Analysis 4 Problem Set 6

Allen Fang and Xu Yuan

1 Exercise 1

1.1 Exercise 1.1

We will show that the power series $\sum z^n$ and $\sum n^2 z^n$ both have radius of convergence $R = 1$.

We first recognize that $\sum z^n$ is just a geometric series, for which we know that the radius of convergence is 1. To calculate the radius of convergence for $\sum n^2 z^n$, consider $(n^2)^{\frac{1}{n}}$.

$$(n^2)^{\frac{1}{n}} = e^{2\frac{\log n}{n}}$$
$$\lim_{n\to\infty} (n^2)^{\frac{1}{n}} = 1.$$

Thus, the radius of convergence of $\sum n^2 z^n$ is also 1.

1.2 Exercise 1.2

By direct computation, we have

$$f(x) = \frac{1}{1-x}, \quad f'(x) = \frac{1}{(1-x)^2} \quad \text{and} \quad f''(x) = \frac{2}{(1-x)^3} \quad \text{for all } x \in (-1,1).$$

We take the derivatives term by term

$$f'(x) = \sum_{n=1}^{+\infty} n x^{n-1} \quad \text{and} \quad f''(x) = \sum_{n=2}^{+\infty} n(n-1)x^{n-2} \quad \text{for all } x \in (-1,1).$$

Then, we see that

$$\sum_{n=0}^{+\infty} n^2 x^n = x^2 f'' + x f' = 2x^2(1-x)^{-3} + x(1-x)^{-2} = \frac{x(1+x)}{(1-x)^3}.$$

2 Exericse 2

2.1 Exercise 2.1

Since

$$\lim_{n\to\infty} (2n+1)^{-\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(2n+1)^{\frac{1}{n}}} = \lim_{n\to\infty} e^{-\frac{\log(2n+1)}{n}} = 1.$$

Using the definition of radius of convergence, we obtain $R = 1$.

2.2 Exercise 2.2

Using the formula for $f(x)$, we can see directly that

$$g(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1} \quad \text{for all } x \in (-1,1).$$

Thus

$$g'(x) = \sum_{n=0}^{+\infty} x^{2n} = \frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) \quad \text{for all } x \in (-1,1). \tag{1}$$

Note that $g(0) = 0$ and using (1), we have

$$g(x) = g(0) + \int_0^x g'(s)ds = \frac{1}{2}\int_0^x \left(\frac{1}{1+s} + \frac{1}{1-s}\right) ds = \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right). \tag{2}$$

Using the definition of $g(x)$ and (2), we obtain

$$f(x) = \frac{g(\sqrt{x})}{\sqrt{x}} = \frac{\ln(1+\sqrt{x}) - \ln(1-\sqrt{x})}{2\sqrt{x}}.$$

3 Exercise 3

3.1 Exercise 3.1

First, it is clear that for $n = 0$, the relation holds since

$$f(x) = e^{-\frac{1}{x^2}},$$

and 1 is a polynomial.

Then, assume for the sake of induction that for $n \in \mathbb{N}$ there exists a polynomial P_n such that

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}.$$

Then,

$$\begin{aligned} f^{(n+1)}(x) &= -\frac{1}{x^2} P_n'\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} + \frac{2}{x^3} P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} \\ &= \left(-\frac{1}{x^2} P_n'\left(\frac{1}{x}\right) + \frac{2}{x^3} P_n\left(\frac{1}{x}\right)\right) e^{-\frac{1}{x^2}}. \end{aligned}$$

But then, if P_n is a polynomial, so is P_n', and clearly $\frac{2}{x^3}, \frac{1}{x^2}$ are polynomials in $\frac{1}{x}$. Using the fact that sums and products of polynomials are still polynomials (that the space of polynomials is closed under addition and multiplication), we see that we have shown that

$$f^{(n+1)}(x) = P_{n+1}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$$

where

$$P_{n+1}(\frac{1}{x}) = \left(-\frac{1}{x^2} P_n'\left(\frac{1}{x}\right) + \frac{2}{x^3} P_n\left(\frac{1}{x}\right)\right).$$

Then, we conclude by induction.

