The Abstract and the Application of the Basel Problem

How can the Basel Problem be efficiently solved and modeled?

IB Extended Essay

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Introduction

Abstract, Aim, and Rationale

The constant π is one of the most perplexing topics in mathematics, being an irrational and patternless number that manages to appear in several branches of the subject ranging from geometry and trigonometry to calculus and complex numbers. It even emerges as a term in numerous mathematical formulae that appear to have little to do with π on the surface level. One such formula is the Basel Problem. Originated by Euler, it is an infinite series of reciprocal squares whose sum becomes $\pi^2/6$ (Dunham, 1999). The unusual transition from a simple harmonic series to an irrational value leads me to ask the question: How can the Basel Problem be efficiently solved and modeled? In this investigation, I will analyze the mathematical procedures required to reach this result by developing proofs with different methods. Ultimately, the aim of this investigation is to create visualizations of the Basel Problem from the proofs and explanations explored to best understand its practical application. I find this investigation to be a valuable exploration into mathematics because finding ways to model this unique problem uncovers applicable and relevant connections between π and algebra—connecting the abstract to the representational and uncovering the beauty and elegance of mathematical proof.

Methodology

The first step in my research process was a 'blind' investigation into the problem, prior to consulting any sources about the Basel Problem. I approached solving the problem utilizing two methods: first with improper integrals, then with odd-even grouping—sorting the terms based on whether they came at an odd or even position in the series. Because the blind approach failed to prove the theorem, I then went ahead to analyze the original proof and supplementary sources to observe the use of different mathematical procedures. I was able to compare the work they did to my own, understanding where my blind investigation was successful and where it fell short. Then, I utilized my findings to create a plausible model from the research I conducted. Lastly, I investigated an additional proof that tied together my blind investigation with the visual models. Except when explicitly cited, all charts, graphs, drawings, and equations are of my own work or in extrapolation and analysis of the work of other mathematicians. All of the graphs I created were made using the online graphing software *Desmos* (2015).

Blind Investigation

Simple Integration Method

Below is the Basel Problem in both the summation notation and the expanded series (Hawking, 2005):

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$$

Having recently studied introductory integral calculus, I began investigating the sum in terms of an integral. Instead of taking the sum of the individual terms, I used an integral of the same expression ranging from one to infinity. This type of integral is called an improper integral because one end is bounded while the other continues unbounded. In order to evaluate the unbounded end, I will need to use limits.

$$\int_{1}^{\infty} \frac{1}{x^2} dx$$

The antiderivative of $\frac{1}{x^2}$ is $\frac{-1}{x}$.

$$= \left[\frac{-1}{x}\right]_{1}^{\infty} = \left(\lim_{x \to \infty} \frac{-1}{x}\right) - \frac{-1}{1}$$

Evaluating the limit as *x* approaches infinity, it is clear that the value of the expression is zero.

$$= 0 + 1 = 1$$

The answer to the improper integral alone does not match up with the answer to the Basel Problem, nor provide insight into the procedure in solving the series. Therefore, I decided to solve the integral gradually, increasing the upper bound by one such that it aligns with each partial sum of the Basel Problem. The first integral will be bounded between one and one, the next will be bounded between one and two, and so forth. Below is a chart I organized comparing the integral values to the actual sum:

Integral Sum	**Basel Problem Sum**
$\int_1^1 \frac{1}{x^2} dx = [\frac{-1}{x}]_1^1 = 1 - 1 = 0$	$\sum_{n=1}^{1} \frac{1}{n^2} = 1$
$\int_1^2 \frac{1}{x^2} dx = [\frac{-1}{x}]_1^2 = \frac{-1}{2} - 1 = 0.5$	$\sum_{n=1}^{2} \frac{1}{n^2} = 1 + \frac{1}{4} = 1.25$
$\int_1^3 \frac{1}{x^2} dx = [\frac{-1}{x}]_1^3 = \frac{-1}{3} - 1 = 0.667$	$\sum_{n=1}^{3} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} = 1.361$
$\int_1^4 \frac{1}{x^2} dx = [\frac{-1}{x}]_1^4 = \frac{-1}{4} - 1 = 0.75$	$\sum_{n=1}^{4} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = 1.424$
$\int_1^5 \frac{1}{x^2} dx = [\frac{-1}{x}]_1^5 = \frac{-1}{5} - 1 = 0.8$	$\sum_{n=1}^{5} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = 1.464$

Figure 1. Chart of first set of terms for integral function and Basel Problem.

