Week 6, March 21th: Power series

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1 Important exercises

Exercise 1.

1. Show that the power series $\sum z^n$ and $\sum n^2 z^n$ have the same radius of convergence, that we call R.
2. We note $f :]-R, R[\to \mathbb{R}, \quad x \mapsto \sum_{n=0}^{\infty} x^n$. Calculate f, f' and f''. Deduce a formula for $\sum n^2 x^n$, for $x \in]-R, R[$.

Exercise 2.

1. Show that the radius of convergence of the series $\sum \frac{z^n}{2n+1}$ is equal to 1. We note $f :]-1, 1[\to \mathbb{R}, \quad x \mapsto \sum_{n=0}^{\infty} \frac{z^n}{2n+1}$.
2. Let $g(x) = xf(x^2)$. Calculate g'. Deduce a formula for $\sum \frac{x^n}{2n+1}$ and $x > 0$.

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f = 0$ for $x \leq 0$ and $f(x) = e^{-\frac{1}{x^2}}$.

1. Show by recurrence that for all n, there exists a polynomial P_n such that for $n \geq 0$
$$f^{(n)}(x) = P_n(x) e^{-\frac{1}{x^2}}.$$
2. Show that the function is C^∞, but can not be written as a power series in a neighbourhood of zero.

Exercise 4. Let $\alpha \in \mathbb{R} \setminus \mathbb{N}$. For all n, we set $a_n = \frac{\alpha(\alpha-1)..(\alpha-n+1)}{n!}$.

1. Show that the radius of convergence of the power series $\sum a_n z^n$ is equal to 1.
2. Show that for all $|z| < 1$ we have
$$(1+z) \sum_{n=0}^{\infty} (n+1) a_n z^n = \sum_{n=0}^{\infty} b_n z^n,$$
where $b_n = \alpha a_n$.
3. Show that the function $g(x) = \sum_{n=0}^{\infty} a_n x^n$, defined for $x \in]-1, 1[$ satisfies the differential equation
$$y' = \frac{\alpha}{1+x} y.$$
4. Calculate $\sum a_n z^n$.

Exercise 5. Let $a, b \in \mathbb{C}$. We define the sequence (u_n) by $u_0 = 0$, $u_1 = 1$ and for $n \geq 0$, $u_{n+2} = au_{n+1} + bu_n$.

1. Let $M = \max(|a|, |b|, \frac{1}{2})$. Show by induction that $|u_n| \leq (2M)^{n-1}$ for all $n \geq 1$.
2. Show that the radius of convergence of the power series $\sum u_n z^n$, R, is different from zero.
3. Show that for all $x \in]-R, R[$ we have

$$\sum_{n=0}^{\infty} u_n x^n = \frac{x}{1 - ax - bx^2}.$$

2 More involved exercises

Exercise 6. Let $\sum a_n x^n$ and $\sum b_n x^n$ be two power series with radius of convergence greater or equal to 1. Assume that $b_n > 0$ and that the series $\sum b_n$ is divergent. We note $A_N = \sum_{n=0}^{N} a_n$ and $B_N = \sum_{n=0}^{N} b_n$. In this exercise, we consider only $x \in \mathbb{R}$.

1. Show that $\sum_{n=0}^{\infty} b_n x^n \to +\infty$ as $x \to 1$ with $x < 1$.
2. We assume that there exists $l \in \mathbb{C}$ such that $\lim_{n\to\infty} \frac{a_n}{b_n} = l$. Show that $\frac{\sum_{n=0}^{\infty} a_n z^n}{\sum_{n=0}^{\infty} b_n z^n} \to l$ as $x \to 1$ with $x < 1$.
3. We assume that there exists $l \in \mathbb{C}$ such that $\lim_{n\to\infty} \frac{A_n}{B_n} = l$. Show that $\frac{\sum_{n=0}^{\infty} a_n z^n}{\sum_{n=0}^{\infty} b_n z^n} \to l$ as $x \to 1$ with $x < 1$.
4. We assume that there exists $l \in \mathbb{C}$ such that $\lim_{n\to\infty} \frac{A_0 + A_1 + .. + A_{n-1}}{n} = l$. Show that $\sum_{n=0}^{\infty} a_n x^n$ tend to l as $x \to 1$ with $x < 1$.
5. Show the following equivalence, as $x \to 1$

$$\sum_{x=0}^{\infty} x^{a^n} \sim -\frac{\ln(1-x)}{\ln(a)}, \text{ for } a \in \mathbb{N}, \quad a \geq 2, \qquad \sum_{x=0}^{\infty} (-1)^n x^{4n+1} \sim \frac{1}{2}.$$