Analysis 4 Problem Set 13

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1 Exercise 1

1.1 Exercise 1.1

The ODE system can be written as

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A(t)\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + B(t) \quad \text{where} \quad A(t) = \begin{pmatrix} 0 & 4 \\ -\frac{1}{t^2} & \frac{4}{t} \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} 2t \\ e^t \end{pmatrix}.$$

Note that $A(t)$ and $B(t)$ are continuous on $(0,+\infty)$, from Theorem 3.4 in Lecture notes, we know that there exists a unique solution $\psi : (0,+\infty) \to \mathbb{R}$.

1.2 Exericise 1.2

By direct computaion, we have

$$\frac{d}{dt}\psi_1(t) = \begin{pmatrix} 4t^3 \\ 3t^2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -\frac{1}{t^2} & \frac{4}{t} \end{pmatrix}\begin{pmatrix} t^4 \\ t^3 \end{pmatrix} \quad \text{and} \quad \frac{d}{dt}\psi_2(t) = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -\frac{1}{t^2} & \frac{4}{t} \end{pmatrix}\begin{pmatrix} 4t \\ 1 \end{pmatrix}$$

So that $\psi_1(t)$ and $\psi_2(t)$ are solutions to the homogeneous equation.

1.3 Exercise 1.3

We look for solutions of the form $\psi_0(t) = \sum_{k=1,2} \lambda_k(t)\psi_k(t)$. By direct computation, we obtain

$$\frac{d}{dt}\psi_0(t) = \sum_{k=1,2} \lambda_k'(t)\psi_k(t) + \sum_{k=1,2} \lambda_k(t)\psi_k'(t) = \sum_{k=1,2} \lambda_k'(t)\psi_k(t) + A(t)\psi_0(t) = A(t)\psi_0(t) + B(t)$$

It follows that

$$\sum_{k=1,2} \lambda_k'(t)\psi_k(t) = B(t) \quad \text{for all} \quad t \in (0,+\infty).$$

So that the space of solutions of (1) is $\left\{\psi_0(t) + c_1\psi_1(t) + c_2\psi_2(t) \middle| (c_1,c_2) \in \mathbb{R}^2\right\}$ where $\lambda_1(t)$ and $\lambda_2(t)$ satisfies

$$t^4\lambda_1' + 4t\lambda_2' = 2t \quad \text{and} \quad t^3\lambda_1'(t) + \lambda_2' = e^t.$$

2 Exericise 2

2.1 Exercise 2.1

The ODE system can be written as

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \vec{F}(t,y,z) \quad \text{where} \quad \vec{F}(t,y,z) = \begin{pmatrix} F_1(t,y,z) \\ F_2(t,y,z) \end{pmatrix} = \begin{pmatrix} \sqrt{1+z^2} - \cos(t)z \\ \arctan(yz) + 3y \end{pmatrix}. \tag{2.1}$$

First, $\vec{F}(t,y,z)$ is Locally Lipschitz in y and z, using Theorem 2.6, we obtain there exists unique locally solution $\phi = (y,z)$.

Second, we prove that the solution $\phi(t)$ is global solution on $\mathbb{R}$. By direct computation, we obtain

$$\|\vec{F}(t,y,z)\|^2 = \sum_{k=1,2} F_k^2(t,y,z) \leq 10\left(y^2+z^2\right) + 100. \tag{2.2}$$

From (2.1) and (2.2), we know that

$$\left|\frac{d}{dt}\left(y^2+z^2\right)\right| = 2\left|\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \cdot \vec{F}(t,y,z)\right| \leq \left\|\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}\right\|^2 + \|\vec{F}(t,y,z)\|^2 \leq 11\left(y^2+z^2\right) + 100. \tag{2.3}$$

Using (2.3) and Corollary 3.3, we obtain

$$y^2(t) + z^2(t) \leq \left(y^2(0) + z^2(0)\right)e^{11t} + \frac{100}{11}\left(e^{11t} - 1\right).$$

It follows that the solution $\phi(t)$ is well-defined on $\mathbb{R}$.

3 Exercise 3

3.1 Exericise 3.1

Let

$$\psi_1(t) = t^2 \quad \text{and} \quad \psi_2(t) = \frac{1}{t}.$$

By direct computation, we know that ψ_1 and ψ_2 are two indepent solutions to the homogeneous equation $t^2y'' - 2y = 0$.

3.2 Exercise 3.2

Let

$$y(t) = \lambda(t)t^2 + \mu(t)t^{-1} \quad \text{for all} \quad t \in (0,+\infty).$$

By direct computaion, we obtain

$$\lambda' t^2 + \mu' t = 0 \quad \text{and} \quad 2t\lambda' - \mu' t^2 = 3.$$

It follows that $\lambda' = t^{-1}$ and $\mu' = -t^2$. So that we obtain a particular solution

$$y(t) = t^2 \log t - \frac{t^2}{3} \quad \text{for all } t > 0.$$

So that the space of solutions is $\left\{c_1t^2 + \frac{c_2}{t} + t^2\log t - \frac{t^2}{3} \middle| (c_1,c_2) \in \mathbb{R}^2\right\}$.

3.3 Exercise 3.3

Using the same argument with Exercise 3.2, we also obtain $\lambda' = t^{-1}$ and $\mu' = -t^2$. It follows that the space of solutions on $(-\infty, 0)$ is $\left\{c_1t^2 + \frac{c_2}{t} + t^2\log|t| - \frac{t^2}{3} \middle| (c_1,c_2) \in \mathbb{R}^2\right\}$.