3.2 Exercise 3.2

Assume for the sake of contradiction that there was a power series approximation for $f(x) = \sum a_n x^n$. It is easy to calculate that $f^{(n)}(0) = 0$ for all n. But then, using Corollary 3.3 from the notes, we have that $a_n = 0$ for all n. However, this would imply that on the domain of convergence, i.e. a neighborhood of the origin, $f(x)$ is identically zero, which we know is false.

4 Exercise 4

4.1 Exercise 4.1

First, we claim following estimate
$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 1. \tag{3}$$
Note that
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\alpha-n}{n+1}\right| = 1.$$
Let $0 < \varepsilon < \frac{1}{2}$. We know that there exists $N \in \mathbb{N}$ such that, for each $n > N$, $1-\varepsilon \le \left|\frac{a_{n+1}}{a_n}\right| < 1+\varepsilon$. Now, for $n > N$, we remark that $|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \cdot \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \cdot |a_N|$. We hence get $|a_N|(1-\varepsilon)^{n-N} < |a_n| < |a_N|(1+\varepsilon)^{n-N}$ for $n > N$. In other words, for all $n > N$, we have
$$\sqrt[n]{|a_N|(1-\varepsilon)^{-N}}(1-\varepsilon) < \sqrt[n]{|a_n|} < \sqrt[n]{|a_N|(1+\varepsilon)^{-N}}(1+\varepsilon).$$
Let $n \to +\infty$, the left side converges to $1-\varepsilon$ and the right side to $1+\varepsilon$, which concludes (3). Thus, the radius of convergence of the series is 1.

4.2 Exercise 4.2

From the definition of a_n, we have
$$\begin{aligned}(n+1)a_{n+1} + na_n &= \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\cdot(n+1) + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\cdot n \\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\left(\frac{\alpha-n}{n+1}\cdot(n+1)+n\right) = \alpha a_n. \end{aligned} \tag{4}$$
By direct computation, we obtain
$$\begin{aligned}(1+z)\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n &= \sum_{n=0}^{+\infty}(n+1)a_{n+1}z^n + \sum_{n=0}^{+\infty} na_n z^n \\ &= \sum_{n=0}^{+\infty}\left((n+1)a_{n+1}+na_n\right)z^n = \sum_{n=0}^{+\infty} b_n z^n.\end{aligned}$$
Thus we have
$$b_n = (n+1)a_{n+1} + na_n \quad \text{for all } n \in \mathbb{N}. \tag{5}$$
Using (4) and (5), we obtain $b_n = \alpha a_n$.

4.3 Exercise 4.3

From Exercise 4.2 and Proposition 3.1 in Lecture notes, we know that for all $x \in (-1,1)$,
$$g'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \frac{1}{1+x}\sum_{n=0}^{+\infty} b_n x^n = \frac{1}{1+x}\sum_{n=0}^{+\infty}\alpha a_n x^n = \frac{\alpha}{1+x}g(x).$$

4.4 Exercise 4.4

From solving the differential equation, we see that $y(x) = (1+x)^{\alpha} + C$. Note that $g(0) = 0$, we obtain $g(x) = (1+x)^{\alpha}$ for all $x \in (-1,1)$.

5 Exercise 5

5.1 Exercise 5.1

Since $u_0 = 0$ and $u_1 = 1$, it is trivial to see that

$$|u_0| = 0 \leq (2M)^{0-1} \quad \text{and} \quad |u_1| = 1 \leq (2M)^{1-1}.$$

Now we assume for the sake of (strong) induction, that for all $1 \leq i \leq n$,

$$|u_n| \leq (2M)^{n-1}.$$

Then from the definition of M, we obtain

$$\begin{aligned} |u_{n+1}| &= |au_n + bu_{n-1}| \\ &\leq |a|(2M)^{n-1} + |b|(2M)^{n-2} \\ &\leq (2M)^n. \end{aligned}$$

Then we can conclude by strong induction.

