

Robótica grupo2

Clase 17

Facultad de Ingeniería UNAM

M.I. Erik Peña Medina

Derechos reservados

Todos los derechos reservados, Facultad de Ingeniería de la Universidad Nacional Autónoma de México © 2020. Quedan estrictamente prohibidos su uso fuera del ámbito académico, alteración, descarga o divulgación por cualquier medio, así como su reproducción parcial o total.

Contenido

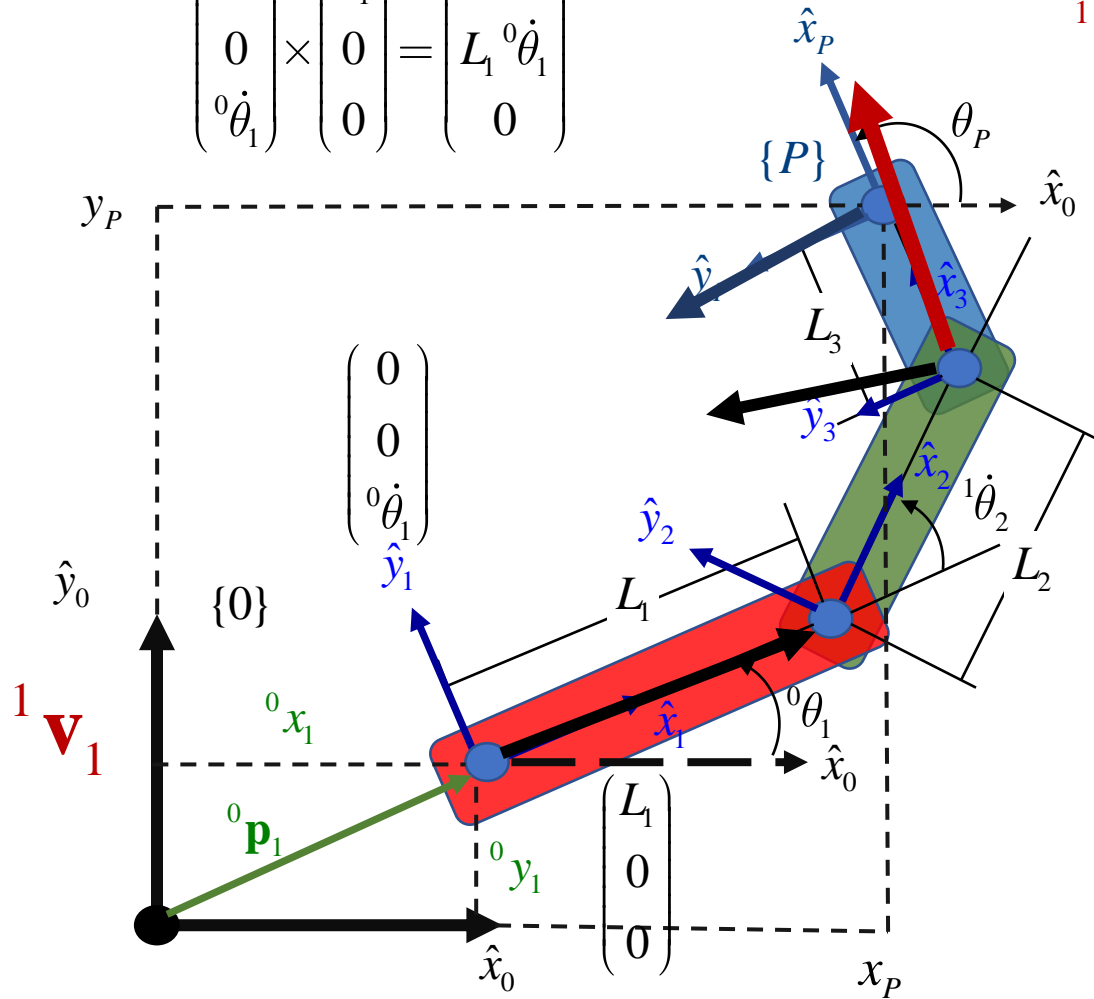
Planteamiento del modelo dinámico de un robot serial RRR en el plano

- Ecuación de Eüler-Lagrange
- Inercia lineal e inercia rotacional
- Propagación de velocidades
- Cálculo del Lagrangeano
- Cálculo de los pares
- Modelo dinámico general

Propagación de velocidades

$$\begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ L_1 {}^0\dot{\theta}_1 \\ 0 \end{pmatrix}$$

$${}^1\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



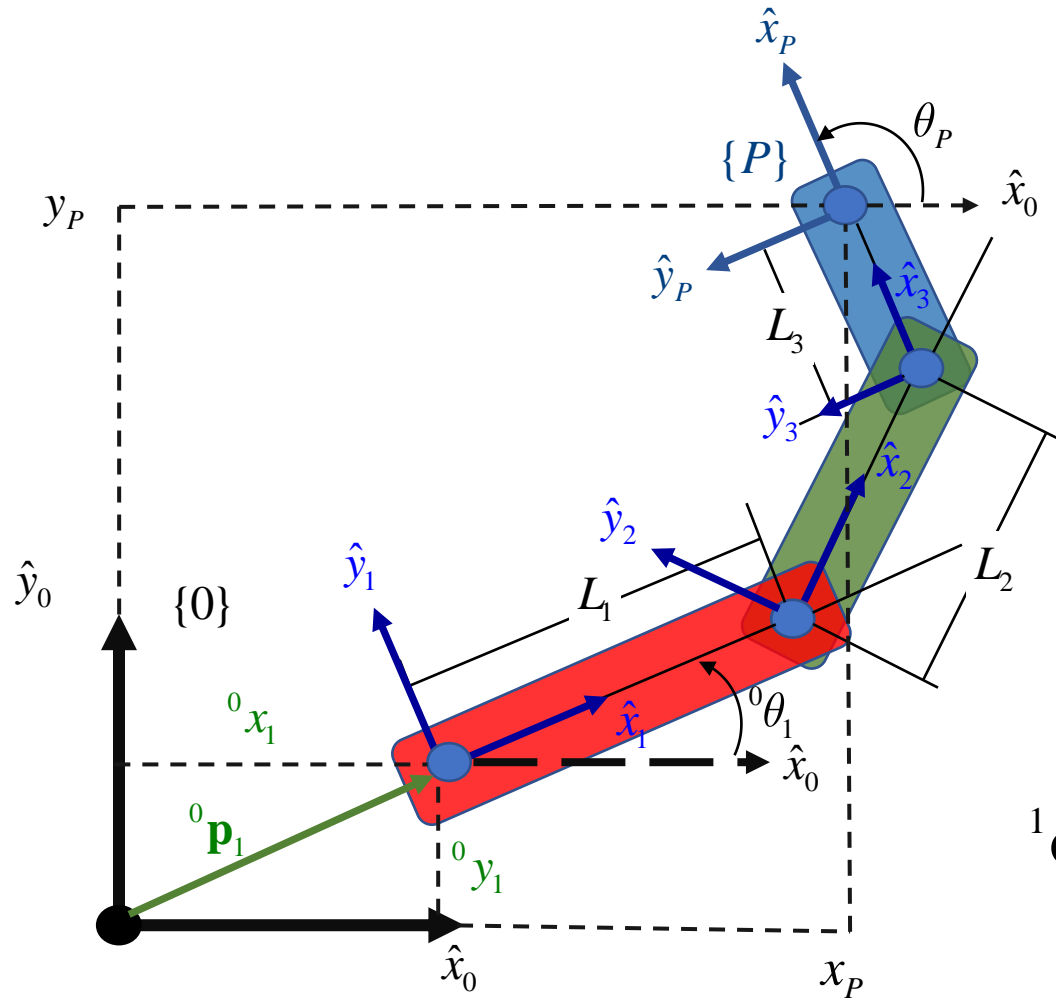
Propagación de velocidades angulares

$${}^{i+1}\boldsymbol{\omega}_{i+1} = {}^{i+1}\mathbf{R}_i {}^i\boldsymbol{\omega}_i + {}^{i+1}\hat{\mathbf{z}}_{i+1} \dot{\theta}_{i+1}$$

Propagación de velocidades lineales

$${}^{i+1}\mathbf{v}_{i+1} = {}^{i+1}\mathbf{R}_i \left({}^i\mathbf{v}_i + {}^i\boldsymbol{\omega}_i \times {}^i\mathbf{p}_{i+1} \right)$$

Propagación de velocidades



Propagación de velocidades entre
del sistema {0} al sistema {1}

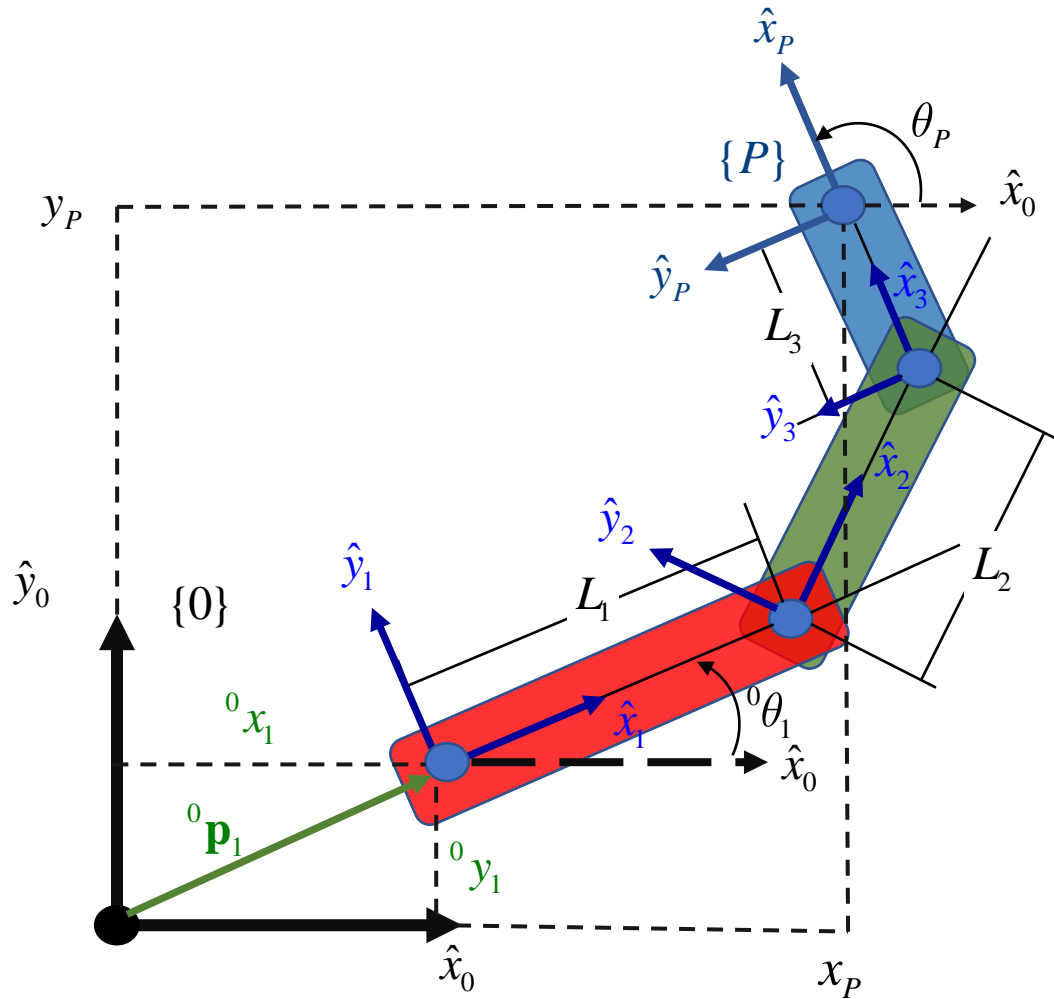
Propagación de velocidades angulares

$${}^1\boldsymbol{\omega}_1 = {}^1\mathbf{R}_0 {}^0\boldsymbol{\omega}_0 + {}^1\hat{\mathbf{z}}_1 \dot{\theta}_1$$

$${}^1\mathbf{R}_0 = {}^0\mathbf{R}_1^{-1} = {}^0\mathbf{R}_1^T$$

$${}^1\boldsymbol{\omega}_1 = \begin{pmatrix} \cos({}^0\theta_1) & \sin({}^0\theta_1) & 0 \\ -\sin({}^0\theta_1) & \cos({}^0\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} {}^0\dot{\theta}_1 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix}$$

Propagación de velocidades



Propagación de velocidades entre del sistema {0} al sistema {1}

Propagación de velocidades lineales

$${}^1\mathbf{v}_1 = {}^1\mathbf{R}_0 \left({}^0\mathbf{v}_0 + {}^0\boldsymbol{\omega}_0 \times {}^0\mathbf{p}_1 \right)$$

$${}^1\mathbf{v}_1 = \begin{pmatrix} \cos({}^0\theta_1) & \sin({}^0\theta_1) & 0 \\ -\sin({}^0\theta_1) & \cos({}^0\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} {}^0x_1 \\ {}^0y_1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Propagación de velocidades

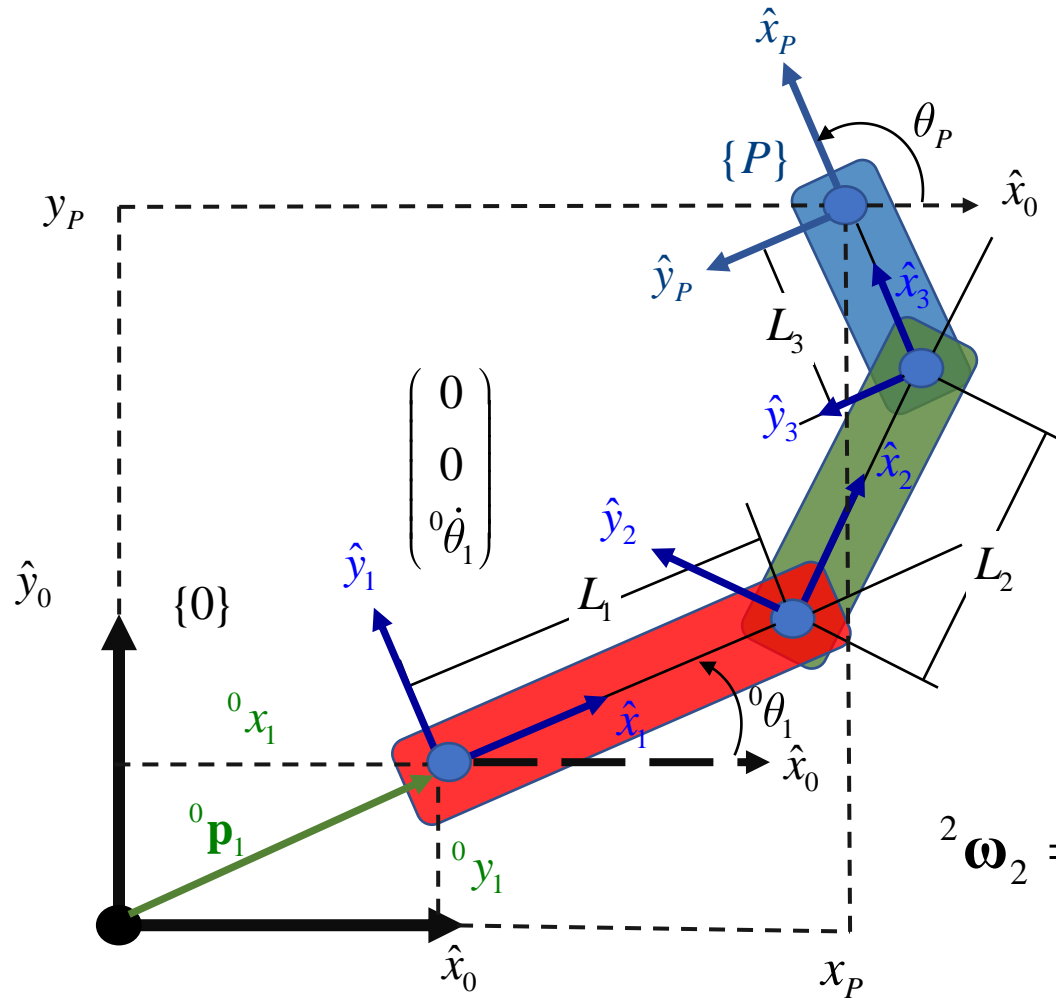
Propagación de velocidades entre
del sistema {1} al sistema {2}

