Exercise Sheet 2

Exercise 1

We start by performing integration by parts on $f_n(x)$:

$$\begin{aligned} f_n(x) &= \int_a^x f(t)\cos(nt)\,dt \\ &= f(t)\frac{\sin(nt)}{n}\bigg|_a^x - \int_a^x f'(t)\frac{\sin(nt)}{n}\,dt \\ &= \frac{1}{n}\left(f(x)\sin(nx) - f(a)\sin(na) - \int_a^x f'(t)\sin(nt)\,dt\right). \end{aligned}$$

Then, since we know that $f : I \to \mathbb{R}$ is a C^1 function, we know that there exists an $M \in \mathbb{R}$ such that $\sup_{x\in I} |f(x)| + |f'(x)| \leq M$. Thus, we have that

$$\begin{aligned} |f_n(x)| &\leq \frac{1}{n}\left(2M + M\int_a^x |\sin(nt)|\,dt\right) \\ &\leq \frac{M}{n}\left(2 + b - a\right). \end{aligned}$$

Now let $\epsilon > 0$, and let $n > \frac{M(2+b-a)}{\epsilon}$. Then we have that $|f_n(x)| \leq \epsilon$. Thus, we see that the sequence of functions f_n converges uniformly to zero.

Exericse 2

Exercise 2.1

Assume for the sake of contradiction that f were not bounded but f_n is a sequence of functions that is uniformly bounded and converges uniformly to f. In particular, let us assume that for all $n \in \mathbb{N}$, $x \in \Omega$, we have that $|f_n(x)| \leq M$. Since f is unbounded, in particular there is an x_0 such that $f(x_0) > M + 1$.

Then, since we assumed that f_n converges uniformly, we also know that, for any $\epsilon \in \mathbb{R}$, there exists $n \in \mathbb{N}$,

$$|f_n(x_0) - f(x_0)| < \epsilon$$

Then, let $\epsilon = \frac{1}{2}$. But then this implies that, there exists $n \in \mathbb{N}$,

$$f_n(x_0) > M + \frac{1}{2}.$$

But this is a contradiction since we assumed that M was the uniform bound for $f_n(x)$.

Exercise 2.2

Let f be the limit of f_n and g be the limit of g_n. Now we show that $(f_n + g_n)(x)$ converges uniformly to $f + g$. Then, since f_n, g_n are uniformly convergent we have that there exists some $N > 0$ such that for all

$n > N$,

$$|f_n(x) - f(x)| < \frac{1}{2}\epsilon$$
$$|g_n(x) - g(x)| < \frac{1}{2}\epsilon$$

for any $\epsilon \in \mathbb{R}, x \in \Omega$. Now, we add the two inequalities above so that for all $n > N$,

$$|f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon.$$

Using the triangle inequality, we have that

$$|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|.$$

But by our inequalities above, we now have that for all $n > N$,

$$|f_n(x) + g_n(x) - (f(x) + g(x))| \leq \epsilon.$$

Thus, we have shown that $f_n + g_n$ is uniformly convergent and that in fact, $f_n + g_n$ uniformly converges to $f + g$.

Exercise 2.3

Let f be the limit of f_n and g be the limit of g_n. Then, we will show that fg is the uniform limit of $f_n g_n$. First we see that since f_n and g_n are both uniformly bounded, say by a constant $M \in \mathbb{R}$, we have that there exists some N_1 such that

$$f_n(x), g_n(x) \leq M \quad \forall x \in \Omega, n > N_1$$

Then since f_n converges uniformly to f and g_n converges uniformly to g, we know that there is some N_2 such that for all $n > N_2$

$$|f_n(x) - f| < \frac{\epsilon}{2M}$$
$$|g_n(x) - g| < \frac{\epsilon}{2M}.$$

Then, by applying the triangle inequality, we have that

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(g_n - g)| + |(f_n - f)g)| \\ &\leq M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} \\ &\leq \epsilon, \end{aligned}$$

for all $n > \max(N_1, N_2)$. Thus, we have shown that $f_n g_n$ converges uniformly to fg.

Exercise 2.4

Consider the sequences $f_n(x) = g_n(x) = x + \frac{1}{n}$. Then, we will prove that f_n (and thus g_n) uniformly converges to x, but that $(f_n g_n)(x) = x^2 + \frac{2x}{n} + \frac{1}{n^2}$ converges pointiwse to x^2, but not uniformly to x^2.

