Robótica grupo2
Clase 18

Facultad de Ingeniería UNAM

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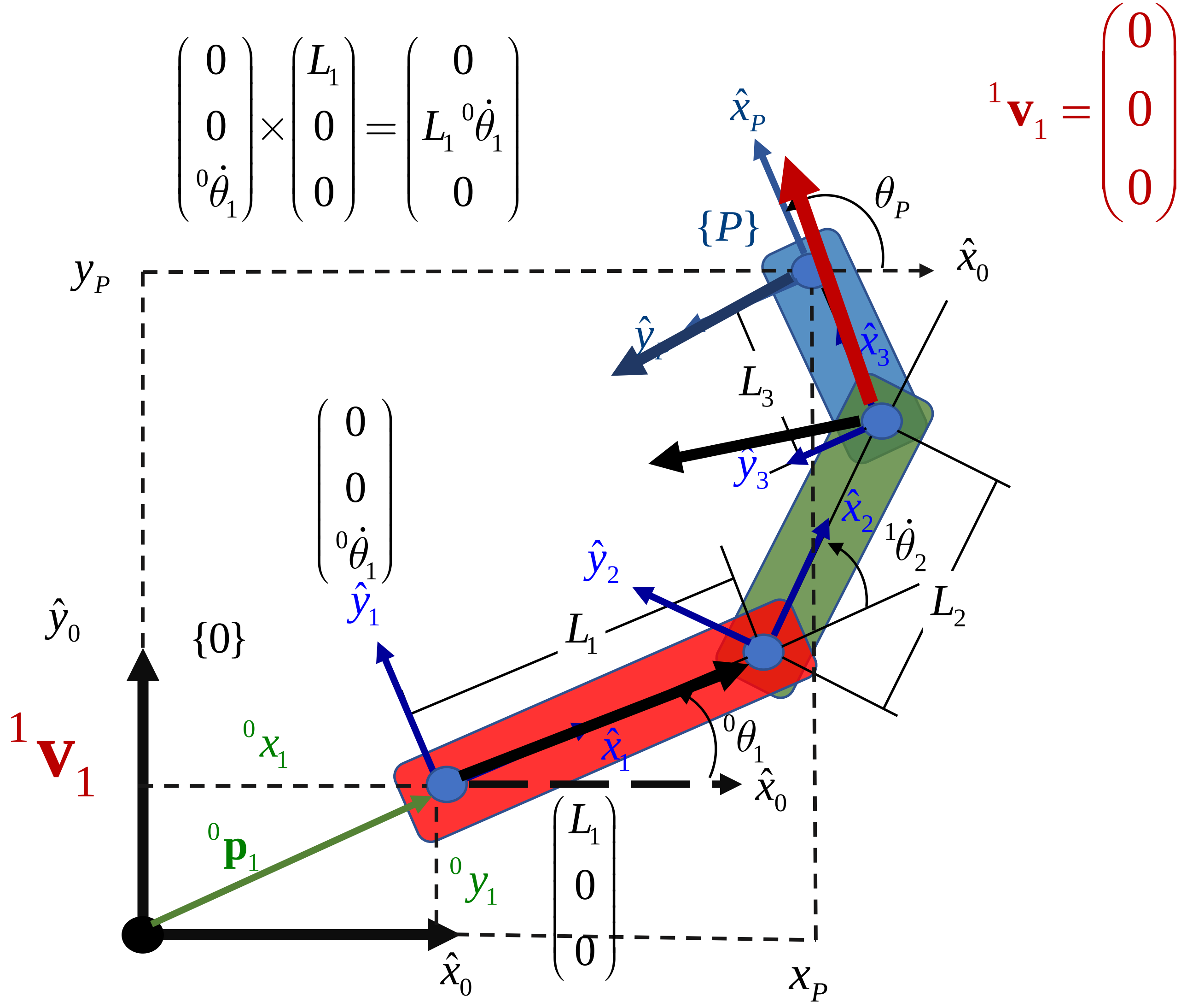
Derechos reservados

Contenido

Planteamiento del modelo dinámico de un robot serial RRR en el plano

- Ecuación de Eüler-Lagrange
- Inercia lineal e inercia rotacional
- Propagación de velocidades
- Cálculo del Lagrangeano
- Cálculo de los pares
- Modelo dinámico general

Propagación de velocidades



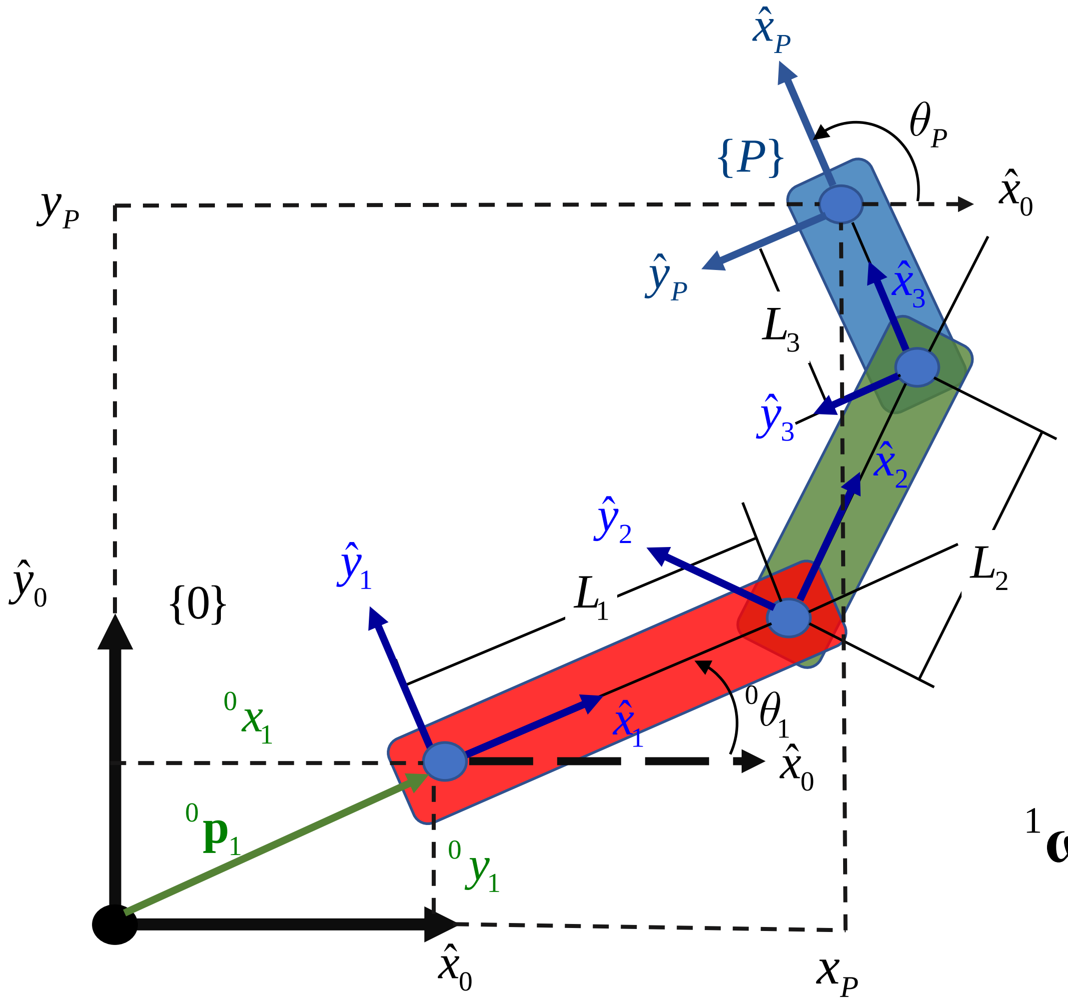
Propagación de velocidades angulares

$$ {}^{i+1}\boldsymbol{\omega}_{i+1} = {}^{i+1}\mathbf{R}_i\,{}^{i}\boldsymbol{\omega}_i + {}^{i+1}\hat{\mathbf{z}}_{i+1}\dot{\theta}_{i+1} $$

Propagación de velocidades lineales

$$ {}^{i+1}\mathbf{v}_{i+1} = {}^{i+1}\mathbf{R}_i\left({}^{i}\mathbf{v}_i + {}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{p}_{i+1}\right) $$

Propagación de velocidades



Propagación de velocidades entre del sistema {0} al sistema {1}

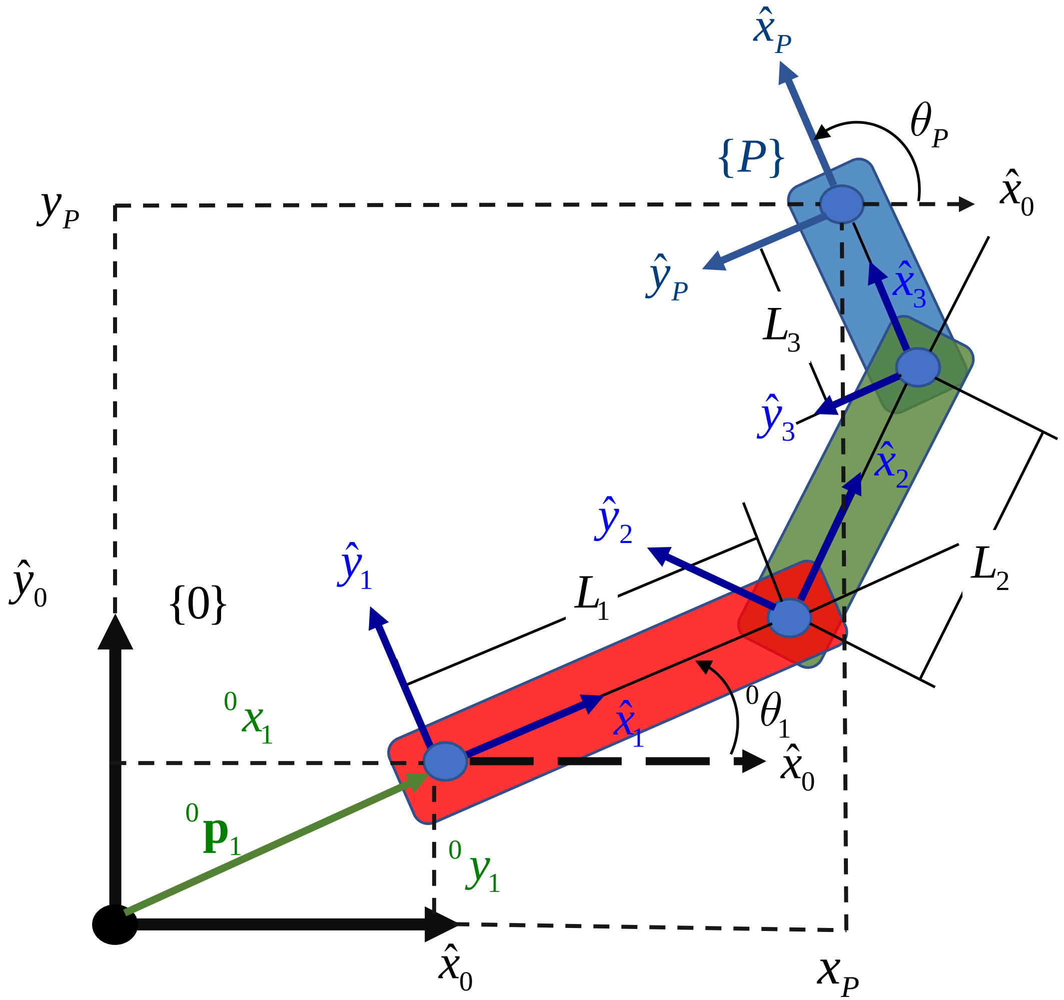
Propagación de velocidades angulares

$$ {}^1\boldsymbol{\omega}_1 = {}^1\mathbf{R}_0 \, {}^0\boldsymbol{\omega}_0 + {}^1\hat{\mathbf{z}}_1 \dot{\theta}_1 $$

$$ {}^1\mathbf{R}_0 = {}^0\mathbf{R}_1^{-1} = {}^0\mathbf{R}_1^{T} $$

$$ {}^1\boldsymbol{\omega}_1 = \begin{pmatrix} \cos({}^0\theta_1) & \sin({}^0\theta_1) & 0 \\ -\sin({}^0\theta_1) & \cos({}^0\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} {}^0\dot{\theta}_1 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix} $$

Propagación de velocidades



Propagación de velocidades entre del sistema {0} al sistema {1}

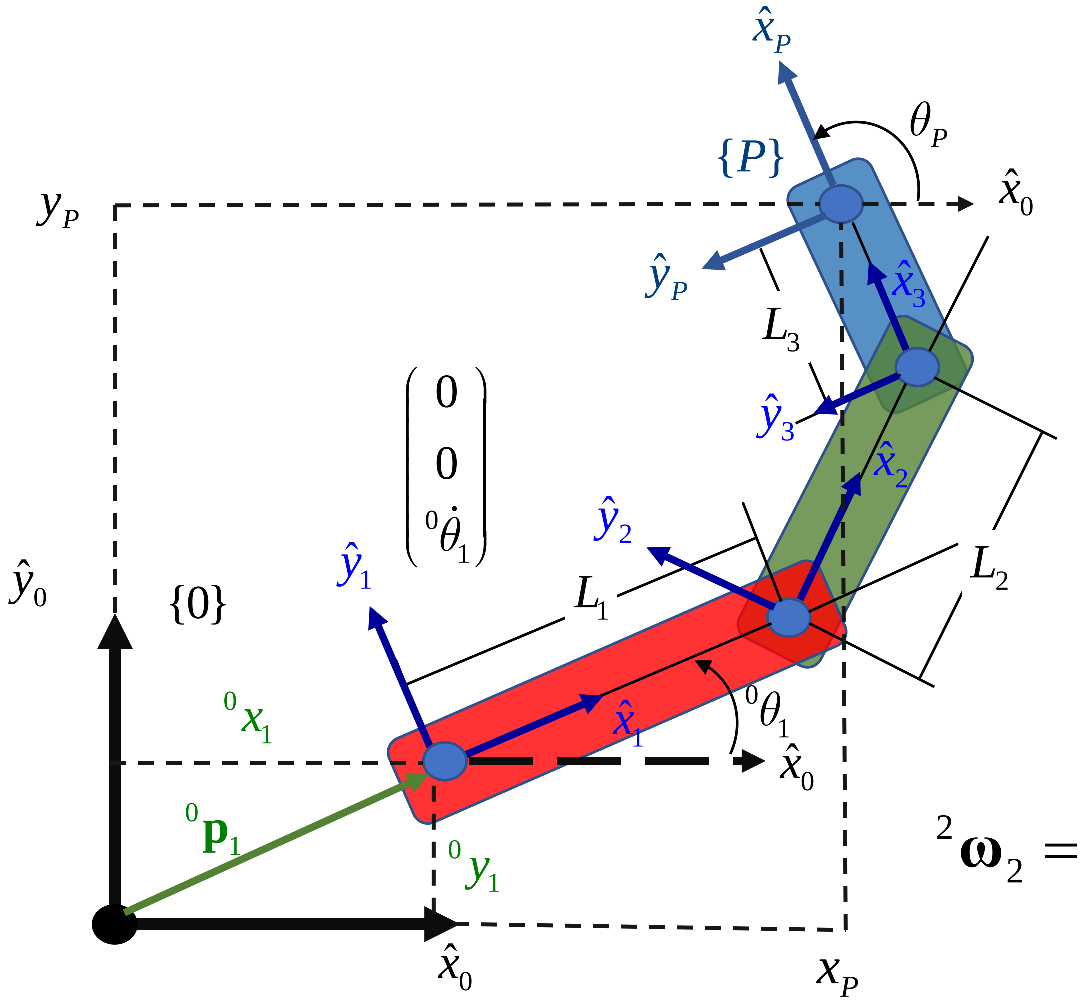
Propagación de velocidades lineales

$$^1\mathbf{v}_1 = {}^1\mathbf{R}_0\left({}^0\mathbf{v}_0 + {}^0\boldsymbol{\omega}_0 \times {}^0\mathbf{p}_1\right)$$

$$^1\mathbf{v}_1 = \begin{pmatrix} \cos(^0\theta_1) & \sin(^0\theta_1) & 0 \\ -\sin(^0\theta_1) & \cos(^0\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} ^0x_1 \\ ^0y_1 \\ 0 \end{pmatrix}\right) =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Propagación de velocidades



