Analysis 4 Problem Set 9

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1 Exercise 1

Recall the definition of the real Fourier coefficients,

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \mathrm{d}x \quad \text{for all } n \in \mathbb{N}, \tag{1}$$

$$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \mathrm{d}x \quad \text{for all } n \in \mathbb{N}^+. \tag{2}$$

(i) The Fourier coefficients $a_n(f_1)$ and $b_n(f_1)$. First, note that $f_1(x) = x$ is odd function on $[-\pi,\pi]$. Therefore the function $f_1(x)\cos nx$ is also odd function on $[-\pi,\pi]$ for all $n \in \mathbb{N}$. It follow that

$$a_n(f_1) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\cos nx \mathrm{d}x = 0 \quad \text{for all } n \in \mathbb{N}.$$

Second, by integration by parts, for all $n \in \mathbb{N}^+$,

$$\begin{aligned} b_n(f_1) &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx \mathrm{d}x \\ &= -\frac{1}{n\pi}x\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos nx \mathrm{d}x \\ &= (-1)^{n+1}\frac{2}{n} + \frac{1}{n^2\pi}\sin nx\Big|_{-\pi}^{\pi} = (-1)^{n+1}\frac{2}{n}. \end{aligned}$$

(ii) The Fourier coefficients $a_n(f_2)$ and $b_n(f_2)$. First, by integration by parts,

$$a_0(f_2) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \mathrm{d}x = \frac{1}{3\pi}x^3\Big|_{-\pi}^{\pi} = \frac{2}{3}\pi^2.$$

Second, for $n \in \mathbb{N}^+$, using again integration by parts,

$$\begin{aligned} a_n(f_2) &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx \mathrm{d}x \\ &= \frac{1}{n\pi}x^2\sin nx\Big|_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin nx \mathrm{d}x \\ &= \frac{2}{n^2\pi}x\cos nx\Big|_{-\pi}^{\pi} - \frac{2}{n^2\pi}\int_{-\pi}^{\pi}\cos nx \mathrm{d}x = (-1)^n\frac{4}{n^2}. \end{aligned}$$

Last, note that $f_2(x) = x^2$ is even function on $[-\pi,\pi]$. Therefore the function $f_2(x)\sin nx$ is odd function on $[-\pi,\pi]$ for all $n \in \mathbb{N}^+$. It follow that

$$b_n(f_2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(x)\sin nx \mathrm{d}x = 0 \quad \text{for all } n \in \mathbb{N}^+.$$

(iii) The Fourier coefficients $a_n(f_3)$ and $b_n(f_3)$. Note that $f_3(x) = \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$. Thus, $f_3(x)$ is a trigonometric polynomials. From Proposition 2.8 in Lecture notes, we obtain

$$f_3(x) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{a_0(f_3)}{2} + \sum_{n=1}^{\infty} a_n(f_3)\cos nx + \sum_{n=1}^{\infty} b_n(f_3)\sin nx.$$

It follows that

$$b_n(f_3) = 0 \quad \text{for all } n \in \mathbb{N}^+,$$

$$a_0(f_3) = 1, \quad a_2(f_3) = -\frac{1}{2}, \quad a_n(f_3) = 0 \quad \text{for all } n \neq 0, 2.$$

(iv) The Fourier coefficients $a_n(f_4)$ and $b_n(f_4)$. First, by integration by parts,

$$a_0(f_4) = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} \sin x \mathrm{d}x = -\frac{2}{\pi}\cos x\Big|_0^{\pi} = \frac{4}{\pi}.$$

Second, note that $\sin x \cos x = \frac{1}{2}\sin 2x$. Thus

$$a_1(f_4) = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos x \mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \sin 2x \mathrm{d}x = 0.$$

Third, note that for $n \in \mathbb{N}^+$ with $n \geq 2$,

$$\sin x \cos nx = \frac{1}{2}\sin((n+1)x) - \frac{1}{2}\sin((n-1)x).$$

Note also that the function $|\sin x| \cos nx$ is even function on $[-\pi, \pi]$. Therefore, for $n \in \mathbb{N}^+$ with $n \geq 2$, using again integration by parts,

$$\begin{aligned}
a_n(f_4) &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos nx \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x)\mathrm{d}x - \frac{1}{\pi}\int_0^{\pi} \sin((n-1)x)\mathrm{d}x \\
&= -\frac{1}{(n+1)\pi}\cos((n+1)x)\Big|_0^{\pi} + \frac{1}{(n-1)\pi}\cos((n-1)x)\Big|_0^{\pi} \\
&= -\frac{1}{(n+1)\pi}(\cos(n+1)\pi - 1) + \frac{1}{(n-1)\pi}(\cos(n-1)\pi - 1) \\
&= \left((-1)^{n+1} - 1\right)\frac{2}{\pi(n^2-1)}.
\end{aligned}$$

