

Exponencial matricial e simulação com representação de estados

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1 Sistema de Segunda Ordem

A equação do movimento escrita na forma padronizada é:

$$\ddot{q} + 2\xi\omega_n\dot{q} + \omega_n^2q = 0 \quad (1)$$

Substituindo pelos valores do enunciado:

$$\ddot{q} + 10\dot{q} + 10^4q = 0 \quad (2)$$

Com isso podemos transformar a equação para a forma matricial:

Definindo um conjunto de variáveis de estado:

$$x_1(t) = q \quad ; \quad x_2(t) = \dot{q}$$
$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Derivando x :

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

Substituindo na eq. original:

$$\begin{cases} \dot{x}_2 + 10x_2 + 10^4x_1 = 0 \\ \dot{x}_1 = x_2 \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = -10x_2 - 10^4x_1 \\ \dot{x}_1 = x_2 \end{cases}$$
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -10^4 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$$

Portanto,

$$A = \begin{bmatrix} 0 & 1 \\ -10^4 & -10 \end{bmatrix}$$

Figura 1: Encontrando a matriz A

Portanto,

$$\dot{x} = Ax + Bu \quad (3)$$

Onde,

$$Bu = 0 \quad (4)$$

$$A = \begin{bmatrix} 0 & 1 \\ -10^4 & -10 \end{bmatrix} \quad (5)$$

2 Calculando a resposta livre

A resposta livre a uma condição inicial é dada por:

$$x(t) = e^{At}x_0 \quad (6)$$

Portanto, precisamos calcular e^{At}

2.1 Por Autovalores

Handwritten mathematical derivation for finding eigenvalues and eigenvectors of matrix A :

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -10^4 & -10 - \lambda \end{vmatrix} = \lambda(10 + \lambda) + 10^4 = 0$$

$$\lambda_1 = -5 + 99,87i$$

$$\lambda_2 = -5 - 99,87i$$

Encontrando os autovetores:

$$\begin{bmatrix} +5 - 99,87i & 1 \\ -10^4 & -10 + 5 - 99,87i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} +5 + 99,87i & 1 \\ -10^4 & -10 + 5 + 99,87i \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = 0$$

$$v_1 = \begin{bmatrix} -5 \cdot 10^{-4} - 1 \cdot 10^{-2}i \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -5 \cdot 10^{-4} + 1 \cdot 10^{-2}i \\ 1 \end{bmatrix}$$

Com isso temos:

$$\Lambda = \begin{bmatrix} -5 + 99,87i & 0 \\ 0 & -5 - 99,87i \end{bmatrix} \Rightarrow e^{\Lambda t} = \begin{bmatrix} e^{(-5 + 99,87i)t} & 0 \\ 0 & e^{(-5 - 99,87i)t} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} -5 \cdot 10^{-4} - 1 \cdot 10^{-2}i & -5 \cdot 10^{-4} + 1 \cdot 10^{-2}i \\ 1 & 1 \end{bmatrix} \Rightarrow \Sigma^{-1} = \begin{bmatrix} 50,07i & 0,5 + 2,5 \cdot 10^{-2}i \\ -50,07i & 0,5 - 2,5 \cdot 10^{-2}i \end{bmatrix}$$

Figura 2: Autovalores

Realizando o cálculo de e^{At} pelo matlab:

$$\begin{aligned} & \text{syms } t; \\ & [U, L] = \text{eig}(A); \\ & e^{At} = U * \text{diag}(\exp(\text{diag}(L * t))) * \text{inv}(U) \end{aligned}$$

$$e^{At} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (7)$$

Onde,

$$a = (\exp(-t*(5+399(1/2)*5i))*(\exp(399(1/2)*t*10i)*(649037107316853271723688571817328-32492496402889698541231938784245i) + 649037107316853363837439069708896 + 32492496402889703152687390203090i))/1298074214633706907132624082305024$$

$$b = -\exp(t*(-5+399(1/2)*5i))*(10001(1/2)/200+3607572128482979i/144115188075855872)*(4611455451418845/9223372036854775808 + 5757109406118223i/576460752303423488) - \exp(-t*(5+399(1/2)*5i))*(10001(1/2)/200-7215144256965959i/288230376151711744)*(4611455451418845/9223372036854775808 - 5757109406118223i/576460752303423488)$$

$$c = (25*10001(1/2)*\exp(-t*(5+399(1/2)*5i))*(\exp(399(1/2)*t*10i)*7046039313443321i-7046039313443322i))/351878905260408832$$

$$d = (100*10001(1/2)*\exp(t*(-5+399(1/2)*5i))*(10001(1/2)/200+3607572128482979i/144115188075855872))/10001+(100*10001(1/2)*\exp(-t*(5+399(1/2)*5i))*(10001(1/2)/200-7215144256965959i/288230376151711744))/10001$$

2.2 Por Laplace

Usaremos a relação $e^{At} = L^{-1}[(sI - A)^{-1}]$

Handwritten derivation for the Laplace transform of the matrix exponential e^{At} .

Step 1: Calculate the inverse of $(sI - A)$.

$$(sI - A) = \begin{bmatrix} s & -1 \\ +10^4 & s+10 \end{bmatrix} \Rightarrow (sI - A)^{-1} = \begin{bmatrix} \frac{s+10}{s^2+10s+10^4} & \frac{1}{s^2+10s+10^4} \\ \frac{-10^4}{s^2+10s+10^4} & \frac{s}{s^2+10s+10^4} \end{bmatrix}$$

Step 2: Use the Laplace transform to find e^{At} .

Com isso, temos:

$$e^{At} = L^{-1}[(sI - A)^{-1}] =$$

$$= \begin{bmatrix} e^{-st} \cdot \cos(99,87t) + 0,05 \cdot \sin(99,87t) & \frac{19,98 \cdot e^{-st} \cdot \sin(99,87t)}{1995} \\ \frac{-4 \cdot 10^4 \cdot e^{-st} \cdot \sin(99,87t)}{399} & \frac{e^{-st} \cdot (\cos(99,87t) - 19,97 \cdot \sin(99,87t))}{399} \end{bmatrix}$$

Figura 3: Laplace

2.3 Por Cayley-Hamilton

Usamos a equação $e^{At} = \sum_{l=0}^{n-1} \alpha_l A^l$