Propagación de velocidades entre del sistema {1} al sistema {2}

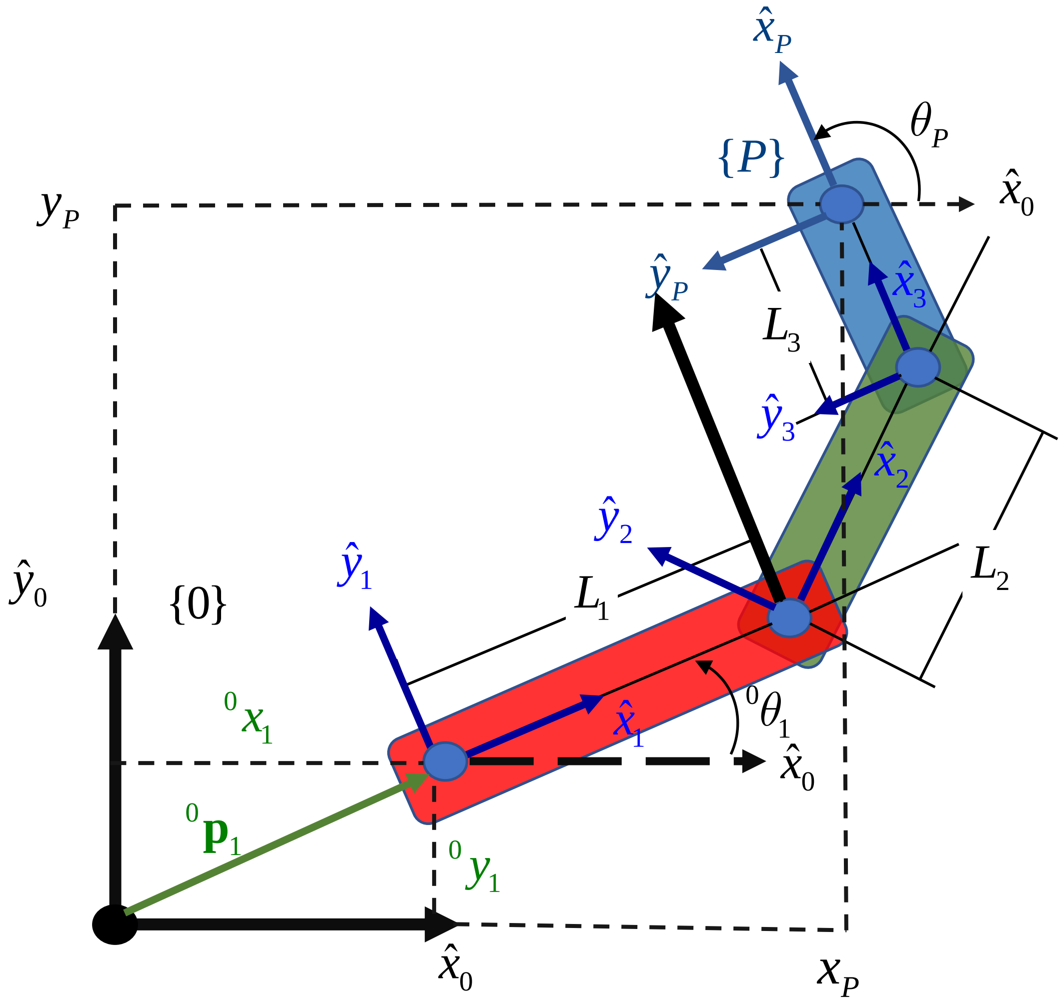
Propagación de velocidades angulares

$$ {}^2\boldsymbol{\omega}_2 = {}^2\mathbf{R}_1\,{}^1\boldsymbol{\omega}_1 + {}^2\hat{\mathbf{z}}_2\,{}^1\dot{\theta}_2 $$

$$ {}^2\mathbf{R}_1 = {}^1\mathbf{R}_2^{-1} = {}^1\mathbf{R}_2^{T} $$

$$ {}^2\boldsymbol{\omega}_2 = \begin{pmatrix} \cos({}^1\theta_2) & \sin({}^1\theta_2) & 0 \\ -\sin({}^1\theta_2) & \cos({}^1\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} {}^1\dot{\theta}_2 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \end{pmatrix} $$

Propagación de velocidades



Propagación de velocidades entre del sistema {1} al sistema {2}

Propagación de velocidades lineales

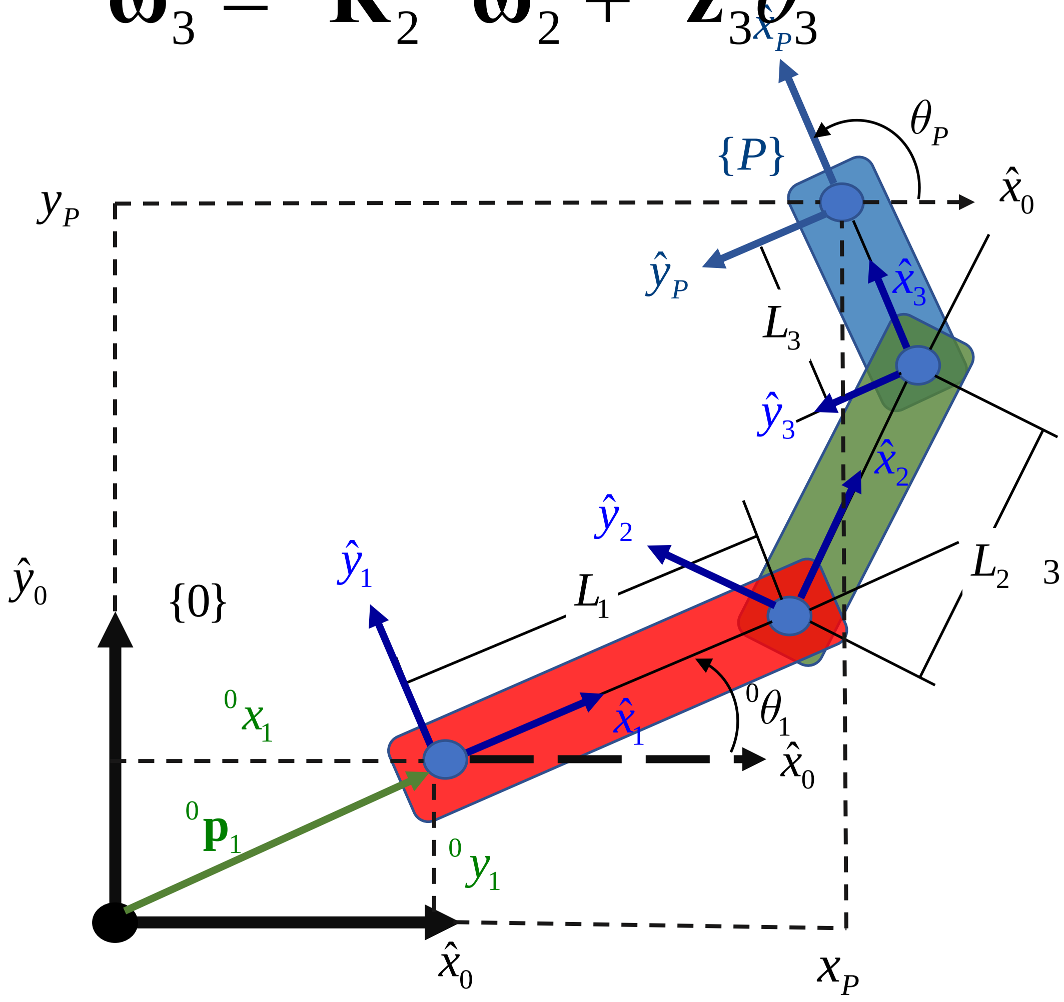
$$ {}^{2}\mathbf{v}_2 = {}^{2}\mathbf{R}_1\left({}^{1}\mathbf{v}_1 + {}^{1}\boldsymbol{\omega}_1 \times {}^{1}\mathbf{p}_2\right) $$

$$ {}^{2}\mathbf{v}_2 = \begin{pmatrix} \cos({}^{1}\theta_2) & \sin({}^{1}\theta_2) & 0 \\ -\sin({}^{1}\theta_2) & \cos({}^{1}\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^{0}\dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix}\right) $$

$$ = \begin{pmatrix} L_1 \sin({}^{1}\theta_2)\, {}^{0}\dot{\theta}_1 \\ L_1 \cos({}^{1}\theta_2)\, {}^{0}\dot{\theta}_1 \\ 0 \end{pmatrix} $$

Propagación de velocidades

$$ {}^{3}\boldsymbol{\omega}_3 = {}^{3}\mathbf{R}_2\, {}^{2}\boldsymbol{\omega}_2 + {}^{3}\hat{\mathbf{z}}_3 \dot{\theta}_3 $$



Propagación de velocidades entre del sistema {2} al sistema {3}

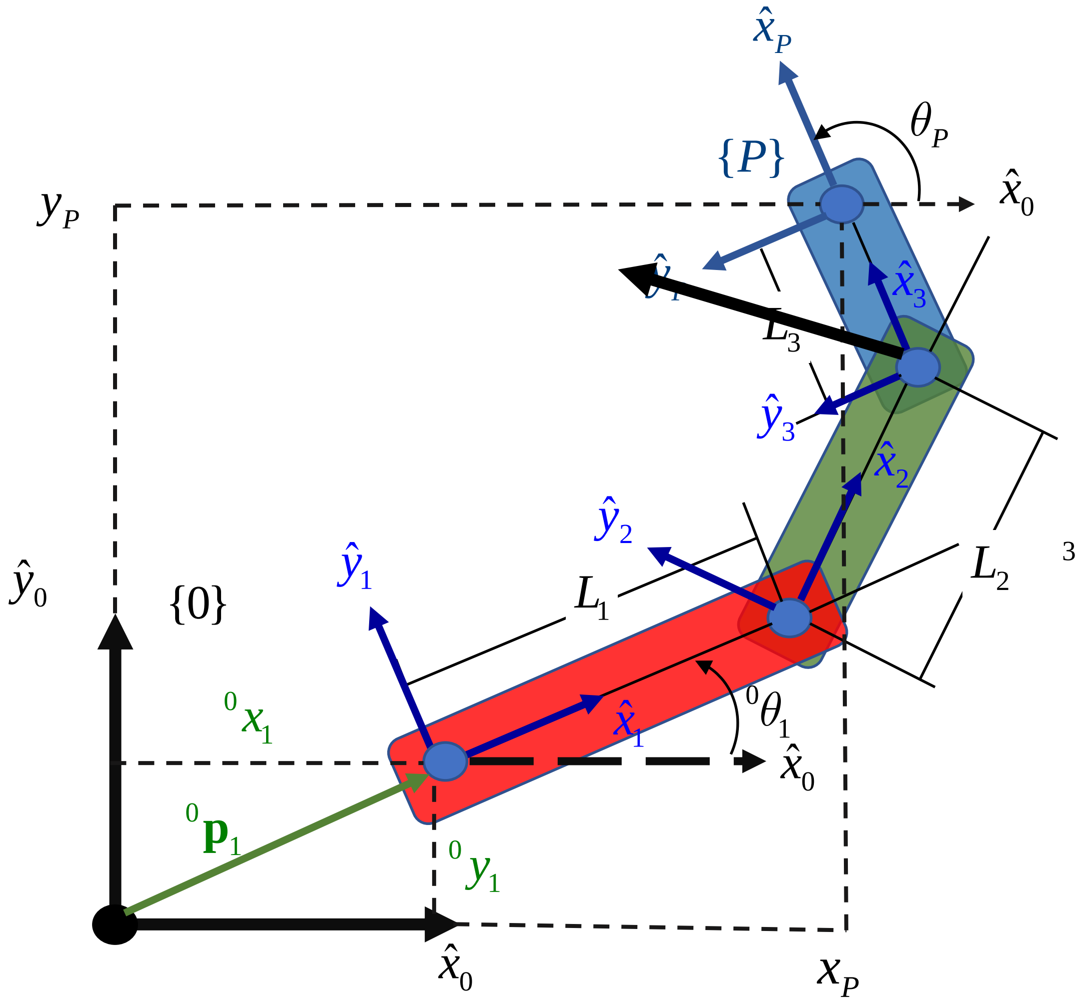
Propagación de velocidades angulares

$$ {}^{3}\boldsymbol{\omega}_3 = {}^{3}\mathbf{R}_2\, {}^{2}\boldsymbol{\omega}_2 + {}^{3}\hat{\mathbf{z}}_3 \dot{\theta}_3 $$

$$ {}^{3}\boldsymbol{\omega}_3 = \begin{pmatrix} \cos({}^{2}\theta_3) & \sin({}^{2}\theta_3) & 0 \\ -\sin({}^{2}\theta_3) & \cos({}^{2}\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^{0}\dot{\theta}_1 + {}^{1}\dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} {}^{2}\dot{\theta}_3 = $$

$$ = \begin{pmatrix} 0 \\ 0 \\ {}^{0}\dot{\theta}_1 + {}^{1}\dot{\theta}_2 + {}^{2}\dot{\theta}_3 \end{pmatrix} $$

Propagación de velocidades



Propagación de velocidades entre del sistema {2} al sistema {3}

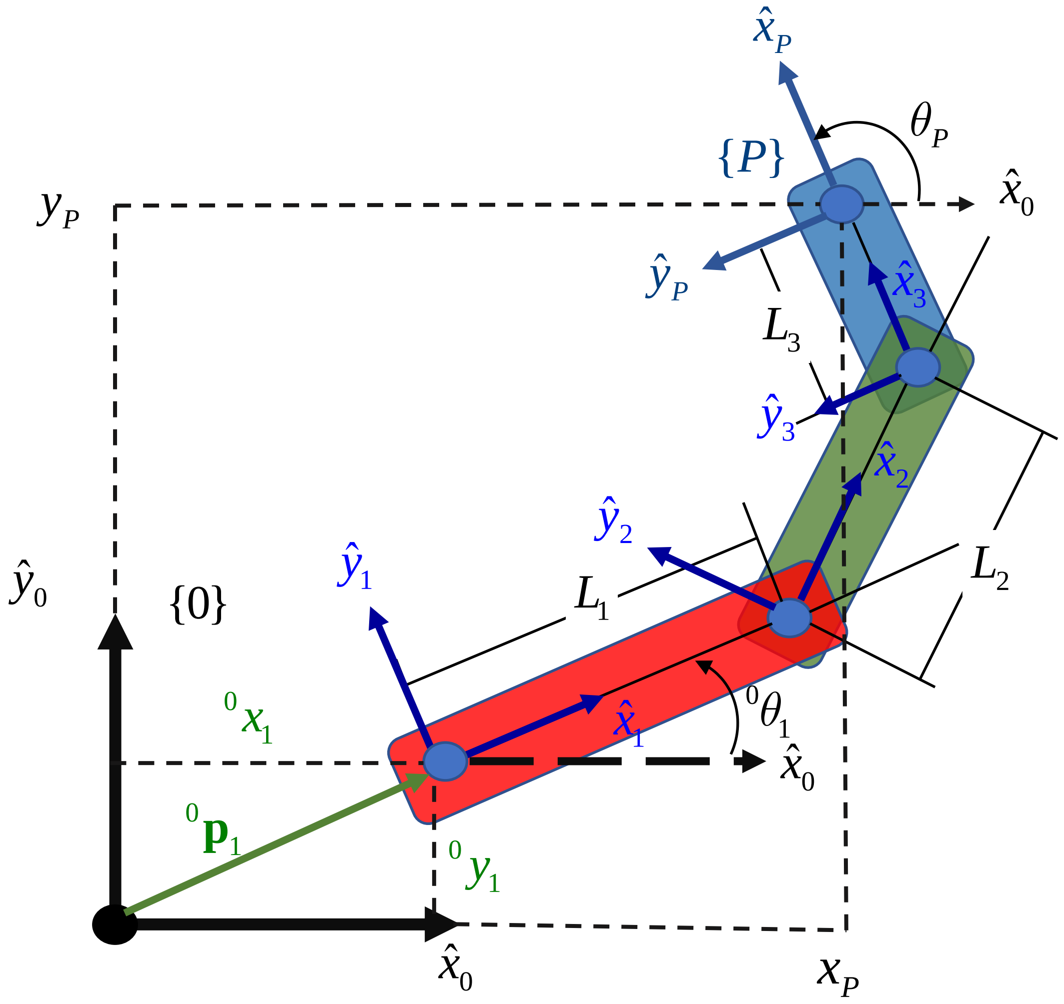
Propagación de velocidades lineales

$$ {}^3\mathbf{v}_3 = {}^3\mathbf{R}_2\left({}^2\mathbf{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\mathbf{p}_3\right) $$

$$ {}^3\mathbf{v}_3 = \begin{pmatrix} \cos({}^2\theta_3) & \sin({}^2\theta_3) & 0 \\ -\sin({}^2\theta_3) & \cos({}^2\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} L_1\sin({}^1\theta_2)\dot{\theta}_1 \\ L_1\cos({}^1\theta_2)\dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} \right) = $$

$$ = \begin{pmatrix} L_2\sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1\sin({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ L_2\cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1\cos({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ 0 \end{pmatrix} $$

