Midterm exam - April 2nd - Duration : 1h30

No document allowed. All electronic devices forbidden.

Exercise 1.

Put $P_0 = 0$, and define, for all $n \in \mathbb{N}$

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

1. Show that for all n, $P_n(x)$ is a polynomial.

2. Show that for all $n \in \mathbb{N}$ and $x \in [-1,1]$,

$$|x| - P_{n+1}(x) = (|x| - P_n(x))\left(1 - \frac{|x| + P_n(x)}{2}\right).$$

3. Deduce that, for all $n \in \mathbb{N}$ and $x \in [-1,1]$,

$$0 \le P_n(x) \le P_{n+1}(x) \le |x|.$$

4. Show that

$$|x| - P_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n \quad \text{for all } x \in [-1,1] \text{ and } n \in \mathbb{N}.$$

5. Show that

$$x\left(1 - \frac{x}{2}\right)^n < \frac{2}{n+1} \quad \text{for all } x \in [0,1] \text{ and } n \in \mathbb{N}.$$

6. Conclude that (P_n) converges uniformly to $|x|$ on $[-1,1]$.

Solution of exercise 1.

1. We proceed by induction. P_0 is a polynomial and if P_n is a polynomial, then $P_n(x) + \frac{x^2 - P_n^2(x)}{2}$ is also a polynomial.

2. We calculate

$$|x| - P_{n+1}(x) = |x| - (P_n(x) + \frac{x^2 - P_n^2(x)}{2}) = |x| - P_n(x) - \frac{1}{2}(|x| - P_n(x))(|x| + P_n(x)) = (|x| - P_n(x))\left(1 - \frac{|x| + P_n(x)}{2}\right).$$

3. We proceed by induction to prove that $P_n(x) \le |x|$. It is true for $n = 0$, and if $n \in \mathbb{N}$ and $P_n(x) \le |x|$ then $\frac{|x| + P_n(x)}{2} \le 1$ for $|x| \le 1$, and consequently $|x| - P_{n+1}(x)$ and $|x| - P_n(x)$ have the same sign, which

is ≥ 0, so we have proven $P_n(x) \leq |x|$. Moreover, since

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}$$

we have $P_{n+1}(x) \geq P_n(x)$. This implies in particular $P_n(x) \geq P_0(x) = 0$.

4. We have

$$|x| - P_{n+1}(x) \leq (|x| - P_n(x))(1 - \frac{|x|}{2}) \leq .. \leq (|x| - P_0(x))(1 - \frac{|x|}{2})^n = |x|(1 - \frac{|x|}{2})^n.$$

5. We study the function $f(x) = x\left(1 - \frac{x}{2}\right)^n$. We have $f'(x) = \left(1 - \frac{x}{2}\right)^{n-1}(1 - \frac{x}{2} - \frac{nx}{2}) = \left(1 - \frac{x}{2}\right)^{n-1}(1 - \frac{(n+1)x}{2})$ so $f'(x) \geq 0$ for $0 \leq x \leq \frac{2}{n+1}$ and $f'(x) \leq 0$ for $\frac{2}{n+1} \leq x \leq 1$. Moreover we have $f(0) = 0$, $f(\frac{2}{n+1}) = \frac{2}{n+1}\left(1 - \frac{1}{n+1}\right)^n$ and $f(1) = \frac{1}{2^n}$. Consequently for $n \geq 1$ $f(x) \leq f(\frac{2}{n+1}) \leq \frac{2}{n+1}$.

6. Let $\varepsilon > 0$. There exists N such that for all $n \geq N$ we have $\frac{2}{n+1} \leq \varepsilon$. Consequently, for all $n \geq N$ and all $x \in [-1, 1]$ we have

$$||x| - P_n(x)| \leq \varepsilon,$$

which means that $P_n(x)$ converges uniformly to $|x|$.

□

Exercise 2. We recall the definition of the ζ function : for $x \in]-1, +\infty[$, $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$.

1. Show that for all $n \geq 1$ we have

$$\int_0^\infty t^2 e^{-nt} dt = \frac{2}{n^3}.$$

2. Deduce the formula

$$\int_0^\infty \frac{t^2}{e^t - 1} dt = 2\zeta(3).$$

Solution of exercise 2.