Propagación de velocidades angulares

$${}^2\boldsymbol{\omega}_2 = {}^2\mathbf{R}_1 {}^1\boldsymbol{\omega}_1 + {}^2\hat{\mathbf{z}}_2 {}^1\dot{\theta}_2$$

$${}^2\mathbf{R}_1 = {}^1\mathbf{R}_2^{-1} = {}^1\mathbf{R}_2^T$$

$${}^2\boldsymbol{\omega}_2 = \begin{pmatrix} \cos({}^1\theta_2) & \sin({}^1\theta_2) & 0 \\ -\sin({}^1\theta_2) & \cos({}^1\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} {}^1\dot{\theta}_2 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \end{pmatrix}$$



Propagación de velocidades

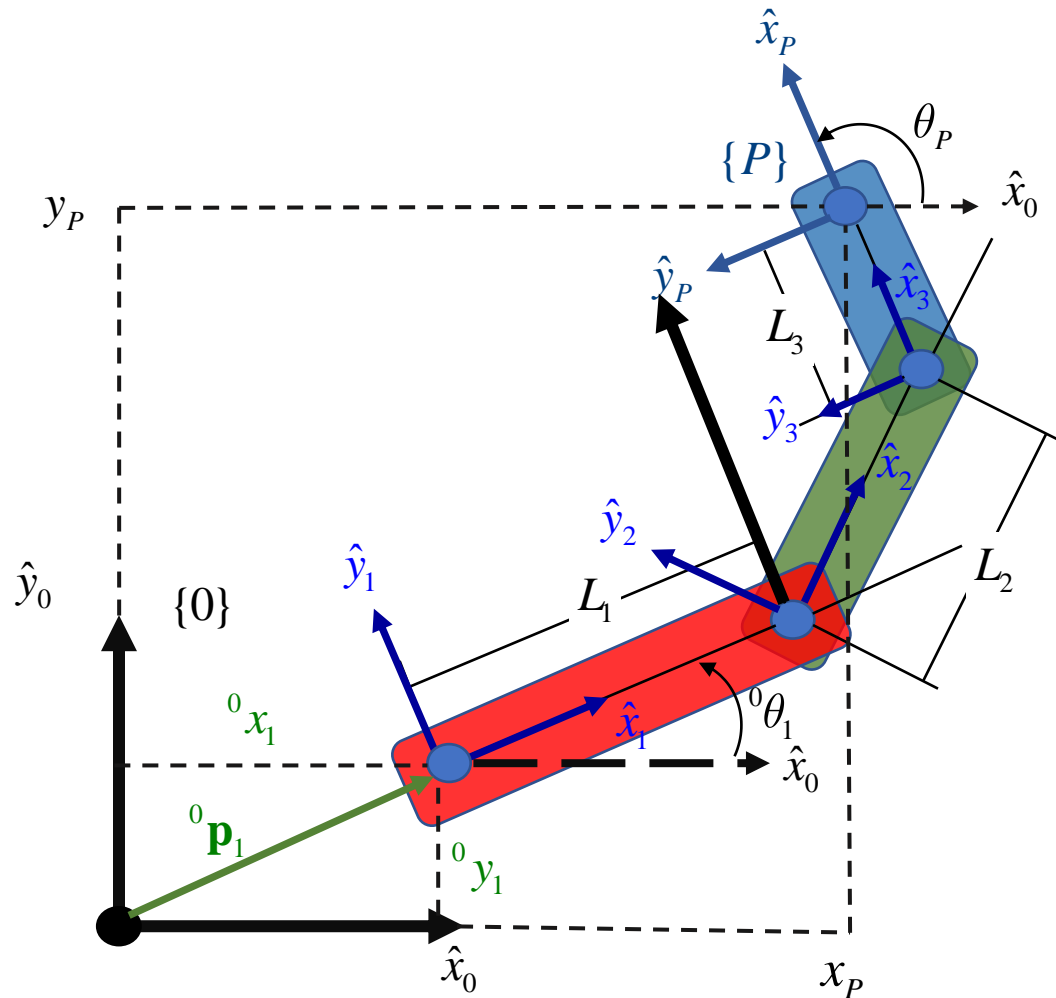
Propagación de velocidades entre
del sistema {1} al sistema {2}

Propagación de velocidades lineales

$${}^2\mathbf{v}_2 = {}^2\mathbf{R}_1 \left({}^1\mathbf{v}_1 + {}^1\boldsymbol{\omega}_1 \times {}^1\mathbf{p}_2 \right)$$

$${}^2\mathbf{v}_2 = \begin{pmatrix} \cos({}^1\theta_2) & \sin({}^1\theta_2) & 0 \\ -\sin({}^1\theta_2) & \cos({}^1\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} L_1 \sin({}^1\theta_2) {}^0\dot{\theta}_1 \\ L_1 \cos({}^1\theta_2) {}^0\dot{\theta}_1 \\ 0 \end{pmatrix}$$



Propagación de velocidades

$${}^3\boldsymbol{\omega}_3 = {}^3\mathbf{R}_2 {}^2\boldsymbol{\omega}_2 + {}^3\hat{\mathbf{z}}_3 \dot{\theta}_3$$

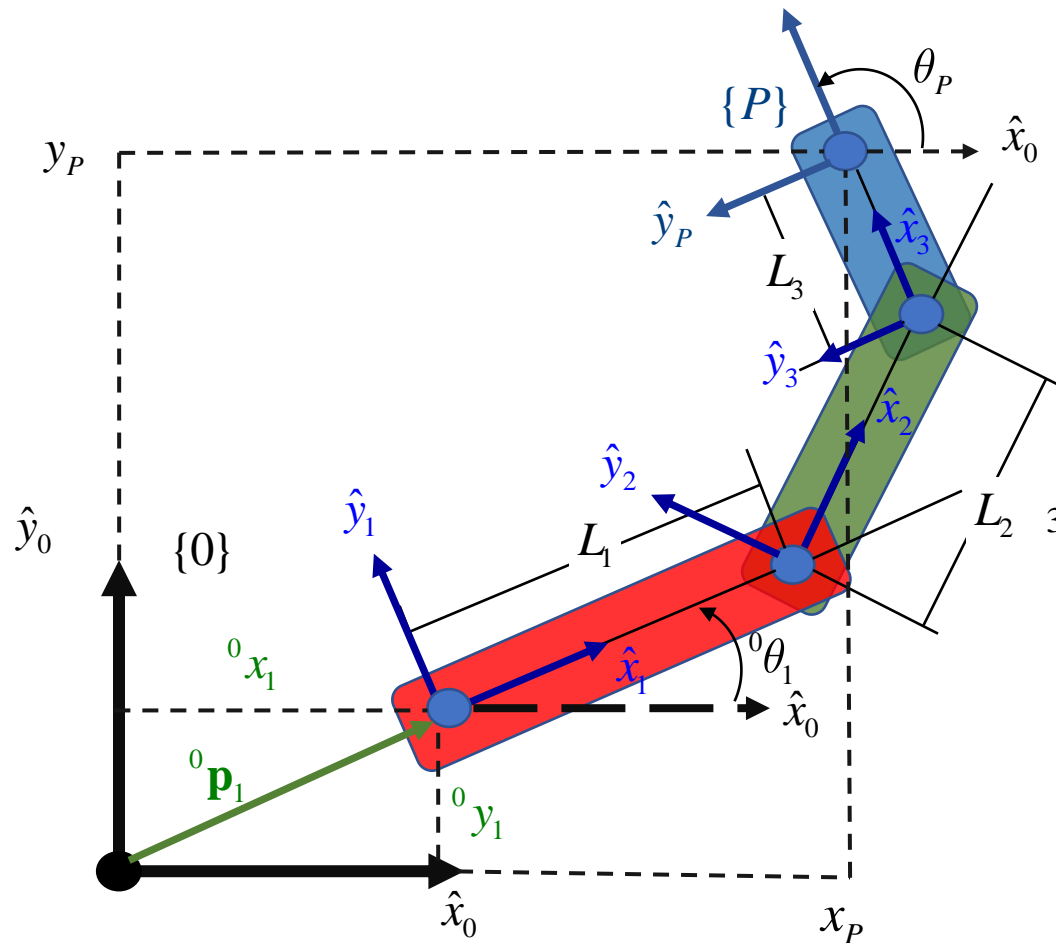
Propagación de velocidades entre
del sistema {2} al sistema {3}

Propagación de velocidades angulares

$${}^3\boldsymbol{\omega}_3 = {}^3\mathbf{R}_2 {}^2\boldsymbol{\omega}_2 + {}^3\hat{\mathbf{z}}_3 \dot{\theta}_3$$

$${}^3\boldsymbol{\omega}_3 = \begin{pmatrix} \cos({}^2\theta_3) & \sin({}^2\theta_3) & 0 \\ -\sin({}^2\theta_3) & \cos({}^2\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} {}^2\dot{\theta}_3 =$$

$$= \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \end{pmatrix}$$

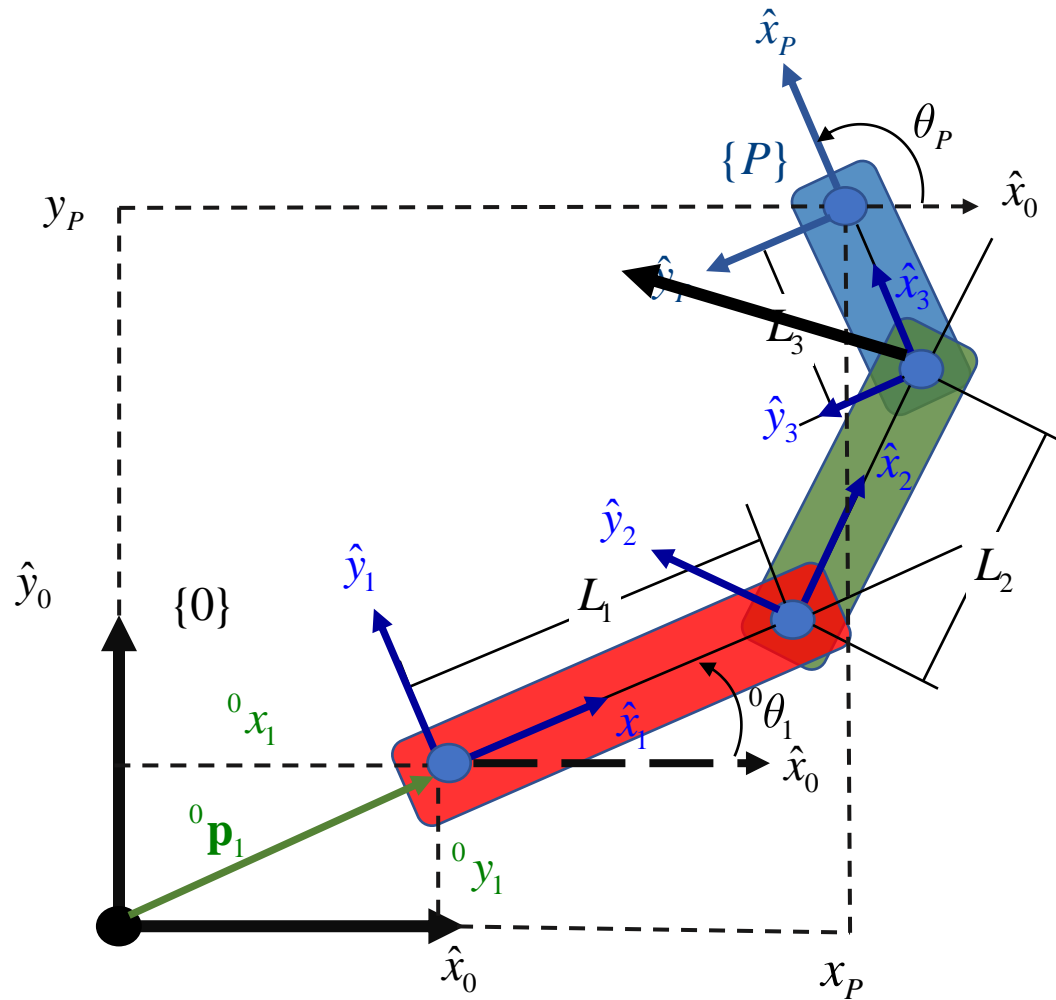


Propagación de velocidades

Propagación de velocidades entre
del sistema {2} al sistema {3}

Propagación de velocidades lineales

$${}^3\mathbf{v}_3 = {}^3\mathbf{R}_2 \left({}^2\mathbf{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\mathbf{p}_3 \right)$$



$${}^3\mathbf{v}_3 = \begin{pmatrix} \cos({}^2\theta_3) & \sin({}^2\theta_3) & 0 \\ -\sin({}^2\theta_3) & \cos({}^2\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} L_1 \sin({}^1\theta_2) \dot{\theta}_1 \\ L_1 \cos({}^1\theta_2) \dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} \right) =$$

$$= \begin{pmatrix} L_2 \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \sin({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ L_2 \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \cos({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ 0 \end{pmatrix}$$