Propagación de velocidades



Propagación de velocidades entre del sistema {2} al sistema {3}

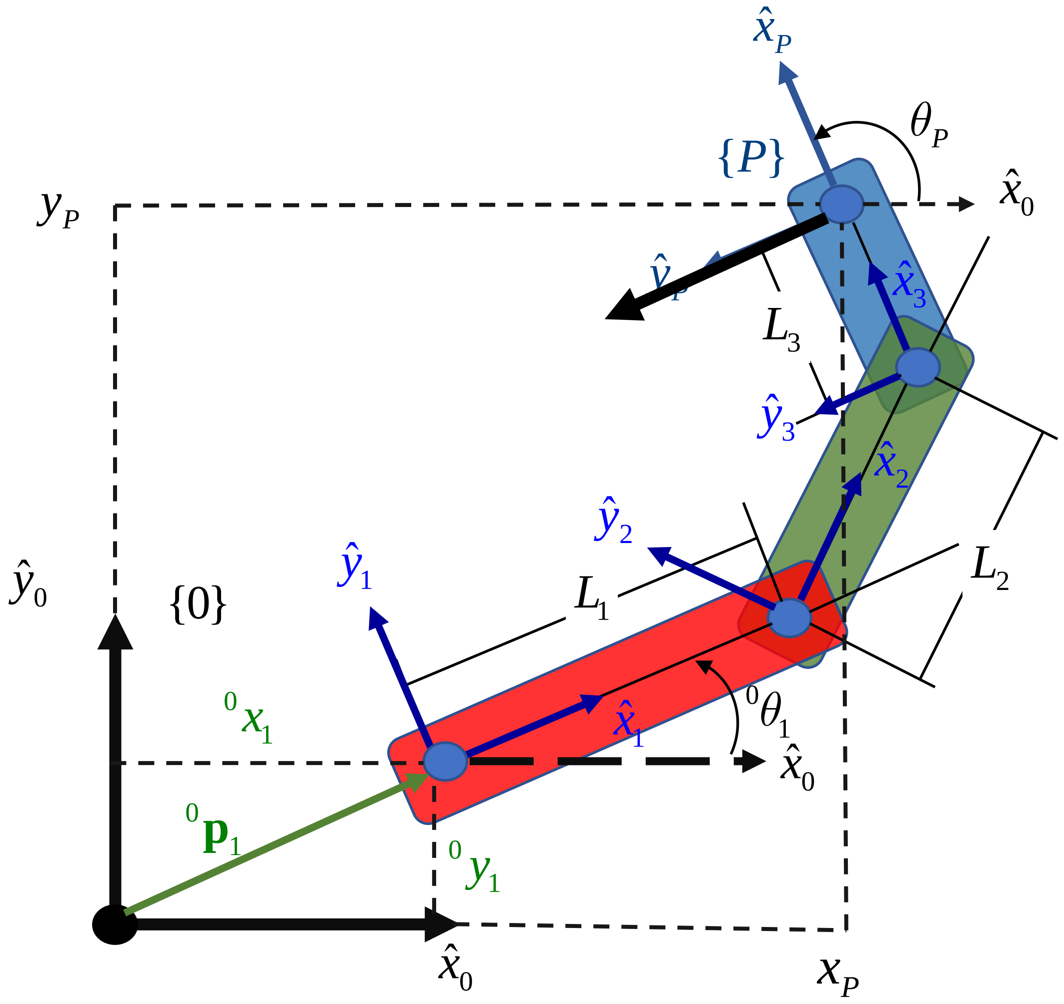
Propagación de velocidades angulares

$$^P\boldsymbol{\omega}_P = {}^P\mathbf{R}_3\,{}^3\boldsymbol{\omega}_3 + {}^P\hat{\mathbf{z}}_P 0$$

$$^P\boldsymbol{\omega}_P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} 0 =$$

$$= \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \end{pmatrix}$$

Propagación de velocidades



Propagación de velocidades entre del sistema {2} al sistema {3}

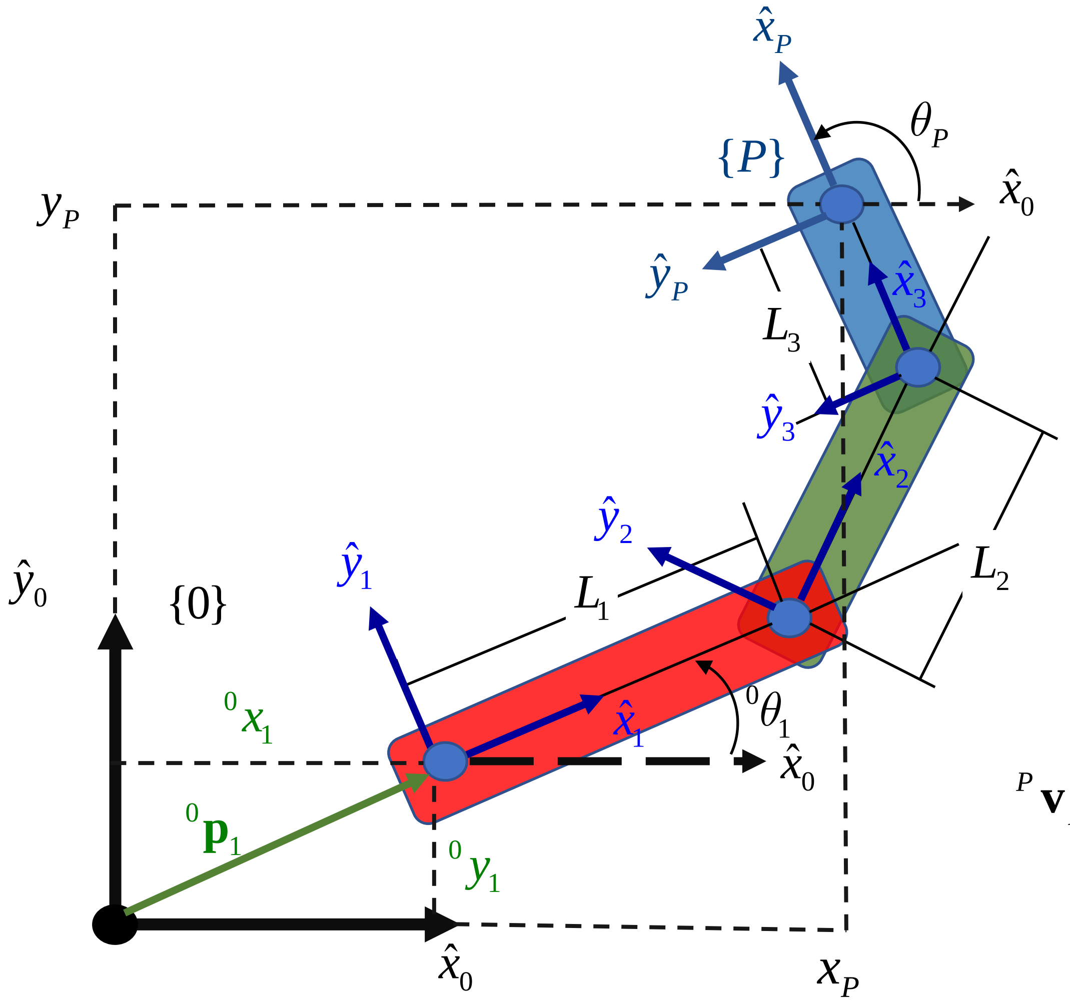
Propagación de velocidades lineales

$$ {}^P\mathbf{v}_P = {}^P\mathbf{R}_3\left({}^3\mathbf{v}_3 + {}^3\boldsymbol{\omega}_3 \times {}^3\mathbf{p}_P \right) $$

$$ {}^P\mathbf{v}_P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{pmatrix} L_2\sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1\sin({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ L_2\cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1\cos({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ 0 \end{pmatrix} + \right. $$

$$ \left. + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \end{pmatrix} \times \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix}\right); $$

Propagación de velocidades



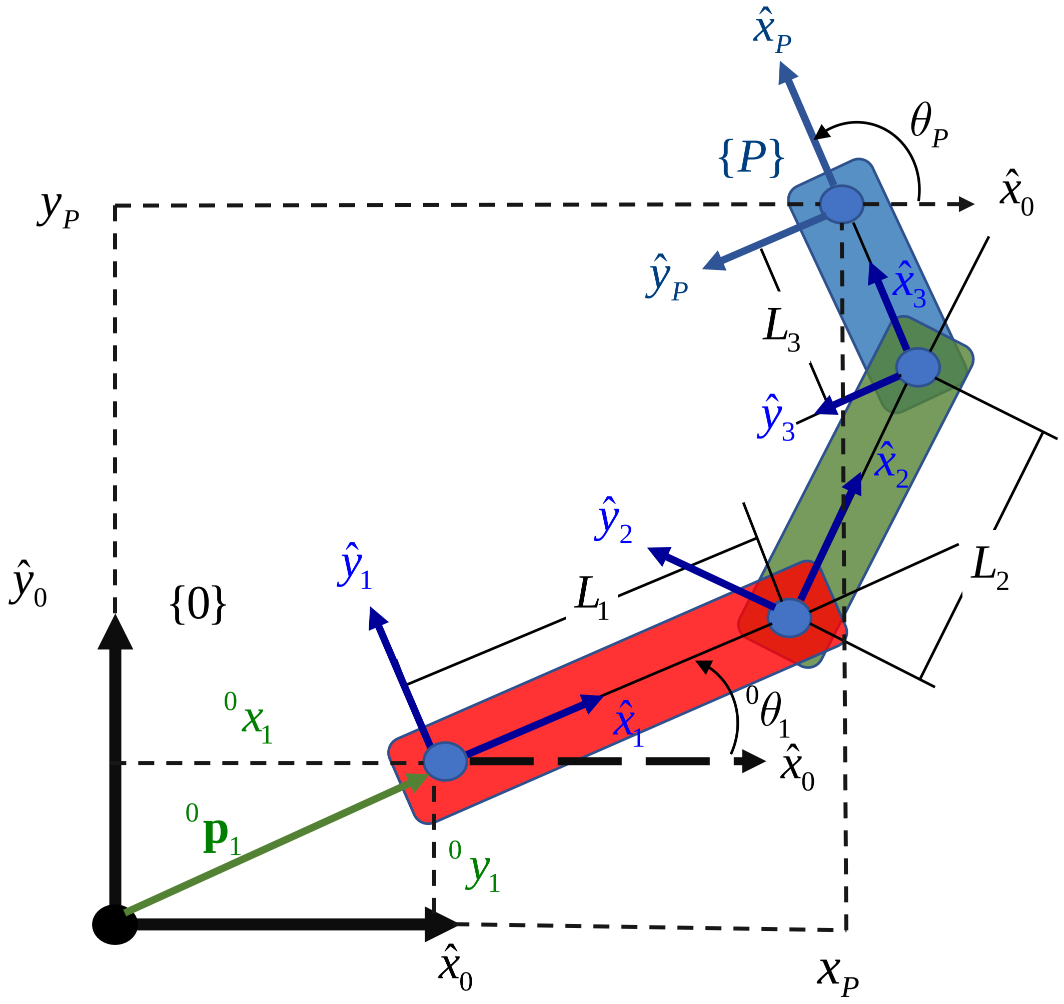
Propagación de velocidades entre del sistema {2} al sistema {3}

Propagación de velocidades lineales

$$ {}^P\mathbf{v}_P = {}^P\mathbf{R}_3\left({}^3\mathbf{v}_3 + {}^3\boldsymbol{\omega}_3 \times {}^3\mathbf{p}_P\right) $$

$$ {}^P\mathbf{v}_P = \begin{pmatrix} L_2\sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1\sin({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ L_2\cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1\cos({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 + L_3({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3) \\ 0 \end{pmatrix} $$

Propagación de velocidades



Propagación de velocidades entre del sistema {P} al sistema {0}

Propagación de velocidades lineales

$${}^0\mathbf{v}_P = {}^0\mathbf{R}_P\, {}^P\mathbf{v}_P$$

$${}^0\mathbf{R}_P = {}^0\mathbf{R}_1\, {}^1\mathbf{R}_2\, {}^2\mathbf{R}_3\, {}^3\mathbf{R}_P =$$

$$= \begin{pmatrix} \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & -\sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & 0 \\ \sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Propagación de velocidades

$$ {}^{0}\mathbf{R}_P = {}^{0}\mathbf{R}_1\, {}^{1}\mathbf{R}_2\, {}^{2}\mathbf{R}_3\, {}^{3}\mathbf{R}_P $$

