Week 14: Differential equations

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Exercise 1. Let Ω be an open set of $\mathbb{R}^2$. Let $f:\Omega\to\mathbb{R}$ be a C^1 function, $\phi:]a,b[\to\mathbb{R}$ be a solution to the differential equation $y'=f(t,y)$ and $t_0\in I$. Let $u:]a,b[\to\mathbb{R}$ be such that for all $a<t<b$ we have $(t,u(t))\in\Omega$.

1. Assume that for all $t\in]a,b[$, $u'(t)>f(t,u(t))$. Show the following statements :
   - If $\phi(t_0)\le u(t_0)$ then $\phi(x)\le u(x)$ for all $x\in[x_0,b[$.
   - If $\phi(t_0)\ge u(t_0)$ then $\phi(x)\ge u(x)$ for all $x\in]a,x_0[$.
2. Assume that for all $t\in]a,b[$, $u'(t)<f(t,u(t))$. Show the following statements :
   - If $\phi(t_0)\ge u(t_0)$ then $\phi(x)\ge u(x)$ for all $x\in[x_0,b[$.
   - If $\phi(t_0)\le u(t_0)$ then $\phi(x)\le u(x)$ for all $x\in]a,x_0[$.

Exercise 2. We admit in this exercise Theorem 4.3 of the notes, which is the above exercise but with $\le$ or $\ge$ signs in the inequalities for u, instead of $<$ or $>$ signs.

We consider the differential equation $y'=-y(y+t)$. The aim of this exercise is to draw the graph $(t,\phi(t))$ of some of the solutions in the (t,y) plane.

1. In which parts of the (t,y) plane are the solutions to the differential equation $y'=-y(y+t)$ increasing ? and decreasing ?
2. Show that there exists a unique maximal $\phi_r:]a,b[\to\mathbb{R}$ such that $\phi_r(0)=r$. What is ϕ_0 ? Show that if $r>0$ then $\phi_r(t)>0$ for all $t\in]a,b[$.
3. We now consider $\phi_1:]a,b[\to\mathbb{R}$. Show that ϕ_1 is decreasing in the region $0\le t<b$. Deduce that $b=+\infty$ and $\phi_1(t)\to 0$ as $t\to\infty$.
4. Let $u(t)=\frac{2}{t+2}$ for $t>-2$. Show that for $t>-2$ we have $u'(t)\ge -u(t)(u(t)+t)$.
5. Show that for all $\max(a,-2)<t<0$ we have $\phi_2(t)\ge u(t)$. Deduce that $-2\le a<0$ and that ϕ_1 is decreasing on $]a,+\infty[$. Draw the graph of ϕ_1.
6. Let $v(t)=\frac{1}{2}-\frac{t}{2}$ and $w(t)=-\frac{3}{t}$. Show that $v'(t)\le -v(t)(v(t)+t)$ for $t\le 0$ and $w'(t)\le -w(t)(w(t)+t)$ for $t\le -2$. Draw the graph of v and w.
7. Let $0<r\le\frac{1}{2}$. Show that ϕ_r is defined on $\mathbb{R}$ and that $\lim_{t\to+\infty}\phi_r(t)=\lim_{t\to-\infty}\phi_r(t)=0$.
8. Show that if ϕ is a solution then $t\mapsto -\phi(-t)$ is also a solution.

9. Draw on a same picture the graph of ϕ_r, for $r = 1, \frac{1}{2}, \frac{1}{4}, -\frac{1}{2}, -1$.

Exercise 3. We consider again the differential equation $y' = -y(y+t)$.

1. Let $t_0 \geq 0$ and $y_0 > 0$. Calculate the maximal solution u of the Cauchy problem $y' = -y^2$, $y(t_0) = y_0$.
2. Let $t_0 \geq 0$ and $\phi :]a,b[\to \mathbb{R}$ a maximal solution such that $\phi(t_0) > 0$. Show that $b = +\infty$ and that $\phi(t) \to 0$ as $t \to \infty$.