Week 7, March 28th: Power series

Instructor: Cécile Huneau (cecile.huneau@polytechnique.edu)
Tutorial Assistants:
– Allen Fang (groups ?, allen.fang@sorbonne-universite.fr)
– Yuan Xu (groups ?, xu.yuan@polytechnique.edu)

1 Important exercises

Exercise 1.

1. Write $\frac{1}{1+x^2}$ as a power series for $|x| < 1$.
2. Deduce that for all $|x| < 1$
$$\arctan(x) = \sum (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Exercise 2.

1. Make the partial fraction decomposition of $\frac{x^2+x-3}{(x-2)^2(2x-1)}$.
2. Write $\frac{1}{x-2}$, $\frac{1}{(x-2)^2}$ and $\frac{1}{(2x-1)}$ as a power series. What are the radius of convergence of the series involved ?
3. Write $\frac{x^2+x-3}{(x-2)^2(2x-1)}$ as a power series, and precise what is the radius of convergence.

Exercise 3.

1. Calculate the coefficients of the power series which is the product $(\sum \frac{z^{2n}}{(2n)!})(\sum \frac{(-1)^n z^{2n}}{(2n)!})$

   Hint : Calculate $(1+i)^{4n}$.
2. Deduce that for all $x \in \mathbb{R}$
$$\cos(x)\cosh(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(4n)!} x^{4n}.$$

Exercise 4. Let $a \in \mathbb{R}$. Show that there exists a unique function $f : \mathbb{R} \to \mathbb{R}$, which can be written as a power series, solution of the differential equation
$$xy'' + y' + xy = 0.$$

Exercise 5.

1. What is the radius of convergence R, of the power series $\sum \frac{(-1)^{n+1}}{n(2n+1)} z^{2n+1}$?

2. Express with usual functions the derivative of $f : x \mapsto \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n(2n+1)} x^{2n+1}$, and deduce an expression for $f(x)$ when $|x| < 1$.

3. Show that
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} = \ln(2) - 2 + \frac{\pi}{2}.$$

Exercise 6.

1. For which $z \in \mathbb{C}$ the series $\sum_{n\geq 1} \frac{z^n}{n^2}$ is convergent ?

2. Let $f : [-1,1] \to \mathbb{R}$, $x \mapsto \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Show that f is continuous on $[-1,1]$ and calculate $f'(x)$ for $x \in]-1,1[$.

3. Show that there exists $c \in \mathbb{R}$ such that for all $x \in]0,1[$ we have
$$f(x) + f(1-x) = c - \ln(x)\ln(1-x).$$

4. Calculate c. Deduce the value of $\sum_{n=1}^{\infty} \frac{1}{2^n n^2}$.

2 More involved exercises

Exercise 7.

1. Show that a power series $\sum a_n z^n$ have a radius of convergence strictly positive if and only if there exists $q > 0$ such that $|u_n| \leq q^n$ for all $n \in \mathbb{N}$.

2. Let $\sum a_n z^n$ be a power series with $a_0 = 1$ and a radius of convergence $R > 0$. We note
$$f : \{z \in \mathbb{C}, |z| < R\} \to \mathbb{C},\ z \mapsto \sum_{n=0}^{\infty} a_n z^n.$$
Show that $\frac{1}{f}$ can be written as a power series on $\{z \in \mathbb{C}, |z| < R\}$.

Exercise 8. Let $\sum a_n z^n$ be a power series with radius of convergence 1. We note $f :]-1,1[\to \mathbb{C}$, $x \mapsto \sum_{n=0}^{\infty} a_n x^n$. We assume that there exists $S \in \mathbb{C}$ such that $f(x) \to S$ as $x \to 1$ with $x < 1$. We assume also that $a_n = o(\frac{1}{n})$. Show that $\sum a_n$ is convergent and that
$$\sum_{n=1}^{\infty} a_n = S.$$