsilon$$

for all $y \in [a, b]$, so we have shown uniform convergence.

6.2 Exercise 6.2

No. Consider $f_n : (0, 1) \to \mathbb{R}$, $f_n(x) = x^n$. Then, we know that since we are only considering the open interval $(0, 1)$, $f_n(x)$ converges pointwise to 0.

To see that f_n is uniformly convergent on any segment $[a, b] \in (0, 1)$, consider $f_n(x) < b^n$. Since $b < 1$, $b^n \to 0$. Thus, there exists some N such that for all $n > N, x \in [a, b]$, $f_n(x) < \epsilon$, and we have that f_n converges uniformly to 0.

However, $f_n(x)$ does not converge uniformly on $(0, 1)$. To see this, assume for the sake of contradiction that actually f_n did converge uniformly to 0. Then, consider some sequence of points in $(0, 1)$ which converge to 1, for example, $x_n = 1 - \frac{1}{n}$. Then, by Exercise 7 from Problem Sheet 1,

$$\lim_{n\to\infty} f_n(x_n) = f(x) = f(1) = 0.$$

However,

$$\begin{aligned}\lim_{n\to\infty} f_n(x_n) &= (1 - \frac{1}{n})^n \\ &= \frac{1}{e}.\end{aligned}$$

This is a contradiction. Thus, we cannot have that $f_n(x)$ converges uniformly.