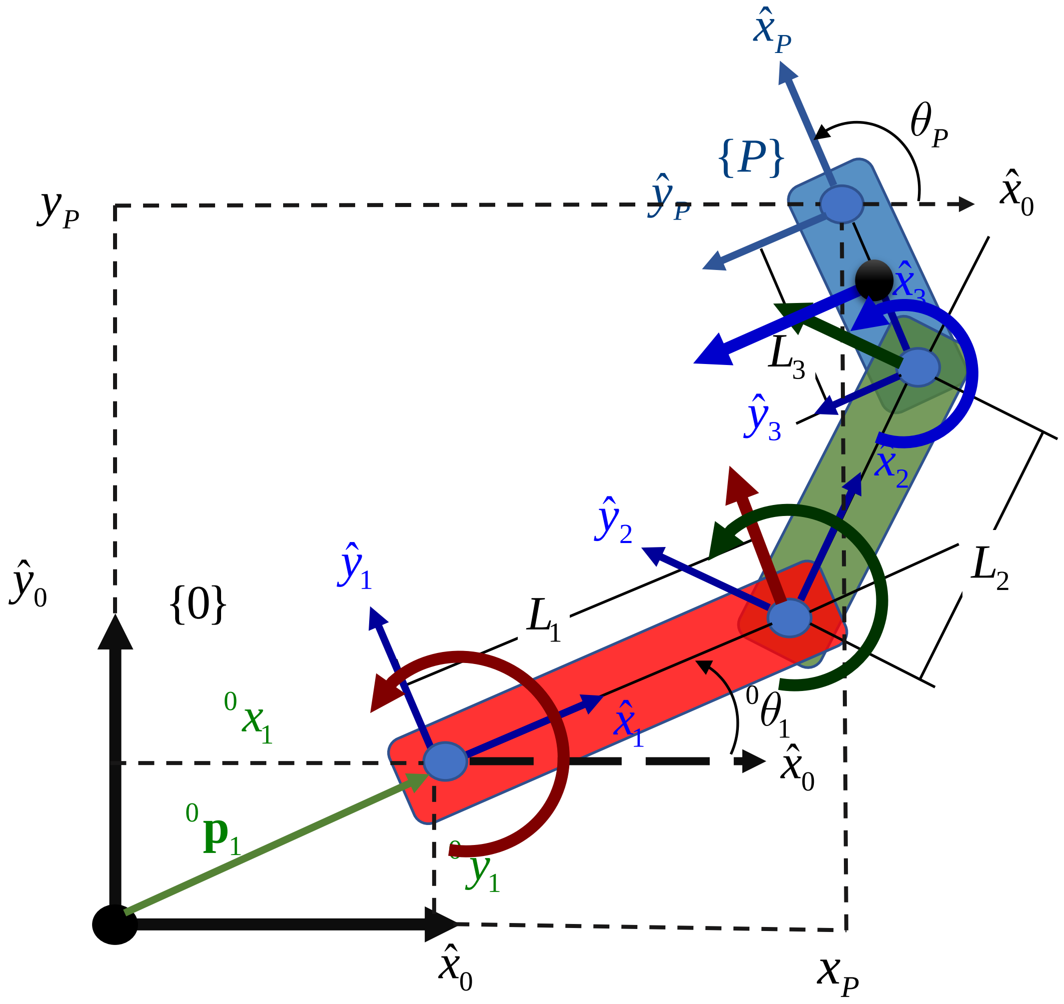
$$ {}^{0}\mathbf{v}_P = {}^{0}\mathbf{R}_P\, {}^{P}\mathbf{v}_P = $$

Propagación de velocidades lineales

$$ {}^{P}\mathbf{v}_P = \begin{pmatrix} L_2 \sin({}^{2}\theta_3)({}^{0}\dot{\theta}_1 + {}^{2}\dot{\theta}_3) + L_1 \sin({}^{1}\theta_2 + {}^{2}\theta_3)\, {}^{0}\dot{\theta}_1 \\ L_2 \cos({}^{2}\theta_3)({}^{0}\dot{\theta}_1 + {}^{2}\dot{\theta}_3) + L_1 \cos({}^{1}\theta_2 + {}^{2}\theta_3)\, {}^{0}\dot{\theta}_1 + L_3({}^{0}\dot{\theta}_1 + {}^{1}\dot{\theta}_2 + {}^{2}\dot{\theta}_3) \\ 0 \end{pmatrix} $$

$$ = \begin{pmatrix} \left(-L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3)\right)\dot{\theta}_1 + \left(-L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3)\right)\dot{\theta}_2 + \left(-L_3 \sin(\theta_1 + \theta_2 + \theta_3)\right)\dot{\theta}_3 \\ \left(L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)\right)\dot{\theta}_1 + \left(L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)\right)\dot{\theta}_2 + \left(L_3 \cos(\theta_1 + \theta_2 + \theta_3)\right)\dot{\theta}_3 \\ 0 \end{pmatrix} $$

Cálculo de los pares



Cálculo del Lagrangeano

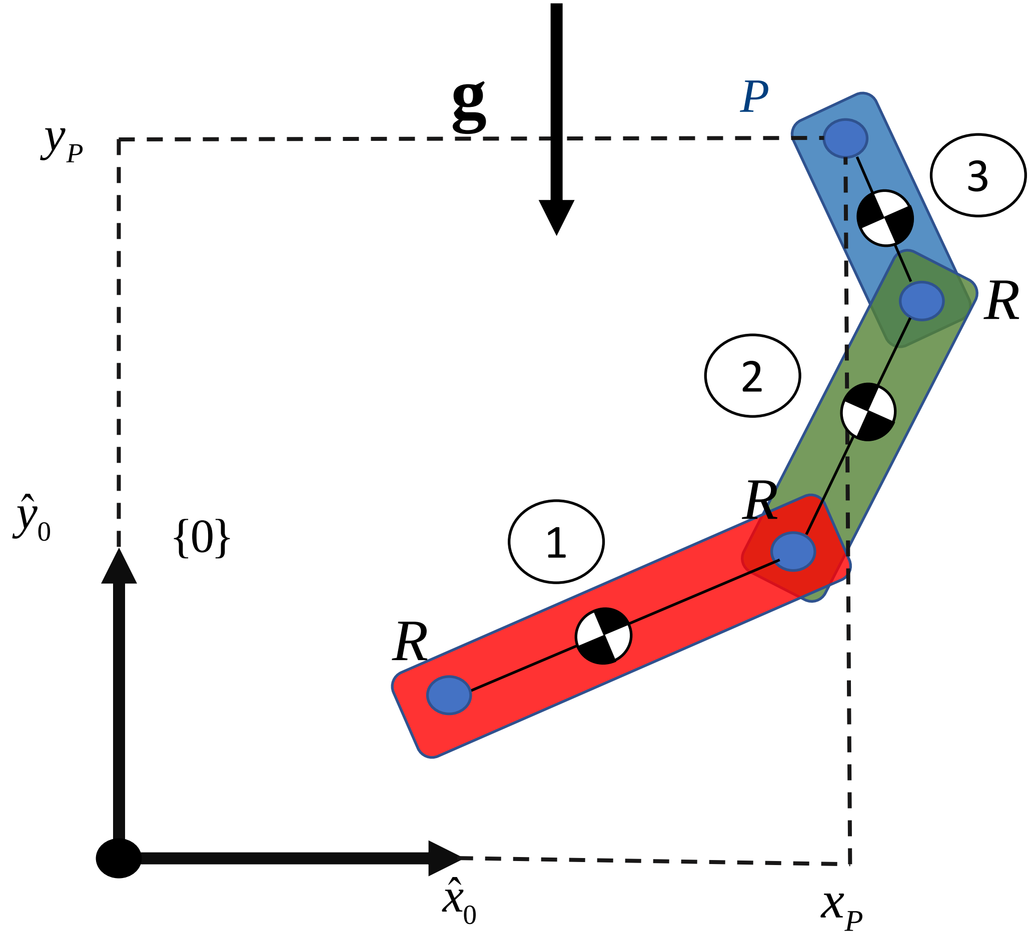
$$\Gamma = \left(k_1 + k_2 + k_3\right) - \left(u_1 + u_2 + u_3\right)$$

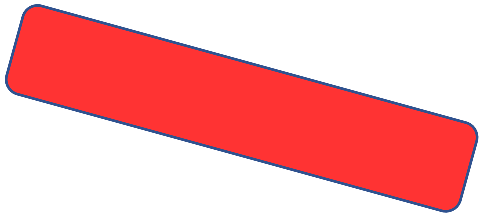
Ecuación del par

$$\tau_i = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\Gamma\right) - \frac{\partial}{\partial q_i}\Gamma$$

$$\boldsymbol{\tau}_i = \begin{pmatrix} \tau_{\theta 1} \\ \tau_{\theta 2} \\ \tau_{\theta 3} \end{pmatrix}$$

Planteamiento del modelo

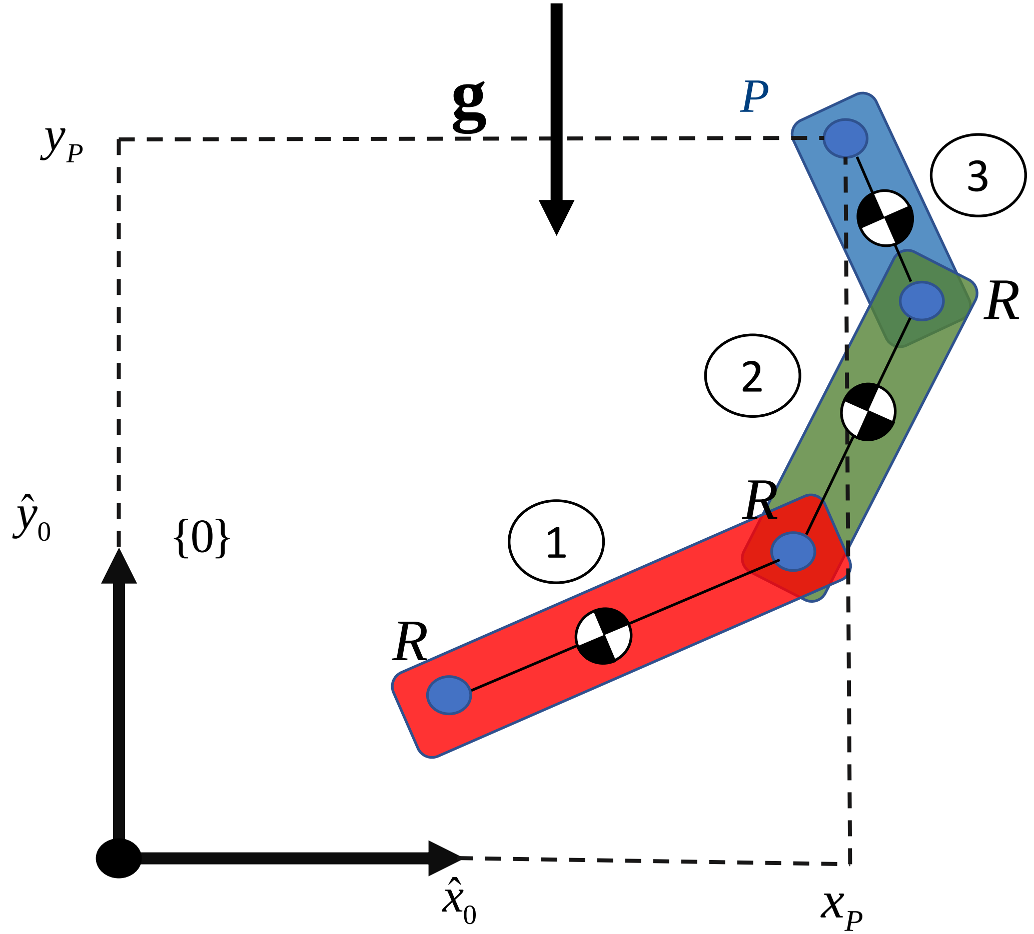


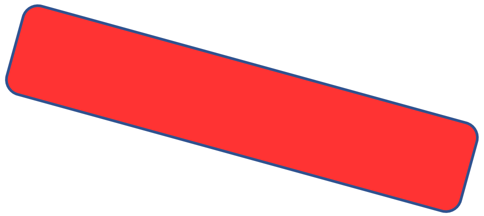


$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}{}^T \mathbf{v}_{c1} + \frac{1}{2} {}^1\boldsymbol{\omega}_1{}^T \mathbf{I}_{c1} {}^1\boldsymbol{\omega}_1$$

$$u_1 = -m_1 \mathbf{g}^T \, {}^0\mathbf{p}_{c1}$$

Planteamiento del modelo

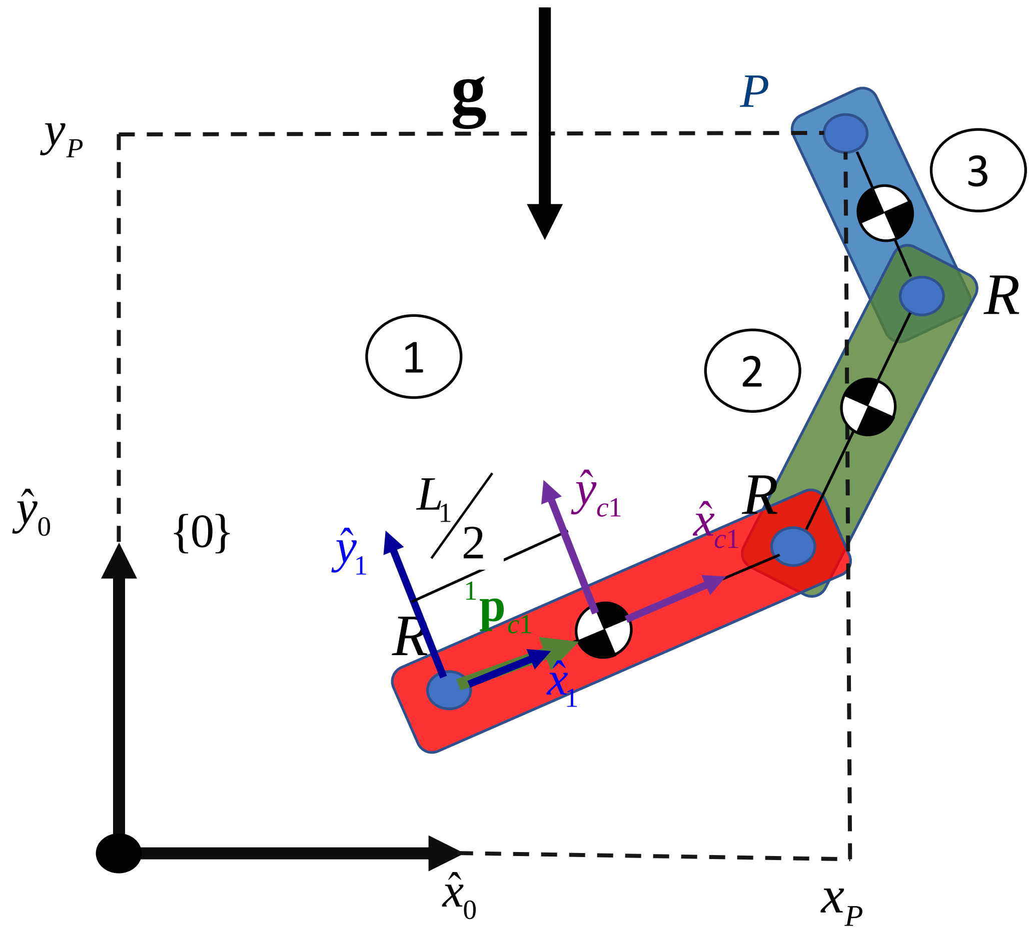




$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}{}^T \mathbf{v}_{c1} + \frac{1}{2} {}^1\boldsymbol{\omega}_1{}^T \mathbf{I}_{c1} {}^1\boldsymbol{\omega}_1$$