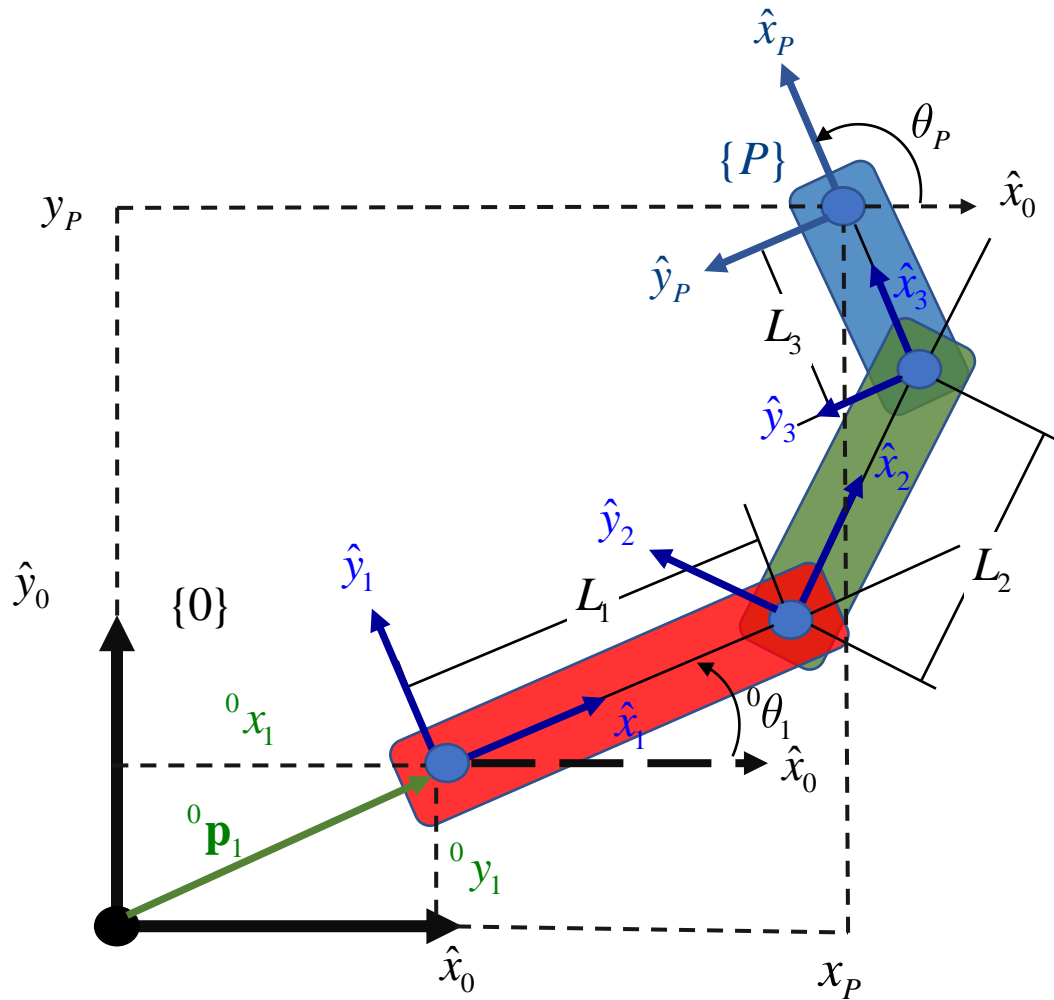
Propagación de velocidades

Propagación de velocidades entre
del sistema {2} al sistema {3}

Propagación de velocidades angulares

$${}^P \boldsymbol{\omega}_P = {}^P \mathbf{R}_3 {}^3 \boldsymbol{\omega}_3 + {}^P \hat{\mathbf{z}}_P 0$$

$$\begin{aligned} {}^P \boldsymbol{\omega}_P &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0 \dot{\theta}_1 + {}^1 \dot{\theta}_2 + {}^2 \dot{\theta}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} 0 = \\ &= \begin{pmatrix} 0 \\ 0 \\ {}^0 \dot{\theta}_1 + {}^1 \dot{\theta}_2 + {}^2 \dot{\theta}_3 \end{pmatrix} \end{aligned}$$



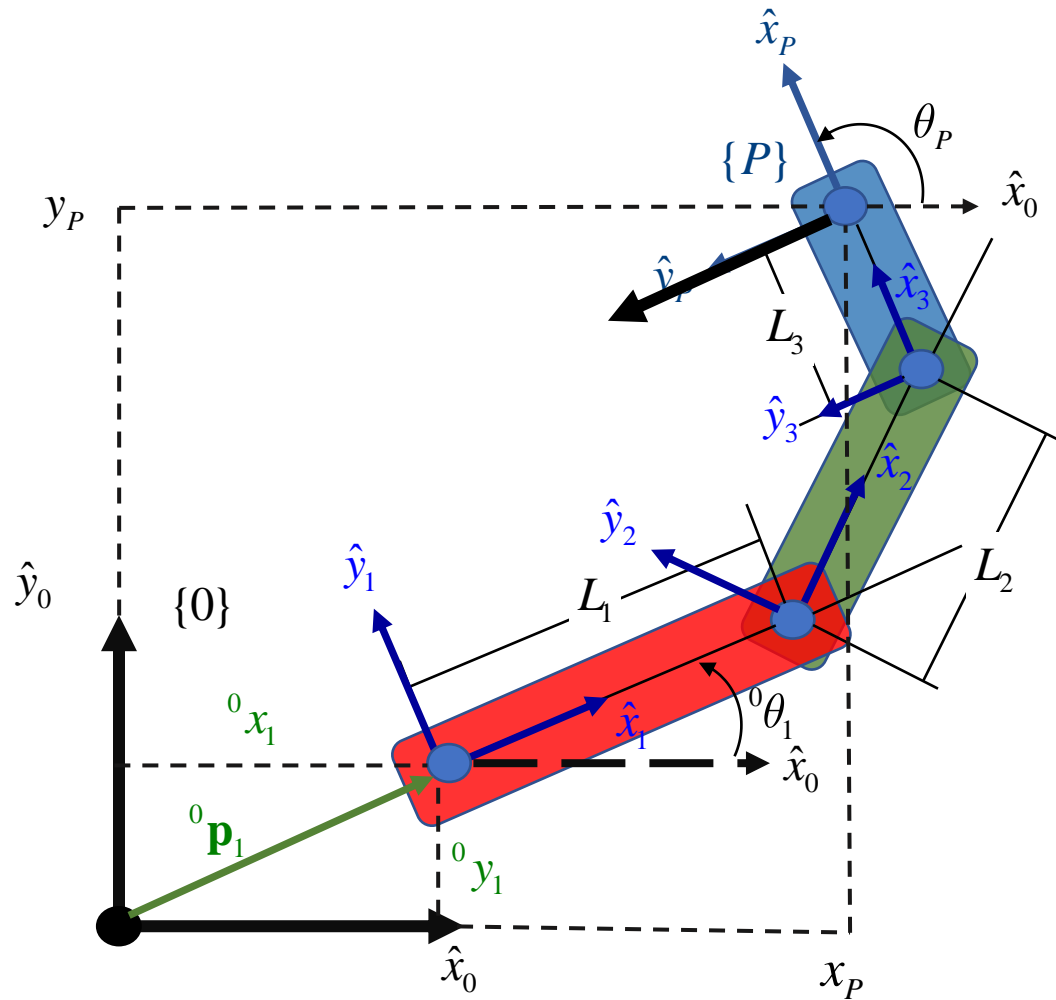
Propagación de velocidades

Propagación de velocidades entre
del sistema {2} al sistema {3}

Propagación de velocidades lineales

$${}^P \mathbf{v}_P = {}^P \mathbf{R}_3 \left({}^3 \mathbf{v}_3 + {}^3 \boldsymbol{\omega}_3 \times {}^3 \mathbf{p}_P \right)$$

$${}^P \mathbf{v}_P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_2 \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \sin({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ L_2 \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \cos({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \end{pmatrix} \times \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix}$$

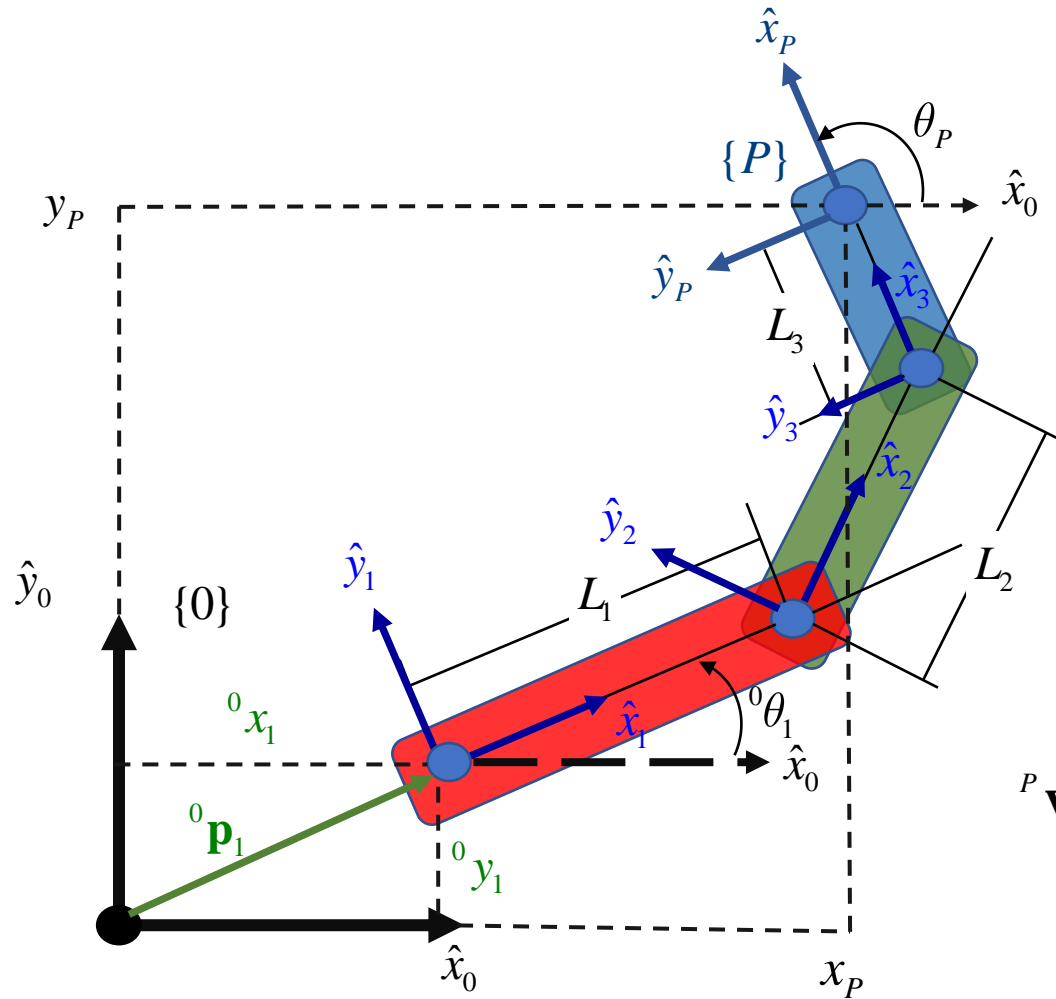


Propagación de velocidades

Propagación de velocidades entre
del sistema {2} al sistema {3}

Propagación de velocidades lineales

$${}^P \mathbf{v}_P = {}^P \mathbf{R}_3 \left({}^3 \mathbf{v}_3 + {}^3 \boldsymbol{\omega}_3 \times {}^3 \mathbf{p}_P \right)$$



$${}^P \mathbf{v}_P = \begin{pmatrix} L_2 \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \sin({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ L_2 \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \cos({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 + L_3({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3) \\ 0 \end{pmatrix}$$

Propagación de velocidades

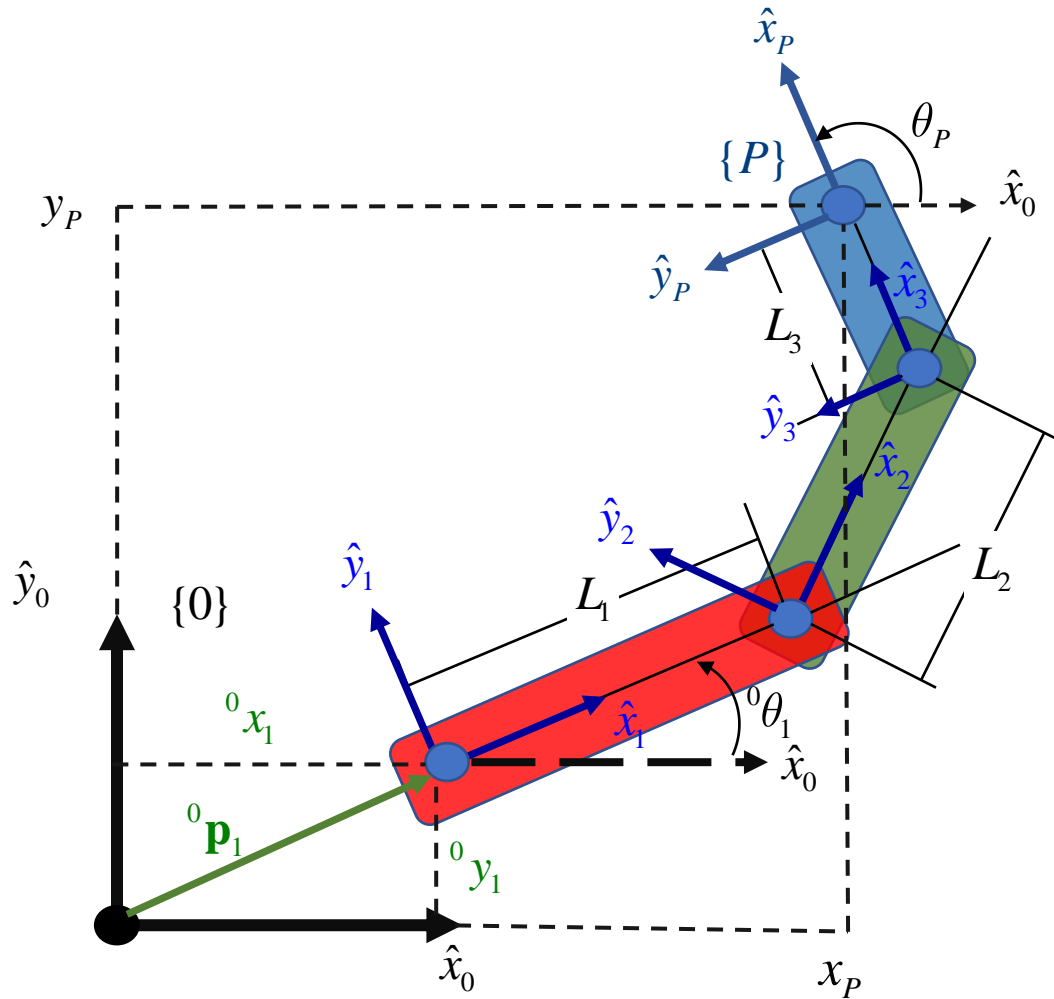
Propagación de velocidades entre
del sistema {P} al sistema {0}

Propagación de velocidades lineales

$${}^0\mathbf{v}_P = {}^0\mathbf{R}_P {}^P\mathbf{v}_P$$

$${}^0\mathbf{R}_P = {}^0\mathbf{R}_1 {}^1\mathbf{R}_2 {}^2\mathbf{R}_3 {}^3\mathbf{R}_P =$$

$$= \begin{pmatrix} \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & -\sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & 0 \\ \sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Propagación de velocidades