Last, note that $f_4(x) = |\sin x|$ is even function on $[-\pi, \pi]$. Therefore the function $f_4(x)\sin nx$ is odd function on $[-\pi, \pi]$ for all $n \in \mathbb{N}^+$. It follow that

$$b_n(f_4) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_4(x)\sin nx \mathrm{d}x = 0 \quad \text{for all } n \in \mathbb{N}^+.$$

(v) The Fourier coefficients $a_n(f_5)$ and $b_n(f_5)$. First, by integration by parts,

$$a_0(f_5) = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \mathrm{d}x = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x = -\frac{2}{\pi}\sin x\Big|_0^{\pi} = \frac{4}{\pi}.$$

Second, by direct computation

$$\begin{aligned}
a_n(f_5) &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \cos nx \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x \cos nx \mathrm{d}x - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos x \cos nx \mathrm{d}x = I_1 - I_2.
\end{aligned}$$

Moreover, by change of variable $x - \pi = y$,

$$I_2 = \int_{-\frac{\pi}{2}}^{0} \cos(y+\pi)\cos(n\pi + ny)\mathrm{d}y = (-1)^{n+1}\int_{-\frac{\pi}{2}}^{0} \cos y \cos ny \mathrm{d}y = (-1)^{n+1} I_1.$$

It follows that

$$a_{2m}(f_5) = 2I_1 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos x \cos nx \mathrm{d}x, \quad a_{2m+1}(f_5) = I_1 - I_2 = 0 \quad \text{for all } n \in \mathbb{N}^+.$$

Note also that

$$\cos x \cos nx = \frac{\cos((n+1)x)}{2} + \frac{\cos((n-1)x)}{2}.$$

Thus

$$\begin{aligned}
a_{2m}(f_5) &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos((2m+1)x) + \cos((2m-1)x)\mathrm{d}x \\
&= \frac{2}{\pi}\left(\frac{1}{2m-1}\sin(2m-1)x\Big|_0^{\frac{\pi}{2}} + \frac{1}{2m+1}\sin(2m+1)x\Big|_0^{\frac{\pi}{2}}\right) \\
&= \frac{2}{\pi}\left(\frac{1}{2m-1}(-1)^{m+1} + \frac{1}{2m+1}(-1)^m\right) = (-1)^m\frac{4}{\pi(1-4m^2)}.
\end{aligned}$$

Last, note that $f_5(x) = |\cos x|$ is even function on $[-\pi,\pi]$. Therefore the function $f_5(x)\sin nx$ is odd function on $[-\pi,\pi]$ for all $n \in \mathbb{N}^+$. It follow that

$$b_n(f_5) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_5(x)\sin nx \mathrm{d}x = 0 \quad \text{for all } n \in \mathbb{N}^+.$$

2 Exercise 2

2.1 Exercise 2.1

Let $c_n(f)$ represent the Fourier coefficients of $f(x)$. Then, we want to show that $c_n = c_n(f)$. To do this, we consider

$$\begin{aligned}
c_n(f) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}\sum_{m\in\mathbb{Z}} c_m e^{imx}\,dx \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_0^{2\pi} c_m e^{-inx}e^{imx}\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} c_n\,dx \\
&= c_n,
\end{aligned}$$

where we used the fact that the trigonometric series converges uniformly to switch the integral and the limit, and then the fact that for $n \neq m$, $\int_0^{2\pi} e^{i(-n+m)x}\,dx = 0$.

2.2 Exercise 2.2

Consider

$$\begin{aligned}
c_n(f) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\,dx \\
&= \frac{1}{2\pi}\left(\frac{1}{-in}e^{-inx}f(x)|_0^{2\pi} - \int_0^{2\pi}\frac{1}{-in}e^{inx}f'(x)\,dx\right) \\
&= \frac{1}{2\pi in}\int_0^{2\pi} e^{-inx}f'(x)\,dx \qquad (3)
\end{aligned}$$

Then, since $f \in C^1$, we know that there exists C such that $|f'(x)| \leq C$. Thus from (3), we have that

$$|c_n(f)| \leq \left|\frac{1}{2\pi in}\right|\left|\int_0^{2\pi} e^{-inx}f'(x)\,dx\right| \leq \frac{1}{2\pi|n|}\int_0^{2\pi}|f'(x)|\,dx \leq \frac{C}{|n|}.$$

2.3 Exercise 2.3

To prove this exercise, we essentially repeat the argument from Exercise 2.2.