$$u_1 = -m_1 \mathbf{g}^T \, {}^0\mathbf{p}_{c1}$$

Planteamiento del modelo



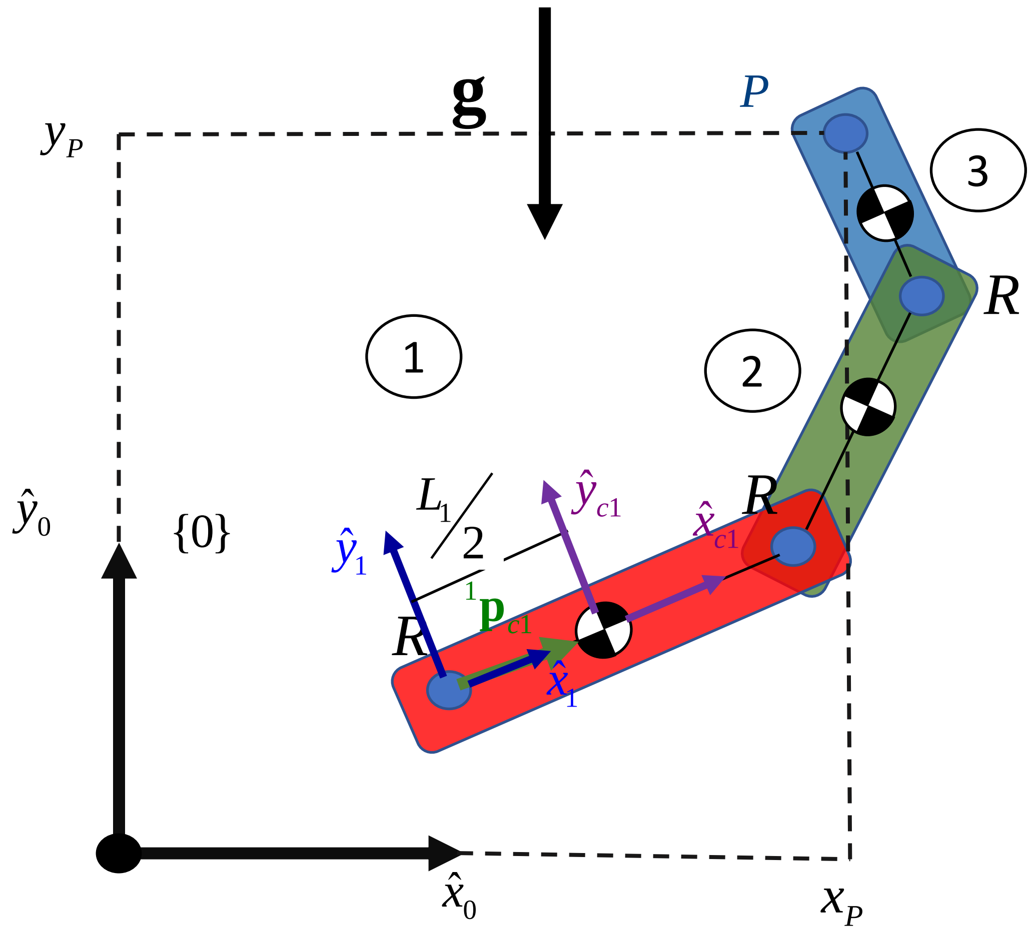
$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}{}^T \mathbf{v}_{c1} + \frac{1}{2} \boldsymbol{\omega}_1{}^T \mathbf{I}_{c1} \boldsymbol{\omega}_1$$

$$^{c1}\mathbf{v}_{c1} = {}^{c1}\mathbf{R}_1 \left({}^1\mathbf{v}_1 + {}^1\boldsymbol{\omega}_1 \times {}^1\mathbf{p}_{c1} \right)$$

$$^{c1}\mathbf{v}_{c1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} L_1/2 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ L_1/2 \; {}^0\dot{\theta}_1 \\ 0 \end{pmatrix}$$

$$^1\boldsymbol{\omega}_1 = \boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix}$$

Planteamiento del modelo

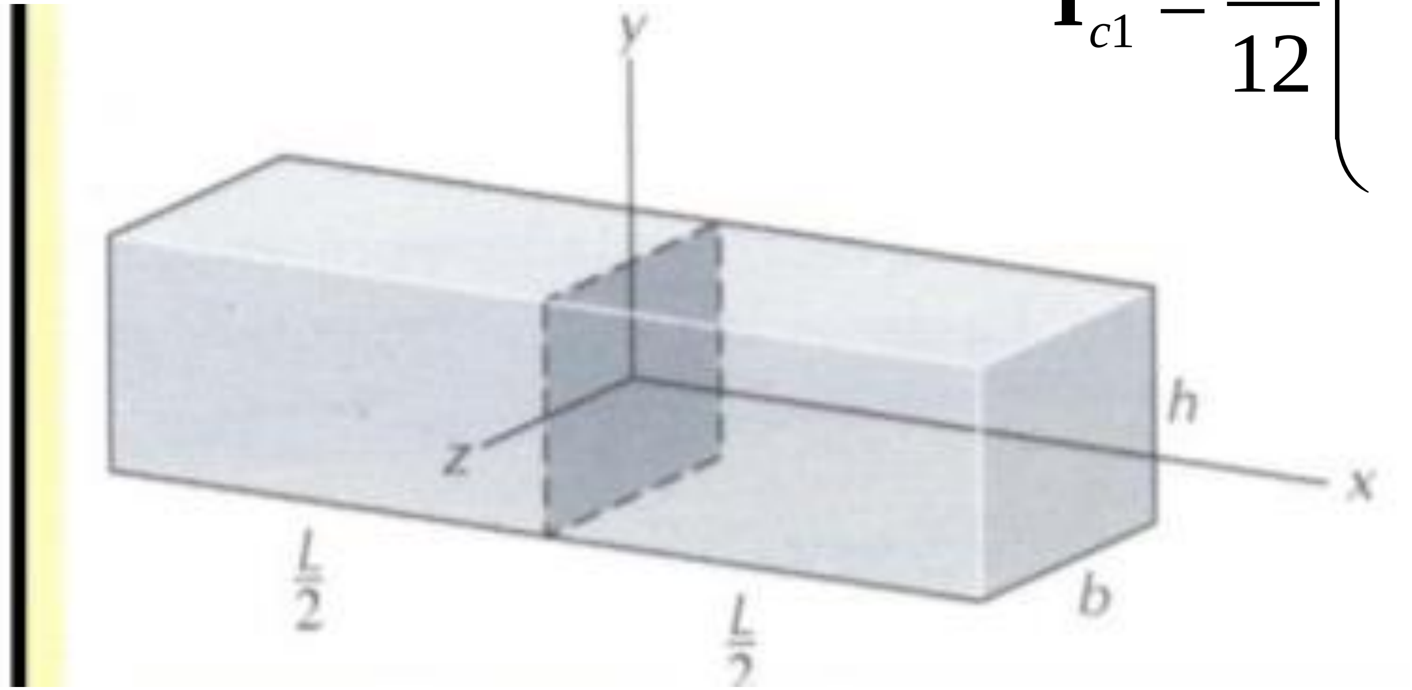


$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}{}^T \mathbf{v}_{c1} + \frac{1}{2} \boldsymbol{\omega}_1{}^T \mathbf{I}_{c1} \boldsymbol{\omega}_1$$

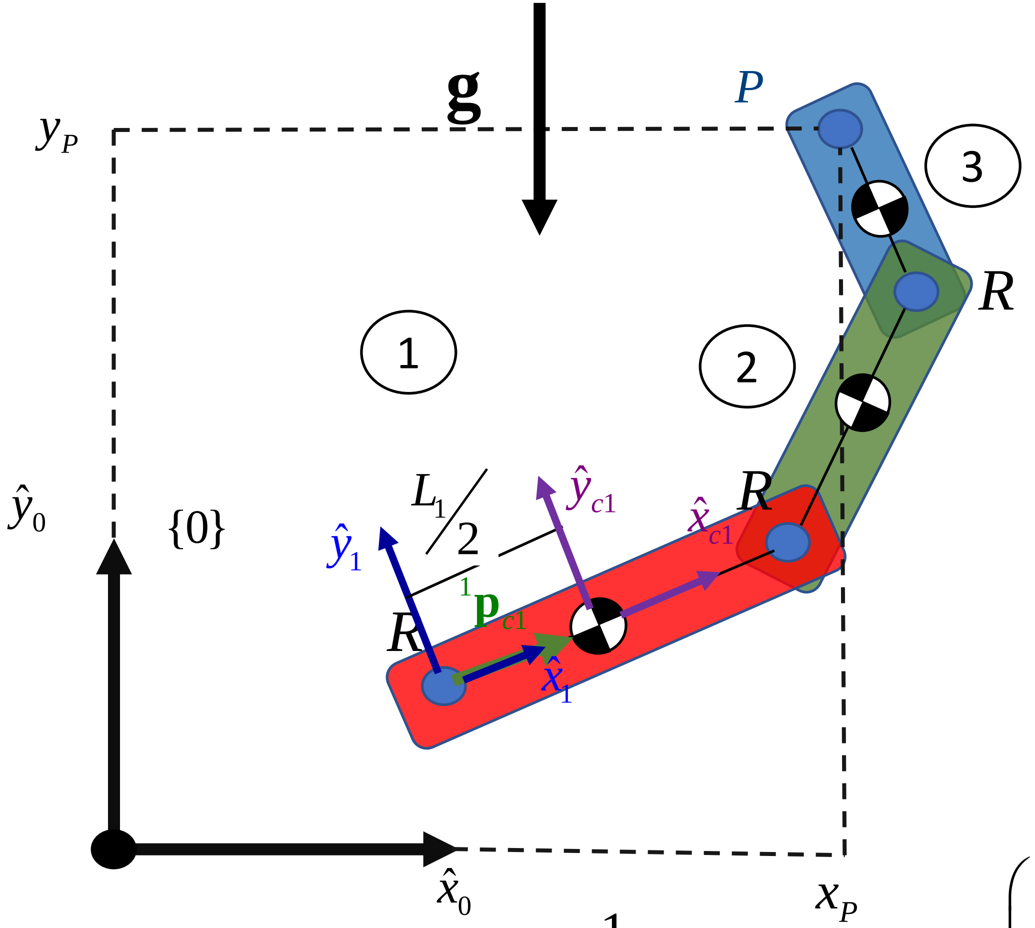
$${}^{c1}\mathbf{v}_{c1} = \begin{pmatrix} 0 \\ L_1 \, {}^0\dot{\theta}_1 \\ 0 \end{pmatrix} \qquad {}^1\boldsymbol{\omega}_1 = \boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_2 \end{pmatrix}$$

$$\mathbf{I}_{c1} = \frac{m_1}{12} \begin{pmatrix} & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & \end{pmatrix}$$

$$\mathbf{I}_{c1} = \frac{m_1}{12}\begin{pmatrix} b_1^{\,2} + h_1^{\,2} & 0 & 0 \\ 0 & L_1^{\,2} + b_1^{\,2} & 0 \\ 0 & 0 & L_1^{\,2} + h_1^{\,2} \end{pmatrix}$$



Planteamiento del modelo

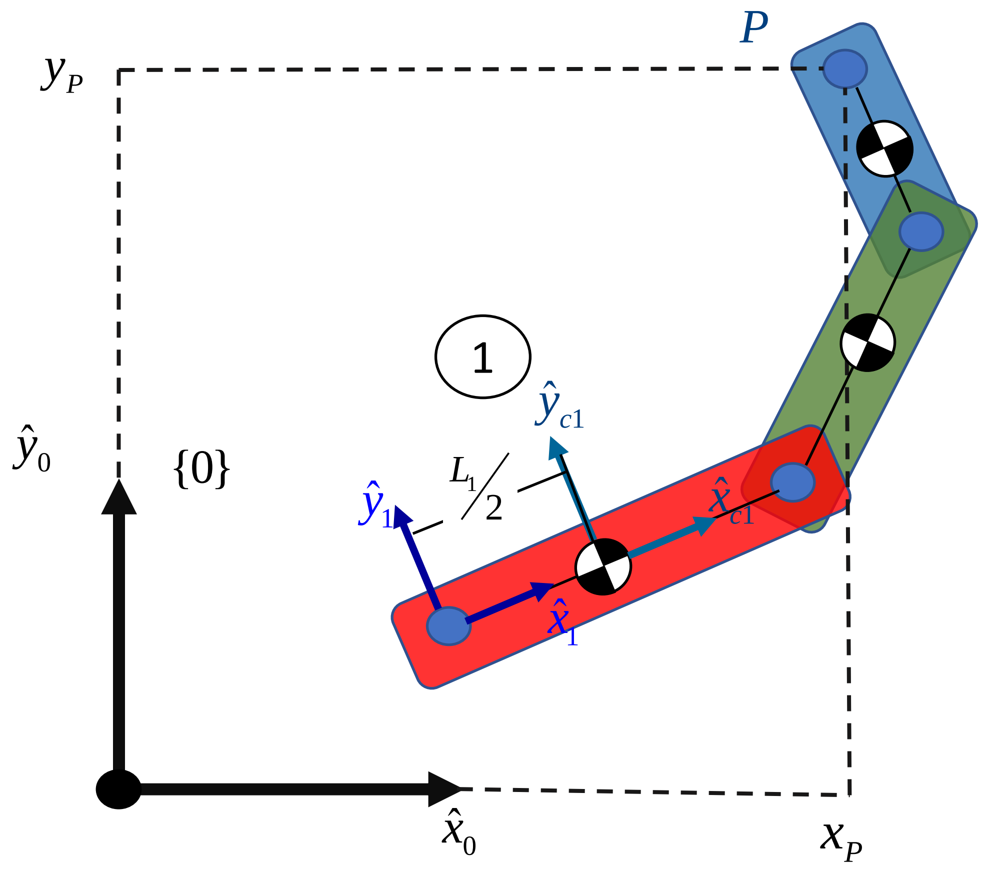


$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}{}^T \mathbf{v}_{c1} + \frac{1}{2} \boldsymbol{\omega}_1{}^T \mathbf{I}_{c1} \boldsymbol{\omega}_1$$

$$^{c1}\mathbf{v}_{c1} = \begin{pmatrix} 0 \\ L_1 \, {}^0\dot{\theta}_1 \\ 0 \end{pmatrix} \qquad {}^1\boldsymbol{\omega}_1 = \boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_2 \end{pmatrix}$$

$$k_1 = \frac{1}{2} m_1 \begin{pmatrix} 0 & L_1 \, {}^0\dot{\theta}_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ L_1 \, {}^0\dot{\theta}_1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & {}^0\dot{\theta}_1 \end{pmatrix} \frac{m_1}{12} \begin{pmatrix} b_1{}^2 + h_1{}^2 & 0 & 0 \\ 0 & L_1{}^2 + b_1{}^2 & 0 \\ 0 & 0 & L_1{}^2 + h_1{}^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix}$$

