Quiz 1 2020

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1 Exercise 1

1. Recalling definition 2.7, if suffices to compute

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-nx}\frac{\pi}{\sin\pi\alpha}e^{i(\pi-x)\alpha} = \frac{1}{n+\alpha}, \quad n \in \mathbb{Z}.$$

2. Use Parseval's with $c_n = \frac{1}{n+\alpha}$ and $f(x) = \frac{\pi}{\sin\pi\alpha}e^{i(\pi-x)\alpha}$.

2 Exercise 2

We can prove inductively that for $f \in C^k$, $c_n(f^{(\ell)}) = (in)^\ell c_n(f)$ for all $\ell < k$. Then, applying Parseval's yields that

$$\sum_{n\in\mathbb{Z}}\left|(in)^\ell c_n(f)\right|^2 < \infty$$

which yields the result.

3 Exercise 3

1. Using u-substitution with $u = x + \frac{\pi}{n}$ and the fact that $1 + e^{i\pi} = 0$ yields the result.
2. Applying the H older condition to the formula for c_n derived in part 1,

$$c_n \le C\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\frac{\pi}{n}\right)^\alpha e^{inx}\,dx.$$

Since α is fixed,

$$\int_{-\pi}^{\pi}\pi^\alpha e^{inx}\,dx \in L^1$$

and thus, $c_n = O(1/|n|^\alpha)$.

3. To verify the H older condition, we write

$$\begin{aligned}|f(x+h)-f(x)| &\le \sum_{k=0}^{\infty}2^{-k\alpha}\left|e^{i2^kh}-1\right|\\ &\le \sum_{2^k|h|\le 1}2^{-k\alpha}2^k|h| + 2\sum_{2^k|h|>1}2^{-k\alpha}.\end{aligned}$$

This splitting follows from the fact that $|1-e^{ix}| \le |x|$ for sufficiently small x and that for any $x, y \in \mathbb{R}$, $|e^{ix}-e^{iy}| \le 2$.

Then, there exists some k_0, $2^{k_0}|h| > 1$, such that $\sum_{2^k|h|>1}2^{-k\alpha} = \sum_{k=k_0}^{\infty}2^{-k\alpha} = \frac{1}{1-2^{-\alpha}}2^{-k_0\alpha} \le \frac{1}{1-2^{-\alpha}}|h|^{-\alpha}$.

To calculate c_{2^k}, it is sufficient to confirm uniform convergence.

4 Exercise 4

1. It is a quick calculation to show that,

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx \\ &= \frac{\pi}{2}. \end{aligned}$$

and

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| e^{inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx \\ &= -\frac{1}{n\pi}\int_0^{\pi} \sin nx\,dx + \frac{1}{n\pi} x\sin nx\Big|_0^{\pi} \\ &= \frac{1}{n^2\pi}\cos nx\Big|_0^{\pi} \\ &= \frac{1}{n^2\pi}((-1)^n - 1). \end{aligned}$$

2. Evaluating the Fourier series for $|x|$ at $x = 0$ yields

$$0 = \frac{\pi}{2} - 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2},$$

which immediately shows the first inequality.

The second equality follows from writing

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

Using the hint, this is equivalent to:

$$\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2},$$

which yields the second equality.

3. Using Parseval's equality shows that

$$\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} + \frac{\pi^2}{4} = \frac{\pi^2}{3},$$

which reduces to the first equation.

The second equality follows from writing

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{n=1}^{\infty}\frac{1}{(2n)^4} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}.$$

This is equivalent to:

$$\frac{15}{16}\sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4},$$

which yields the second equality.

5 Exercise 5

1. The map is clearly invertible, since for $v(t) = C(t,s)y$, $v(s) = y$. Moreover, it is linear, as $C(t,s)y_1 + C(t,s)y_2$ solves

$$y'(t) = A(t)y(t), \quad y(s) = y_1 + y_2,$$

which is the defining equation for $C(t,s)(y_1 + y_2)$.

2. Plugging in $C(t,s)$ verifies that it is a solution.

3. By definition, the map $t \mapsto C(t,u)y_0$ is equal to $C(s,u)y_0$ at the point s, so

$$C(t,u)y_0 = C(t,s)(C(s,u)y_0) = (C(t,s)C(s,u))y_0.$$

Applying this, $C(t,s)C(s,t) = C(t,t) = Id$.

4. $C(t,s) = C(t,u)(C(s,u))^{-1}$. We conclude using the fact that $C(t,s)$ is continuous for fixed s.

5. First, it is immediate that $f(t_0) = y_0$. Then, checking that f satisfies the differential equation follows from an application of Leibniz's integration formula and the property in part 2.