Continuing the equations in the solution for Exercise 2.2

$$\begin{aligned} c_n(f) &= \frac{1}{2\pi in} \int_0^{2\pi} e^{-inx} f'(x)\,dx \\ &= \frac{1}{2\pi in} \left(\frac{1}{-in} e^{-inx} f'(x)|_0^{2\pi} - \int_0^{2\pi} \frac{1}{-in} e^{inx} f''(x)\,dx \right) \\ &= \frac{1}{-2\pi n^2} \int_0^{2\pi} e^{inx} f''(x)\,dx \end{aligned} \tag{4}$$

Then, using the fact that $f \in C^2$ and thus there exists C' such that $|f''(x)| \leq C'$, we have that

$$|c_n(f)| \leq \left| \frac{1}{2\pi n^2} \right| \left| \int_0^{2\pi} e^{-inx} f''(x)\,dx \right| \leq \frac{1}{2\pi n^2} \int_0^{2\pi} |f''(x)|\,dx \leq \frac{C'}{n^2}.$$

3 Exercise 3

3.1 Exercise 3.1

Consider a f piecewise continuous such that it is discontinuous at a finite set of points $\{x_1, \cdots, x_n\}$. Assume without loss of generality that the $\min_{i,j}(x_i - x_j) \geq 1$ (if this is not true we can rescale the function until it is). Then, define f_ϵ the continuous function that agrees with f on $[a,b]$ except for $N(x_i, \epsilon)$, on which it is a monotonic function between $f(x_i - \epsilon), f(x_i + \epsilon)$.

Then,

$$\begin{aligned} \int_a^b |f(x) - f_\epsilon(x)|^2\,dx &\leq \sum_{i=1}^n \int_{N(x_i,\epsilon)} |f(x) - f_\epsilon(x)|^2 \\ &\leq \sum_{i=1}^N \epsilon \sup |f(x) - f_\epsilon(x)|^2 \\ &\leq 2N\epsilon \sup |f(x)|^2. \end{aligned}$$

But then, since ϵ can be made arbitrarily small, in particular, we can choose $\epsilon < \frac{\delta}{2N \sup |f(x)|^2}$ so that $\int_a^b |f(x) - f_\epsilon(x)|^2\,dx < \delta$.

3.2 Exercise 3.2

We first claim that uniform convergence implies L^2 convergence. Then, by the Weierstrass approximation theorem, we know that there exists some sequence $P_n(x)$ such that $||P_n(x) - g(x)||_{L^2} \leq \epsilon$. Then,

$$\begin{aligned} ||f(x) - P_n(x)||_{L^2} &\lesssim ||f(x) - g(x))||_{L^2} + ||g(x) - P_n(x)||_{L^2} \\ &\lesssim \epsilon. \end{aligned}$$

Now we show that uniform convergence implies L^2 convergence. Let f_n converge to f uniformly. Then

$$\begin{aligned} ||f_n - f||_{L^2}^2 &= \int_a^b |f_n(x) - f(x)|^2\,dx \\ &\leq (b-a) \left(\sup_{x \in [a,b]} |f_n(x) - f(x)| \right)^2 \end{aligned}$$

Then, for N sufficiently large, since $f_n(x)$ uniformly converges to f, it also converges to f in L^2.

4 Exercise 4

4.1 Exercise 4.1

Consider f the characteristic function of an interval $I = [c,d] \subset [a,b]$, denoted χ_I. Then,

$$\begin{aligned}\lim_{n\to\infty}\int_a^b \chi_I(t)\phi(nt)\,dt &= \lim_{n\to\infty}\int_I \phi(nt)\,dt \\ &= \frac{1}{n}\int_{cn}^{dn}\phi(t)\,dt.\end{aligned}$$

Then, we can furthermore bound

$$\frac{n(d-c)}{T}\int_0^T \phi(t)\,dt - MT \leq \int_{cn}^{dn}\phi(t)\,dt \leq \frac{n(d-c)}{T}\int_0^T\phi(t)\,dt + MT,$$

where M is the bound of $\phi(t)$. Taking the limit as $n \to \infty$ and using the Sandwich Theorem yields that

$$\lim_{n\to\infty}\int_a^b \chi_I(t)\phi(nt)\,dt = \frac{|I|}{T}\int_0^T \phi(t)\,dt,$$

which is exactly the result in the case of a characteristic function.

