Analysis 4 Problem Set 10

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1 Exercise 1

1.1 Exercise 1.1

For $n \in \mathbb{Z}$, we use the definition of $c_n(f)$, and integrate by parts to get

$$\begin{aligned} c_n(f) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)x} dx \\ &= \frac{1}{2\pi(1-in)} \left(e^{\pi} e^{-in\pi} - e^{-\pi} e^{in\pi} \right) = \frac{(-1)^n \left(e^{\pi} - e^{-\pi} \right)}{2\pi(1-in)}. \end{aligned} \tag{1}$$

1.2 Exercise 1.2

By direct computation, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2x} dx = \frac{1}{4\pi} \left(e^{2\pi} - e^{-2\pi} \right) = \frac{1}{4\pi} \left(e^{\pi} - e^{-\pi} \right) \left(e^{\pi} + e^{-\pi} \right). \tag{2}$$

Using Parseval's formula, (**??**) and (**??**), we obtain

$$\sum_{-\infty}^{\infty} |c_n(f)|^2 = \sum_{-\infty}^{\infty} \frac{\left(e^{\pi} - e^{-\pi} \right)^2}{4\pi^2 \left(1 + n^2\right)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{4\pi} \left(e^{\pi} - e^{-\pi} \right) \left(e^{\pi} + e^{-\pi} \right).$$

It follows that

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2} = \frac{1 + \pi \coth \pi}{2} \quad \text{where } \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

2 Exercise 2

2.1 Exercise 2.1

Note that $f(0) = \frac{\pi^2}{4} = f(2\pi)$, it follows that f is continuos. Now we calculate its real Fourier coefficients,

$$a_0(f) = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x^2}{4} = \frac{\pi^2}{6}. \tag{3}$$

For $n \geq 1$, we have

$$\begin{aligned} a_n(f) &= \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} \cos nx dx \\ &= \frac{1}{\pi n} \frac{(\pi - x)^2}{4} \sin nx \Big|_0^{2\pi} + \frac{1}{\pi n} \int_0^{2\pi} \sin nx \frac{(\pi - x)}{2} dx \\ &= \frac{1}{\pi n^2} \frac{x - \pi}{2} \cos nx \Big|_0^{2\pi} - \frac{1}{2\pi n^2} \int_0^{2\pi} \cos nx dx = \frac{1}{n^2} \end{aligned} \tag{4}$$

and

$$b_n(f) = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{(x)^2}{4} \sin nx dx = 0. \tag{5}$$

2.2 Exercise 2.2

From (**??**), (**??**) and (**??**), we know that the trigonometric series $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$ converges uniformly. From Proposition 2.9, we obtain f is equal to its Fourier series everywhere. Let $x=0$, we have

$$\frac{\pi^2}{4} = f(0) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Then Euler's formula follows.

2.3 Exercise 2.3

We denote

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx = \frac{\pi^2}{12} + \sum_{k=1}^{n} \frac{\cos kx}{k^2} \quad \text{for } n \in \mathbb{N}.$$

By direct computaion, we have

$$S_n'(x) = -\sum_{k=1}^{n} \frac{\sin kx}{k} \quad \text{for } x \in [\delta, 2\pi - \delta].$$

Recall that the series $\sum_{k=1}^{n} \frac{\sin kx}{k}$ converges uniformly on $[\delta, 2\pi-\delta]$. Using Proposition in Lecture notes, we obtain te Fourier series of f can be differentiated term by term in all segment $[\delta, 2\pi-\delta]$ for $0<\delta<2\pi$. Let $\delta \to 0$ and using $f'(x) = -\frac{\pi - x}{2}$, we obtain

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad \text{for all } x \in (0, 2\pi).$$

2.4 Exericise 2.4

By direct computaion,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx = \frac{1}{32\pi}\int_0^{2\pi} (x-\pi)^4 dx = \frac{\pi^4}{80}. \tag{6}$$

Using Parseval's formula and (**??**), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} |a_n(f)|^2 = \frac{1}{\pi}\int_0^{2\pi} |f(x)|^2\, dx - \frac{1}{2}|a_0(f)|^2 = \frac{\pi^4}{40} - \frac{\pi^4}{72} = \frac{\pi^4}{90}.$$