Propagación de velocidades lineales

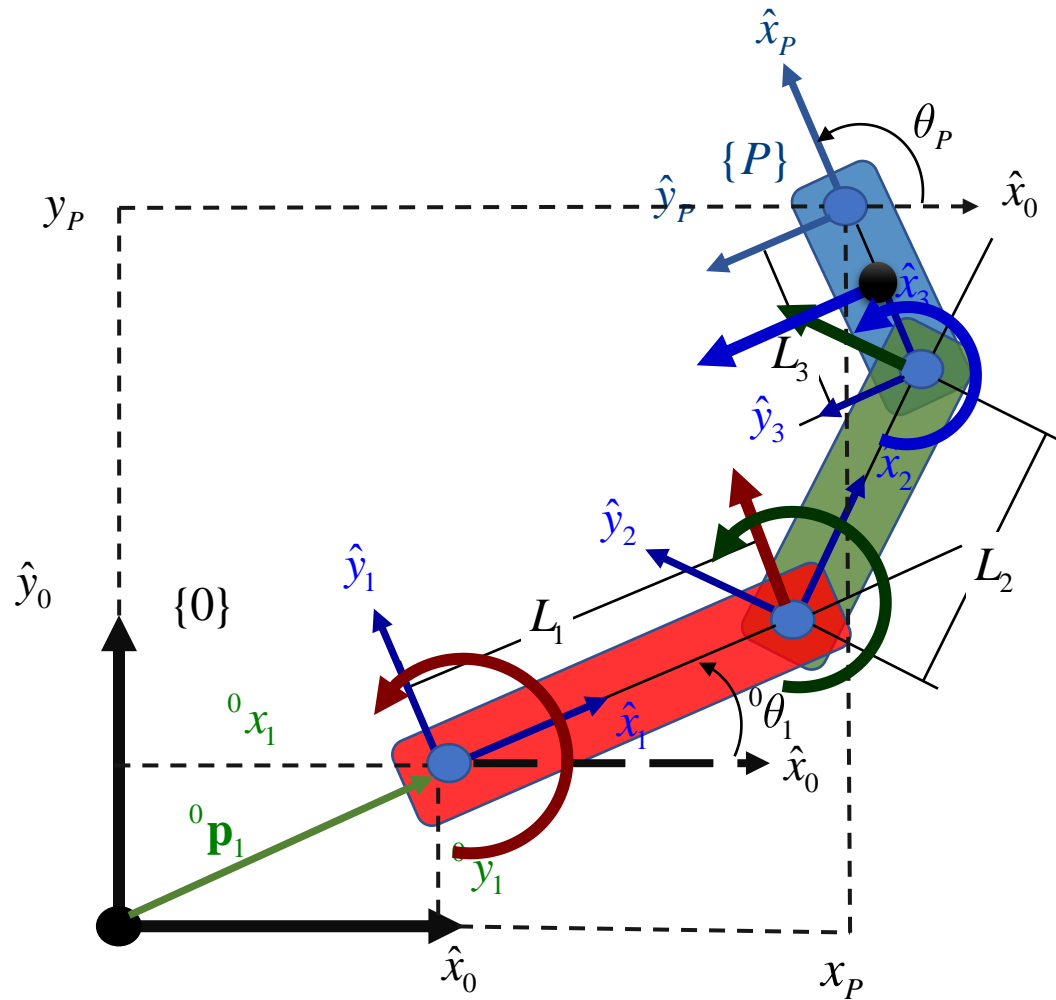
$${}^0\mathbf{R}_P = {}^0\mathbf{R}_1 {}^1\mathbf{R}_2 {}^2\mathbf{R}_3 {}^3\mathbf{R}_P$$

$${}^0\mathbf{v}_P = {}^0\mathbf{R}_P {}^P\mathbf{v}_P =$$

$${}^P\mathbf{v}_P = \begin{pmatrix} L_2 \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \sin({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ L_2 \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \cos({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 + L_3({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} (-L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3))\dot{\theta}_1 + (-L_2 \sin(\theta_1 + \theta_2) - L_3 \sin(\theta_1 + \theta_2 + \theta_3))\dot{\theta}_2 + (-L_3 \sin(\theta_1 + \theta_2 + \theta_3))\dot{\theta}_3 \\ (L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3))\dot{\theta}_1 + (L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3))\dot{\theta}_2 + (L_3 \cos(\theta_1 + \theta_2 + \theta_3))\dot{\theta}_3 \\ 0 \end{pmatrix}$$

Cálculo de los pares



Cálculo del Lagrangeano

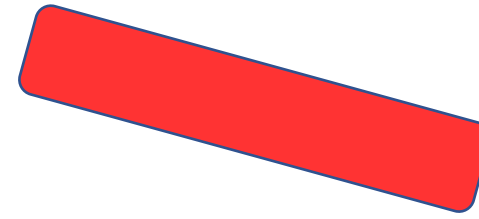
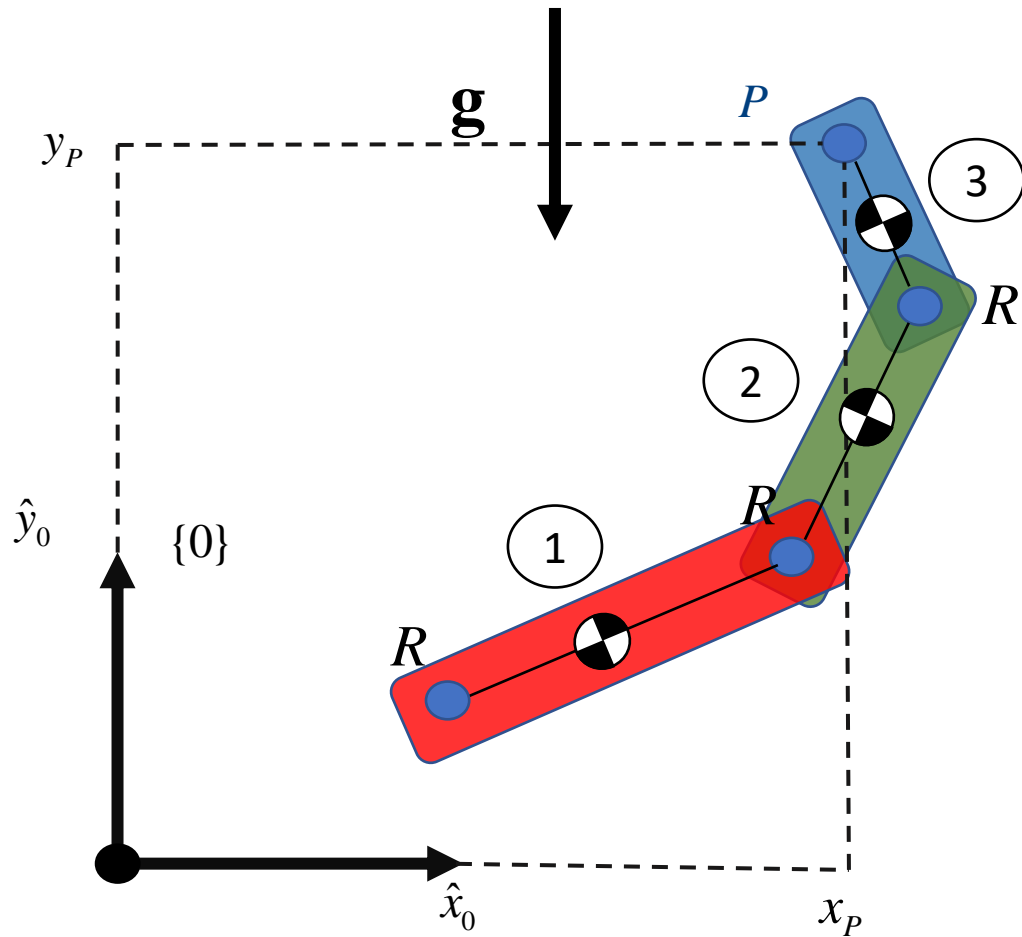
$$\Gamma = (k_1 + k_2 + k_3) - (u_1 + u_2 + u_3)$$

Ecuación del par

$$\tau_i = \frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_i} \Gamma \right) - \frac{\partial}{\partial q_i} \Gamma$$

$$\boldsymbol{\tau}_i = \begin{pmatrix} \tau_{\theta 1} \\ \tau_{\theta 2} \\ \tau_{\theta 3} \end{pmatrix}$$

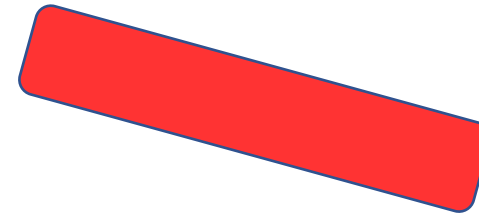
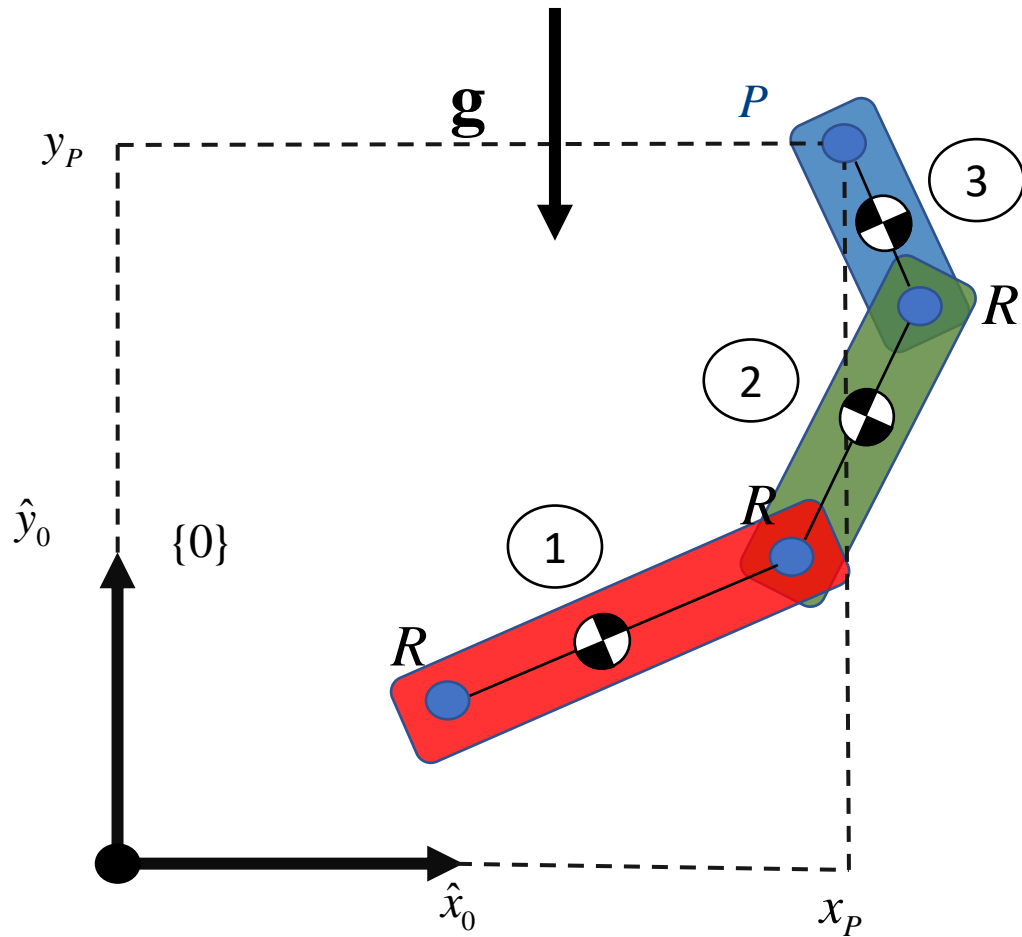
Planteamiento del modelo



$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}^T \mathbf{v}_{c1} + \frac{1}{2} {}^1\boldsymbol{\omega}_1^T \mathbf{I}_{c1} {}^1\boldsymbol{\omega}_1$$

$$u_1 = -m_1 \mathbf{g}^{T0} \mathbf{p}_{c1}$$

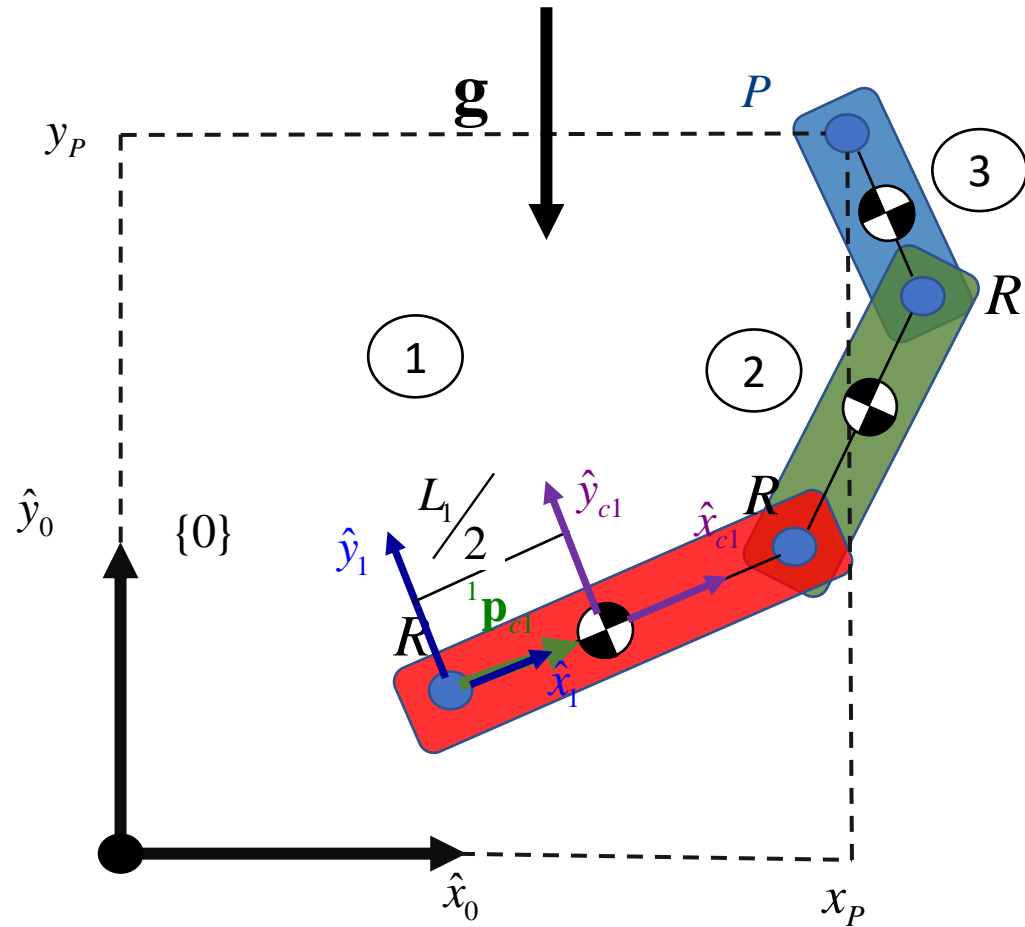
Planteamiento del modelo



$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}^T \mathbf{v}_{c1} + \frac{1}{2} {}^1\boldsymbol{\omega}_1^T \mathbf{I}_{c1} {}^1\boldsymbol{\omega}_1$$

$$u_1 = -m_1 \mathbf{g}^{T0} \mathbf{p}_{c1}$$

Planteamiento del modelo



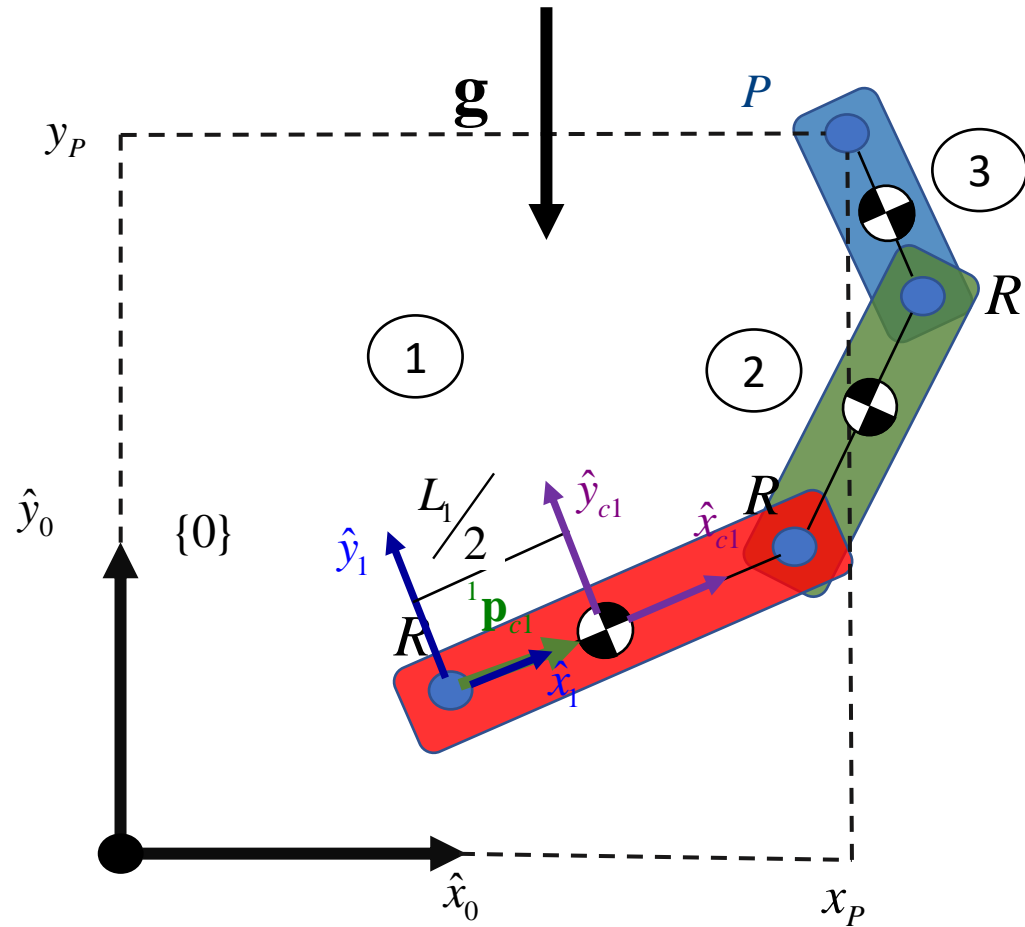
$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}^T \mathbf{v}_{c1} + \frac{1}{2} \boldsymbol{\omega}_1^T \mathbf{I}_{c1} \boldsymbol{\omega}_1$$

$${}^{c1}\mathbf{v}_{c1} = {}^{c1}\mathbf{R}_1 \left({}^1\mathbf{v}_1 + {}^1\boldsymbol{\omega}_1 \times {}^1\mathbf{p}_{c1} \right)$$

$${}^{c1}\mathbf{v}_{c1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} L_1/2 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ L_1/2 \cdot {}^0\dot{\theta}_1 \\ 0 \end{pmatrix}$$

$${}^1\boldsymbol{\omega}_1 = \boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix}$$

