Week 9, April 11th: Fourier series

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1 Important exercises

Exercise 1. We consider 2π periodic functions defined on the interval $[-\pi, \pi[$ by

$$f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = \sin^2(x), \quad f_4(x) = |\sin(x)|, \quad f_5(x) = |\cos(x)|.$$

For $i = 1,..,5$, compute the real Fourier coefficients $a_n(f_i)$ and $b_n(f_i)$.

Exercise 2.

1. Prove the following proposition : if the trigonometric series $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ converges uniformly, and if we note $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, then f is equal to its Fourier series everywhere.
2. Let f be a C^1 2π periodic function. Show that there exists a constant C such that $|c_n(f)| \leq \frac{C}{|n|}$ for all $n \neq 0$.

   Hint : Integrate by parts.
3. If now f is C^2 2π periodic, show that there exists a constant C' such that $|c_n(f)| \leq \frac{C'}{n^2}$ for all $n \neq 0$.
4. Deduce that if f is C^2 and 2π periodic, f is equal to its Fourier series everywhere.

Exercise 3.

1. Let $f : [a, b] \to \mathbb{R}$ be a piecewise continuous function. Show that for all $\varepsilon > 0$ there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that
$$\int_a^b |f(x) - g(x)|^2 dx \leq \varepsilon.$$
2. Show that there exists a sequence of polynomials P_n such that
$$\int_a^b |P_n(x) - f(x)|^2 dx \to 0, \quad n \to \infty.$$

Exercise 4.

1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous, T periodic function, and $f : [a,b] \to \mathbb{R}$ be a piecewies continuous function. Show that

$$\lim_{n\to\infty} \int_a^b f(t)\phi(nt)dt = \frac{1}{T}(\int_0^T \phi(t)dt)(\int_a^b f(t)dt)$$

   Hint : Start by considering the case where f is the characteristic function of an interval, then the case where f is a step function.

2. Show the Riemann-Lebesgue lemma : $\lim_{n\to\infty} \int_a^b f(t)e^{int}dt = 0$ for f a piecewise continuous function.

2 More involved exercises

Exercise 5.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function such that $f(x) = O\left(\frac{1}{|x|^2}\right)$ and $f'(x) = O\left(\frac{1}{|x|^2}\right)$ as $|x| \to \infty$. Show that for all $x \in \mathbb{R}$

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum f^*(n)e^{2i\pi nx},$$

   where $f^*(n) = \int_{-\infty}^{\infty} f(t)e^{-2i\pi nt}dt$.

2. Let $I(x) = \int_{-\infty}^{\infty} e^{-u^2}e^{-2i\pi ux}du$. Show that $I'(x) = -2\pi^2 xI(x)$. We recall that $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$. Calculate I.

3. Show that for all $s > 0$

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 s} = s^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{s}}.$$