Analysis 4 Problem Set 11

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1 Exercise 1

1.1 Exercise 1.1

Let us consider the diferential equation

$$y' = F(t,y) \quad \text{where} \quad F(t,y) = y + \ln y - t.$$

From the definition of the domain of the differential equation, we know that the domain of this equation is $\mathbb{R} \times (0, +\infty)$.

1.2 Exericise 1.2

Let $y = e^t$. By direct computation, we know that

$$y' = e^t \quad \text{and} \quad y + \ln y - t = e^t + t - t = e^t. \tag{1.1}$$

From (1.1), we obtain the function $\mathbb{R} \to \mathbb{R}$, $t \to e^t$ is a solution.

2 Exericise 2

2.1 Exercise 2.1

The solution of the first order linear equation is

$$y(t) = y_0 e^{\frac{1}{2}(t^2 - t_0^2)} \qquad \text{that satisfies the initial condition} \quad y(t_0) = y_0.$$

We obtain the space of solutions of the differential equation is $\left\{ y(t) = y_0 e^{\frac{1}{2}(t^2 - t_0^2)} \middle| (t_0, y_0) \in \mathbb{R}^2 \right\}$.

2.2 Exercise 2.2

Let $f(t) = y(t) e^{-\frac{1}{2}t^2}$. By direct computation, we have

$$\frac{d}{dt} f(t) = (y' - ty) e^{-\frac{1}{2}t^2} = (-\sin t - t\cos t)\, e^{-\frac{1}{2}t^2}. \tag{2.1}$$

Integratin (2.1) on $[0,t]$ and using $f(0) = y(0)$, we obtain

$$\begin{aligned} f(t) &= y(0) - \int_0^t \sin s e^{-\frac{1}{2}s^2} ds - \int_0^t s\cos s e^{-\frac{1}{2}s^2} ds \\ &= y(0) + \cos s e^{-\frac{1}{2}s^2}\Big|_0^t + \int_0^t s\cos s e^{-\frac{1}{2}s^2} ds - \int_0^t s \cos s e^{-\frac{1}{2}s^2} ds = (y(0) - 1) + \cos t e^{-\frac{1}{2}t^2}. \end{aligned} \tag{2.2}$$

Using the definition of $f(t)$, (2.2) and $y(0) = 2$, we obtain

$$y(t) = e^{\frac{1}{2}t^2} f(t) = e^{\frac{1}{2}t^2} \left((y(0) - 1) + \cos t e^{-\frac{1}{2}t^2} \right) = e^{\frac{1}{2}t^2} + \cos t.$$

3 Exercise 3

3.1 Exericise 3.1

Let us consider the diferential equation

$$y' = F(t, y) \quad \text{where} \quad F(t, y) = \frac{y}{1+t} + 1 + t.$$

From the definition of the domain of the differential equation, we know that the domain of this equation is $(-1, +\infty) \times \mathbb{R}$.

3.2 Exercise 3.2

Let $f(t) = \frac{y(t)}{1+t}$. By direct computation, we have

$$\frac{d}{dt} f(t) = \frac{1}{1+t}\left(y'(t) - \frac{y(t)}{1+t}\right) = 1. \tag{3.1}$$

Integratin (3.1) on $[0, t]$ and using $f(0) = y(0)$, we obtain

$$f(t) = y(0) - \int_0^s 1 ds = y(0) + t. \tag{3.2}$$

Using the definition of $f(t)$, (3.2) and $y'(0) = -2$, we obtain

$$y(t) = (1+t)f(t) = (1+t)(y(0)+t) = (t+1)(t-3).$$

4 Exercise 4

4.1 Exercise 4.1

The characteristic polynomial of this ODE is

$$P(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

Thus, the space of solutions of the ODE is $\left\{ c_1 e^{2t} + c_2 e^{-t} \middle| (c_1 \times c_2) \in \mathbb{R}^2 \right\}$.

4.2 Exercise 4.2

The second order ODE can be written as

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

4.3 Exericise 4.3

The second order ODE can be written as

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix} \tag{4.1}$$

Let

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \lambda(t)\begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + \mu(t)\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \tag{4.2}$$

From (4.1) and (4.2), we know that

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} &= \lambda'(t)\begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + \mu'(t)\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + \lambda(t)\begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} + \mu(t)\begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ e^{t} \end{pmatrix} = \lambda(t)\begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} + \mu(t)\begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} + \begin{pmatrix} 0 \\ e^{t} \end{pmatrix}.\end{aligned} \tag{4.3}$$

From (4.3) and $y(0) = 1$, $y'(0) = 0$, we obtain

$$\begin{cases} \lambda'(t)e^{2t} + \mu'(t)e^{-t} = 0 \\ 2\lambda'(t)e^{2t} - \mu'(t)e^{-t} = e^{t} \end{cases} \quad \text{and} \quad \begin{cases} \lambda(0) + \mu(0) = 1 \\ 2\lambda(0) - \mu(0) = 0 \end{cases}$$

It follows that

$$y(t) = \frac{2}{3}e^{2t} - \frac{1}{2}e^{t} + \frac{5}{6}e^{-t}.$$

5 Exercise 5

5.1 Exercise 5.1

We denote

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \tag{5.1}$$

By direct computaion, the characteristic polynomial of A is $P(\lambda) = (\lambda - (2+i))\,(\lambda - (2-i))$ and

$$\operatorname{Ker}(A - (2+i)I) = \mathbb{C}\begin{pmatrix} 1+i \\ 1 \end{pmatrix} \quad \text{and} \quad \operatorname{Ker}(A - (2-i)I) = \mathbb{C}\begin{pmatrix} 1-i \\ 1 \end{pmatrix}.$$