Now we show that statement continues to hold for linear combinations of characteristic functions. That is, consider $u\chi_I, v\chi_J$, where $I, J \subset [a,b]$ and $u, v \in \mathbb{R}$. Then,

$$\begin{aligned}\lim_{n\to\infty}\int_a^b (u\chi_I(t) + v\chi_J(t))\phi(nt)\,dt &= \lim_{n\to\infty} u\int_a^b \chi_I(t)\phi(nt)\,dt + v\int_a^b \chi_J(t))\phi(nt)\,dt \\ &= \frac{u}{T}\int_0^T\phi(t)\,dt\int_a^b\chi_I(t)\,dt + \frac{v}{T}\int_0^T\phi(t)\,dt\int_a^b\chi_J(t)\,dt \\ &= \frac{1}{T}\int_0^T\phi(t)\,dt\int_a^b (u\chi_I(t) + v\chi_J(t))\,dt\end{aligned}$$

Thus, the statement in the problem holds for any simple function $f(t)$. Now, since we know that the simple functions are dense in the space of piecewise continuous functions, let $g_m(t)$ be a sequence of simple functions that converges uniformly to $f(t)$. Then,

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_a^b g_m(t)\phi(nt)\,dt = \lim_{m\to\infty}\frac{1}{T}\int_0^T\phi(t)\,dt\int_a^b g_m(t)\,dt$$

Then passing to the limit in m on both sides yields the result.

4.2 Exercise 4.2

This follows directly from applying Exercise 4.1 where we use that $\phi = e^{int}$.

5 Exercise 5

5.1 Exercise 5.1

Suppose for the sake of contradiction that this were not true. Then there exists $\delta > 0$ and a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of integers such that $\rho_{n_k} \geq \delta$ for all k. Restricting further to a subsequence, we can assume that $n_{k+1} > 6n_k$ for all k.

Then, consider

$$I_1 = \left[\frac{1}{n_1}\left(\theta_{n_1} - \frac{\pi}{3}\right), \frac{1}{n_1}\left(\theta_{n_1} + \frac{\pi}{3}\right)\right],$$

such that for all $t \in I_1$, $\cos(n_1 t - \theta_{n_1}) \geq \frac{1}{2}$. As t varies in I_1, $n_2 t - \theta_{n_2}$ varies in an inteveral of length $n_2 \cdot \frac{2\pi}{3n_1} \geq 4\pi$.

Thus, we can find a segment $I_2 \subset I_1$ of length $\frac{2\pi}{3n_2}$ such that $\cos(n_2 t - \theta_{n_2}) \geq \frac{1}{2}$ for all $t \in I_2$. Continuing to iterate in this way, we construct for all k a segment $I_k \subset I_{k-1}$ of length $\frac{2\pi}{3n_k}$ such that

$$\forall t \in I_k, \qquad \cos(n_k t - \theta n_k) \geq \frac{1}{2}.$$

Then, by construction we have that

$$|\rho_{n_k} \cos(n_k t - \theta_{n_k})| \geq \frac{\delta}{2}.$$

5.2 Exercise 5.2

Assume that $\rho_n \cos(nt - \theta_n) \to 0$ as $n \to \infty$ but $\rho_n \not\to 0$ for the sake of contradiction, then using Exercise 5.1, we know that there exists $\xi \in \mathbb{R}$ such that $\bigcap_{k \in \mathbb{N}} I_k = \{\xi\}$. For all k, we have $\rho_{n_k} \cos(n_k \xi - \theta_k) \geq \frac{\delta}{2}$, which contradicts our assumption that $\rho_n \cos(nt - \theta_n) \to 0$.

5.3 Exercise 5.3

We can write $c_n e^{int} + c_{-n} e^{-int}$ in the form $a_n \cos nt + b_n \sin nt) + i(a'_n \cos nt + b'_n \sin nt)$, where $a_n, b_n, a'_n, b'_n \in \mathbb{R}$. Then WLOG, we have to show that the series a_n, b_n tend to zero. For all $t \in \mathbb{R}$, we know that $\lim_{n \to \infty}(a_n \cos nt + b_n \sin nt) = 0$. Then define $\rho_n = \sqrt{a_n^2 + b_n^2}$. Then there exists for all n a real $\theta_n \in [0, 2\pi]$ such that $a_n \cos nt + b_n \sin nt = \rho_n \cos(nt - \theta_n)$ for all $t \in \mathbb{R}$. So we have reduced our problem to proving the following: if for all $t \in \mathbb{R}$, $\lim_{n \to \infty} \rho_n \cos(nt - \theta_n) = 0$, then the sequence (ρ_n) tends to 0. But this is exactly the statement in Exercise 5.2. Therefore, we are done.