5.2 Exercise 5.2

From Exercise 5.1, we have

$$\begin{aligned} \limsup_{n\to\infty} |u_n|^{\frac{1}{n}} &\leq \lim_{n\to\infty} (2M)^{1-\frac{1}{n}} \\ &= 2M. \end{aligned}$$

Thus, since $|a|, |b|$ are finite, $R \neq 0$

5.3 Exercise 5.3

By direct computation, we have

$$\begin{aligned} (1 - ax - bx^2) \sum_{n=0}^{+\infty} u_n x^n &= \sum_{n=0}^{+\infty} u_n x^n - \sum_{n=0}^{+\infty} a u_n x^{n+1} - \sum_{n=0}^{+\infty} b u_n x^{n+2} \\ &= u_0 + u_1 x - a u_0 x + \sum_{n=2}^{+\infty} (u_n - a u_{n-1} - b u_{n-2})\, x^n \end{aligned} \tag{6}$$

From $u_0 = 0$, $u_1 = 1$ and for $n \geq 0$, $u_{n+2} = au_{n+1} + bu_n$ and (6), we obtain

$$\sum_{n=0}^{+\infty} u_n x^n = \frac{x}{1 - ax - bx^2} \quad \text{for all } x \in (-R, R).$$

6 Exercise 6

6.1 Exercise 6.1

Fix some large M. From $\sum_{n=0}^{+\infty} b_n$ divergent, we obtain there exists $N \in \mathbb{N}$ such that $\sum_{n=0}^{N} b_n > M$. From $b_n > 0$ for all $n \in \mathbb{N}$, we have

$$\liminf_{x\to 1} \sum_{n=0}^{+\infty} b_n x^n \geq \liminf_{x\to 1} \sum_{n=0}^{N} b_n x^n = \sum_{n=0}^{N} b_n > M.$$

Let $M \to +\infty$, we are done.

6.2 Exercise 6.2

Fix $\epsilon > 0$. Then, we know that there exists N such that $\left|\frac{a_n}{b_n} - \ell\right| < \frac{\epsilon}{2}$ for $n > N$. Then for all $0 < x < 1$,

$$\begin{aligned}\left|\frac{\sum a_n x^n}{\sum b_n x^n} - \ell\right| &\leq \left|\frac{\sum_{n=0}^{N}(a_n - \ell b_n)x^n}{\sum b_n x^n}\right| + \left|\frac{\sum_{n=N+1}^{+\infty}(a_n - \ell b_n)x^n}{\sum b_n x^n}\right| \\ &\leq \frac{\sum_{n=0}^{N}\left(|a_n| + |\ell b_n|\right)}{\sum b_n x^n} + \frac{\epsilon}{2}\left|\frac{\sum_{n=N+1}^{+\infty} b_n x^n}{\sum b_n x^n}\right| \\ &\leq \frac{\sum_{n=0}^{N}\left(|a_n| + |\ell b_n|\right)}{\sum b_n x^n} + \frac{\epsilon}{2}.\end{aligned}$$

Using $\sum_{n=0}^{+\infty} b_n x^n \to +\infty$ as $x \to 1$, we obtain

$$\limsup_{x\to 1}\left|\frac{\sum a_n x^n}{\sum b_n x^n} - \ell\right| \leq \frac{\epsilon}{2}.$$

Let $\epsilon \to 0$, we obtain $\frac{\sum a_n x^n}{\sum b_n x^n} \to \ell$ as $x \to 1$.

6.3 Exercise 6.3

From Proposition 2.4 and Proposition 4.1 in Lecture notes, we obtain

$$\frac{\sum_{n=0}^{+\infty} a_n x^n}{1-x} = \sum_{n=0}^{+\infty} A_n x^n \quad \text{and} \quad \frac{\sum_{n=0}^{+\infty} b_n x^n}{1-x} = \sum_{n=0}^{+\infty} B_n x^n. \tag{7}$$

Using (7), Exercise 6.2 and $\frac{A_n}{B_n} \to \ell$ as $n \to +\infty$, we obtain

$$\frac{\sum_{n=0}^{+\infty} a_n x^n}{\sum_{n=0}^{+\infty} b_n x^n} = \frac{\sum_{n=0}^{+\infty} a_n x^n}{1-x} \cdot \left(\frac{\sum_{n=0}^{+\infty} b_n x^n}{1-x}\right)^{-1} = \frac{\sum_{n=0}^{+\infty} A_n x^n}{\sum_{n=0}^{+\infty} B_n x^n} \to \ell \quad \text{as } x \to 1.$$