3 Exercise 3

3.1 Exercise 3.1

Fix $M>0$, we claim that the series $\sum_{n\in\mathbb{Z}} f(x+n)$ converges normally on $[-M, M]$. From the fact that $f(x)$ is $O(\frac{1}{|x|^2})$ as $|x| \to +\infty$, we obtain that there exists K such that

$$|f(x)| \le \frac{K}{|x|^2} \quad \text{for all } |x| \ge 1. \tag{7}$$

Using (**??**), we obtain for all $x \in [-M, M]$

$$|f(x+n)| \le \frac{K}{|n+x|^2} \le \frac{K}{(|n|-M)^2} \quad \text{for all } |n| > M+1.$$

It follows that $\sum_{n\in\mathbb{Z}} f(x+n)$ converges normally on $[-M, M]$. We denote by $F(x)$ the limit.

3.2 Exercise 3.2

Using a similar argument as in Exericise 3.1, we obtain $F(x)$ is C^2. It is not difficult to check that $F(x)$ is 1 periodic function. So we can consider the Fourier series associated to F. From $F(x)$ is C^2, we know that $c_n(F)$ is $O(\frac{1}{n^2})$ as $|n| \to +\infty$. It follows that the Fourier series associated to F converges uniformly, and that it is equal to its Fourier series.

3.3 Exericise 3.3

By direct computation, we have

$$\begin{aligned} c_n(F) &= \int_0^1 F(x)e^{-2\pi inx}dx \\ &= \int_0^1 \sum_{k\in\mathbb{Z}} f(x+k)e^{-2\pi inx}dx \\ &= \sum_{k\in\mathbb{Z}} \int_0^1 f(x+k)e^{-2\pi inx}dx = \sum_{k\in\mathbb{Z}} \int_k^{k+1} f(x)e^{-2\pi inx}dx = f^*(n) \end{aligned}$$

where $f^*(n) = \int_{-\infty}^{\infty} f(t)e^{-2i\pi nt}dt$.

3.4 Exericise 3.4

Using Exercise 3.2 and Exericise 3.3, we deduce that

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} f^*(n)e^{2i\pi nx} \quad \text{for all } x \in \mathbb{R}.$$

3.5 Exercise 3.5

By direct computation,

$$I'(x) = -2i\pi \int_{-\infty}^{\infty} ue^{-u^2}e^{-2i\pi ux}du. \tag{8}$$

Using integration by parts and (**??**), we obtain

$$I(x) = -\frac{1}{-2i\pi x}e^{-u^2}e^{-2i\pi xu}\Big|_{-\infty}^{\infty} - \frac{1}{2i\pi x}\int_{-\infty}^{\infty} 2ue^{-u^2}e^{-2i\pi xu}du = \frac{I'(x)}{-2\pi^2 x}$$

It follows that $I'(x) = -2\pi^2 xI(x)$. By solving this ODE and using initial data $I(0) = \int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$, we obtain $I(x) = \sqrt{\pi}e^{-\pi^2 x^2}$.

3.6 Exersice 3.6

Fix $s > 0$, we consider $f(x) = e^{-\pi s x^2}$. By direct computation, we obtain

$$f^*(n) = \int_{-\infty}^{\infty} e^{-\pi s u^2}e^{-2i\pi nu}du = \frac{1}{\sqrt{\pi s}}\int_{-\infty}^{\infty} e^{-u^2}e^{-2i\pi\frac{n}{\sqrt{\pi s}}u}du = \frac{1}{\sqrt{\pi s}}I\left(\frac{n}{\sqrt{\pi s}}\right) = s^{-\frac{1}{2}}e^{-\frac{\pi n^2}{s}}. \tag{9}$$

Letting $x = 0$ in Exercise 3.4 and using (**??**), we obtain

$$\sum_{-\infty}^{\infty} e^{-\pi n^2 s} = s^{-\frac{1}{2}}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{s}} \quad \text{for all } s > 0.$$

4 Exercise 4

4.1 Exercise 4.1

For $n = 0$, we have

$$c_0(f) = \frac{1}{2\pi}\int_0^\pi 1dx = \frac{1}{2}$$

For $n \neq 0$, we use the definition of $c_n(f)$, and integrate by parts to get

$$c_n(f) = \frac{1}{2\pi}\int_0^\pi e^{-inx}dx = \frac{1}{2\pi in}\left(1 - e^{-in\pi}\right) = \frac{1}{2\pi in}\left(1 - (-1)^{n+1}\right).$$

