Week 3, February 21th: Series of functions

Instructor: Cécile Huneau (cecile.huneau@polytechnique.edu)
Tutorial Assistants:
– Allen Fang (groups ?, allen.fang@sorbonne-universite.fr)
– Yuan Xu (groups ?, xu.yuan@polytechnique.edu)

1 Important exercises

Exercise 1. For $n \geq 1$, we define $f_n : [0, \infty[\to \mathbb{R}, \quad x \mapsto \frac{x}{n(x+n)}$.

1. Show that the series $\sum f_n$ is pointwise convergent. We note $f = \sum_{n=1}^{\infty} f_n$.
2. Show that for all $a > 0$, the series $\sum f_n$ is normally convergent on $[0, a]$. Deduce that f is continuous on $[0, \infty[$.
3. Show that f is an increasing function.
4. Show that for all integers $1 \leq p \leq N$
$$\sum_{n=1}^{N} \frac{p}{n(p+n)} = \sum_{n=1}^{p} \frac{1}{n} - \sum_{n=N+1}^{N+p} \frac{1}{n},$$
and calculate $f(p)$ for each integer $p \geq 1$.
5. Deduce that $\lim_{x \to +\infty} f(x) = +\infty$.
6. Show that the series $\sum \frac{f_n}{x}$ is normally convergent on $[0, \infty[$ and that $\lim_{x \to +\infty} \frac{f(x)}{x} = 0$.

Exercise 2. For $n \geq 1$ define $f_n : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \ln\left(1 + \frac{x^2}{n^2}\right)$.

1. Show that the series $\sum f_n$ is pointwise convergent. We note $f = \sum_{n=1}^{\infty} f_n$.
2. Show that f is continuous.
3. Show that f is even, that f is increasing on $[0, +\infty[$ and that $\lim_{x \to \infty} f(x) = +\infty$.
4. Show that the series of functions $\sum f_n'$ is normally convergent on all segment of $\mathbb{R}$ and that f is derivable.

Exercise 3. For $n \geq 1$ define $f_n : [0, +\infty[\to \mathbb{R}, \quad x \mapsto \frac{(-1)^n}{n(1+nx)}$.

1. Show that the series $\sum f_n$ is uniformly convergent on $[0, +\infty[$. Define $f = \sum_{n=1}^{\infty} f_n$.
2. Show that f is continuous.
3. Calculate $\lim_{x \to 0} f(x)$ and $\lim_{x \to +\infty} f(x)$

Exercise 4. Let I be a bounded interval of $\mathbb{R}$ and $a \in I$. Let $h_n : I \to \mathbb{R}$ be continuous functions such that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $h : I \to \mathbb{R}$.

1. Show that h is continuous.

2. For all $n \in \mathbb{N}$ we define $H_n : I \to \mathbb{R}$ by $H_n(x) = \int_a^x h_n(t)dt$. Show that the sequence of functions (H_n) converges uniformly to the function $H(x) = \int_a^x h(t)dt$.

3. We assume that now that I is an open interval and that (f_n) is a sequence of C^1 functions $I \to \mathbb{R}$. We assume that the sequence of functions (f_n') converges uniformly to a function $g : I \to \mathbb{R}$ and that the sequence $(f_n(a))_{n\in\mathbb{N}}$ has a limit $l \in \mathbb{R}$.

   (a) Show that g is continuous.

   (b) Show that the sequence f_n converges uniformly to the functions $f : x \mapsto l + \int_a^x g(t)dt$. Deduce that f is C^1 and that $g' = f$.

2 More involved exercises

Exercise 5. Let f_n and g be continuous functions on $]0,\infty[$. We assume that $|f_n| \le |g|$ and that (f_n) converges uniformly on all compact set of $]0,\infty[$ to a function f. We assume also that $\int_0^\infty g(x)dx < \infty$. Show that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx.$$

Exercise 6. Let (a_n) be a decreasing sequence in $\mathbb{R}$ converging to zero.

1. Let $0 < x < 2\pi$ and $z = \cos(x) + i\sin(x)$. Show that for all $0 \le p < q$

$$|a_p z^p + a_{p+1} z^{p+1} + \ldots + a_q z^q| \le \frac{a_p}{\sin(\frac{x}{2})}.$$

2. Deduce that the series of function $\sum a_n \sin(nx)$ is pointwise converging on $]0,2\pi[$. Let $f(x) = \sum_{n=1}^\infty a_n \sin(nx)$.

3. Let $0 < u < \pi$. Show that the series $\sum a_n \sin(nx)$ is uniformly convergent on $[u, 2\pi - u]$. Deduce that f is continuous on on $]0,2\pi[$.