1. We do integration by parts

$$\begin{aligned}
\int_0^R t^2 e^{-nt} dt &= \int_0^R \frac{2t}{n} e^{-nt} dt + [-\frac{t^2}{n} e^{-nt}]_0^R \\
&= \int_0^R \frac{2}{n^2} e^{-nt} dt + [\frac{2t}{n^2} e^{-nt}]_0^R + [-\frac{t^2}{n} e^{-nt}]_0^R \\
&= \frac{2}{n^3} - \frac{2}{n^3} e^{-nR} - \frac{2R}{n^2} e^{-nR} - \frac{R^2}{n} e^{-nR}.
\end{aligned}$$

Consequently, taking the limit as R tend to infinity we obtain

$$\int_0^\infty t^2 e^{-nt} dt = \frac{2}{n^3}.$$

2. We consider the series of function $\sum t^2 e^{-nt}$. This is a geometric series, which is normally convergent on all segment $[a,b]$ with $0 < a < b$. Moreover, the series $\sum \int_0^\infty |f_n(t)| dt$ is the series $\sum \frac{2}{n^3}$ so it is convergent. Consequently, we can apply Proposition 3.2 of the first chapter :

$$\sum_{n=1}^\infty \frac{2}{n^3} = \sum_{n=1}^\infty \int_0^\infty t^2 e^{-nt} dt = \int_0^\infty \sum_{n=1}^\infty t^2 e^{-nt} dt = \int_0^t \frac{t^2}{e^t - 1} dt.$$

□

Exercise 3.

1. Give the partial fraction decomposition of $\frac{2x^2-x+4}{(x-1)^2(3x+2)}$.
2. Write $\frac{1}{(x-1)^2}$ and $\frac{1}{3x+2}$ as a power series. What are their radius of convergence ?
3. Write $\frac{2x^2-x+4}{(x-1)^2(3x+2)}$ as a power series, and give the radius of convergence.

Solution of exercise 3.

1. We have
$$\frac{2x^2 - x + 4}{(x-1)^2(3x+2)} = \frac{1}{(x-1)^2} + \frac{2}{3x+2}.$$

2. We expand in power series : we have for $|x| < 1$
$$(1-x)^{-2} = \sum_{k=0}^\infty \frac{-2(-2-1)..(-2-k+1)}{k!}(-x)^k = \sum_{k=0}^n (k+1)x^k.$$
And we have, for $|x| < \frac{2}{3}$
$$\frac{2}{3x+2} = \frac{1}{1+\frac{3x}{2}} = \sum (-\frac{3}{2})^k x^k.$$

3. If we sum a power series with radius of convergence 1 and one with radius of convergence $\frac{2}{3}$, we obtain a power series with radius of convergence $\frac{2}{3}$ and we can write, for $|x| < \frac{2}{3}$ $\frac{2x^2-x+4}{(x-1)^2(3x+2)} = \sum a_n x^n$ with $a_n = (n+1) + (-\frac{3}{2})^n$.

□

Exercise 4. Let $a > 0$ and $f :]-a,a[\to \mathbb{R}$ be a C^∞ function such that for all $k \in \mathbb{N}$ and all $x \in]-a,a[$ we have $f^{(2k)}(x) \geq 0$. The aim of this exercise is to show that f can be writen as a power series on $]-a,a[$.

1. Let $g :]-a, a[\to \mathbb{R}$ be a C^∞ function. Show with induction and integration by part that for all $n \in \mathbb{N}$, and all $x \in]-a, a[$

$$g(x) = g(0) + xg'(0) + .. + \frac{x^n}{n!}g^{(n)}(0) + \int_0^x \frac{(x-t)^n}{n!}g^{(n+1)}(t)dt.$$

2. Let $F : x \mapsto f(x) + f(-x)$. Show that we can write, for all $n \in \mathbb{N}$ and $x \in [0, b]$ with $b < a$

$$F(x) = F(0) + \frac{x^2}{2}F^{(2)}(0) + .. + \frac{x^{(2n)}}{(2n)!}F^{(2n)}(0) + R_n(x),$$

with $0 \leq R_n(x) \leq \left(\frac{x}{b}\right)^{2n+1} F(b)$.