4.2 Exercise 4.2

We denote the partial sums of Fourier series $S_n = \sum_{k=-n}^{n} c_n(f)e^{inx}$. Note that S_{2n-1} can be written as

$$\begin{aligned} S_{2n-1}(x) &= \frac{1}{2} + \sum_{k=-(2n-1)}^{2n-1} c_k(f)e^{ikx} \\ &= \frac{1}{2} + \sum_{k=-(2n-1)}^{2n-1} \frac{1}{2\pi ik}\left(1-(-1)^{k+1}\right)(\cos kx + i\sin kx) \\ &= \frac{1}{2} + \sum_{k=1}^{n}\frac{2}{\pi(2k-1)}\sin(2k-1)x. \end{aligned} \tag{10}$$

Using Dirichlet's test, (**??**), and the fact that $|c_n| \to 0$ as $n \to +\infty$, we obtain that the Fourier series associated to f converges uniformly on all compact $[\delta, \pi - \delta]$.

4.3 Exercise 4.3

By direct computation, we obtain

$$\begin{aligned} S'_{2n-1}(x) &= \frac{2}{\pi}\sum_{k=1}^{n}\cos(2k-1)x \\ &= \frac{1}{\pi}\sum_{k=1}^{n}\left(e^{i(2k-1)\pi} + e^{-i(2k-1)\pi}\right) \\ &= \frac{1}{\pi}\left(e^{ix}\sum_{k=0}^{n-1}e^{(2ix)k} + e^{-ix}\sum_{k=0}^{n-1}e^{-(2ix)k}\right) \\ &= \frac{1}{\pi}\left(e^{ix}\frac{1-e^{2inx}}{1-e^{2ix}} + e^{-ix}\frac{1-e^{-2inx}}{1-e^{-2ix}}\right) \\ &= \frac{1}{\pi}\left(\frac{1-e^{2inx}}{e^{-ix}-e^{ix}} + \frac{1-e^{-2inx}}{e^{ix}-e^{-ix}}\right) = \frac{1}{\pi}\frac{\sin 2nx}{\sin x}. \end{aligned} \tag{11}$$

Integrating (**??**) on $[0, x]$ and using $S_{2n-1}(0) = \frac{1}{2}$, we obtain

$$S_{2n-1}(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^x \frac{\sin 2ns}{\sin s}ds. \tag{12}$$

4.4 Exercise 4.4

By direct argument and using (**??**), we know that the function S_{2n-1} has $2n$ critical points on $[0, \pi]$, which are exactly $x_k = \frac{k\pi}{2n}$, $1 \leq k \leq 2n$.

4.5 Exercise 4.5

Note that $S'_{2n-1}(x) = \frac{1}{\pi}\frac{\sin 2nx}{\sin x} < 0$ on (x_{2k-1}, x_{2k}). Then we obtain

$$S_{2n-1}(x_{2k}) < S_{2n-1}(x_{2k-1}) \quad \text{for all } 1 \leq k \leq n.$$

4.6 Exercise 4.6

We claim the following inequality

$$S_{2n-1}(x_{2k+1}) - S_{2n-1}(x_{2k}) < S_{2n-1}(x_{2k-1}) - S_{2n-1}(x_{2k}). \tag{13}$$

Note that from (**??**), it follows that $S_{2n-1}(x_{2k+1}) < S_{2n-1}(x_{2k-1})$. Furthermore,

$$\begin{aligned} S_{2n-1}(x_{2k-1}) - S_{2n-1}(x_{2k}) &= -\frac{1}{\pi}\int_{x_{2k-1}}^{x_{2k}} \frac{\sin 2ns}{\sin s} ds \\ &= -\frac{1}{\pi}\int_{\frac{(2k-1)\pi}{2n}}^{\frac{k\pi}{n}} \frac{\sin 2ns}{\sin s} ds \\ &= -\frac{1}{2n\pi}\int_{(2k-1)\pi}^{2k\pi} \frac{\sin s}{\sin \frac{s}{2n}} ds \\ &= \frac{1}{2n\pi}\int_{2k\pi}^{(2k+1)\pi} \frac{\sin s}{\sin(\frac{s}{2n} - \frac{\pi}{2n})} ds \end{aligned} \tag{14}$$

and

$$\begin{aligned} S_{2n-1}(x_{2k+1}) - S_{2n-1}(x_{2k}) &= \frac{1}{\pi}\int_{x_{2k}}^{x_{2k+1}} \frac{\sin 2ns}{\sin s} ds \\ &= -\frac{1}{\pi}\int_{\frac{k\pi}{n}}^{\frac{(2k+1)\pi}{2n}} \frac{\sin 2ns}{\sin s} ds \\ &= \frac{1}{2n\pi}\int_{2k\pi}^{(2k+1)\pi} \frac{\sin s}{\sin \frac{s}{2n}} ds \end{aligned} \tag{15}$$

Using (**??**), (**??**), and the fact that $\sin x$ is increasing on $[0, \frac{\pi}{2}]$, we obtain (**??**).