First we show that f_n is uniformly convergent to x. To see this, consider

$$|f_n(x) - x| = \frac{1}{n}.$$

Then, for $\epsilon > 0$, consider $N > \frac{1}{\epsilon}$. Then, for any $n > N$, we have that $|f_n(x) - x| < \epsilon$ and thus, $x + \frac{1}{n}$ converges uniformly to x.

Next, we show that $f_n g_n$ is pointwise convergent to x^2. To see this, consider some fixed x. Now,

$$\begin{aligned}|(f_n g_n)(x) - x^2| &= \left|\frac{2x}{n} + \frac{1}{n^2}\right| \\ &\leq \left|\frac{2x}{n}\right| + \left|\frac{1}{n^2}\right|.\end{aligned}$$

Then, it is not hard to see that the limit of the right hand side in the inequality above is 0, allowing us to conclude that indeed, $f_n g_n$ converges pointwise to x^2.

Finally, we show that $f_n g_n$ is not uniformly convergent. Assume for the sake of contradiction that $f_n g_n$ is uniformly convergent to x. Then, let $\epsilon < 1$. Since $f_n g_n$ is uniformly convergent, there must exist some N such that for all $n > N$,

$$|(f_n g_n)(x) - x^2| = \left|\frac{2x}{n} + \frac{1}{n^2}\right| \leq \epsilon.$$

Then, let $n > N$, $x > n$. But in this case,

$$|(f_n g_n)(x) - x^2| = \left|\frac{2x}{n} + \frac{1}{n^2}\right| > 2 > \epsilon.$$

Thus, we have a contradiction and indeed, $f_n g_n$ does not uniformly converge to x^2.

Exercise 3

Exercise 3.1

First, notice that $(-1)^{2k} a_{2k} + (-1)^{2k+1} a_{2k+1} > 0$, and similarly $(-1)^{2k+1} a_{2k+1} + (-1)^{2k} < 0$. Thus, $(S_{2p})_{p \in \mathbb{N}} = a_0 + \sum_{i=1}^{p} (-a_{2i-1} + a_{2i})$ is monotonic decreasing and bounded above by a_0. At the same time however, we see that $(S_{2p})_{p \in \mathbb{N}} = a_{2p} + \sum_{i=0}^{p-1} (a_{2i} - a_{2i+1})$ is bounded below by a_{2p}. Thus, since $a_n \geq 0$ for all n, we know that $(S_{2p}) \geq 0$ Thus, we know that $(S_{2p})_{p \in \mathbb{N}}$ is a monotonic and bounded sequence and thus, by monotonic convergence, is convergent. Now let S_1 be the limit of $(S_{2p})_{p \in \mathbb{N}}$.

To see that (S_{2p+1}) converges to the same limit, we can argue in a similar fashion as above to se that (S_{2p+1}) converges to some limit S_2. Then, we argue that $S_1 = S_2$. To see this, it is sufficient to show that $S_{2p+1} - S_{2p}$ converges to zero. But $S_{2p+1} - S_{2p} = -a_{2p+1}$. Since we know that a_k converges to 0, there exists some N such that for all $2n + 1 > N$, $a_{2p+1} < \epsilon$. Thus, we have shown that$S_{2p+1} - S_{2p}$ converges to 0, and thus, that $(S_{2p}), (S_{2p+1})$ have the same limit.

Exercise 3.2

If $n = 2p$ is even,

$$\begin{aligned}|S - S_n| &= |-a_{2p+1} + \sum_{i=p+1}^{\infty} (a_{2i} - a_{2i+1})| \\ &\leq a_{2p+1} = a_{n+1}.\end{aligned}$$

If $n = 2p + 1$ is odd,

$$\begin{aligned}|S - S_n| &= |a_{2p+2} + \sum_{i=p+1}^{\infty} (-a_{2i+1} + a_{2(i+1)})| \\ &\leq a_{2p+2} = a_{n+1}.\end{aligned}$$

Exercise 4

Notice that because $\cos(nx) \leq 1$, we have that $|f_n(x)| < \frac{1}{n^2}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Thus, we have that $\sum f_n$ is normally convergent. Using Theorem 2.8, we have that $\sum f_n$ is also uniformly convergent. Then, since each f_n is also continuous, we know by $\sum_{i=1}^{n} f_i$ is also continuous for each n. Then, since $f = \sum_{i=1}^{\infty} f_i$ is the limit of a uniformly convergent series of continuous functions, using Corollary 2.4, it must also be continuous.