As I have already solved, the integral does not converge onto the desired $\pi^2/6$ value. However, I can also graph the integral against the summation series along each integer step to examine if the functions share a relationship that could be used to prove the Basel Problem. In creating the graph of the integral, the function from the parent $F(x) = \frac{-1}{x}$ should intercept the *x*-axis when *x* equals one because it is the lower bound of the integral it derives from:

$$F(1) = \frac{-1}{1} + b = 0$$

$$b = \frac{1}{1} = 1$$

Hence, the formula for the integral model would be:

$$F(x) = \frac{-1}{x} + 1$$

To be able to graph the summation series and present only the integer values, I assigned the summation to the function G(x) and rewrote the upper bound as the floor function of the variable:

$$G(x) = \sum_{n=1}^{floor(x)} \frac{1}{n^2}$$

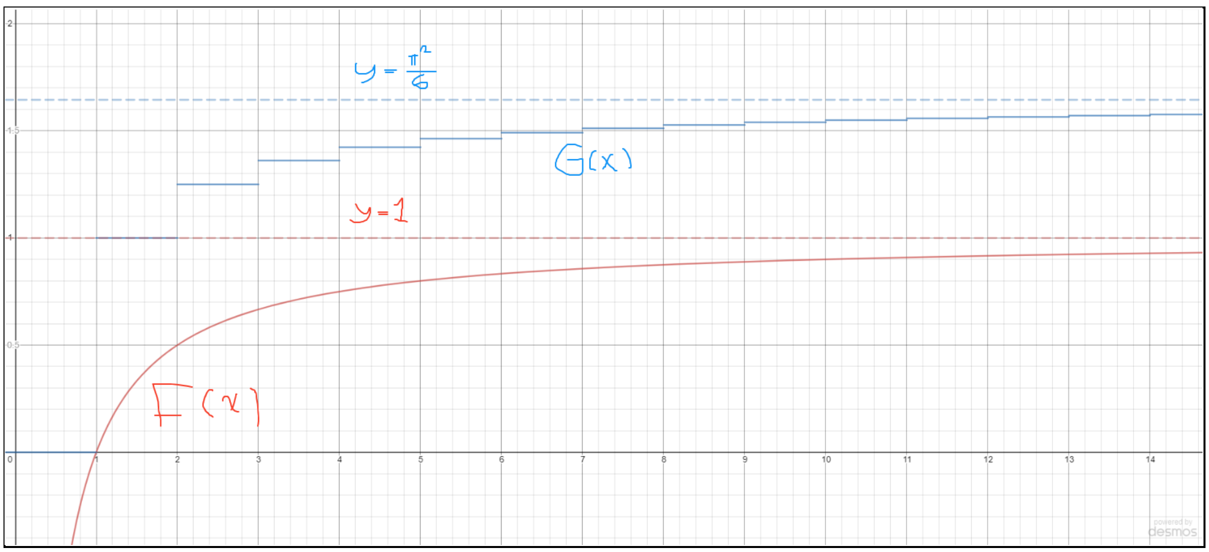

Figure 2. Graphs of F(x) and G(x) with asymptotes.

As expected, both the floor function and the integral resemble a similar trend: increasing but concave down, approaching an asymptote. I wanted to construct a function that would calculate the difference between the functions to see if I could uncover an algebraically-solvable sequence that leads to the desired $\pi^2/6$ sum. To do so, I converted the equation of F(x) into another floor function and subtracted the two functions:

$$F_2(x) = F(floor(x)) = \frac{-1}{floor(x)} + 1$$

Let function H represent the remainder of the subtraction:

$$H(x) = G(x) - F_2(x)$$

$$H(x) = \left(\sum_{n=1}^{floor(x)} \frac{1}{n^2}\right) - \frac{1}{floor(x)} + 1$$

The asymptote of the new floor function would hence be the asymptote x-value of G subtracted by the asymptote x-value of F:

