# Quiz 1 2020

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## 1 Exercise 1

*Proof.* (i). For all $x \in [0,1)$, we have

$$|f_n(x)| \leq n^2 x^{n-1} + n^2 x^n \to 0, \quad \text{as } n \to \infty.$$

Note also that $f_n(1) = 0$ for all $n \in \mathbb{N}^+$. Therefore, we obtain $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

(ii). Note that, for $n = 1$, we have

$$\int_0^1 f_1(x) = \int_0^1 f_1(x)\mathrm{d}x = \int_0^1 (1-x)\mathrm{d}x = \frac{1}{2}.$$

By direct computation, for all $n \in \mathbb{N}^+$,

$$\int_0^1 f_n(x)\mathrm{d}x = n^2 \left( \int_0^1 x^{n-1}\mathrm{d}x - \int_0^1 x^n \mathrm{d}x \right) = \frac{n}{n-1}. \tag{1}$$

(iii). Let $n \to \infty$ in (1), we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x = 1.$$

(iv). From (1), (3) and Proposition 3.1 in Lecture notes, we proved (iv). ☐

## 2 Exercise 2

*Proof.* (i). From the Taylor-Lagrange formula, taking $\delta_1$ small enough, for any $x \in [-\delta_1, \delta_1]$, there exist $c_x \in [0,x]$ or $[x,0]$, such that

$$f(x) = f(0) + f'(0)x + \frac{f''(c)}{2}x^2$$

Therefore, from $f(0) = 0$ and $0 < f'(0) < 1$, we obtain

$$|f(x)| \leq f'(0)|x| + \frac{1}{2}M|x|^2 \leq \left( f'(0) + \frac{1}{2}M\delta_1 \right)|x|. \tag{2}$$

(ii). Taking $0 < \delta_1$ small enough, we have

$$0 < f'(0) + \frac{1}{2}M\delta_1 < 1.$$

Set $q = f'(0) + \frac{1}{2}M\delta_1$. From (2), we proved (ii).

(iii). We prove (iii) by induction. For $n = 1$, from (2), we have

$$|f_1(x)| = |f(f(x))| < q|f(x)| < q^2|x| < q|x|.$$

We assume (iii) is true for $n = k$. Now, we prove that also true for $n = k+1$. Using (2) again,

$$|f_{k+1}(x)| = |f(f_k(x))| < q|f_k(x)| < q^{k+1}|x|.$$

Therefore, (iii) is also true for $n = k+1$. By induction, we have proven (iii) for $n \in \mathbb{N}^+$. Last, from $\sum_{n=1}^{\infty} q^n < \infty$, we obtain that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[-\delta_2, \delta]$. ☐

## 3 Exercise 3

(i). By direct computation,

$$|a_n|^{\frac{1}{n}} = \frac{2}{n^{\frac{1}{2}}} \to 2 \quad \text{as } n \to \infty.$$

Therefore, from the definition of the radius of convergence $R$,

$$R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = \frac{1}{2}.$$

(ii). By direct computation,

$$|a_n|^{\frac{1}{n}} = \frac{n^{\frac{1}{2}}}{e} \to \frac{1}{e} \quad \text{as } n \to \infty.$$

Therefore, from the definition of the radius of convergence $R$,

$$R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = e.$$

(iii). By direct computation,

$$|a_n|^{\frac{1}{n}} = \frac{\pi}{n^{\frac{\pi}{n}}} \to \pi \quad \text{as } n \to \infty.$$

Therefore, from the definition of the radius of convergence $R$,

$$R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = \frac{1}{\pi}.$$

(iv). Note that

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2((2n)!)}{(n!)^2((2n+2)!)} = \frac{(n+1)^2}{(2n+1)(2n+2)} \to \frac{1}{4} \quad \text{as } n \to \infty.$$

It follows that

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4} \quad \text{as } n \to \infty.$$

Therefore, from the definition of the radius of convergence $R$,

$$R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = 4.$$

(v). (iv) is same as (v).

## 4 Exercise 4

Recall that,

$$f_n(x) < f_{n+1}(x) \quad \text{for all } n \in \mathbb{N}^+ \quad \text{and} \quad f_n(x) \to e^x \quad \text{as } n \to \infty.$$

Set

$$g_n(x) = e^x - f_n(x) \quad \text{for all } x \in [0,1] \text{ and } n \in \mathbb{N}^+.$$

From Dini's theorem, we obtain that the sequence $(g_n)_{n\in\mathbb{N}^+}$ converges uniformly to 0 on $[0,1]$. It follows that, the sequence $(f_n)_{n\in\mathbb{N}^+}$ converges uniformly to $e^x$ on $[0,1]$.

## 5 Exercise 5

(i) By direct computation,

$$\frac{5}{(x^2+4)(x^2-1)} = -\frac{1}{x^2+4} + \frac{1}{x^2-1} = -\frac{1}{x^2+4} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}.$$

(ii) Recall that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{and} \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x| < 1.$$

Therefore, we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{and} \quad \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \quad \text{if } |x| < 1.$$

(iii) From (ii), we know that

$$\frac{1}{x^2+4} = \frac{1}{4}\frac{1}{1+(\frac{x}{2})^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^{n+1}}$$

Thus, from (i) and (ii), we have

$$\frac{5}{(x^2+4)(x^2-1)} = -\frac{1}{x^2+4} - \frac{1}{1-x^2} = -\sum_{n=0}^{\infty} \left(1 + (-1)^n \frac{1}{4^{n+1}}\right) x^{2n}.$$