3. Show that F can be written as a power series on $]-b, b[$.

4. Show that for all $n \in \mathbb{N}$ and $x \in]-b, b[$ we can write

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f^{(2)}(0) + ... + \frac{x^{(2n+1)}}{(2n+1)!}f^{(2n+1)}(0) + r_n(x),$$

with $|r_n(x)| \leq R_n(|x|)$.

5. Show that f can be written as a power series on $]-b, b[$ and conclude.

Solution of exercise 4.

1. For $n = 0$ the formula is just

$$g(x) = g(0) + \int_0^x g'(y)dy.$$

We assume the formula is true for some n. Then we calculate

$$\begin{aligned}
\int_0^x \frac{(x-t)^{n+1}}{(n+1)!}g^{(n+2)}(t)dt &= \int_0^x \frac{(x-t)^n}{(n)!}g^{(n+1)}(t)dt + [\frac{(x-t)^{n+1}}{(n+1)!}g^{(n+1)}(t)]_0^x \\
&= \int_0^x \frac{(x-t)^n}{(n)!}g^{(n+1)}(t)dt - \frac{x^{n+1}}{(n+1)!}g^{(n+1)} \\
&= -g(0) - xg'(0) - .. - \frac{x^n}{n!}g^{(n)}(0) + g(x) - \frac{x^{n+1}}{(n+1)!}g^{(n+1)}
\end{aligned}$$

so we obtain the formula for $n+1$.

2. F is even so $F^{(2k+1)}(0) = 0$. We write the Taylor formula for F

$$F(x) = F(0) + \frac{x^2}{2}F^{(2)}(0) + .. + \frac{x^{(2n)}}{(2n)!}F^{(2n)}(0) + R_n(x),$$

with

$$R_n(x) = \int_0^x \frac{(x-t)^{(2n+1)}}{(2n+1)!}F^{(2k+2)}(t)dt$$

Moreover, we know that $F^{(2k)}(x) = f^{(2k)}(x) + f^{(2k)}(-x) \geq 0$ for all x, so $0 \leq R_n(x) \leq F(x)$ and for $0 \leq x < b < a$ we have

$$R_n(x) = \int_0^x \left(\frac{x-t}{x-b}\right)^{2n+1} \frac{(x-b)^{(2n+1)}}{(2n+1)!} F^{(2k+2)}(t)dt \leq \left(\frac{x}{b}\right)^{2n+1} \int_0^b \frac{(x-b)^{2n+1}}{(2n+1)!} F^{(2k+2)}(t)dt$$
$$\leq \left(\frac{x}{b}\right)^{2n+1} R_n(b) \leq \left(\frac{x}{b}\right)^{2n+1} F(b).$$

where we have used $\frac{x-t}{x-b} \leq \frac{x}{b}$ for $t \leq x < b$.

3. For $0 < x < b$ we have $R_n(x) \to 0$ as $n \to \infty$, which means that the power series $\sum \frac{F^{(2k)}(0)}{(2k)!} x^k$ is converging, and its limit is $F(x)$, for all $x 1[0, b[$. We conclude on the whole $]-b, b[$ using the fact that F is even.

4. We write the Taylor formula for f

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f^{(2)}(0) + \ldots + \frac{x^{(2n+1)}}{(2n+1)!} f^{(2n+1)}(0) + r_n(x),$$

with

$$r_n(x) = \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!} f^{(2n+2)}(t)dt.$$

We have $f^{(2n+2)}(t) \leq f^{(2n+2)}(t) + f^{(2n+2)}(-t) \leq F^{(2n+2)}(t)$ so

$$|r_n(x)| \leq \int_0^{|x|} \frac{(x-t)^{2n+1}}{(2n+1)!} F^{(2n+2)}(t)dt = R_n(|x|).$$

5. We write $S_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$. We have proved that $S_{2n+1}(x) \to f(x)$ for all $x \in]-b, b[$. Moreover

$$S_{2n+2}(x) - S_{2n+1}(x) = \frac{f^{(2n+2)}(0)}{(2n+2)!} x^{2n+2} = \frac{F^{(2n+2)(0)}}{2(2n+2)!} x^{2n+2}.$$

Since $\frac{F^{(2n+2)(0)}}{(2n+2)!} x^{2n+2}$ is the term of a converging series, we obtain that $S_{2n+2}(x)$ and $S_{2n+1}(x)$ have the same limit. Consequently we can write, for all $x \in]-b, b[$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

□