$$\lim_{x \to \infty} H(x) = y = \frac{\pi^2}{6} - 1$$

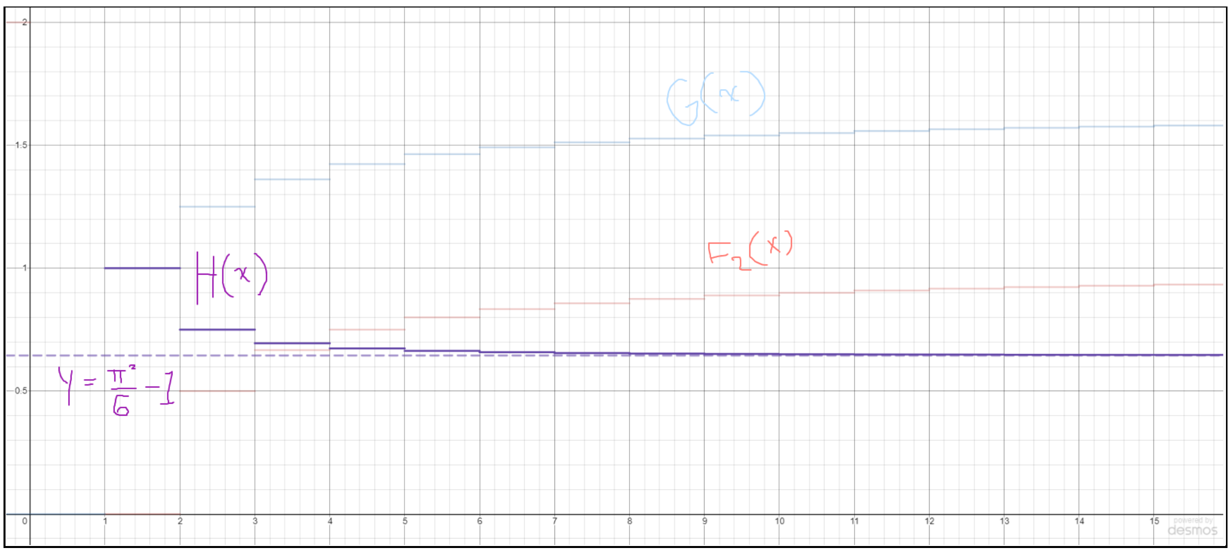

Figure 3. Graph of H(x) with asymptote.

If I solve each term of the new function algebraically, I may be able to rewrite the function and take its limit as it approaches infinity, hence solving the Basel Problem:

Term	**F_2(x)**	**G(x)**	**H(x) Value**
1	0	1	$H(1) = 1 - 0 = 1$
2	1/2	5/4	$H(2) = 5/4 - 1/2 = 3/4$
3	2/3	49/36	$H(3) = 49/36 - 2/3 = 25/36$
4	3/4	201/144	$H(4) = 201/144 - 3/4 = 93/144 = 31/48$
5	4/5	5269/3600	$H(5) = 5269/3600 - 4/5 = 2389/3600$

Figure 4. Chart of first terms of function H.

In experimenting with the created sequence {1, 3/4, 25/36, 31/48, 2389/3600…}, I failed to find any possible expression to represent these terms of H(x) aside from reverting the series back into its F_2 and G subparts. Unfortunately, this method of investigating was unsuccessful.

I believe this method failed because combining F_2(x) and G(x) did not help in untangling the summation but instead added more procedures to solving it. The solution to H(x) would have been a byproduct of solving for the Basel Problem and, therefore, would only complicate the solution. However, this step in the investigation was necessary because while I learned about its relation to integration and reciprocal expressions, it pointed my research towards other methods of disassembling the problem.

Grouping Method

I changed my focus instead to arithmetic manipulation by means of grouping. Without distributing the squares of reciprocals, the Basel Problem is written as:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$$

I can then rearrange the series into two groups: odd and even terms.

$$= (\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...) + (\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + ...)$$

All of the even terms share a factor of one fourth, since every denominator in the subseries is a square of an even number and hence has a factor of two squared.

$$= (\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...) + (\frac{1}{4} (\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...))$$

Interestingly, the resulting series after factoring the even group is the original Basel Problem.

$$= (\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...) + (\frac{1}{4} (\sum_{n=1}^{\infty} \frac{1}{n^2}))$$

This means that the even and odd groupings are recursive to the Basel Problem, containing a simple fraction of itself within each.

$$Let\ k = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$k = (\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...) + \frac{1}{4}k,\ \frac{1}{4}k = k_{even}$$

By gathering the *k* terms to the left side and simplifying, it is shown that the odd group is three-fourths of the value of the Basel Problem:

$$\frac{3}{4}k = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = k_{odd}$$

While the recursive property of grouping odds and evens was an interesting find in my investigation, it unfortunately means I reached an indefinite proof loop in the Basel Problem, meaning that this method alone cannot produce the answer to the series:

$$k = k_{odd} + \frac{1}{4}(k_{odd} + \frac{1}{4}(k_{odd} + \ldots)), \text{ k remains unknown.}$$

The value of k could be any number and the statement would hold true. This series, however, does produce an interesting new expression to write π when assuming the Basel Problem equation to be true:

$$k_{odd} = \frac{3}{4}k, k_{odd} = \frac{3}{4} * \frac{\pi^2}{6}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$

$$\sqrt{8 + \frac{8}{3^2} + \frac{8}{5^2} + \frac{8}{7^2} + \ldots} = \pi$$

Conclusion

Once again, the blind investigation fell short because I was unable to find a solution without entering a 'proof loop.' The reason for this is because both groups of terms are a multiple of the whole series and hence cycles the proof back to the original problem. However, discovering this recursion proves useful later in my investigation because it plays a major role in proving the Basel Problem using a different method.