Exericse 5

Exercise 5.1

Fix x, then because we have that $\sum \frac{1}{n^2}$ converges, we know that there exists some N be such that $\sum_{n=p}^{q} \frac{1}{n^2} < \frac{\epsilon}{|x|+1}$ for all $q > p > N$. Then, we have

$$\begin{aligned}\left|\sum_{n=p}^{q} \frac{x}{n^2+x^2}\right| &< |x|\left|\sum_{n=p}^{q} \frac{1}{n^2}\right| \\ &\leq \epsilon,\end{aligned}$$

for $n > N$. This shows pointwise convergence using the Cauchy criterion.

Exercise 5.2

Consider $x = p$. Then, $\frac{x}{n^2+x^2} = \frac{1}{\frac{n^2}{p}+p}$. For $p+1 \leq n \leq 2p$, $\frac{1}{\frac{n^2}{p}+p} \geq \frac{1}{5p}$.

Thus,

$$\sum_{n=p+1}^{2p} \frac{x}{n^2+x^2} \geq p \cdot \frac{1}{5p} \geq \frac{1}{5}.$$

To see that this implies that the series is not uniformly convergent, assume for the sake of contradiction that it were uniformly convergent. Then for $\epsilon < \frac{1}{5}$, there must exist some N such that for all $m > n > N$, $\sum_{i=n}^{m} \frac{x}{i^2+x^2} < \epsilon$. But then let $n = p+1 > N$, and $m = 2p$. Then, by the work we have done above, $\sum_{n=p+1}^{2p} \frac{x}{n^2+x^2} \geq \frac{1}{5} \geq \epsilon$, which is a contradiction.

Exercise 5.3

To see that the limit function $f = \sum_{n=1}^{\infty} \frac{x}{n^2+x^2}$ is continuous, consider

$$\begin{aligned}|f(x)-f(y)| &= \left|\sum_{n=1}^{\infty} \frac{x}{n^2+x^2} - \sum_{n=1}^{\infty} \frac{y}{n^2+y^2}\right| \\ &\leq \sum_{n=1}^{\infty}\left|\frac{x}{n^2+x^2} - \frac{y}{n^2+y^2}\right| \\ &\leq \sum_{n=1}^{\infty}\left|\frac{x-y}{n^2+x^2}\right| + \sum_{n=1}^{\infty}\left|\frac{y}{n^2+x^2} - \frac{x}{n^2+y^2}\right| + \sum_{n=1}^{\infty}\left|\frac{x-y}{n^2+y^2}\right| \\ &\leq 2\sum_{n=1}^{\infty}\left|\frac{x-y}{n^2}\right| + \sum_{n=1}^{\infty}\left|\frac{yn^2+y^3-xn^2-x^3}{(n^2+x^2)(n^2+y^2)}\right| \\ &\leq 3|x-y|\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}|x-y|\frac{x^2+xy+y^2}{(n^2+x^2)(n^2+y^2)} \\ &\leq 5|x-y|\sum_{n=1}^{\infty}\frac{1}{n^2}.\end{aligned}$$

Thus, for $|x-y| < \frac{\epsilon}{5}\sum_{n=1}^{\infty}\frac{1}{n^2}$, $|f(x)-f(y)| < \epsilon$.

Exercise 5.4

First we notice that for all $x \in \mathbb{R}$, $f_{n+1}(x) \leq f_n(x)$. Then, we can also see that

$$\begin{aligned}f_n(x) &= \frac{x}{n^2+x^2} \\ &= \frac{1}{\frac{n^2}{x}+x} \\ &\leq \frac{1}{2n}.\end{aligned}$$

Where we used Cauchy-Schwartz in the last inequality. Therefore, we can apply Proposition 2.9 to conclude that indeed, $\sum(-1)^n f_n$ is uniformly convergent.

Exercise 6

Exercise 6.1

We first show that f_n converges pointwise. For any x, consider $n > \frac{1}{x}$. Then, by the definition of $f_n(x)$, $f_n(x) = 0$. This shows that $f_n(x)$ converges pointwise to 0.

Now we show that f_n does not converge uniformly to 0. First we note that for even $n = 2k$, let $x_n = \frac{1}{\frac{1}{2}+2k}$. Then,

$$\sin^2\left(\frac{\pi}{x_n}\right) = \sin^2\left(\frac{\pi}{2} + 2\pi k\right) = 1.$$

Now, for the sake of contradiction, assume that f_n does converge uniformly to zero. Then, let $\epsilon = \frac{1}{2}$. Then, there exists some N such that $f_n(x) < \frac{1}{2}$ for all $n > N$ and all $x \in \mathbb{R}$. However, we already showed that for any even n, we can find an x_n such that $f_n(x) = 1$. Thus, f_n cannot converge uniformly to 0.