3.4 Exercise 3.4

There exists no C^2 solutions to the equation $t^2y'' - 2y = 3t^2$ which are defined on $\mathbb{R}$. The reason is $t^2\log|t| \notin C^2(\mathbb{R})$.

4 Exercise 4

4.1 Exercise 4.1

We denote the space of solutions to the differential equation $Y' = A(t)Y$ by S_H. From Theorem 3.5, we know that S_H is a sub-vector space of dimension n of C^1. Note that A is T periodic, so that $Y_T = Y(t+T) \in S_H$ for all $Y(t) \in S_H$. Now we conside following linera map on S_H,

$$\Phi(Y) = Y_T \quad \text{for all } Y \in S_H$$

Because S_H is a sub-vector space of dimension n of C^1 and Φ is linear map, so that there exists $\lambda \in \mathbb{C}$ and $V(t) \in S_H$ such that $\Phi(V) = \lambda V$. From the definition of Φ, we obtain $V(t+T) = \lambda V(t)$. for all $t \in \mathbb{R}$.

4.2 Exercise 4.2

Because $V_1, \cdots, V_n$ are n linearly independent solutions, we know that the dimension of S_H is n and the base of S_H are $V_1, \cdots, V_n$. Let $B = (b_{i,j})_{n\times n}$ be the matrix of Φ under the base $V_1, \cdots, V_n$ i.e.

$$V_{i,T}(t) = V_i(t+T) = \sum_{j=1}^{n} b_{i,j} V_j(t) \quad \text{for all } i \in \{1, \cdots, n\} \tag{4.1}$$

Note that $\mathrm{Ker}(\Phi) = 0$, so that B is invertible matrix. Let $V_{i,k}$ be the k-th component of V_i, from (4.1), we obtain

$$V_{i,k,T}(t) = \sum_{j=1}^{n} b_{i,j} V_{j,k}(t) \quad \text{for all } i, k \in \{1, \cdots, n\}. \tag{4.2}$$

From (4.2) we know that $M^{\mathrm{t}}(t+T) = BM^{\mathrm{t}}(t)$, it follows that $M(t+T) = M(t)C$.

5 Exercise 5

5.1 Exercise 5.1

It is not difficult to check the map is linear map, so we only prove that the map is invertible. From the ODE system exists a unique solution $v(t)$ with initial data $v(s) = y$, we obtain the kernal of the map is $\{0\}$. We conclude that the map is invertible. We note it by $C(t,x)$.

5.2 Exercise 5.2

From the definition of $C(t,x)$, we know that $C(s,s) = \mathrm{id}$. Now we prove that

$$\frac{d}{dt} C(t,s) = A(t)C(t,s) \quad \text{for all } t \in \mathbb{R}. \tag{5.1}$$

Fix $y \in \mathbb{R}^d$, from the defintion of $C(t,s)$, we know that $C(t,s)y$ satisfies for all $t \in \mathbb{R}$,

$$\frac{d}{dt}\left(C(t,s)y\right) = A(t)C(t,s)y \quad \text{and} \quad \frac{d}{dt}\left(C(t,s)y\right) = C'(t,s)y. \tag{5.2}$$

From the arbitrary choose of y and (5.2), we obtain (5.1).

5.3 Exercise 5.3

From the ODE system exists a unique solution $v(t)$ with initial data $v(s) = y$, we know that,

$$C(t,s)y = C(t,u)\left(C(u,s)y\right) = C(t,u)C(u,s)y \quad \text{for all } y \in \mathbb{R}^d. \tag{5.3}$$

From the arbitrary choose of y and (5.3), we obtain

$$C(t,s) = C(t,u)C(u,s) \quad \text{for all } s,t,u \in \mathbb{R}. \tag{5.4}$$

Using (5.4) and $C(t,t) = \mathrm{id}$ for all $t \in \mathbb{R}$, we have

$$C(t,s)C(s,t) = C(t,t) = \mathrm{id} \quad \text{for all t, s} \in \mathbb{R}. \tag{5.5}$$

5.4 Exercise 5.4

Fix $s \in \mathbb{R}$, from the definition of $C(t,s)$, we know that $t \to C(t,s)$ is continuous. Fix $t \in \mathbb{R}$, we have $C(t,s) = \left(C(s,t)\right)^{-1}$ is also continuous with s.

5.5 Exercise 5.5

Using $C(t,t) = \mathrm{id}$ for all $t \in \mathbb{R}$, we have $f(t_0)$ satisfies the initial condition i.e.

$$f(t_0) = C(t_0,t_0)y_0 + \int_{t_0}^{t_0} C(t,s)\psi(s)ds = y_0.$$

Now we prove that the function $f(t)$ satisfies the ODE $y'(t) = A(t)y(t) + \psi(t)$. By direct computaion, we have

$$\begin{aligned}\frac{d}{dt}f(t) &= \frac{d}{dt}C(t,t_0)y_0 + \frac{d}{dt}\left(\int_{t_0}^{t} C(t,s)\psi(s)ds\right)\\ &= A(t)C(t,t_0)y_0 + C(t,t)\psi(t) + \int_{t_0}^{t} A(t)C(t,s)\psi(s)ds\\ &= A(t)\left(C(t,t_0)y_0 + \int_{t_0}^{t} C(t,s)\psi(s)ds\right) + \psi(t) = A(t)f(t) + \psi(t).\end{aligned}$$

We conclude that $f(t)$ is the Cauchy problem's solution.