4.7 Exericise 4.7

From Exercise 4.4, 4.5 and 4.6, we deduce that S_{2n-1} attains its maximum in x_1.

4.8 Exercise 4.8

We denote by M_n this maximum. Using (**??**), we obtain

$$M_n = \frac{1}{2} + \frac{1}{\pi}\int_0^{\frac{\pi}{2n}} \frac{\sin 2ns}{\sin s} ds = \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin s}{2n \sin \frac{s}{2n}} ds \to M = \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin s}{s} ds$$

4.9 Exercise 4.9

Using the definition of f and Exericise 4.8, we conclude that the Fourier series associated with f doesn't converge uniformly.

5 Exercise 5

5.1 Exercise 5.1

Suppose for the sake of contradiction that this were not true. Then there exists $\delta > 0$ and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of integers such that $\rho_{n_k} \geq \delta$ for all k. Restricting further to a subsequence, we can assume that $n_{k+1} > 6n_k$ for all k.

Then, consider

$$I_1 = \left[\frac{1}{n_1}\left(\theta_{n_1} - \frac{\pi}{3}\right), \frac{1}{n_1}\left(\theta_{n_1} + \frac{\pi}{3}\right)\right],$$

such that for all $t \in I_1$, $\cos(n_1 t - \theta_{n_1}) \geq \frac{1}{2}$. As t varies in I_1, $n_2 t - \theta_{n_2}$ varies in an inteveral of length $n_2 \cdot \frac{2\pi}{3n_1} \geq 4\pi$.

Thus, we can find a segment $I_2 \subset I_1$ of length $\frac{2\pi}{3n_2}$ such that $\cos(n_2 t - \theta_{n_2}) \geq \frac{1}{2}$ for all $t \in I_2$. Continuing to iterate in this way, we construct for all k a segment $I_k \subset I_{k-1}$ of length $\frac{2\pi}{3n_k}$ such that

$$\forall t \in I_k, \qquad \cos(n_k t - \theta n_k) \geq \frac{1}{2}.$$

Then, by construction we have that

$$|\rho_{n_k} \cos(n_k t - \theta_{n_k})| \geq \frac{\delta}{2}.$$

5.2 Exercise 5.2

Assume that $\rho_n \cos(nt - \theta_n) \to 0$ as $n \to \infty$ but $\rho_n \not\to 0$ for the sake of contradiction, then using Exercise 5.1, we know that there exists $\xi \in \mathbb{R}$ such that $\bigcap_{k \in \mathbb{N}} I_k = \{\xi\}$. For all k, we have $\rho_{n_k} \cos(n_k \xi - \theta_k) \geq \frac{\delta}{2}$, which contradicts our assumption that $\rho_n \cos(nt - \theta_n) \to 0$.

5.3 Exercise 5.3

We can write $c_n e^{int} + c_{-n} e^{-int}$ in the form $a_n \cos nt + b_n \sin nt) + i(a'_n \cos nt + b'_n \sin nt)$, where $a_n, b_n, a'_n, b'_n \in \mathbb{R}$. Then WLOG, we have to show that the series a_n, b_n tend to zero. For all $t \in \mathbb{R}$, we know that $\lim_{n\to\infty}(a_n \cos nt + b_n \sin nt) = 0$. Then define $\rho_n = \sqrt{a_n^2 + b_n^2}$. Then there exists for all n a real $\theta_n \in [0, 2\pi]$ such that $a_n \cos nt + b_n \sin nt = \rho_n \cos(nt - \theta_n)$ for all $t \in \mathbb{R}$. So we have reduced our problem to proving the following: if for all $t \in \mathbb{R}$, $\lim_{n\to\infty} \rho_n \cos(nt - \theta_n) = 0$, then the sequence (ρ_n) tends to 0. But this is exactly the statement in Exercise 5.2. Therefore, we are done.