# Week 10, April 18th: Fourier series

Instructor: Cécile Huneau (cecile.huneau@polytechnique.edu)
Tutorial Assistants:
– Allen Fang (groups ?, allen.fang@sorbonne-universite.fr)
– Yuan Xu (groups ?, xu.yuan@polytechnique.edu)

## 1 Important exercises

*Exercise 1.* We consider the $2\pi$ periodic function defined on $[-\pi, \pi[$ by $f(x) = e^x$.

1. Calculate the Fourier coefficients $c_n(f)$.
2. Use Parseval's formula to calculate $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$.

*Exercise 2.*

1. We consider the $2\pi$ periodic function defined fy $f(x) = \frac{(\pi-x)^2}{4}$ for $x \in [0, 2\pi[$. Show that $f$ is continuous, and calculate its real Fourier coefficients.
2. Show that $f$ is equal to its power series. Deduce the Euler formula $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
3. Show that the Fourier series of $f$ can be differentiated term by term in all segment $[\delta, 2\pi - \delta]$ for $0 < \delta < \pi$, and deduce that for all $x \in ]0, 2\pi[$
$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$
4. Thanks to Parseval's formula, calculate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

*Exercise 3.*

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2$ function such that $f(x)$, $f'(x)$ and $f''(x)$ are $O\left(\frac{1}{|x|^2}\right)$ as $|x| \to \infty$. Show that the series of functions $\sum_{n\in\mathbb{N}} f(x+n)$ converges pointwise on $\mathbb{R}$. We note $F(x)$ the limit.
2. Show that $F$ is $C^2$. Recall why the Fourier series associated to $F$ converges uniformly, and that it is equal to its Fourier series (you will see later in the course that the hypothesis that $F$ is $C^2$ is not necessary to prove that.)
3. Calculate the Fourier coefficients of $F$.
4. Deduce that for all $x \in \mathbb{R}$
$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum f^*(n) e^{2i\pi nx},$$
where $f^*(n) = \int_{-\infty}^{\infty} f(t) e^{-2i\pi nt} dt$.

5. Let $I(x)=\int_{-\infty}^{\infty}e^{-u^2}e^{-2i\pi ux}du$. Show that $I'(x)=-2\pi^2 xI(x)$. We recall that $\int_{-\infty}^{\infty}e^{-u^2}du=\sqrt{\pi}$. Calculate $I$.

6. Show that for all $s>0$
$$\sum_{n=-\infty}^{\infty}e^{-\pi n^2 s}=s^{-\frac{1}{2}}\sum_{n=-\infty}^{\infty}e^{-\frac{\pi n^2}{s}}.$$

## 2 More involved exercises

*Exercise 4.*

1. Calculate the Fourier series associated to the "square" signal, which is the $2\pi$ periodic function defined by $f(x)=1$ for $x\in]0,\pi[$, $f(x)=0$ for $x\in]\pi,2\pi[$ and $f(0)=f(\pi)=\frac{1}{2}$.

2. Show that the Fourier series associated to f converges uniformly on all compact $[\delta,\pi-\delta]$ for $0<\delta<\frac{\pi}{2}$.

3. Show that the partial sums $S_{2n-1}(x)=\sum_{k=-(2n-1)}^{2n-1}c_k(f)e^{ikx}$ can be written
$$S_{2n-1}(x)=\frac{1}{2}+\frac{1}{\pi}\int_0^x\frac{\sin(2ns)}{\sin(s)}ds.$$

4. Show that the function $S_{2n-1}$ has $2n$ critical points on $[0,\pi]$, which are $x_k=\frac{k\pi}{2n}$, $1\le k\le 2n$.

5. Show that $S_{2n-1}(x_{2k})<S_{2n-1}(x_{2k-1})$ for all $1\le k\le n$.

6. Show that $S_{2n-1}(x_{2k+1})<S_{2n-1}(x_{2k-1})$ for all $1\le k<n$.

7. Deduce that $S_{2n-1}$ attains its maximum in $x_1$. We note $M_n$ this maximum.

8. Show that $M_n$ converges has $n\to\infty$ to
$$M=\frac{1}{2}+\frac{1}{\pi}\int_0^{\pi}\frac{\sin(s)}{s}ds.$$

9. We admit that $M\approx 1,089$ at the order $10^{-3}$. Conclude.

*Exercise 5.*

1. Let $\rho_n$ and $\theta_n$ be two sequences in $\mathbb{R}$ such that for all $t\in\mathbb{R}$, $\rho_n\cos(nt-\theta_n)\to 0$ as $n\to\infty$. Show that if $\rho_n$ does not tend to 0 as $n\to\infty$, then you can construct $\delta>0$, a strictly increasing sequence of integer $n_k$ and closed segments $I_k$, with $I_{k+1}\subset I_k$ such that for all $t\in I_k$ you have
$$|\rho_{n_k}\cos(n_k t-\theta_{n_k})|\ge\delta.$$

2. Conclude that if $\rho_n\cos(nt-\theta_n)\to 0$ as $n\to\infty$ for all $t$ then $\rho_n\to 0$.

3. Show that if the trigonometric series $\sum_{n\in\mathbb{Z}}c_ne^{inx}$ converges pointwise on $\mathbb{R}$ then $c_n\to 0$ as $|n|\to\infty$.