Week 11, May 9th: Differential equations

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1 Important exercises

Exercise 1.

1. What is the domain of definition of the differential equation $y' - y - \ln(y) = -t$?
2. Check that the function $\mathbb{R} \to \mathbb{R}$, $t \mapsto e^t$ is a solution.

Exercise 2.

1. What is the space of solutions of the differential equation $y' - ty = 0$?
2. Solve the Cauchy problem $y' - ty = -\sin(t) - t\cos(t)$, $y'(0) = 2$.

Exercise 3.

1. What is the domain of definition of the differential equation $y' - \frac{y}{1+t} = 1 + t$?
2. Solve the Cauchy problem $y' - \frac{y}{1+t} = 1 + t$, $y'(0) = -2$.

Exercise 4.

1. What is the space of solutions of the differential equation $y'' - y' - 2y = 0$?
2. Write the equation $y'' - y' - 2y = 0$ as a first order equation.
3. Solve the Cauchy problem $y'' - y' - 2y = e^t$, $y(0) = 1$, $y'(0) = 0$.

Exercise 5.

1. What is the space of solutions of the system of differential equations

$$\begin{cases} x' = 3x - 2y \\ y' = x + y \end{cases}$$

2. Solve the Cauchy problem

$$\begin{cases} x' - (3x - 2y) = t \\ y' - (x + y) = 1 \\ x(0) = 1,\ y(0) = 2 \end{cases}$$

Exercise 6.

1. Let $F : \mathbb{R} \to \mathbb{R}$ defined by $F(y) = 0$ for $y \leq 0$ and $F(y) = \sqrt{y}$ for $y \geq 0$. Is F locally Lipschitz on $\mathbb{R}$?

2. We consider the differential equation $y' = F(y)$. Show that if ϕ is solution, then for all c, the function $\phi_c : t \mapsto \phi(t-c)$ is also a solution.

3. Show that their is not unicity of solutions to the Cauchy problem $y' = F(y)$, $y(0) = 0$.

2 More involved exercises

Exercise 7. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function. We look at the differential equation $y'' = F(t, y, y')$. We assume that for all $t \in \mathbb{R}$, $F(t, 0, 0) = 0$.

1. Show that the function $t \mapsto 0$ is a solution.

2. Show that all solution to $y'' = F(t, y, y')$ which is not identiquely zero has isolated zeros.

Exercise 8. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two solutions to the differential equation $y' = F(t, y)$. We assume that there exists $t_0 \in \mathbb{R}$ such that $f(t_0) < g(t_0)$. Show that for all $t \in \mathbb{R}$, $f(t) < g(t)$.