Quiz 1B

1 Exercise 1

For $n \geq 1$ define $f_n : \mathbb{R} \to \mathbb{R}$, $x \to \frac{x}{1+nx^2}$.

1. Show that (f_n) is pointwise convergent. We note $f(x) = \lim_{n\to\infty} f_n(x)$.
2. Show that (f_n) converges uniformly to f.
3. Show that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

2 Exercise 2

For $n \geq 1$ define $f_n : [0, \frac{\pi}{2}] \to \mathbb{R}$, $x \to \frac{\cos nx}{n^2}$.

1. Show that the series $\sum_1^\infty f_n$ converges uniformly on $[0, \frac{\pi}{2}]$.
2. Take as given that

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} \qquad \text{for } 0 \leq x \leq \frac{\pi}{2}.$$

Show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. Show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos nx}{n^2} dx = \frac{(-1)^{k+1}}{(2k-1)^3} \quad \text{for } n = 2k-1$$

and

$$\int_0^{\frac{\pi}{2}} \frac{\cos nx}{n^2} dx = 0 \quad \text{for } n = 2k$$

where $k \in \mathbb{N}$.

4. Show that

$$\int_0^{\frac{\pi}{2}} \left(\frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} \right) dx = \frac{\pi^3}{32}.$$

5. Conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$