6.4 Exercise 6.4

Let us denote by $S_n = \sum_{k=0}^{n} A_k$. Then, since

$$\sum_{n=0}^{\infty} S_n x^n = (1-x)^{-1}\sum_{n=0}^{\infty} A_n x^n = (1-x)^{-2}\sum_{n=0} a_n x^n,$$

we want to prove that

$$\lim_{x\to 1}(1-x)^2\sum_{n=0}^{\infty} S_n x^n = \ell.$$

Using the fact that $\sum(n+1)x^n = (1-x)^{-2}$, we can write the following two identities:

$$\begin{aligned}(1-x)^2\sum_{n=0}^{\infty} S_n x^n &= \sum_{n=0}^{\infty}(n+1)\frac{S_n}{n+1}(1-x)^2 x^n \\ 1 &= \sum_{n=0}^{\infty}(n+1)(1-x)^2 x^n.\end{aligned}$$

Thus, we have that

$$\begin{aligned}\left|\sum_{n=0}^{\infty} a_n x^n - \ell\right| &\leq \left|(1-x)^2\sum_{n=0}^{\infty} S_n x^n - \ell\right| \\ &\leq \sum_{n=0}^{N}(n+1)(1-x)^2 x^n\left|\frac{S_n}{n+1} - \ell\right| + \sum_{n=N+1}^{\infty}(n+1)(1-x)^2 x^n\left|\frac{S_n}{n+1} - \ell\right|.\end{aligned}$$

Then, since we know that $\lim_{n\to\infty}\frac{S_n}{n+1} = \ell$, we can choose some N such that for all $n > N$, $\left|\frac{S_n}{n+1} - \ell\right| < \frac{\epsilon}{2}$.

Then, the second term in the inequality above is less thatn $\frac{\epsilon}{2}$ by choice of N independent of x. The first term can be controlled by having x sufficiently close to 1, and we are done.

6.5 Exercise 6.5

Consider $\sum_{x=0}^{\infty} x^{a^n}$. Then from Taylor series, we know that

$$-\ln(1-x) = \sum_{x=0}^{\infty} \frac{x^n}{n}.$$

We will use Exercise 6.3 to prove that

$$\frac{\sum_{n=0}^{\infty} x^{a^n}}{\sum_{x=0}^{\infty} \frac{x^n}{n}} = (\ln a)^{-1}.$$

Then, it is not hard to see that $(\log_a n) - 1 \leq A_n \leq \log_a n$. On the other hand, $B_n \sum_{k=0}^{n} \frac{1}{k}$, and thus $\log n \leq B_n \leq \log n + 1$.

Then, we can calculate that

$$\begin{aligned}
\lim_{n\to\infty} \frac{A_n}{B_n} &\leq \lim_{n\to\infty} \frac{\log_a n - 1}{\log n} \\
&= \lim_{n\to\infty} \frac{1}{\log a} - \frac{1}{\log n} \\
&= \frac{1}{\log a}. \\
\lim_{n\to\infty} \frac{A_n}{B_n} &\geq \lim_{n\to\infty} \frac{\log_a n}{\log n + 1} \\
&= \lim_{n\to\infty} \frac{1}{\log a} \frac{\log n}{\log n + 1} \\
&= \frac{1}{\log a}.
\end{aligned}$$

Hence, by applying Exercise 6.4, we are done.

Now consider $\sum_{x=0}^{\infty} (-1)^n x^{4n+1}$. Then, let $C_n = \frac{1}{n} \sum_{k=0}^{n-1} A_k$. Then, it is not hard to calculate that

$$\begin{aligned}
C_{4n} &= \frac{1}{2} \\
C_{4n+j} &= \frac{1}{2} + \frac{j}{4n+j}, \quad 1 \leq j \leq 3.
\end{aligned}$$

Then, $\lim_{n\to\infty} C_n = \frac{1}{2}$. Thus, using exercise 6.4, we see that

$$\sum_{x=0}^{\infty} (-1)^n x^{4n+1} \to \frac{1}{2}$$

as $x \to 1$ from below.