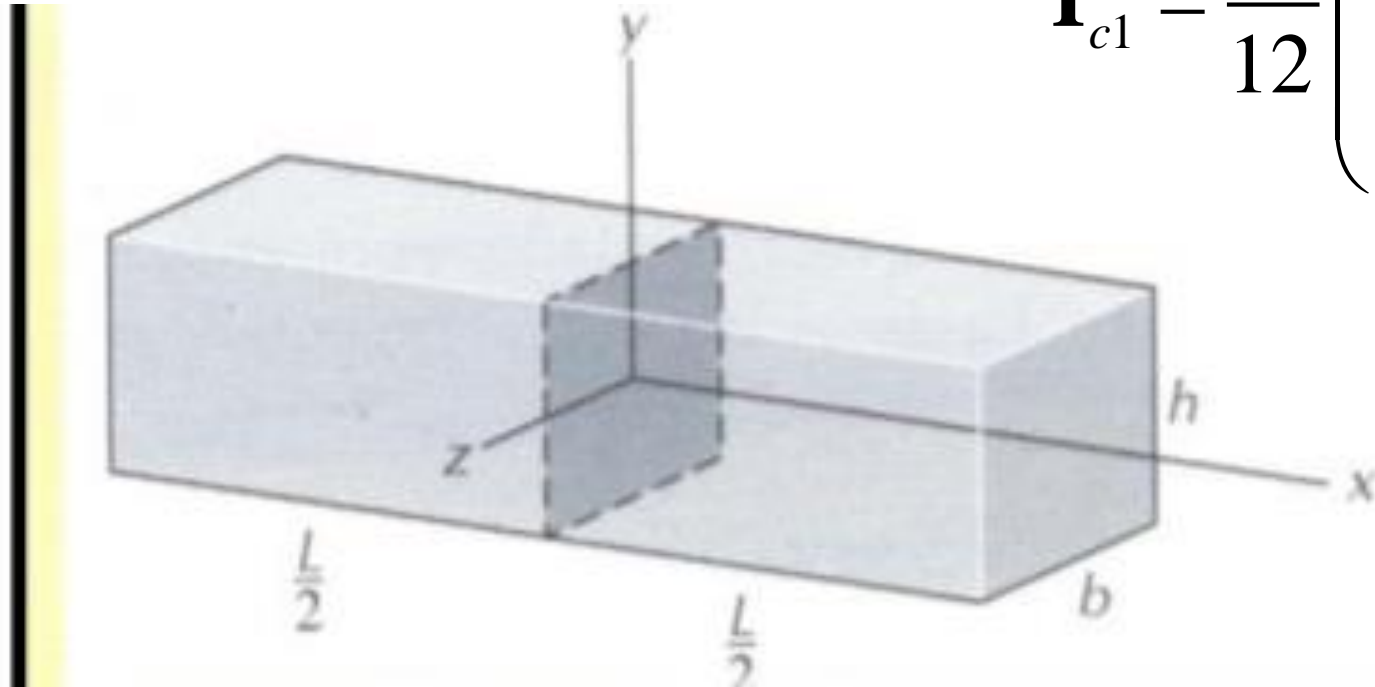
Planteamiento del modelo



$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}^T \mathbf{v}_{c1} + \frac{1}{2} \boldsymbol{\omega}_1^T \mathbf{I}_{c1} \boldsymbol{\omega}_1$$

$${}^{c1}\mathbf{v}_{c1} = \begin{pmatrix} 0 \\ L_1 {}^0\dot{\theta}_1 \\ 0 \end{pmatrix} \quad {}^1\boldsymbol{\omega}_1 = \boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_2 \end{pmatrix}$$

$$\mathbf{I}_{c1} = \frac{m_1}{12} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

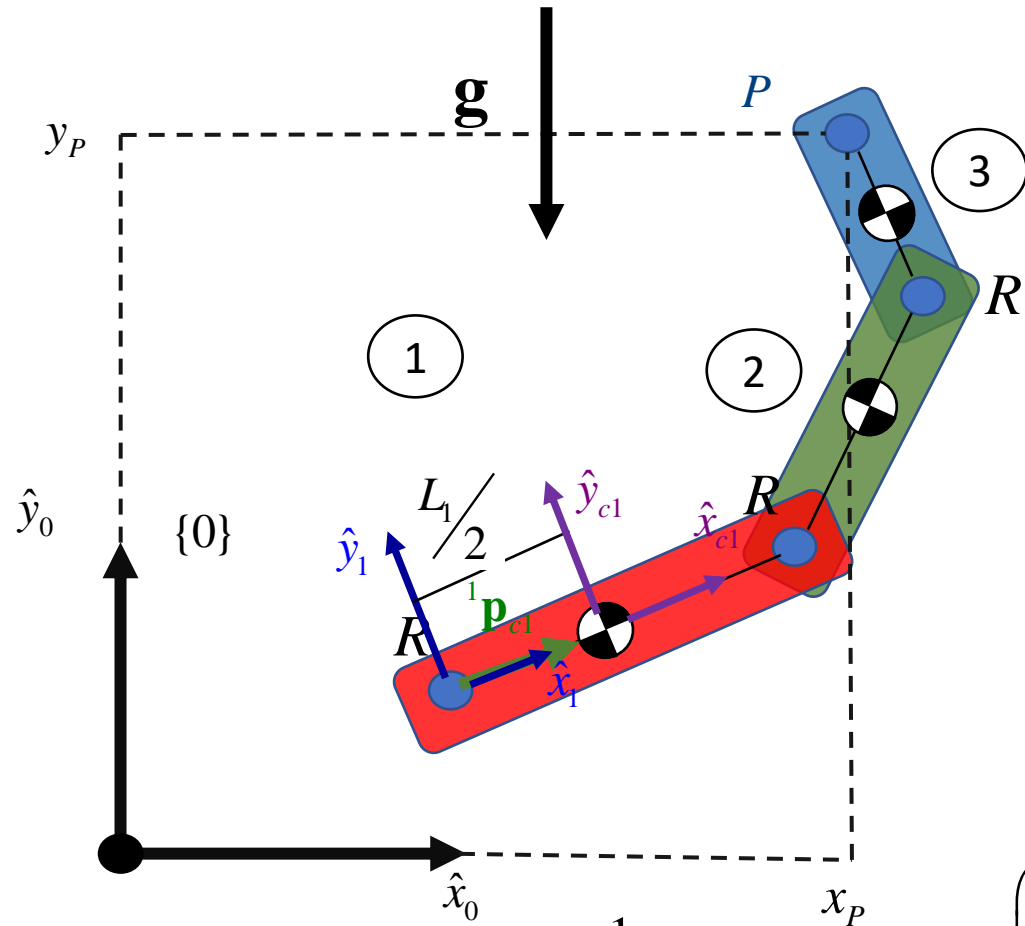


$$\mathbf{I}_{c1} = \frac{m_1}{12} \begin{pmatrix} b_1^2 + h_1^2 & 0 & 0 \\ 0 & L_1^2 + b_1^2 & 0 \\ 0 & 0 & L_1^2 + h_1^2 \end{pmatrix}$$

Planteamiento del modelo

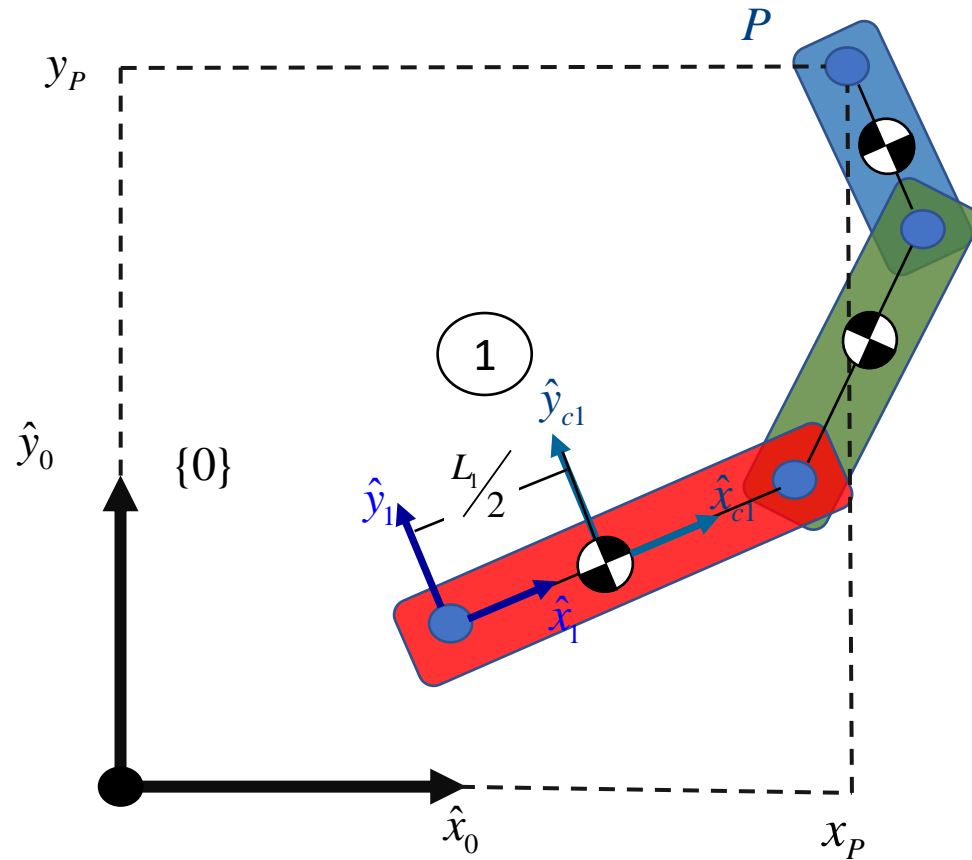
$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}^T \mathbf{v}_{c1} + \frac{1}{2} \boldsymbol{\omega}_1^T \mathbf{I}_{c1} \boldsymbol{\omega}_1$$

$${}^{c1}\mathbf{v}_{c1} = \begin{pmatrix} 0 \\ L_1 {}^0\dot{\theta}_1 \\ 0 \end{pmatrix} \quad {}^1\boldsymbol{\omega}_1 = \boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_2 \end{pmatrix}$$



$$k_1 = \frac{1}{2} m_1 \begin{pmatrix} 0 & L_1 {}^0\dot{\theta}_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ L_1 {}^0\dot{\theta}_1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & {}^0\dot{\theta}_2 \end{pmatrix} \frac{m_1}{12} \begin{pmatrix} b_1^2 + h_1^2 & 0 & 0 \\ 0 & L_1^2 + b_1^2 & 0 \\ 0 & 0 & L_1^2 + h_1^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 \end{pmatrix}$$

Planteamiento del modelo



Energía cinética para el primer eslabón

$$k_1 = \frac{1}{2} m_1 \mathbf{v}_{c1}^T \mathbf{v}_{c1} + \frac{1}{2} \boldsymbol{\omega}_1^T \mathbf{I}_{c1} \boldsymbol{\omega}_1$$

$$k_1 = \frac{L_1^2 m_1}{8} {}^0 \dot{\theta}_1^2 + \frac{m_1 (L_i^2 + h_i^2)}{24} {}^0 \dot{\theta}_1^2 =$$

$$= \frac{m_1 (4L_i^2 + h_i^2)}{24} {}^0 \dot{\theta}_1^2$$

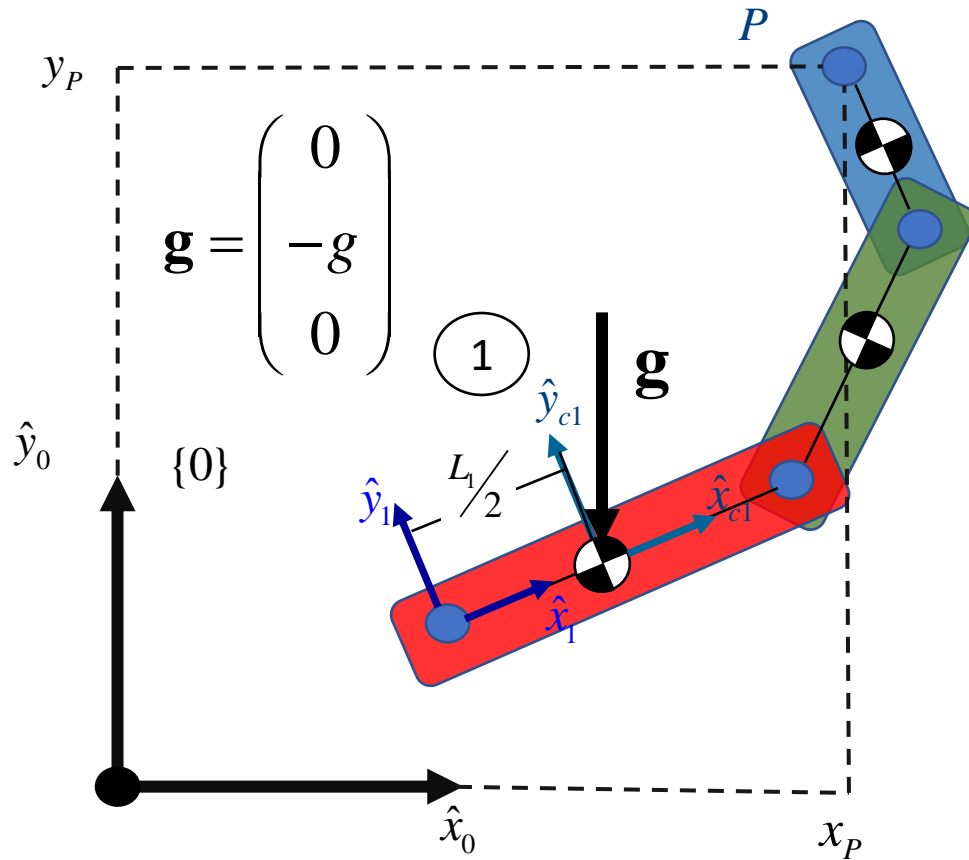
Cálculo del Lagrangeano

Energía potencial para el primer eslabón

$$u_1 = -m_1 \mathbf{g}^T {}^0 \mathbf{p}_{c1}$$

$${}^0 \mathbf{T}_{c1} = {}^0 \mathbf{T}_1 {}^1 \mathbf{T}_{c1} = \begin{pmatrix} {}^0 \mathbf{R}_{c1} & {}^0 \mathbf{p}_{c1} \\ \mathbf{0}^T & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos({}^0\theta_1) & -\sin({}^0\theta_1) & 0 & {}^0x_1 \\ \sin({}^0\theta_1) & \cos({}^0\theta_1) & 0 & {}^0y_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \frac{L_1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