Planteamiento del modelo



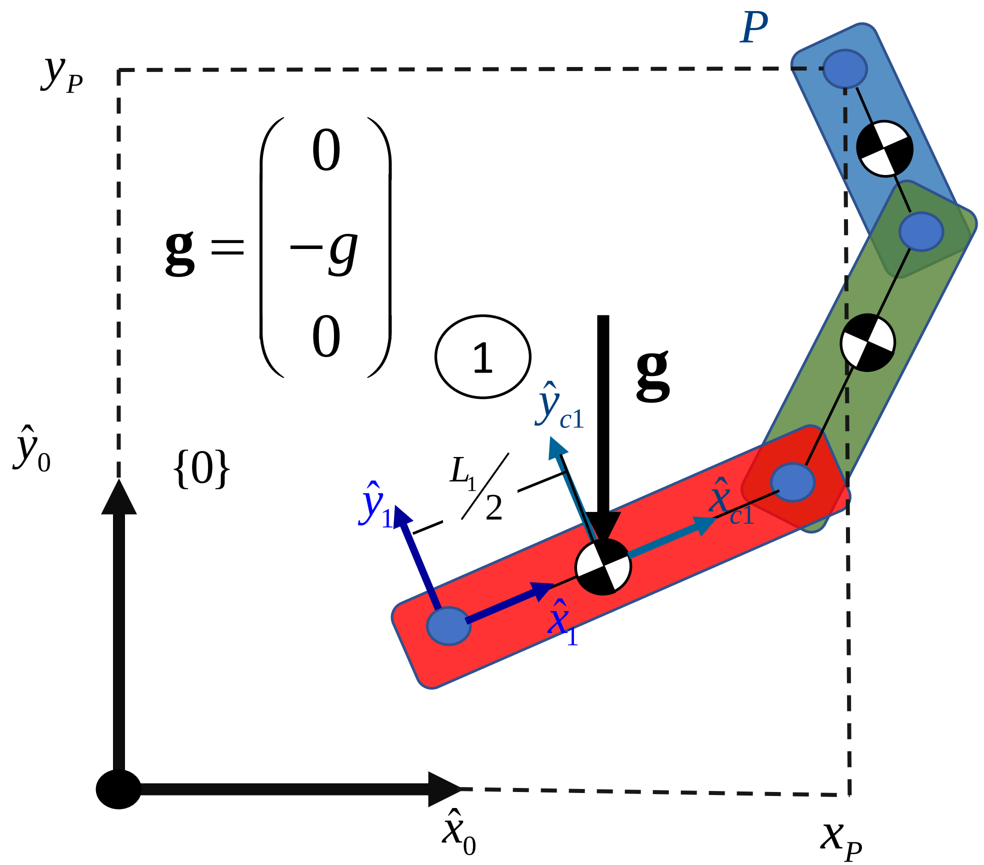
Energía cinética para el primer eslabón

$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}{}^T \mathbf{v}_{c1} + \frac{1}{2} \boldsymbol{\omega}_1{}^T \mathbf{I}_{c1} \boldsymbol{\omega}_1$$

$$k_1 = \frac{L_1{}^2 m_1}{8} {}^0\dot{\theta}_1{}^2 + \frac{m_1\left(L_i{}^2 + h_i{}^2\right)}{24} {}^0\dot{\theta}_1{}^2 =$$

$$= \frac{m_1\left(4L_i{}^2 + h_i{}^2\right)}{24} {}^0\dot{\theta}_1{}^2$$

Cálculo del Lagrangeano



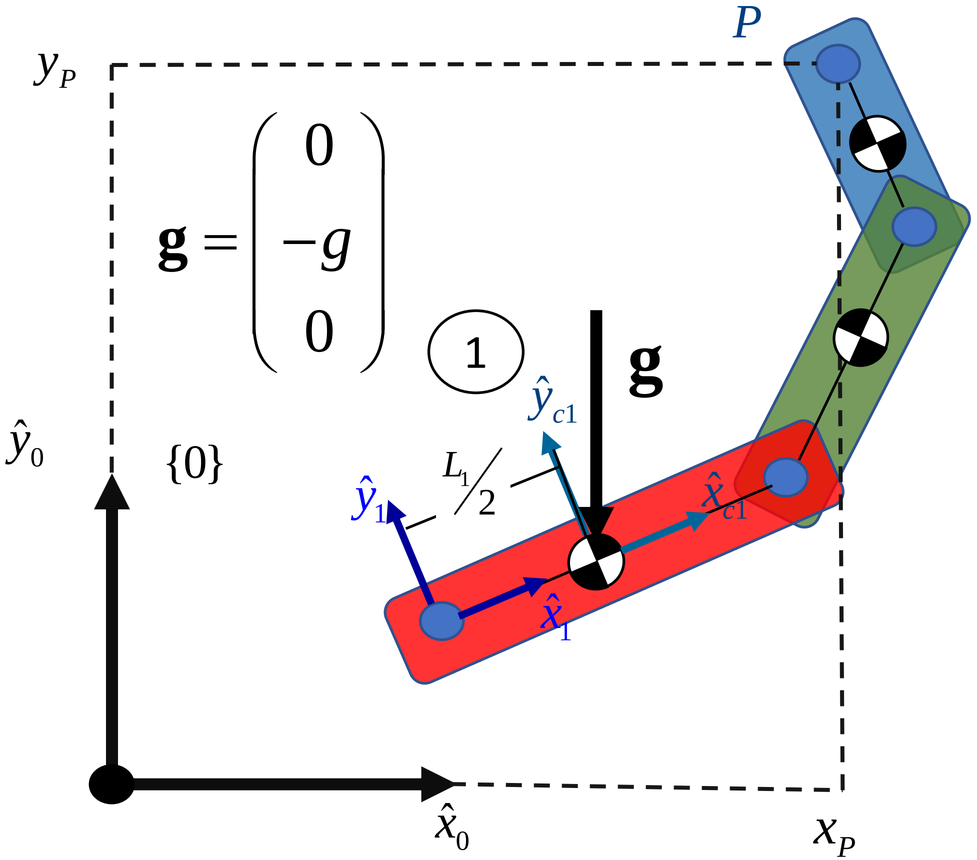
Energía potencial para el primer eslabón

$$u_1 = -m_1 \mathbf{g}^T \, {}^0\mathbf{p}_{c1}$$

$$ {}^0\mathbf{T}_{c1} = {}^0\mathbf{T}_1 \, {}^1\mathbf{T}_{c1} = \begin{pmatrix} {}^0\mathbf{R}_{c1} & {}^0\mathbf{p}_{c1} \\ \mathbf{0}^T & 1 \end{pmatrix} = $$

$$ = \begin{pmatrix} \cos({}^0\theta_1) & -\sin({}^0\theta_1) & 0 & {}^0x_1 \\ \sin({}^0\theta_1) & \cos({}^0\theta_1) & 0 & {}^0y_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \frac{L_1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Cálculo del Lagrangeano



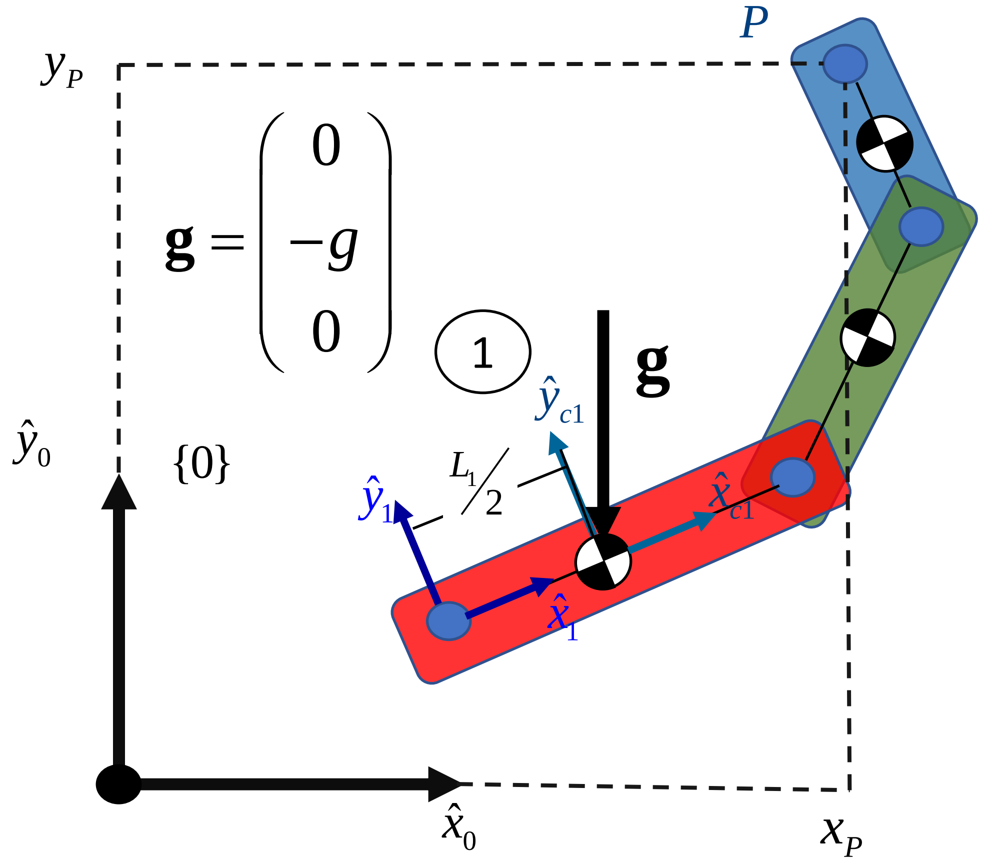
Energía potencial para el primer eslabón

$$u_1 = -m_1 \mathbf{g}^T\, {}^0\mathbf{p}_{c1}$$

$${}^0\mathbf{T}_{c1} = {}^0\mathbf{T}_1\, {}^1\mathbf{T}_{c1} = \begin{pmatrix} {}^0\mathbf{R}_{c1} & {}^0\mathbf{p}_{c1} \\ \mathbf{0}^T & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos({}^0\theta_1) & -\sin({}^0\theta_1) & 0 & {}^0x_1 + \frac{L_1}{2}\cos({}^0\theta_1) \\ \sin({}^0\theta_1) & \cos({}^0\theta_1) & 0 & {}^0y_1 + \frac{L_1}{2}\sin({}^0\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Cálculo del Lagrangeano



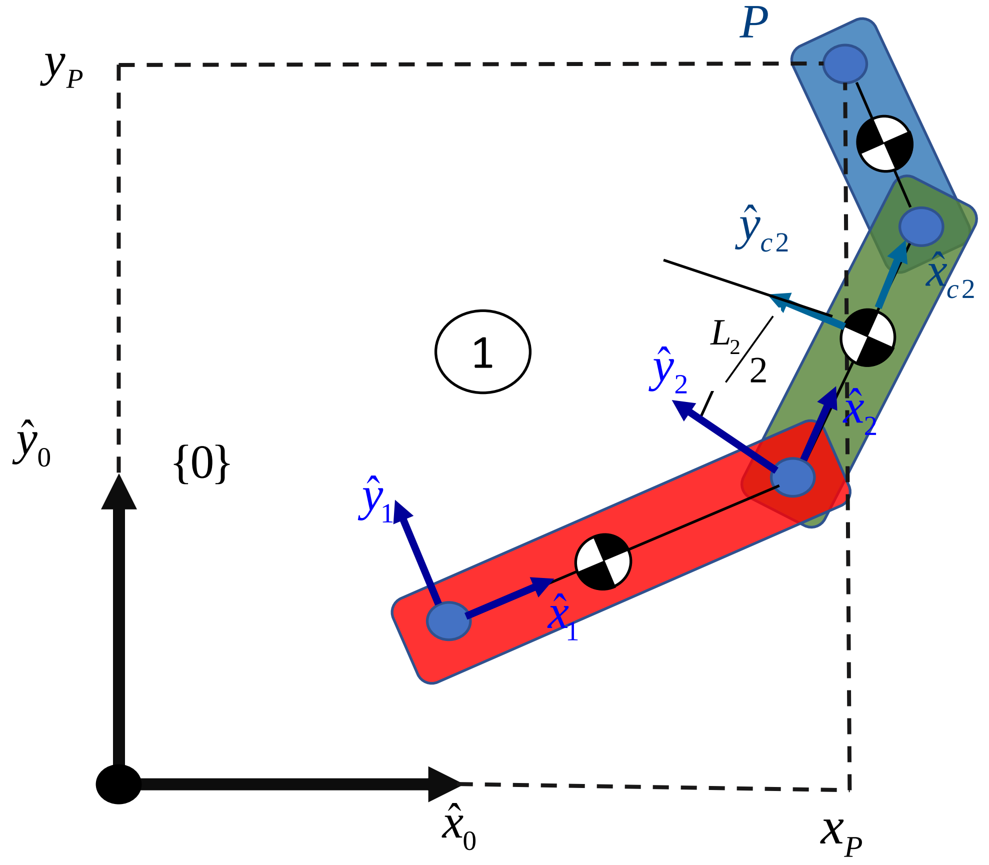
Energía potencial para el primer eslabón

$$u_1 = -m_1 \mathbf{g}^T \, {}^0\mathbf{p}_{c1}$$

$$= -m_1 \begin{pmatrix} 0 & -g & 0 \end{pmatrix} \begin{pmatrix} {}^0x_1 + \frac{L_1}{2}\cos({}^0\theta_1) \\ {}^0y_1 + \frac{L_1}{2}\sin({}^0\theta_1) \\ 0 \end{pmatrix} =$$

$$= m_1 g\left({}^0y_1 + \frac{L_1}{2}\sin({}^0\theta_1)\right)$$

Cálculo del Lagrangeano



Energía cinética para el segundo eslabón

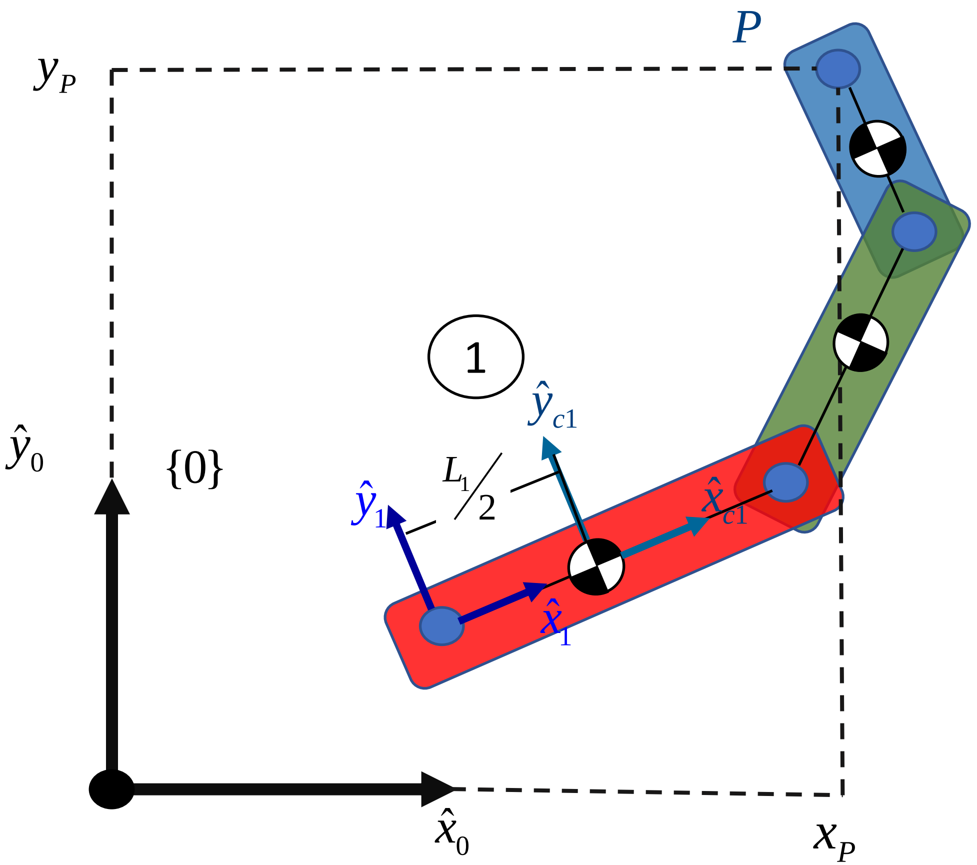
$$k_2 = \frac{1}{2} m_2 \mathbf{v}_{c2}{}^T \mathbf{v}_{c2} + \frac{1}{2} \boldsymbol{\omega}_2{}^T \mathbf{I}_{c2} \boldsymbol{\omega}_2$$

Energía cinética de la parte lineal

$$\mathbf{v}_{c2} = {}^{c2}\mathbf{R}_2 \left({}^2\mathbf{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\mathbf{p}_{c2} \right) =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} L_1 \sin({}^1\theta_2)\,{}^0\dot{\theta}_1 \\ L_1 \cos({}^1\theta_2)\,{}^0\dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} \frac{L_2}{2} \\ 0 \\ 0 \end{pmatrix} \right) =$$

Cálculo del Lagrangeano



Energía cinética para el segundo eslabón

$$k_2 = \frac{1}{2} m_2 \mathbf{v}_{c2}{}^T \mathbf{v}_{c2} + \frac{1}{2} \boldsymbol{\omega}_2{}^T \mathbf{I}_{c2} \boldsymbol{\omega}_2$$

Energía cinética de la parte lineal

$$\mathbf{v}_{c2} = {}^{c2}\mathbf{R}_2 \left({}^2\mathbf{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\mathbf{p}_{c2} \right) =$$

$$= \begin{pmatrix} \frac{L_2}{2} \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) + L_1 \sin({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ \frac{L_2}{2} \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) + L_1 \cos({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ 0 \end{pmatrix}$$

