Week 4, March 7th: Series of functions

Instructor: Cécile Huneau (cecile.huneau@polytechnique.edu)
Tutorial Assistants:
– Allen Fang (groups ?, allen.fang@sorbonne-universite.fr)
– Yuan Xu (groups ?, xu.yuan@polytechnique.edu)

1 Important exercises

Exercise 1. Let $f_n : \mathbb{R} \to \mathbb{R}$, defined by $f_n(x) = e^{-xn^2}$.

1. Let $a > 0$. Show that the series $\sum f_n$ is normally convergent on $[a, +\infty[$, and pointwise convergent on $]0, +\infty[$. We write $f = \sum_{n=0}^{\infty} f_n$.
2. Show that f is continuous and decreasing on $]0, \infty[$. Calculate $\lim_{x\to+\infty} f(x)$.
3. Show that the function f is not bounded, and calculate $\lim_{x\to 0} f(x)$.
4. Let $x > 0$. Show that $\int_0^\infty e^{-xt^2} dt < \infty$ and
$$\int_0^\infty e^{-xt^2} dt = \frac{1}{\sqrt{x}} \int_0^\infty e^{-t^2} dt.$$
5. Show that for all $n \geq 0$ and $x > 0$ we have
$$e^{-xn^2} + \int_0^n e^{-xt^2} dt \leq \sum_{k=0}^{n} e^{-xk^2} \leq 1 + \int_0^n e^{-xt^2} dt.$$
Deduce that $\lim_{x\to 0} \sqrt{x} f(x) = \int_0^\infty e^{-t^2} dt$.

Exercise 2. Let $H : \mathbb{R} \to \mathbb{R}$ be defined by $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 1$. Let x_n be a sequence of distinct points of $]a, b[$ and $\sum \alpha_n$ an absolutely convergent numerical series.

1. Show that the series $\sum \alpha_n H(x - x_n)$ converges uniformly on $]a, b[$. We note f the limit.
2. Show that f is continuous for all $x \neq x_n$.

Exercise 3. Show that if f is continuous on $[0, 1]$ and if $\int_0^1 f(x)x^n dx = 0$ for all $n \in \mathbb{N}$, then $f(x) = 0$ for all $x \in [0, 1]$.

Tip : Use Weierstrass theorem to prove that $\int_0^1 f^2(x) dx = 0$.

Exercise 4. **Approximation by Bernstein polynomials.**

For all continuous function $f : [0, 1] \to \mathbb{C}$ and $n \in \mathbb{N}$ we note
$$B_n(f) : [0, 1] \to \mathbb{C}, \quad x \mapsto \sum_{k=0}^{n} f(\frac{k}{n}) b_n^k(x),$$
where $b_n^k(x) = C_n^k x^k (1 - x)^{n-k}$.

1. Calculate $B_n(1)$, $B_n(x)$ and $B_n(x^2)$.

2. Give a simplified expression for $\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 b_n^k$ and show that for all $\eta > 0$ and $x \in [0,1]$ we have

$$\sum_{k,\,|\frac{k}{n}-x|\geq\eta} b_n^k(x) \leq \frac{1}{nk^2}.$$

3. Show that for all continuous function $f : [0,1] \to \mathbb{C}$, $B_n(f)$ converges uniformly to f on $[0,1]$, and deduce Weierstrass theorem.

2 More involved exercises

Exercise 5. **Second theorem of Dini.**

Let $f_n : [a,b] \to \mathbb{R}$. Assume that for all n, f_n is continuous and increasing, and that the sequence (f_n) converges pointwise to a function f which is continuous. Show that the convergence is uniform.

Tip : We recall Heine's theorem : a function which is continuous on a compact interval $[a,b]$ is uniformly continuous.

Exercise 6.

1. Let $f_n : [a,b] \to \mathbb{R}$. We assume that there exists K such that for all n, the function f_n is K-Lipschitz continuous. Show that pointwise convergence on $[a,b]$ implies uniform convergence.

2. Let $f_n :]a,b[\to \mathbb{R}$ be a sequence of convex functions, which converges pointwise to a function f. Show that (f_n) is uniformly convergent on all segment $[a',b'] \subset]a,b[$. Do we have that (f_n) converges uniformly on $]a,b[$?

Tip : Consider the sequence $f_n :]0,1[\to \mathbb{R}$, $x \mapsto x^n$.