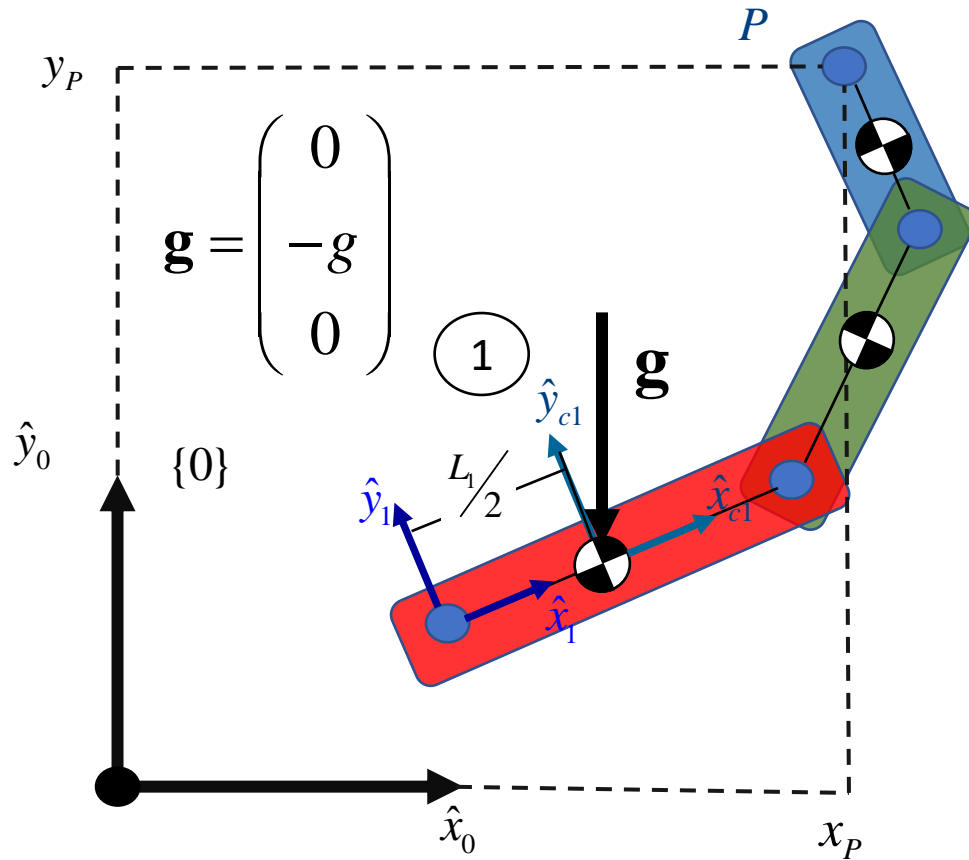
Cálculo del Lagrangeano

Energía potencial para el primer eslabón

$$u_1 = -m_1 \mathbf{g}^T {}^0 \mathbf{p}_{c1}$$

$${}^0 \mathbf{T}_{c1} = {}^0 \mathbf{T}_1 {}^1 \mathbf{T}_{c1} = \begin{pmatrix} {}^0 \mathbf{R}_{c1} & {}^0 \mathbf{p}_{c1} \\ \mathbf{0}^T & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos({}^0\theta_1) & -\sin({}^0\theta_1) & 0 & {}^0x_1 + \frac{L_1}{2}\cos({}^0\theta_1) \\ \sin({}^0\theta_1) & \cos({}^0\theta_1) & 0 & {}^0y_1 + \frac{L_1}{2}\sin({}^0\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



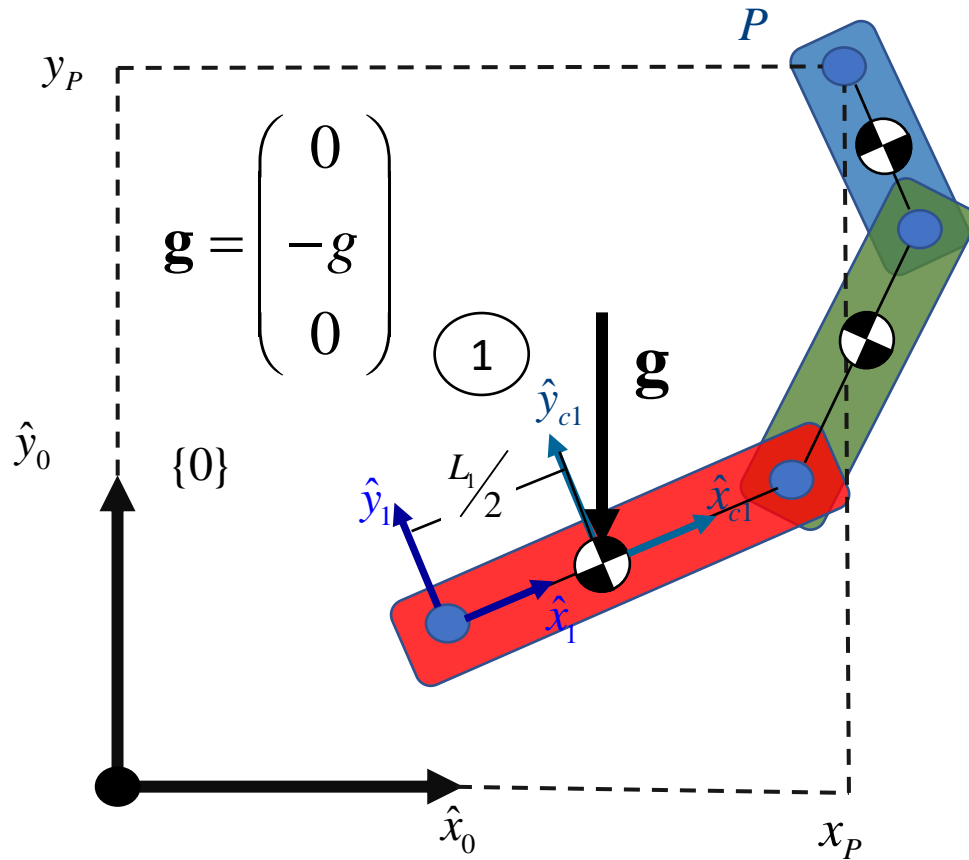
Cálculo del Lagrangeano

Energía potencial para el primer eslabón

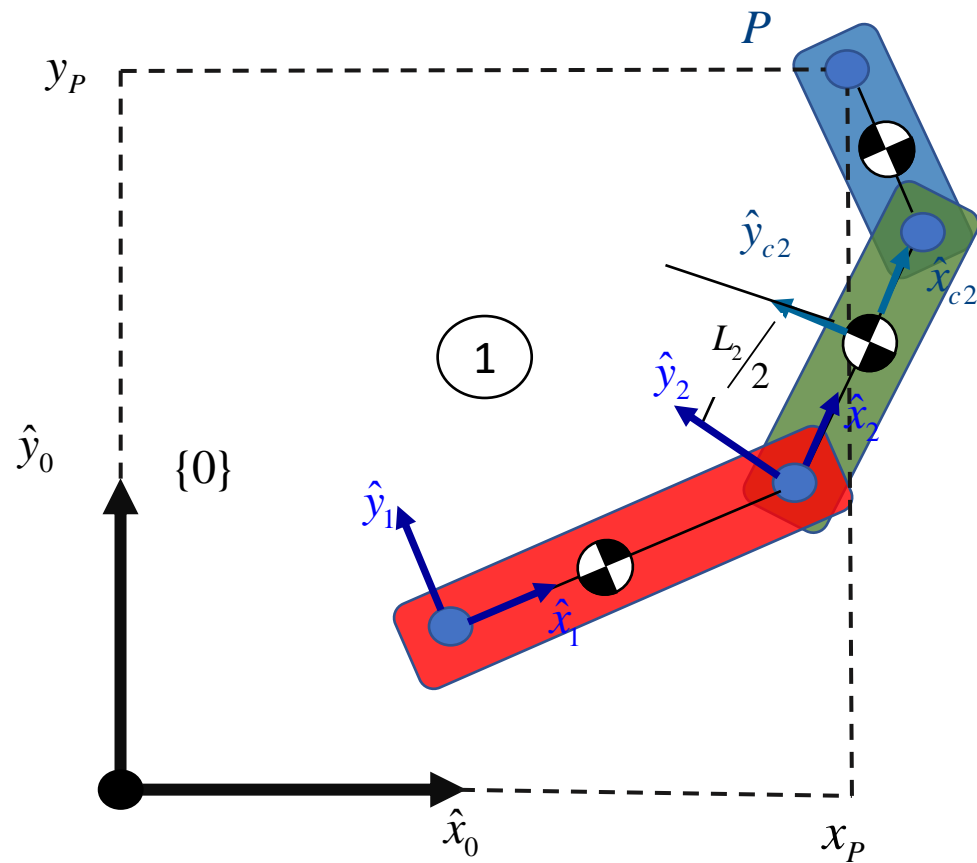
$$u_1 = -m_1 \mathbf{g}^T {}^0 \mathbf{p}_{c1}$$

$$= -m_1 \begin{pmatrix} 0 & -g & 0 \end{pmatrix} \begin{pmatrix} {}^0 x_1 + \frac{L_1}{2} \cos({}^0 \theta_1) \\ {}^0 y_1 + \frac{L_1}{2} \sin({}^0 \theta_1) \\ 0 \end{pmatrix} =$$

$$= m_1 g \left({}^0 y_1 + \frac{L_1}{2} \sin({}^0 \theta_1) \right)$$



Cálculo del Lagrangeano



Energía cinética para el segundo eslabón

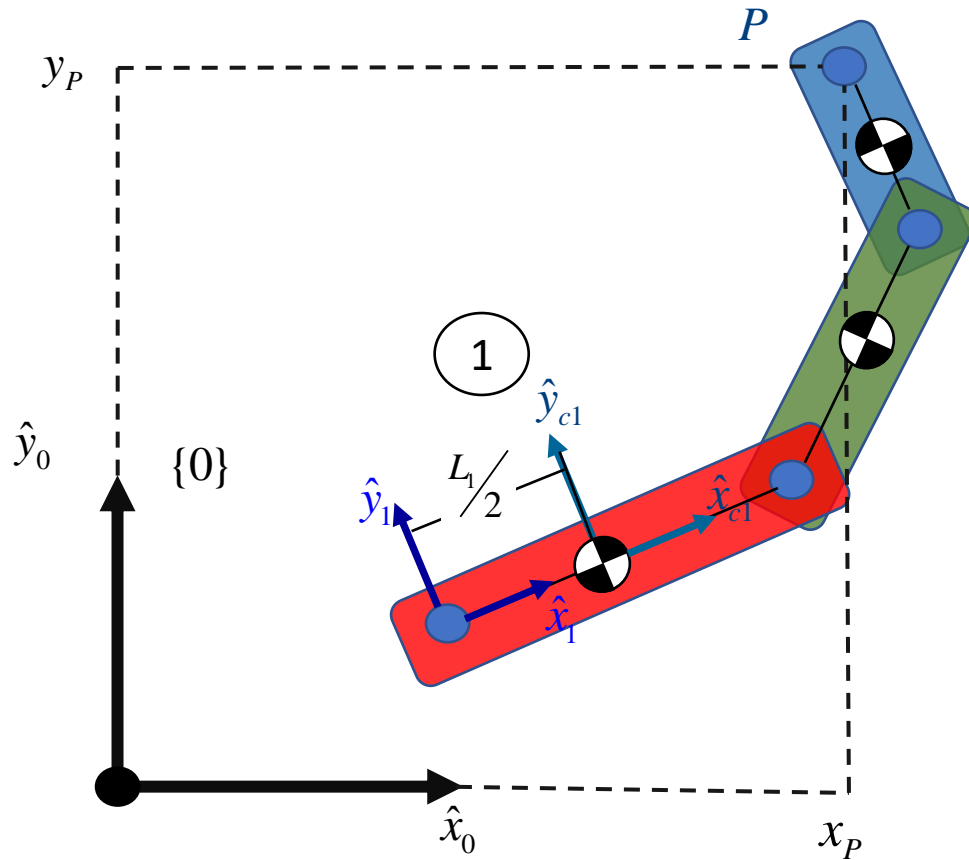
$$k_2 = \frac{1}{2} m_2 \mathbf{v}_{c2}^T \mathbf{v}_{c2} + \frac{1}{2} \boldsymbol{\omega}_2^T \mathbf{I}_{c2} \boldsymbol{\omega}_2$$

Energía cinética de la parte lineal

$$\mathbf{v}_{c2} = {}^{c2}\mathbf{R}_2 \left({}^2\mathbf{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\mathbf{p}_{c2} \right) =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} L_1 \sin({}^1\theta_2) {}^0\dot{\theta}_1 \\ L_1 \cos({}^1\theta_2) {}^0\dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} \frac{L_2}{2} \\ 0 \\ 0 \end{pmatrix} \right) =$$

Cálculo del Lagrangeano



Energía cinética para el segundo eslabón

$$k_2 = \frac{1}{2} m_2 \mathbf{v}_{c2}^T \mathbf{v}_{c2} + \frac{1}{2} \boldsymbol{\omega}_2^T \mathbf{I}_{c2} \boldsymbol{\omega}_2$$

Energía cinética de la parte lineal

$$\mathbf{v}_{c2} = {}^c \mathbf{R}_2 \left({}^2 \mathbf{v}_2 + {}^2 \boldsymbol{\omega}_2 \times {}^2 \mathbf{p}_{c2} \right) =$$

$$= \begin{pmatrix} \frac{L_2}{2} \sin({}^2 \theta_3) ({}^0 \dot{\theta}_1 + {}^1 \dot{\theta}_2) + L_1 \sin({}^1 \theta_2 + {}^2 \theta_3) {}^0 \dot{\theta}_1 \\ \frac{L_2}{2} \cos({}^2 \theta_3) ({}^0 \dot{\theta}_1 + {}^1 \dot{\theta}_2) + L_1 \cos({}^1 \theta_2 + {}^2 \theta_3) {}^0 \dot{\theta}_1 \\ 0 \end{pmatrix}$$