Cálculo del Lagrangeano

Energía cinética para el segundo eslabón

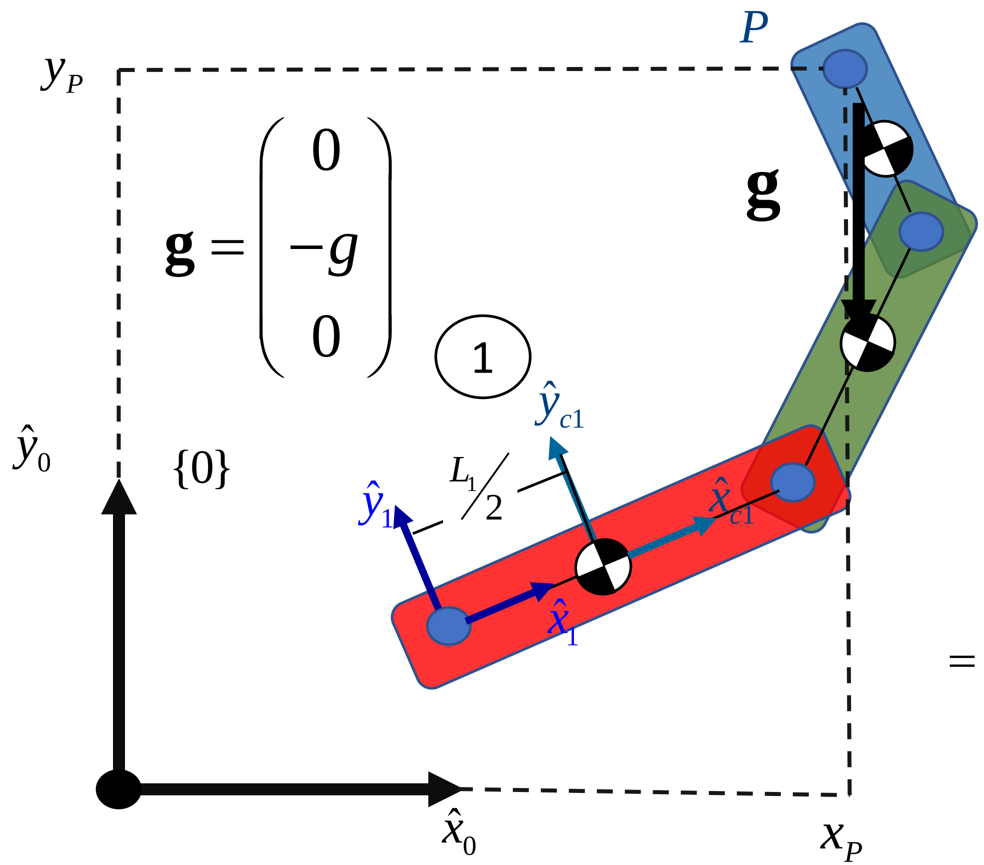
$$k_2 = \frac{1}{2} m_2 \mathbf{v}_{c2}{}^T \mathbf{v}_{c2} + \frac{1}{2} \boldsymbol{\omega}_2{}^T \mathbf{I}_{c2} \boldsymbol{\omega}_2 =$$

$$= \frac{1}{2} m_2 \begin{pmatrix} \frac{L_2}{2}\sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) + L_1 \sin({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 & \frac{L_2}{2}\cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) + L_1 \cos({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 & 0 \end{pmatrix} \begin{pmatrix} \frac{L_2}{2}\sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) + L_1 \sin({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ \frac{L_2}{2}\cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) + L_1 \cos({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ 0 \end{pmatrix} +$$

$$+ \frac{1}{2}\begin{pmatrix} 0 & 0 & {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \end{pmatrix} \frac{m_2}{12} \begin{pmatrix} h_2{}^2 + b_2{}^2 & 0 & 0 \\ 0 & L_2{}^2 + b_2{}^2 & 0 \\ 0 & 0 & L_2{}^2 + h_2{}^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \end{pmatrix} =$$

$$= \frac{m_2}{24}\left(h_2{}^2 \left({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2\right)^2 + 4\left(3L_1{}^2\, {}^0\dot{\theta}_1{}^2 + L_2{}^2 \left({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2\right)^2\right) + 12 L_1 L_2\, {}^0\dot{\theta}_1 \left({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2\right)\cos({}^1\theta_2)\right)$$

Cálculo del Lagrangeano



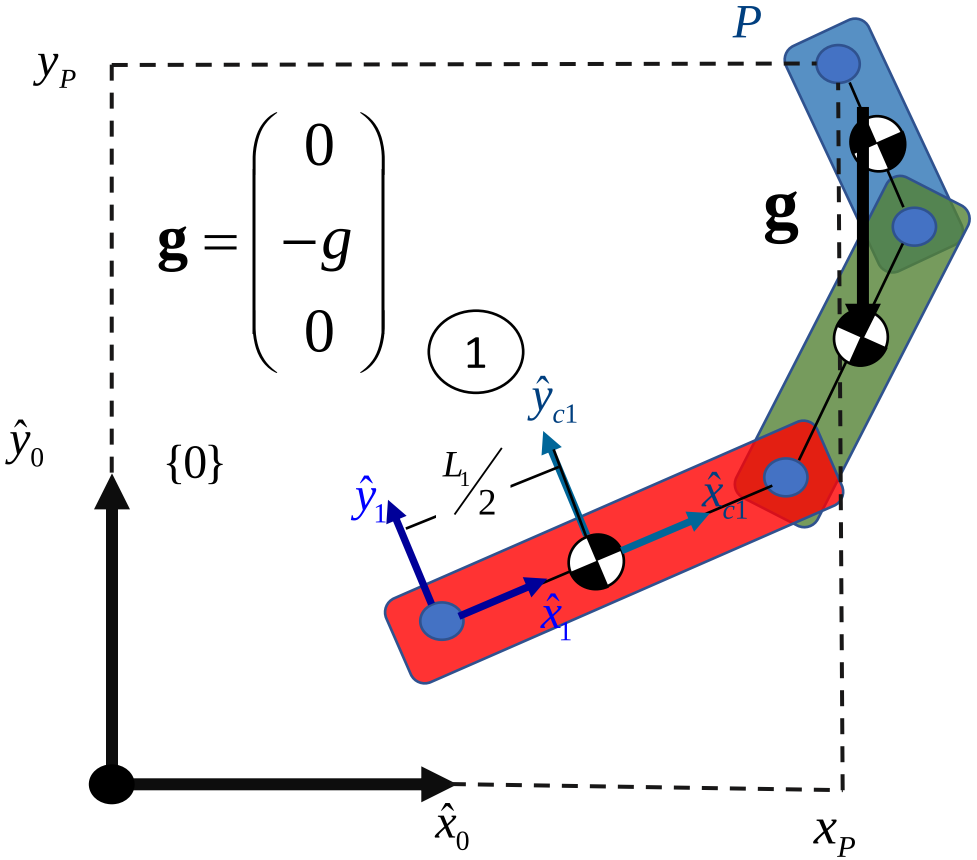
Energía potencial para el primer eslabón

$$u_2 = -m_2 \mathbf{g}^T\, {}^0\mathbf{p}_{c2}$$

$$ {}^0\mathbf{T}_{c2} = {}^0\mathbf{T}_1\, {}^1\mathbf{T}_2\, {}^2\mathbf{T}_{c2} = \begin{pmatrix} {}^0\mathbf{R}_{c2} & {}^0\mathbf{p}_{c2} \\ \mathbf{0}^T & 1 \end{pmatrix} = $$

$$ = \begin{pmatrix} \cos({}^0\theta_1 + {}^1\theta_2) & -\sin({}^0\theta_1 + {}^1\theta_2) & 0 & {}^0x_1 + L_1\cos({}^0\theta_1) + \frac{L_2}{2}\cos({}^0\theta_1 + {}^1\theta_2) \\ \sin({}^0\theta_1 + {}^1\theta_2) & \cos({}^0\theta_1 + {}^1\theta_2) & 0 & {}^0y_1 + L_1\sin({}^0\theta_1) + \frac{L_2}{2}\sin({}^0\theta_1 + {}^1\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Cálculo del Lagrangeano



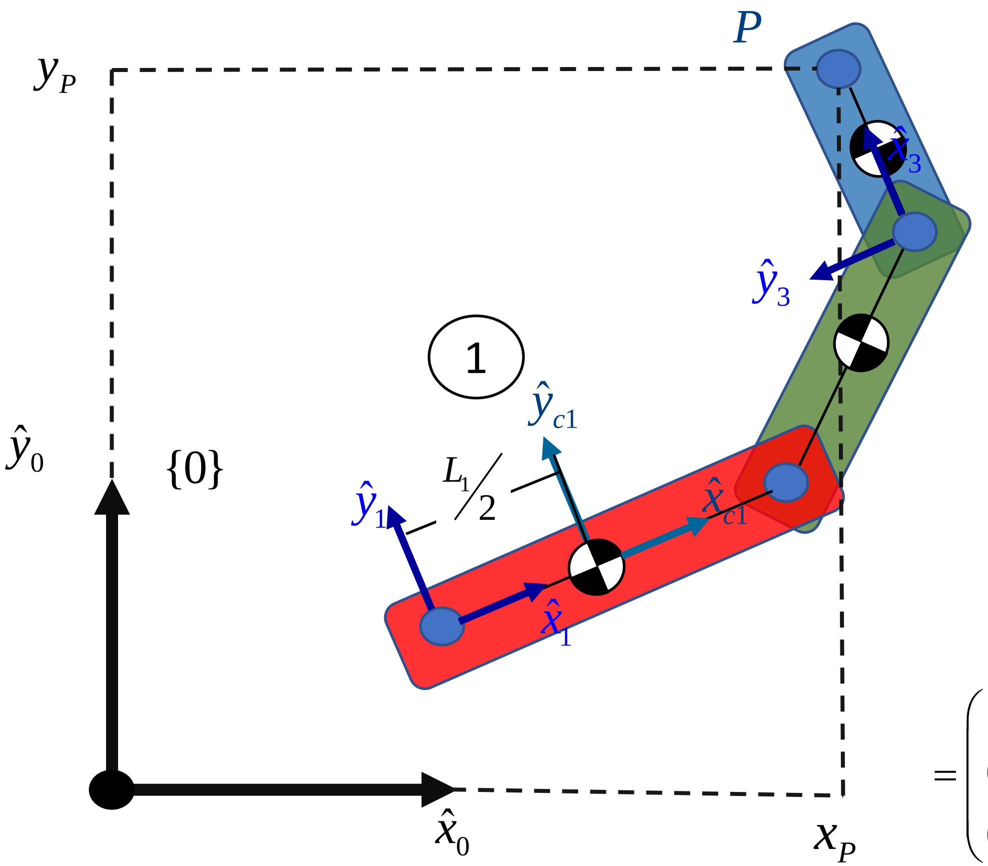
Energía potencial para el primer eslabón

$$u_2 = -m_2 \mathbf{g}^T \, {}^0\mathbf{p}_{c2}$$

$$= -m_2 \begin{pmatrix} 0 & -g & 0 \end{pmatrix} \begin{pmatrix} {}^0x_1 + L_1 \cos({}^0\theta_1) + \frac{L_2}{2} \cos({}^0\theta_1 + {}^1\theta_2) \\ {}^0y_1 + L_1 \sin({}^0\theta_1) + \frac{L_2}{2} \sin({}^0\theta_1 + {}^1\theta_2) \\ 0 \end{pmatrix} =$$

$$= m_2 g \left({}^0y_1 + L_1 \sin({}^0\theta_1) + \frac{L_2}{2} \sin({}^0\theta_1 + {}^1\theta_2) \right)$$

Cálculo del Lagrangeano



Energía cinética para el tercer eslabón

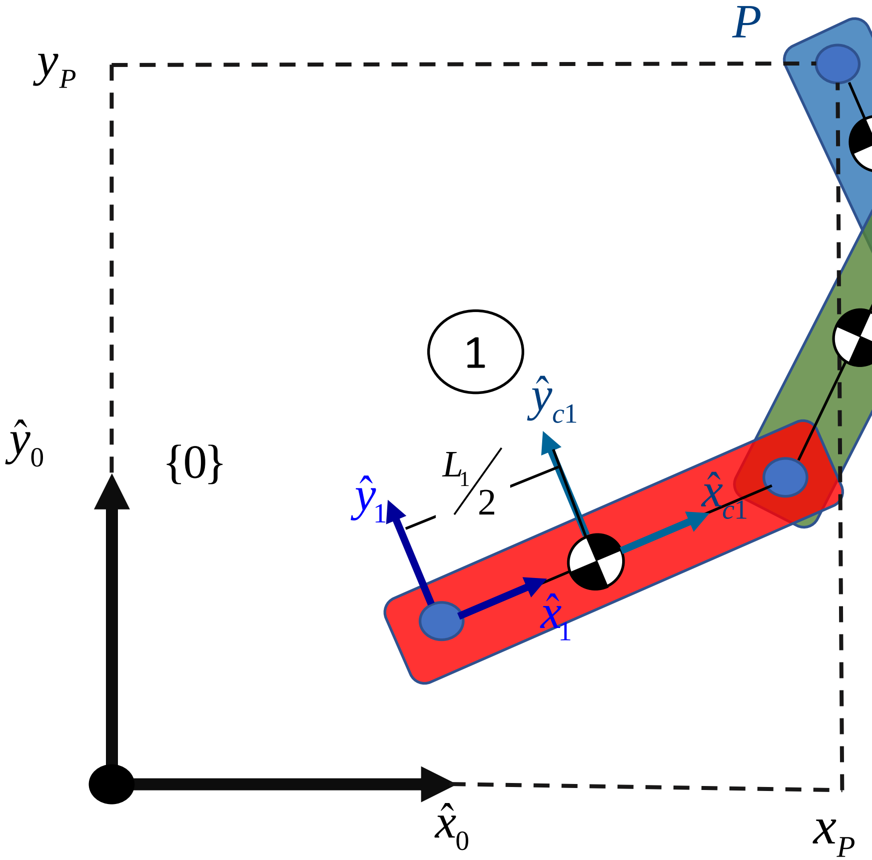
$$k_3 = \frac{1}{2} m_3 \mathbf{v}_{c3}{}^T \mathbf{v}_{c3} + \frac{1}{2} \boldsymbol{\omega}_3{}^T \mathbf{I}_{c3} \boldsymbol{\omega}_3$$

Energía cinética de la parte lineal

$$\mathbf{v}_{c3} = {}^{c3}\mathbf{R}_3 \left({}^3\mathbf{v}_3 + {}^3\boldsymbol{\omega}_3 \times {}^3\mathbf{p}_{c3} \right) =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} L_2 \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \sin({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ L_2 \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \cos({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \end{pmatrix} \times \begin{pmatrix} \frac{L_3}{2} \\ 0 \\ 0 \end{pmatrix} \right) =$$

Cálculo del Lagrangeano



Energía cinética para el tercer eslabón

$$k_3 = \frac{1}{2} m_3 \mathbf{v}_{c3}{}^T \mathbf{v}_{c3} + \frac{1}{2} \boldsymbol{\omega}_3{}^T \mathbf{I}_{c3} \boldsymbol{\omega}_3$$

Energía cinética de la parte lineal

$$\mathbf{v}_{c3} = {}^{c3}\mathbf{R}_3 \left({}^3\mathbf{v}_3 + {}^3\boldsymbol{\omega}_3 \times {}^3\mathbf{p}_{c3} \right) =$$

$$= \begin{pmatrix} L_2 \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \sin({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 \\ L_2 \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \cos({}^1\theta_2 + {}^2\theta_3)\,{}^0\dot{\theta}_1 + \frac{L_3}{2}({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3) \\ 0 \end{pmatrix}$$