Exercise 6.2

We show that $\sum |f_n(x)|$ converges pointwise to $\sin^2\left(\frac{\pi}{x}\right)$ for $x > 0$, and 0 for $x \le 0$. For any $x < 0$, $f_n(x) = 0$ for all n. Thus, we automatically have that $\sum |f_n|$ converges pointwise to zero for non-positive x.

Now we consider the case with $x > 0$. In this case, there is exactly one n such that $\frac{1}{n+1} \le x \le \frac{1}{n}$. Thus, let $N(x)$ denote this n. Then, for each $x > 0$, for all $n > N(x)$, $f_n(x) = \sin^2\left(\frac{\pi}{x}\right)$.

Now we show that the series $\sum f_n$ does not converge uniformly. To see this, we use the Cauchy Criterion for uniform convergence. Assume for the sake of contradiction that $\sum f_n$ converges uniformly. Then consider some $\epsilon < 1$. By uniform convergence we must have that there exists some N such that for all $m > n > N$,

$$\left|\sum_{i=n}^{m} f_i(x)\right| < \epsilon < 1 \quad \forall x \in \mathbb{R}$$

But then as we calculated before, for $n = 2k$ even integer, $f_n(x_n) = 1$ where $x_n = \frac{1}{\frac{1}{2}+2k}$. In particular, it's clear that $\sum_{i=1}^{j} f_i(x_n) = 0$ for $j < 2k$, $\sum_{i=1}^{j} f_i(x_{2k}) = 1$ for $j > 2k$. Then, for $N < n < 2k < m$,

$$\left|\sum_{i=n}^{m} f_i(x_{2k})\right| = 1 > \epsilon$$

so we have a contradiction.

Exercise 7

We first note that expanding the term given in the hint yields

$$\sum_{n=1}^{q}\left(\sum_{k=1}^{n} a_k\right)(b_n - b_{n+1}) + \left(\sum_{n=1}^{q} a_n\right) b_{q+1} = \sum_{n=1}^{q} a_n b_n$$

Now, letting $f_n(x) = a_n$, $g_n(x) = b_n$ in the hint, we have that

$$\sum_{n=1}^{q} f_n(x) g_n(x) = \sum_{n=1}^{q}\left(\sum_{k=1}^{n} f_n(x)\right)(g_n - g_{n+1}) + \left(\sum_{n=1}^{q} f_n\right) g_{q+1}.$$

Now, to show that $\sum f_n g_n$ converges uniformly, we use the Cauchy condition for uniform convergence. Since we know that the partial sums $\sum_{n=1}^{N}$ are uniform bounded, we know that there exists an $M \in \mathbb{R}$ such that $\sup_{N\in\mathbb{N},x\in\Omega}\left|\sum_{n=1}^N f_n(x)\right| \le M$. In what follows, assume that $p < q$. Consider

$$\left|\sum_{n=1}^{p} f_n(x)g_n(x) - \sum_{n=1}^{q} f_n(x)g_n(x)\right| = \left|\sum_{n=p}^{q}\left(\sum_{k=1}^{n} f_k(x)\right)(g_n(x) - g_{n+1}(x))\right| + \left|\left(\sum_{n=1}^{p} f_n\right)g_{p+1}\right| + \left|\left(\sum_{n=1}^{q} f_n\right)g_{q+1}\right|$$
$$\le \sum_{n=p}^{q} M|g_n(x) - g_{n+1}(x)| + M|g_{p+1}(x)| + M|g_{q+1}(x)|.$$

Then, since $g_n(x)$ is decreasing for all $x \in \Omega$, then

$$\left|\sum_{n=1}^{p} f_n(x)g_n(x) - \sum_{n=1}^{q} f_n(x)g_n(x)\right| \le M\sum_{n=p}^{q}(g_n(x) - g_{n+1}(x)) + M|g_{p+1}(x)| + M|g_{q+1}(x)|.$$

But then this is exactly a telescoping series, so we actually have that

$$\left|\sum_{n=1}^{p} f_n(x)g_n(x) - \sum_{n=1}^{q} f_n(x)g_n(x)\right| \le M(g_p(x) - g_{q+1}(x)) + M|g_{p+1}(x)| + M|g_{q+1}(x)|.$$

Then, since g_n converges uniformly to zero on Ω, so there exists some N such that for all $n > N$, $|g_n(x)| < \frac{\epsilon}{4M}$, so we finally have that, for all $q > p > N$,

$$\left|\sum_{n=1}^{p} f_n(x)g_n(x) - \sum_{n=1}^{q} f_n(x)g_n(x)\right| \le \epsilon,$$

and thus we can conclude that $\sum f_n g_n$ converges uniformly on Ω.