Cálculo del Lagrangeano

Energía cinética para el segundo eslabón

$$\begin{aligned}
 k_2 &= \frac{1}{2} m_2 \mathbf{v}_{c2}^T \mathbf{v}_{c2} + \frac{1}{2} \boldsymbol{\omega}_2^T \mathbf{I}_{c2} \boldsymbol{\omega}_2 = \\
 &= \frac{1}{2} m_2 \left(\frac{L_2}{2} \sin({}^2\theta_3) ({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) + L_1 \sin({}^1\theta_2 + {}^2\theta_3) {}^0\dot{\theta}_1 \right)^2 \\
 &\quad + \frac{1}{2} (0 \quad 0 \quad {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) \frac{m_2}{12} \begin{pmatrix} h_2^2 + b_2^2 & 0 & 0 \\ 0 & L_2^2 + b_2^2 & 0 \\ 0 & 0 & L_2^2 + h_2^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 \end{pmatrix} = \\
 &= \frac{m_2}{24} \left(h_2^2 ({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2)^2 + 4 \left(3L_1^2 {}^0\dot{\theta}_1^2 + L_2^2 ({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2)^2 \right) + 12L_1L_2 {}^0\dot{\theta}_1 ({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2) \cos({}^1\theta_2) \right)
 \end{aligned}$$

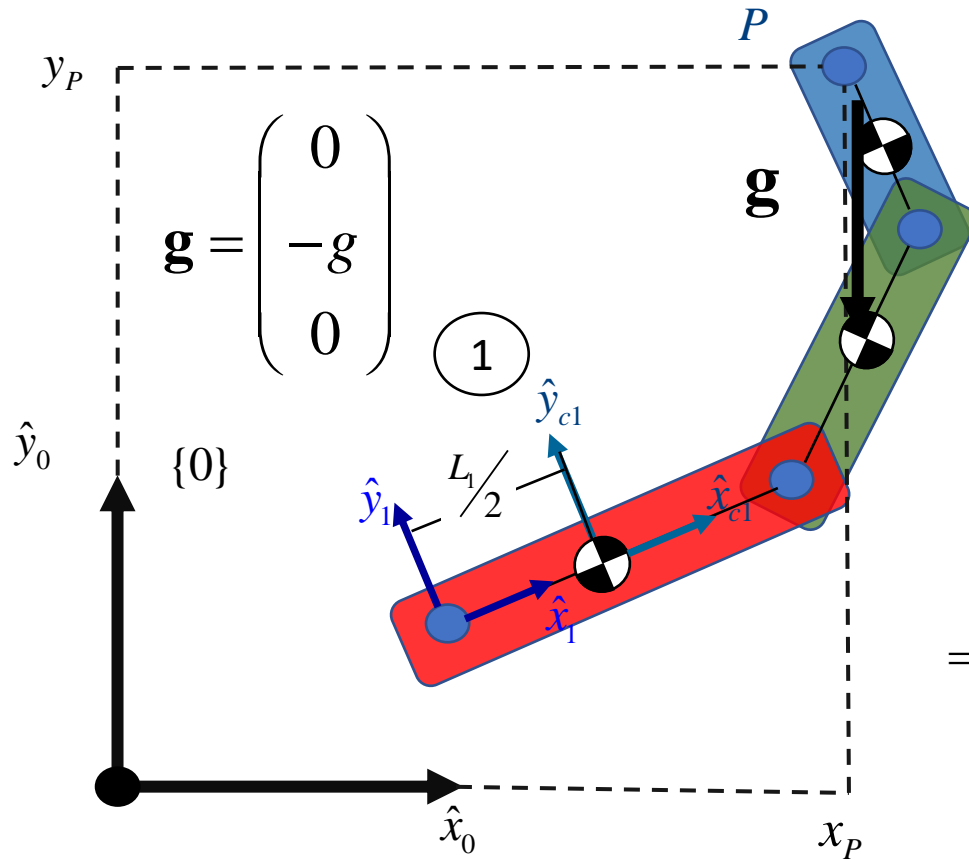
Cálculo del Lagrangeano

Energía potencial para el primer eslabón

$$u_2 = -m_2 \mathbf{g}^T {}^0 \mathbf{p}_{c2}$$

$${}^0 \mathbf{T}_{c2} = {}^0 \mathbf{T}_1 {}^1 \mathbf{T}_2 {}^2 \mathbf{T}_{c2} = \begin{pmatrix} {}^0 \mathbf{R}_{c2} & {}^0 \mathbf{p}_{c2} \\ \mathbf{0}^T & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos({}^0\theta_1 + {}^1\theta_2) & -\sin({}^0\theta_1 + {}^1\theta_2) & 0 & {}^0x_1 + L_1 \cos({}^0\theta_1) + \frac{L_2}{2} \cos({}^0\theta_1 + {}^1\theta_2) \\ \sin({}^0\theta_1 + {}^1\theta_2) & \cos({}^0\theta_1 + {}^1\theta_2) & 0 & {}^0y_1 + L_1 \sin({}^0\theta_1) + \frac{L_2}{2} \sin({}^0\theta_1 + {}^1\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



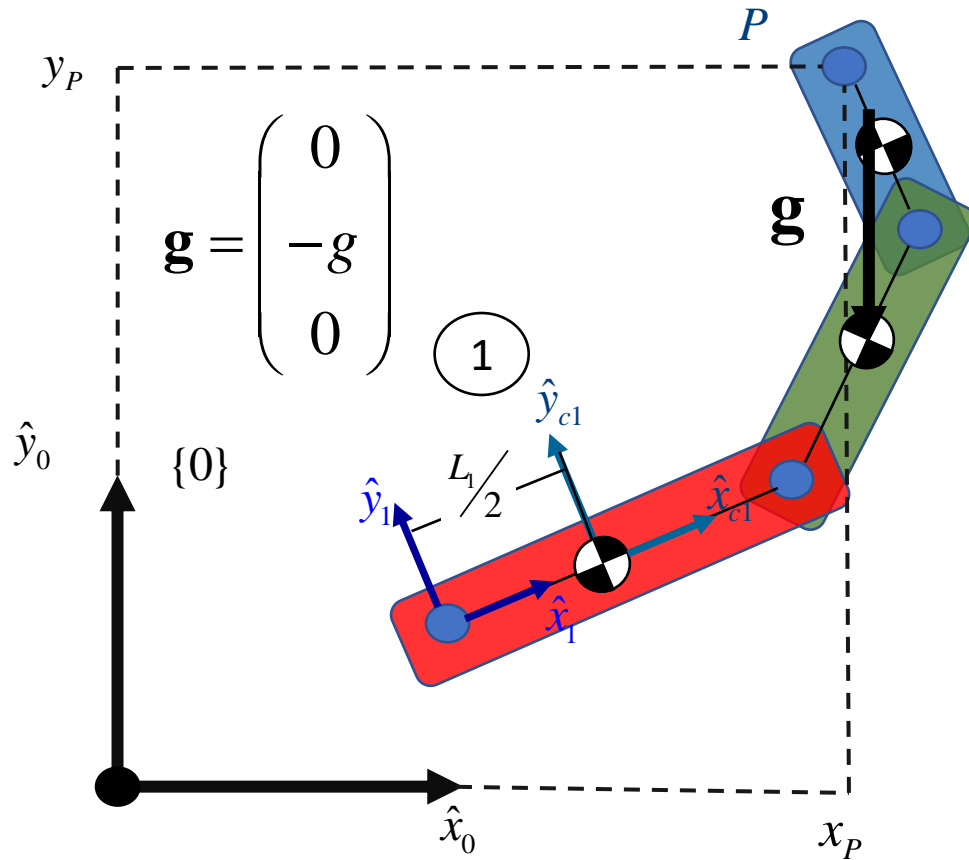
Cálculo del Lagrangeano

Energía potencial para el primer eslabón

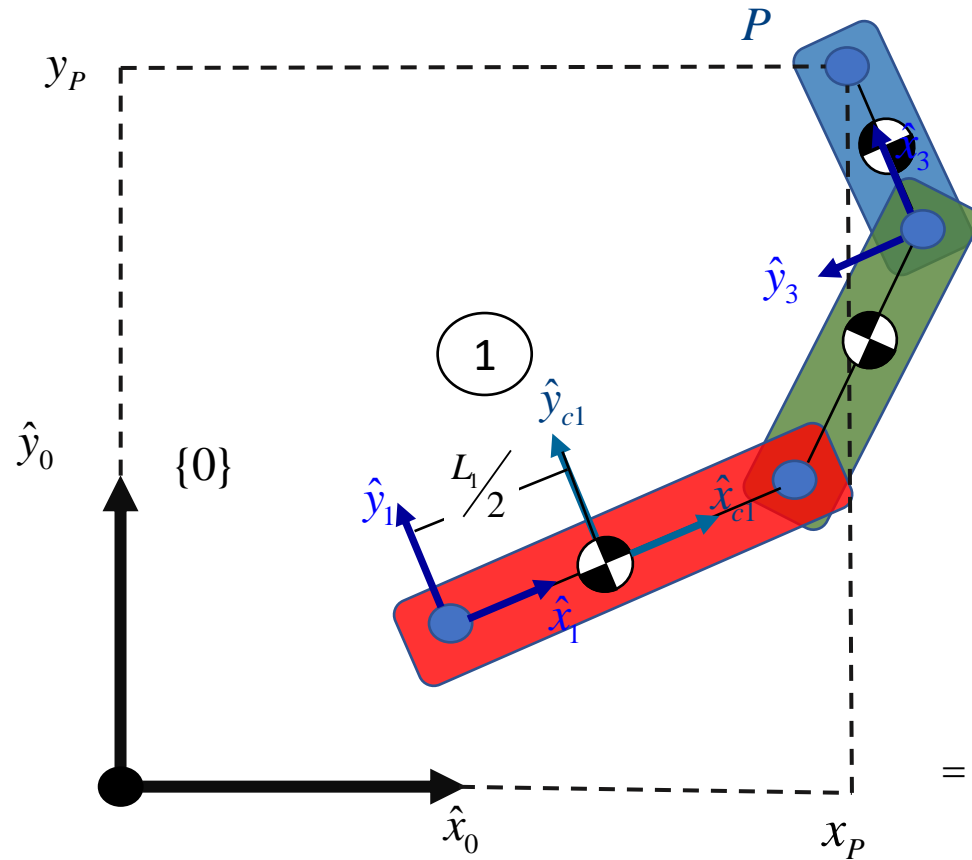
$$u_2 = -m_2 \mathbf{g}^T {}^0 \mathbf{p}_{c2}$$

$$= -m_2 \begin{pmatrix} 0 & -g & 0 \end{pmatrix} \begin{pmatrix} {}^0 x_1 + L_1 \cos({}^0 \theta_1) + \frac{L_2}{2} \cos({}^0 \theta_1 + {}^1 \theta_2) \\ {}^0 y_1 + L_1 \sin({}^0 \theta_1) + \frac{L_2}{2} \sin({}^0 \theta_1 + {}^1 \theta_2) \\ 0 \end{pmatrix} =$$

$$= m_2 g \left({}^0 y_1 + L_1 \sin({}^0 \theta_1) + \frac{L_2}{2} \sin({}^0 \theta_1 + {}^1 \theta_2) \right)$$



Cálculo del Lagrangeano



Energía cinética para el tercer eslabón

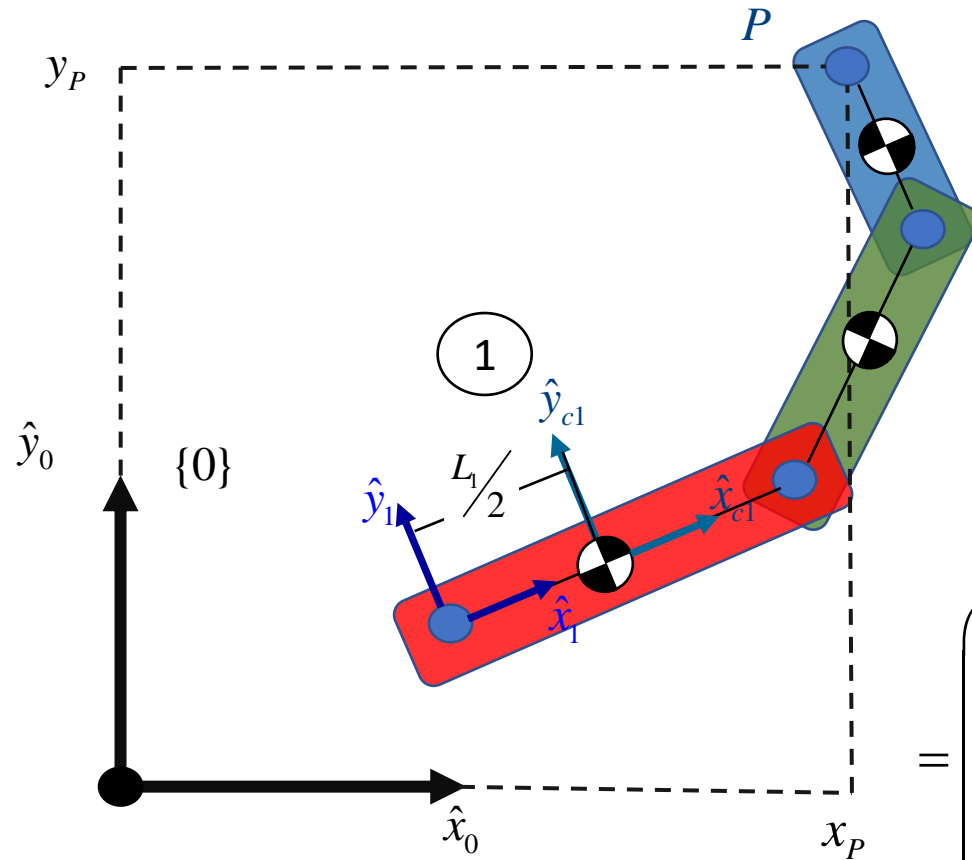
$$k_3 = \frac{1}{2} m_3 \mathbf{v}_{c3}^T \mathbf{v}_{c3} + \frac{1}{2} \boldsymbol{\omega}_3^T \mathbf{I}_{c3} \boldsymbol{\omega}_3$$

Energía cinética de la parte lineal

$$\mathbf{v}_{c3} = {}^c\mathbf{R}_3 \left({}^3\mathbf{v}_3 + {}^3\boldsymbol{\omega}_3 \times {}^3\mathbf{p}_{c3} \right) =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} L_2 \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \sin({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ L_2 \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \cos({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ {}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3 \end{pmatrix} \times \begin{pmatrix} \frac{L_3}{2} \\ 0 \\ 0 \end{pmatrix} \right) =$$

Cálculo del Lagrangeano



Energía cinética para el tercer eslabón

$$k_3 = \frac{1}{2} m_3 \mathbf{v}_{c3}^T \mathbf{v}_{c3} + \frac{1}{2} \boldsymbol{\omega}_3^T \mathbf{I}_{c3} \boldsymbol{\omega}_3$$

Energía cinética de la parte lineal

$$\mathbf{v}_{c3} = {}^{c3}\mathbf{R}_3 \left({}^3\mathbf{v}_3 + {}^3\boldsymbol{\omega}_3 \times {}^3\mathbf{p}_{c3} \right) =$$

$$= \begin{pmatrix} L_2 \sin({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \sin({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 \\ L_2 \cos({}^2\theta_3)({}^0\dot{\theta}_1 + {}^2\dot{\theta}_3) + L_1 \cos({}^1\theta_2 + {}^2\theta_3){}^0\dot{\theta}_1 + \frac{L_3}{2}({}^0\dot{\theta}_1 + {}^1\dot{\theta}_2 + {}^2\dot{\theta}_3) \\ 0 \end{pmatrix}$$