It follows that the complex space of solutions of the ODE system is

$$t \to \alpha e^{(2+i)t}\begin{pmatrix} 1+i \\ 1 \end{pmatrix} + \beta e^{(2-i)t}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \quad \text{where } \alpha, \beta \in \mathbb{C}.$$

It is equivalent that the real space of solutions of the ODE system is

$$t \to c_1 e^{2t}\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix} \quad \text{where } c_1,\ c_2 \in \mathbb{R}.$$

5.2 Exercise 5.2

Let

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1(t)e^{2t}\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2(t)e^{2t}\begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}$$

From Exericise 5.1, we know that

$$\begin{cases} c_1'(t)\,(\cos t - \sin t) + c_2'(t)\,(\cos t + \sin t) = te^{-2t} \\ c_1'(t)\cos t + c_2'(t)\sin t = e^{-2t} \end{cases}$$

It is equivalent that

$$\begin{cases} c_1'(t) = ((1-t)\sin t + \cos t)\,e^{-2t} \\ c_2'(t) = ((t-1)\cos t + \sin t)\,e^{-2t} \end{cases} \tag{5.2}$$

From $x(0) = 1$, $y(0) = 2$ and (5.2), we obtain

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1(t)e^{2t}\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2(t)e^{2t}\begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}$$

where

$$\begin{cases} c_1(t) = 2 + \int_0^t ((1-s)\sin s + \cos s)\, e^{-2s} ds \\ c_2(t) = -1 + \int_0^t ((s-1)\cos s + \sin s)\, e^{-2s} ds \end{cases}$$

6 Exercise 6

6.1 Exercise 6.1

F is not locally Lipschitz on $\mathbb{R}$. Because we consider $y = 0$ and $x > 0$,

$$\frac{|F(x) - F(y)|}{|x-y|} = \frac{\sqrt{x}}{x} \to +\infty \quad \text{as} \quad x \to 0^+.$$

This contradicts$F(x)$ being locally Lipschitz at $x = 0$.

6.2 Exercise 6.2

From ϕ is solution of $t' = F(y)$, we have

$$\phi'(t) = F(\phi(t)) \quad \text{for all } t \in \mathbb{R}. \tag{6.1}$$

Fix $c \in \mathbb{R}$, from (6.1), we obtain

$$\phi_c'(t) = \phi'(t-c) = F(\phi(t-c)) = F(\phi_c(t))$$

It follows that the function $\phi_c : t \to \phi(t-c)$ is also a solution.

6.3 Exericise 6.3

Let

$$\phi(t) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{t^2}{4} & \text{if } x > 0 \end{cases}$$

It is not difficult to check $\phi(t)$ is solution of the Cauchy problem $y' = F(y)$, $y(0) = 0$. From Exericise 6.2, we know that for all $c > 0$, $\phi_c(t)$ is also a solution of the Cauchy problem $y' = F(y)$, $y(0) = 0$. It follows that there is no unique solution to the Cauchy problem $y' = F(y)$, $y(0) = 0$.

7 Exercise 7

7.1 Exercise 7.1

We denote $g(t) = 0$ for all $t \in \mathbb{R}$. From $F(t,0,0) = 0$ for all $t \in \mathbb{R}$, we know that

$$g''(t) = 0 = F(t,0,0) = F(t, g(t), g'(t)) \quad \text{for all } t \in \mathbb{R}.$$

It follows that the function $t \to 0$ is a solution.

7.2 Exericise 7.2

The second order ODE can be written as

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ F(t,y,y') \end{pmatrix} = \vec{F}(t,y,y'). \tag{7.1}$$

Let y be a solution of $y'' = F(t, y, y')$ which is not identically zero. If $y(t_0) = 0$, we claim that $y'(t_0) \neq 0$. Ayctually, if $y'(t_0) = 0$, we have $y(t_0) = y'(t_0) = 0$. From $\vec{F}(t, y, y')$ is locally Lipschitz, we know that the Cauchy Problem to the (7.1) has only one solution. From Exericise 7.1, we have know that the function $t \to 0$ is a solution. So we have $y(t) = 0$ for all $t \in \mathbb{R}$. This is contradictory with the fact that $y(t)$ is not identically zero. From the Taylor Formula and $y'(t_0) \neq 0$, we obtain that there exists $\delta > 0$ such that

$$|y(t)| = |y(0) + (t - t_0)y'(t_0) + o(t - t_0)| \geq \frac{1}{2}|y'(t_0)||t - t_0| > 0 \quad \text{for all } t \in (t_0 - \delta, t_0) \bigcup (t_0, t_0 + \delta)\,.$$

Thus, $y(t)$ only has isolated zeros.

8 Exercise 8

We prove Exericise 8 by contradiction. Assume that there exists $t_0 < T$ such that $f(T) \geq g(T)$. Let

$$t^* = \inf\{t_0 < t : f(t) \geq g(t)\}$$

From $f(t_0) < g(t_0)$ and that there exists $t_0 < T$ such that $f(T) \geq g(T)$, we know that t^* is well-defined and $t_0 < t^*$. Note that f and g are continuous, we obtain

$$f(t^*) = g(t^*) \quad \text{and} \quad f(t) < g(t) \quad \text{for all } t \in (t_0, t^*). \tag{8.1}$$

Because F is locally Lipschitz and $f(t^*) = g(t^*)$, we know that there exists $\delta > 0$ such that

$$f(t) = g(t) \quad \text{for all } t \in (t_0 - \delta, t_0 + \delta).$$

This is contradictory with (8.1).