Quiz 1A solution

1 Exercise 1

1.1 Exercise 1.1

Let

$$a_n(x) = (-1)^n \frac{x^2}{n^2} \quad \text{and} \quad b_n = (-1)^n \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Note that $f_n = a_n + b_n$ for all $n \in \mathbb{N}$. Fix x, then because we have that $\sum \frac{1}{n^2}$ and $\sum (-1)^n \frac{1}{n}$ converges, we know that for all $\varepsilon > 0$ there exists some N be such that

$$\left| \sum_{p=n}^{q} \frac{1}{n^2} \right| < \varepsilon \quad \text{and} \quad \left| \sum_{p=n}^{q} (-1)^n \frac{1}{n} \right| < \varepsilon \quad \text{for all } q > p > N. \tag{1}$$

Using (1), we have for all $q > p > N$,

$$\left| \sum_{p=n}^{q} f_n(x) \right| \le \left| \sum_{p=n}^{q} a_n(x) \right| + \left| \sum_{p=n}^{q} b_n \right| \le \left(|x|^2 + 1 \right) \varepsilon.$$

This shows pointwise convergence using the Cauchy criterion.

1.2 Exercise 1.2

Let $I = [a, b]$ be a bounded interval of $\mathbb{R}$. From (1), we know that for all $q > p > N$,

$$\sup_{x \in [a,b]} \left| \sum_{p=n}^{q} f_n(x) \right| \le \sup_{x \in [a,b]} \left| \sum_{p=n}^{q} a_n(x) \right| + \left| \sum_{p=n}^{q} b_n \right| \le \left(|a|^2 + |b|^2 + 1 \right) \varepsilon.$$

This shows converges uniformly using the Cauchy criterion. Using Corollary 2.4, f is continuous on $[a, b]$, for any $a, b \in \mathbb{R}$. Let $a \to -\infty$ and $b \to +\infty$, we obtain f is continuous on $\mathbb{R}$.

2 Exercise 2

2.1 Exercise 2.1

Because we have that $\sum \frac{1}{n^2}$converges, we know that for all $\varepsilon > 0$ there exists some N be such that

$$\sup_{x \in [0,\infty)} \left| \sum_{p=n}^{q} f_n(x) \right| = \sup_{x \in [0,\infty)} \left| \sum_{p=n}^{q} \frac{1}{x + n^2} \right| \le \sum_{p=n}^{q} \frac{1}{n^2} \le \varepsilon \quad \text{for all } q > p > N.$$

This shows converges uniformly using the Cauchy criterion.

2.2 Exercise 2.2

Using Corollary 2.4 and $\sum_1^\infty f_n$ converges uniformly, we obtain f is continuous on $[0, \infty)$.

2.3 Exercise 2.3

By direct computation, we obtain

$$f_n'(x) = -\frac{1}{(x+n^2)^2} \quad \text{for all } n \in \mathbb{N}.$$

Because we have that $\sum \frac{1}{n^4}$ converges, we know that for all $\varepsilon > 0$ there exists some N be such that

$$\sup_{x\in[0,\infty)} \left| \sum_{p=n}^{q} f_n(x) \right| = \sup_{x\in[0,\infty)} \left| \sum_{p=n}^{q} \frac{1}{(x+n^2)^2} \right| \le \sum_{p=n}^{q} \frac{1}{n^4} \le \varepsilon \quad \text{for all } q > p > N.$$

This shows uniformly convergent on $[0,\infty)$ using the Cauchy criterion.

2.4 Exercise 2.4

Using Theorem 3.2, Exercise 2.1 and Exercise 2.3, we know that $f(x)$ is derivable on $[0,\infty)$ and $f'(x) = \sum_1^\infty f_n'(x)$. Using Corollary 2.4 and $\sum_1^\infty f_n'$ converges uniformly, we obtain $\sum_1^\infty f_n'(x)$ is continuous on $[0,\infty)$. We conclude that f is C^1 on $[0,\infty)$.