Planteamiento del modelo

Energía cinética para el tercer eslabón

$$\begin{aligned} k_3 &= \frac{1}{2} m_3 \mathbf{v}_{c3}^T \mathbf{v}_{c3} + \frac{1}{2} \boldsymbol{\omega}_3^T \mathbf{I}_{c3} \boldsymbol{\omega}_3 = \\ &= \frac{m_3}{24} \left((L_3^2 + h_2^2) (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + 12 \left(\frac{1}{2} L_3 (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) + L_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_3) + L_1 \dot{\theta}_1 \cos(\theta_2 + \theta_3) \right)^2 + \right. \\ &\quad \left. + \left(L_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_3) + L_1 \dot{\theta}_1 \sin(\theta_2 + \theta_3) \right)^2 \right) \end{aligned}$$

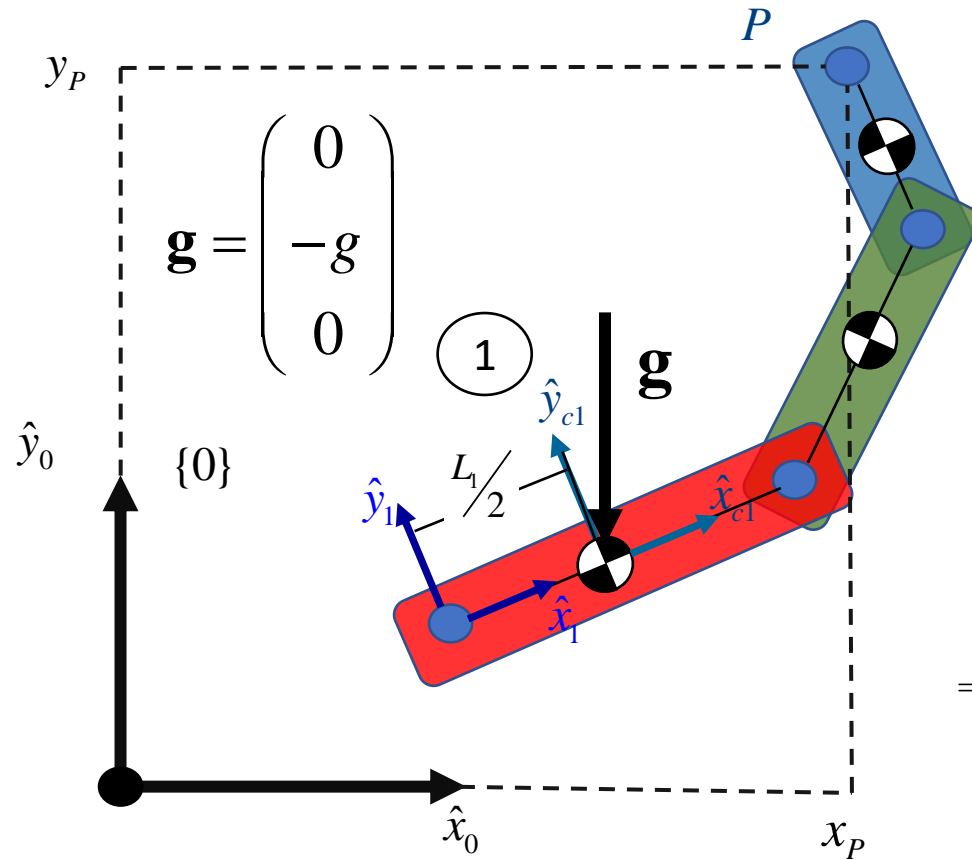
Planteamiento del modelo

Energía potencial para el primer eslabón

$$u_3 = -m_3 \mathbf{g}^T {}^0 \mathbf{p}_{c3}$$

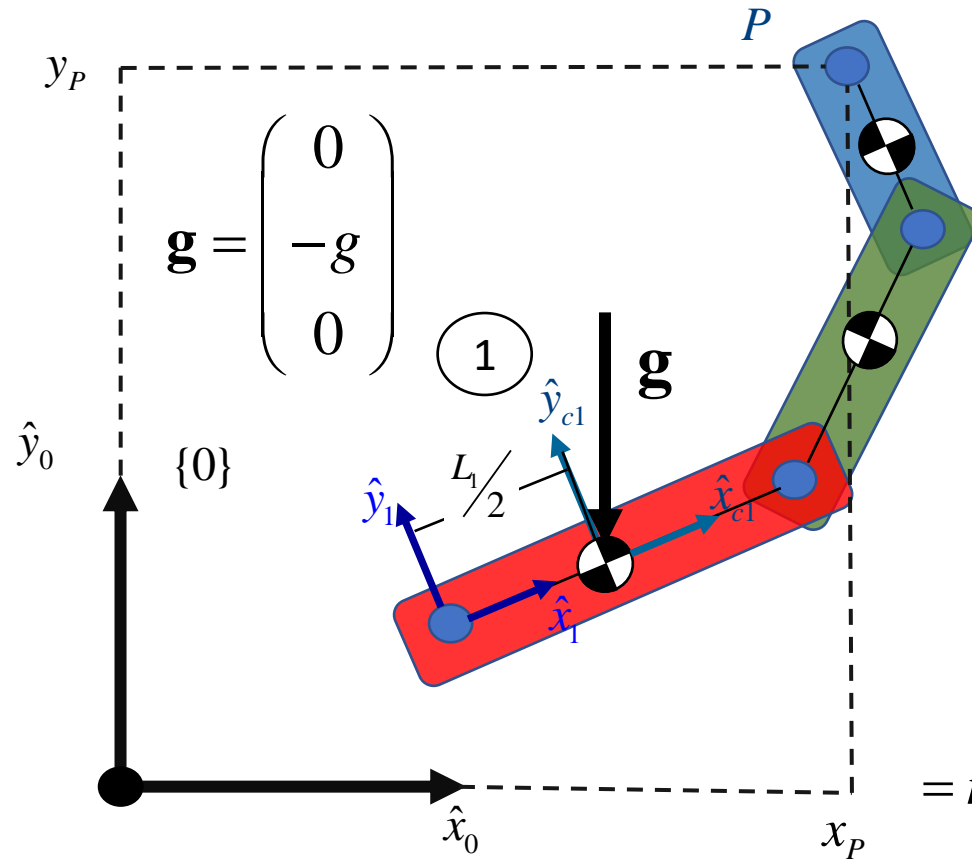
$${}^0 \mathbf{T}_{c3} = {}^0 \mathbf{T}_1 {}^1 \mathbf{T}_2 {}^2 \mathbf{T}_3 {}^3 \mathbf{T}_{c3} = \begin{pmatrix} {}^0 \mathbf{R}_{c3} & {}^0 \mathbf{p}_{c3} \\ \mathbf{0}^T & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & -\sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & 0 & {}^0x_1 + L_1 \cos({}^0\theta_1) + L_2 \cos({}^0\theta_1 + {}^1\theta_2) + \frac{L_3}{2} \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) \\ \sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & \cos({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) & 0 & {}^0y_1 + L_1 \sin({}^0\theta_1) + L_2 \sin({}^0\theta_1 + {}^1\theta_2) + \frac{L_3}{2} \sin({}^0\theta_1 + {}^1\theta_2 + {}^2\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Planteamiento del modelo

Energía potencial para el primer eslabón

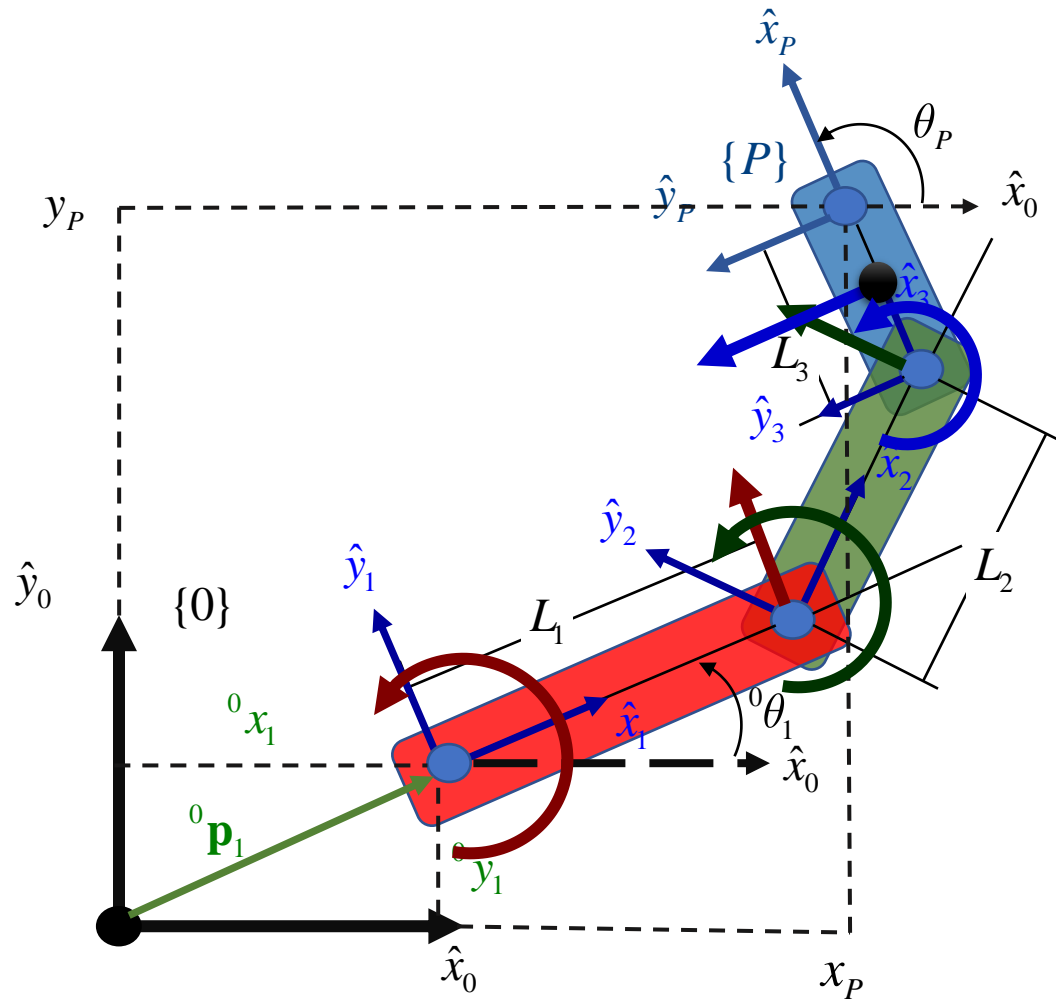


$$u_3 = -m_3 \mathbf{g}^T {}^0 \mathbf{p}_{c3}$$

$$= -m_3 \begin{pmatrix} 0 & -g & 0 \end{pmatrix} \begin{pmatrix} {}^0 x_1 + L_1 \cos({}^0 \theta_1) + L_2 \cos({}^0 \theta_1 + {}^1 \theta_2) + \frac{L_3}{2} \cos({}^0 \theta_1 + {}^1 \theta_2 + {}^2 \theta_3) \\ {}^0 y_1 + L_1 \sin({}^0 \theta_1) + L_2 \sin({}^0 \theta_1 + {}^1 \theta_2) + \frac{L_3}{2} \sin({}^0 \theta_1 + {}^1 \theta_2 + {}^2 \theta_3) \\ 0 \end{pmatrix} =$$

$$= m_3 g \left({}^0 y_1 + L_1 \sin({}^0 \theta_1) + L_2 \sin({}^0 \theta_1 + {}^1 \theta_2) + \frac{L_3}{2} \sin({}^0 \theta_1 + {}^1 \theta_2 + {}^2 \theta_3) \right)$$

Cálculo de los pares



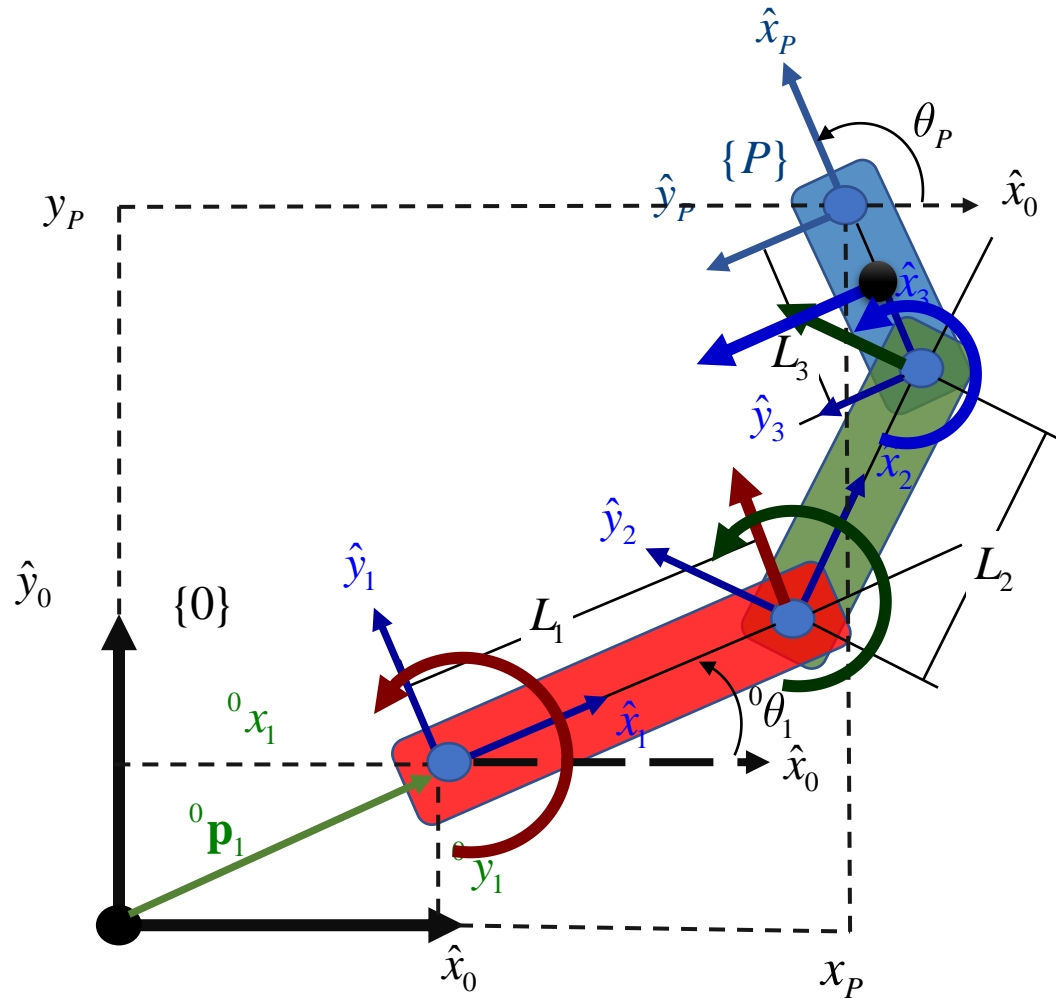
Cálculo del Lagrangeano

$$\Gamma = (k_1 + k_2 + k_3) - (u_1 + u_2 + u_3)$$

Ecuación del par

$$\tau_i = \frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_i} \Gamma \right) - \frac{\partial}{\partial q_i} \Gamma$$

Cálculo de los pares



Cálculo del Lagrangeano

$$\Gamma = (k_1 + k_2 + k_3) - (u_1 + u_2 + u_3)$$

Ecuación del par

$$\boldsymbol{\tau}_\theta = \mathbf{M}(q)\ddot{\mathbf{q}} + \mathbf{V}(q, \dot{q}) + \mathbf{G}(q)$$