# Quiz 1B solution

## 1 Exercise 1

### 1.1 Exercise 1.1

For $x = 0$, it is clear that $f_n(0) = 0$ for all $n \in \mathbb{N}$. For $x \neq 0$, we have

$$|f_n(x)| = \left|\frac{1}{x^{-1} + nx}\right| \leq \frac{2}{\sqrt{n}} \to 0. \tag{1}$$

We conclude that $(f_n)$ is pointwise convergent.

### 1.2 Exercise 1.2

Using (1) and $f_n(0) = 0$ for all $n \in \mathbb{N}$, we have

$$\sup_{x \in \mathbb{R}} |f_n(x)| = \sup_{x \neq 0} |f_n(x)| \leq \frac{2}{\sqrt{n}} \to 0.$$

We conclude that $(f_n)$ converges uniformly to 0.

### 1.3 Exercise 1.3

By direct computation, we have

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2} \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

Let $n \to \infty$ in (2), we obtain

$$\lim_{n \to \infty} f_n'(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

We conclude that $f'(x) = \lim_{n \to \infty} f_n'(x)$ is correct if $x \neq 0$, but false if $x = 0$.

## 2 Exercise 2

### 2.1 Exercise 2.1

Because we have that $\sum \frac{1}{n^2}$ converges, we know that for all $\varepsilon > 0$ there exists some $N$ be such that

$$\sup_{x \in [0, \frac{\pi}{2}]} \left|\sum_{n=p}^{q} f_n(x)\right| \leq \sum_{n=p}^{q} \frac{1}{n^2} \leq \varepsilon \quad \text{for all } q > p > N.$$

This shows uniformly convergent on $[0, \frac{\pi}{2}]$ using the Cauchy criterion.

## 2.2 Exercise 2.2

Let $x = 0$ in identity

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}. \tag{3}$$

We obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

## 2.3 Exercise 2.3

For $n = 2k - 1$, using integral by parts,

$$\int_0^{\frac{\pi}{2}} \frac{\cos nx}{n^2} dx = \frac{1}{(2k-1)^2} \int_0^{\frac{\pi}{2}} \cos\left((2k-1)x\right) dx = \frac{1}{(2k-1)^3} \int_0^{k\pi - \frac{\pi}{2}} \cos x dx = \frac{(-1)^{k+1}}{(2k-1)^3}. \tag{4}$$

For $n = 2k$, using similar argument,

$$\int_0^{\frac{\pi}{2}} \frac{\cos nx}{n^2} dx = \frac{1}{(2k)^2} \int_0^{\frac{\pi}{2}} \cos\left((2k)x\right) dx = \frac{1}{(2k)^3} \int_0^{k\pi} \cos x dx = 0. \tag{5}$$

## 2.4 Exercise 2.4

By direct computation, we have

$$\int_0^{\frac{\pi}{2}} \left( \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} \right) dx = \left( \frac{x^3}{12} - \frac{\pi x^2}{4} + \frac{\pi^2 x}{6} \right) \Bigg|_0^{\frac{\pi}{2}} = \frac{\pi^3}{96} - \frac{\pi^3}{16} + \frac{\pi^3}{12} = \frac{\pi^3}{32}. \tag{6}$$

## 2.5 Exercise 2.5

From Proposition 3.1, (3) and the series $\sum_1^\infty f_n$ converges uniformly on $[0, \frac{\pi}{2}]$, we obtain

$$\sum_{n=1}^{\infty} \int_0^{\frac{\pi}{2}} \frac{\cos nx}{n^2} dx = \int_0^{\frac{\pi}{2}} \left( \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} \right). \tag{7}$$

Using (4), (5), (6) and (7), we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$