Proof of summation by parts.

$$\sum_{n=1}^{q}\left(\sum_{k=1}^{n} a_k\right)(b_n - b_{n+1}) + \left(\sum_{n=1}^{q} a_n\right)b_{q+1} = \sum_{n=1}^{q} a_n b_n. \tag{1}$$

For two given sequences (a_n) and (b_n), with $n \in \mathbb{N}$. If we define $A_n = \sum_{k=1}^n a_k$, then we have for every $n > 1$, $a_n = A_n - A_{n-1}$ and

$$\begin{aligned}\sum_{n=1}^{q} a_n b_n &= \sum_{n=1}^{q}(A_n - A_{n-1})b_n \\ &= \sum_{n=1}^{q} A_n b_n - \sum_{n=1}^{q-1} A_n b_{n+1} = \sum_{n=1}^{q} A_n b_n + A_q b_{q+1} = \sum_{n=1}^{q}\left(\sum_{k=1}^{n} a_k\right)(b_n - b_{n+1}) + \left(\sum_{n=1}^{q} a_n\right)b_{q+1}.\end{aligned}$$

Exercise 8

Exercise 8.1

We use the notation $\phi_n(x) = \phi(x)$ to emphasize the relationship between ϕ and n. Given that $f_n(x)$ is continuous on $[0,n]$, ϕ_n is also continuous on the same interval. For each ϕ_n. Then, by the Extreme Value Theorem, we know that ϕ_n must achieve a maximum and a minimum on $[0,n]$. Then, it suffices to check that the maximum is not either of the end points. To see this, we calculate that

$$\phi_n(0) = 0 \qquad \phi_n(n) = e^{-n}.$$

Then, it suffices to show that $\phi_n(n)$ is not a maximum. To see this, we can calculate that

$$\phi_n'(n) = -e^{-n} < 0.$$

Thus, there must be some $\alpha \in (0,n)$ such that $\phi_n(\alpha)$ is a maximum by the Extreme Value Theorem. Since α is on the interior and $\phi_n(\alpha)$ is a maximum value, we know also that $\phi_n'(\alpha) = 0$. To see that $\phi_n(x)$ is bounded below by 0, we reference exercise 8.2, where we prove that there is a unique critical point in the interval $[0,n]$, which is a local maximum. The Extreme Value Theorem then tells us that the endpoints of $[0,n]$ must contain the minima of $\phi_n(x)$, that is, that $\phi_n(x) \ge 0$.

Exercise 8.2

We can calculate directly that

$$\phi_n'(x) = -e^{-x} + \left(1 - \frac{x}{n}\right)^{n-1}.$$

Thus, since we know that $\phi_n'(\alpha) = 0$,

$$e^{-\alpha} = \left(1 - \frac{\alpha}{n}\right)^{n-1}.$$

Substituting this into the equation for ϕ_n, we see that

$$\begin{aligned}\phi_n(\alpha) &= e^{-\alpha} - \left(1 - \frac{\alpha}{n}\right)^n \\ &= e^{-\alpha}\left(1 - \left(1 - \frac{\alpha}{n}\right)\right) \\ &= \frac{\alpha}{n}e^{-\alpha}.\end{aligned}$$

Then, since we know that xe^{-x} achieves a maximum value of e^{-1} at $x = 1$, we have here directly that

$$\phi_n(\alpha) < \frac{1}{ne}.$$

Exercise 8.3

We now consider $\epsilon > 0$. Then for $x \in [0, +\infty)$,

$$\begin{aligned}|e^{-x} - f_n(x)| &= \sup(|\phi(x)|, e^{-n}) \\ &\leq \sup(|\phi(\alpha)|, e^{-n}) \\ &\leq \sup(\frac{1}{en}, e^{-n}).\end{aligned}$$

Thus, for $N > \sup(\frac{1}{e\epsilon}, \ln\frac{1}{\epsilon})$, $|e^{-x} - f_n(x)| < \epsilon$ if $n > N$, which shows that $f_n(x)$ uniformly converges to e^{-x}.