Planteamiento del modelo

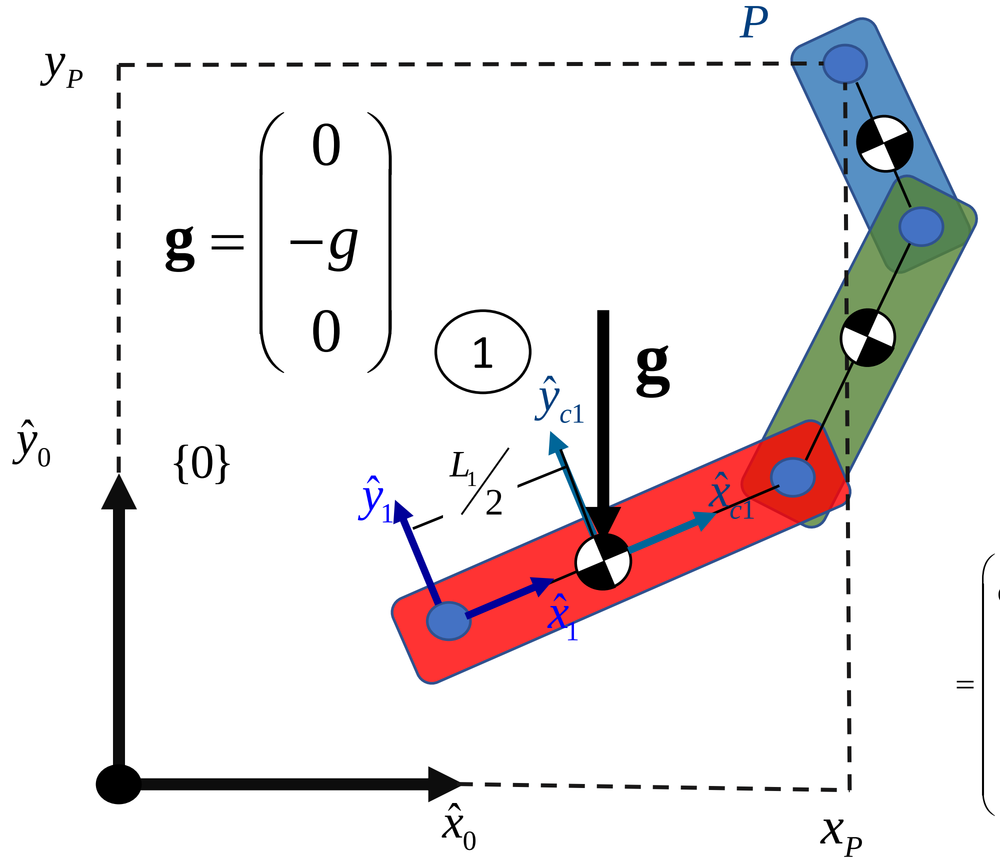
Energía cinética para el tercer eslabón

$$k_3 = \frac{1}{2} m_3 \mathbf{v}_{c3}{}^T \mathbf{v}_{c3} + \frac{1}{2} \boldsymbol{\omega}_3{}^T \mathbf{I}_{c3} \boldsymbol{\omega}_3 =$$

$$= \frac{m_3}{24} \left(\left(L_3{}^2 + h_2{}^2 \right) \left({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \right)^2 + 12 \left(\frac{1}{2} L_3 \left({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \right) + L_2 \left({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \right) \cos({}^2\theta_3) + L_1 \, {}^0\dot{\theta}_1 \cos({}^1\theta_2 + {}^2\theta_3) \right)^2 + \right.$$

$$\left. + \left(L_2 \left({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \right) \sin({}^2\theta_3) + L_1 \, {}^0\dot{\theta}_1 \sin({}^1\theta_2 + {}^2\theta_3) \right)^2 \right)$$

Planteamiento del modelo



$$\mathbf{g} = \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix}$$

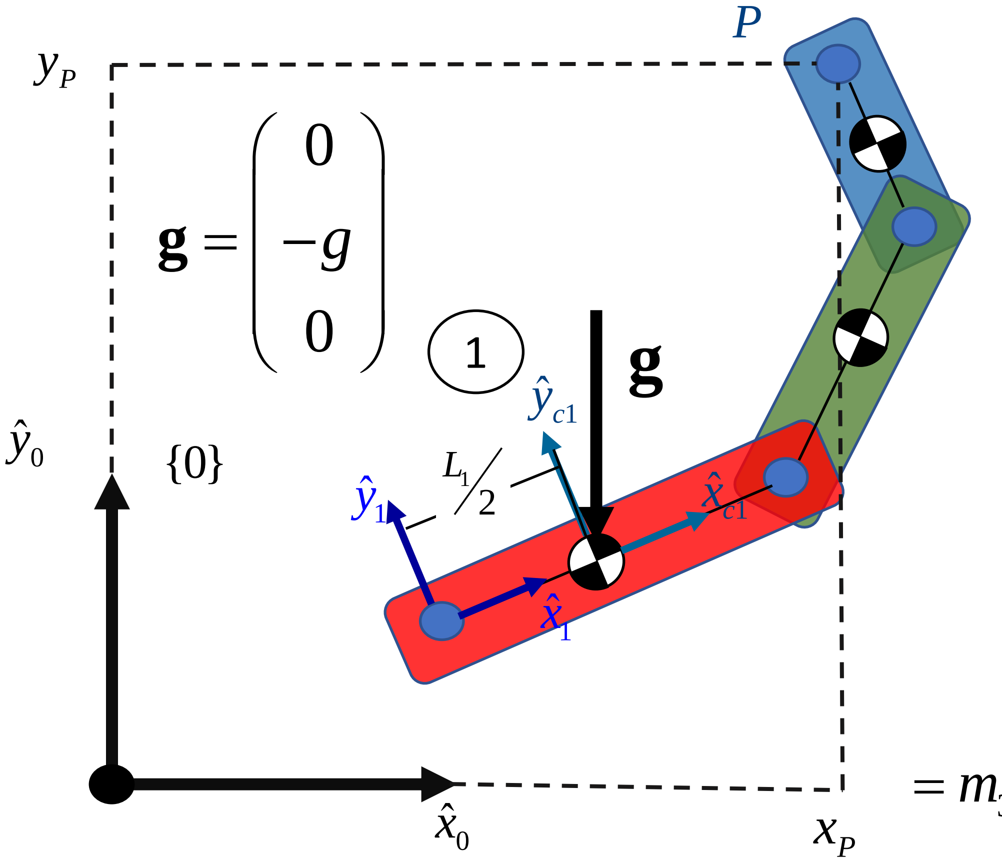
Energía potencial para el primer eslabón

$$u_3 = -m_3 \mathbf{g}^T \, {}^0\mathbf{p}_{c3}$$

$${}^0\mathbf{T}_{c3} = {}^0\mathbf{T}_1 \, {}^1\mathbf{T}_2 \, {}^2\mathbf{T}_3 \, {}^3\mathbf{T}_{c3} = \begin{pmatrix} {}^0\mathbf{R}_{c3} & {}^0\mathbf{p}_{c3} \\ \mathbf{0}^T & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & -\sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & 0 & {}^0x_1 + L_1\cos({}^0\theta_1) + L_2\cos({}^0\theta_1 + {}^1\theta_2) + \frac{L_3}{2}\cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) \\ \sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & 0 & {}^0y_1 + L_1\sin({}^0\theta_1) + L_2\sin({}^0\theta_1 + {}^1\theta_2) + \frac{L_3}{2}\sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Planteamiento del modelo



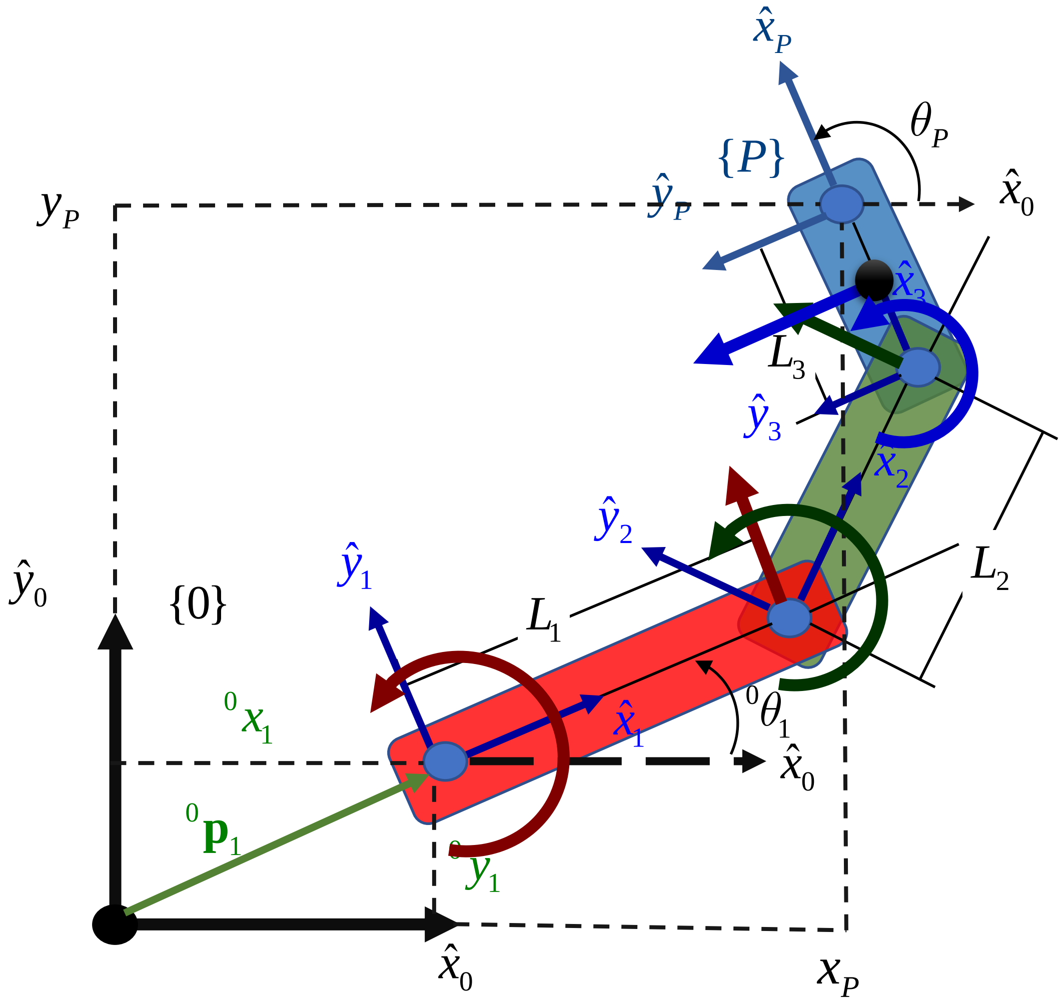
Energía potencial para el primer eslabón

$$u_3 = -m_3 \mathbf{g}^T \, {}^0\mathbf{p}_{c3}$$

$$= -m_3 \begin{pmatrix} 0 & -g & 0 \end{pmatrix} \begin{pmatrix} {}^0x_1 + L_1\cos({}^0\theta_1) + L_2\cos({}^0\theta_1 + {}^1\theta_2) + \frac{L_3}{2}\cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) \\ {}^0y_1 + L_1\sin({}^0\theta_1) + L_2\sin({}^0\theta_1 + {}^1\theta_2) + \frac{L_3}{2}\sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) \\ 0 \end{pmatrix} =$$

$$= m_3 g \left({}^0y_1 + L_1\sin({}^0\theta_1) + L_2\sin({}^0\theta_1 + {}^1\theta_2) + \frac{L_3}{2}\sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) \right)$$

Cálculo de los pares



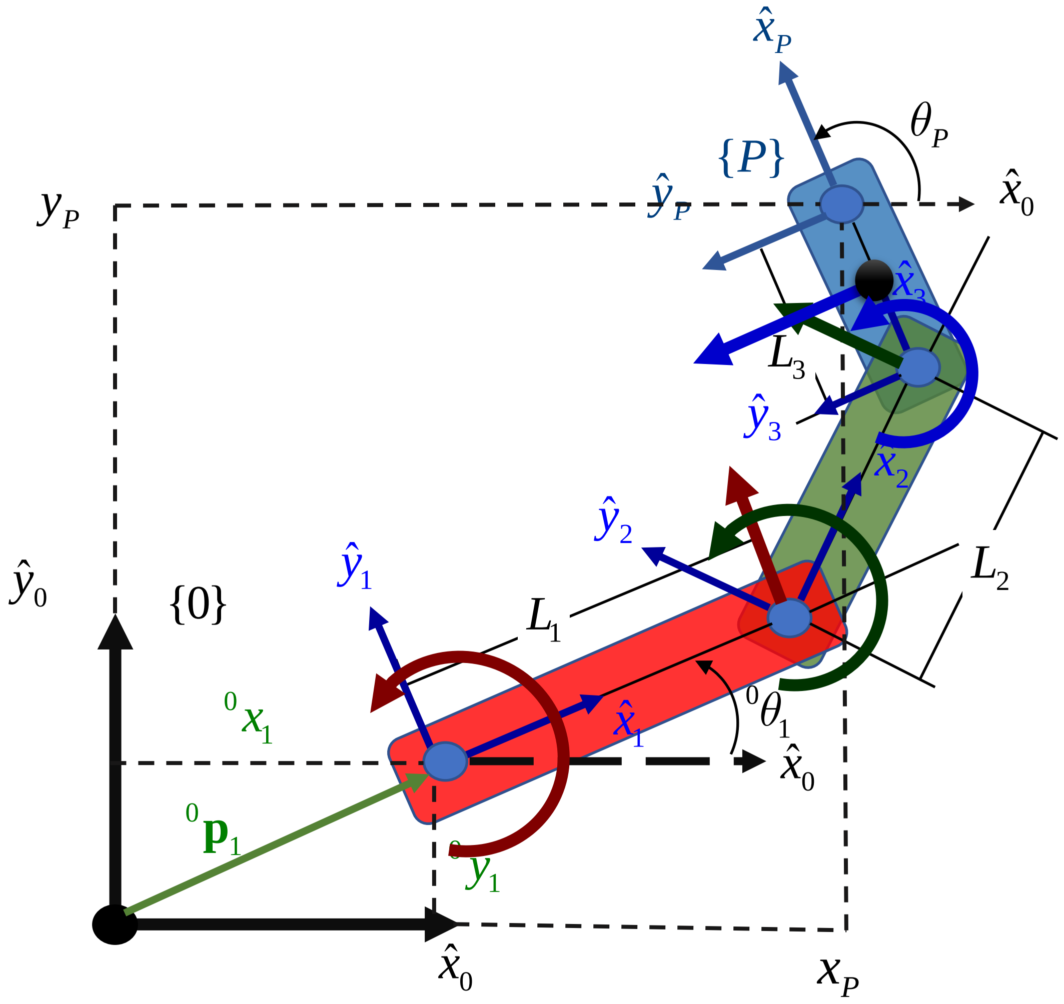
Cálculo del Lagrangeano

$$\Gamma = (k_1 + k_2 + k_3) - (u_1 + u_2 + u_3)$$

Ecuación del par

$$\tau_i = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\Gamma\right) - \frac{\partial}{\partial q_i}\Gamma$$

Cálculo de los pares



Cálculo del Lagrangeano

$$\Gamma = \left(k_1 + k_2 + k_3\right) - \left(u_1 + u_2 + u_3\right)$$

Ecuación del par

$$\boldsymbol{\tau}_\theta = \mathbf{M}(q)\ddot{\mathbf{q}} + \mathbf{V}(q, \dot{q}) + \mathbf{G}(q)$$