Week 12, May 16th: Differential equations

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1 Important exercises

Exercise 1. Solve the following differential equations.

1. $y' = te^y$,
2. $y'' + 3(y')^2 = 0$,
3. $y' = \sqrt{y}t^2$.

Exercise 2. We consider the differential equation $y' = ty^2 - y$.

1. Show that there exists a unique maximal solution $\phi_r :]a, b[\to \mathbb{R}$ such that $\phi_r(0) = r$.
2. What is ϕ_0 ? Show that ϕ_1 is the function $]-1, +\infty[\to \mathbb{R}$, $t \mapsto \frac{1}{1+t}$.
3. For $0 < r < 1$ we consider $\phi_r :]a, b[\to \mathbb{R}$. Show that for all $a < t < b$ we have $0 \leq \phi_r(t) \leq \frac{1}{1+t}$.
4. Assume that $b < \infty$. Show that the solution ϕ_r can be extended to a solution defined on a strictly bigger interval.
5. Show that fot all $0 < r < 1$ the interal of definition of the maximal solution is $]-1, +\infty[$.

Exercise 3. Let $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ be two continuous functions. We consider the Cauchy problem $y' = a(t)y + b(t)y^\alpha$, $y(t_0) = y_0$.

1. Discuss, depending on the choice of α, what is the domain of the equation.
2. In the following, we assume $\alpha \in]0 + \infty[\backslash \{1\}$ and $y_0 > 0$. Let $\phi : I \to \mathbb{R}$ be the maximal solution. Show that $\phi(t) > 0$ for all $t \in I$.
3. We set $\psi = \phi^{1-\alpha}$. What is the differential equation satisfied by ψ ?
4. Give a formula for ϕ, and determine the interval I.
5. Let $c : \mathbb{R} \to \mathbb{R}$ be an other continuous function. We consider now the differential equation $y' = a(t)y + b(t)y^2 + c(t)$. Assume that we know a particular solution $\phi_0 : \mathbb{R} \to \mathbb{R}$. Determine the set of solutions.

2 More involved exercises

Exercise 4. We consider the differential equation $y' + y + y^2 + 1 = 0$.

1. Look for a particular constant solution.

2. Determine all the maximal solutions wich are real valued.

Exercise 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We assume that the differential equation $y' = f(y)$ has a solution $\phi : \mathbb{R} \to \mathbb{R}$ which is bounded. Show that there exists t_0 such that $f(t_0) = 0$.