The Original Proof

Leonhard Euler

The original proof developed by Euler utilizes the Taylor Series of a sine function and properties in factorization. Below is the Taylor Series for sine of x, as discovered by Brook Taylor in 1715 (Hawking, 2005):

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - ...$$

The reason why the Taylor Series function exactly models the sine function is because it matches every value of its derivative functions of any degree. Continuing the proof, Euler then divides each side by x. (Hawking, 2005):

$$\frac{sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - ...$$

It is important to note that the second term in the series now resembles the negative of the solution to the Basel Problem when substituting π for x:

$$\frac{sin(\pi)}{\pi} = 1 - \frac{\pi^2}{3!} + \frac{\pi^4}{5!} - \frac{\pi^6}{7!} + \frac{\pi^8}{9!} - ...$$

$$\frac{-\pi^2}{3!} = \frac{-\pi^2}{6}$$

Furthermore, it is important to establish the value this expression approaches as x approaches zero:

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \frac{0^6}{7!} + \frac{0^8}{9!} - ...$$

Every term containing x cancels to zero, leaving only the constant. Hence:

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1$$

The next steps involve returning to the original sine function and performing algebraic manipulation. Based on the Familiar Factor Theorem, “the polynomial f(x) has a root x = r if and only if it is divisible by (x - r)” (Monks, 7). Since sine of x is a periodic function, it contains an infinite set of roots that are each of equal intervals. Sine of x crosses the x-axis at nπ, where n is an integer. Hence, I could begin writing the factor as x equals 0, -π, π, -2π, 2π, and so on to cover every integer to infinity:

$$sin(x) \; =? \; x(x + \pi)(x - \pi)(x + 2\pi)(x - 2\pi)...$$

However, one needs to take into account that sin(x)/x needs to equal one when x equals zero (only theoretically, given that the real value is undefined due to dividing by zero) (Dunham, 1999):

$$\frac{sin(x)}{x} =? \frac{x(x+\pi)(x-\pi)(x+2\pi)(x-2\pi)...}{x}$$

$$\frac{sin(0)}{0} =? (0 + \pi)(0 - \pi)(0 + 2\pi)(0 - 2\pi)... = \pi * (-\pi) * 2\pi * (-2\pi) *...$$

Evidently, an infinite product of multiples of π diverges towards an oscillating infinite value and therefore cannot equal one, resembling a diverging geometric sequence (Martin, 2010). Euler solves this problem by introducing an alternative application of the Familiar Factor Theorem where each factor is divided by the negative root (Danham, 1999) (Hawking, 2005). This works because it preserves the x-axis intersections as well as the direction of the curve because the factors are in positive and negative pairs. The following is the factor conversion by a professor at Ursinus College (Monks, 2020, pp. 7-8):

$$\text{“}x - r = (-\,r(\frac{x}{-r} + 1)) = (-\,r(1 - \frac{x}{r}))\text{ ”}$$

New factor: $(1 - \frac{x}{r})$

$$\frac{sin(x)}{x} = (1 + \frac{x}{\pi})(1 - \frac{x}{\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{3\pi})(1 - \frac{x}{3\pi})...$$

By plugging in zero for x, the new factorization succeeds in maintaining the constant value because every factor equals one as x approaches zero (Hawking, 2005):

$$\frac{sin(0)}{0} = (1 + \frac{0}{\pi})(1 - \frac{0}{\pi})(1 + \frac{0}{2\pi})(1 - \frac{0}{2\pi})(1 + \frac{0}{3\pi})(1 - \frac{0}{3\pi})...$$

$$\frac{sin(0)}{0} = (1 + 0)(1 - 0)(1 + 0)(1 - 0)(1 + 0)(1 - 0)... = 1$$

The new equation can be condensed using special properties of binomial factors:

$$\frac{sin(x)}{x} = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2})(1 - \frac{x^2}{16\pi^2})...$$

The only constant that appears from distributing the infinite product would come from multiplying the one’s together. Furthermore, the only way to obtain squared terms would be to multiply the x^2 terms in each factor by the one’s from the other factors (which therefore leaves the original term itself) (Hawking, 2005):

$$Let\ t_1\ be\ the\ constant.\ Let\ t_2\ be\ the\ square\ term.$$

$$t_1 = 1 * 1 * 1 *... = 1$$

$$t_2 = ((-\frac{x^2}{\pi^2} * 1 * 1...) - (\frac{x^2}{4\pi^2} * 1 * 1...) - (\frac{x^2}{9\pi^2} * 1 * 1...) - ...)$$

$$t_2 = -(\frac{x^2}{\pi^2} + \frac{x^2}{4\pi^2} + \frac{x^2}{9\pi^2} + \frac{x^2}{16\pi^2} + \frac{x^2}{25\pi^2} + ...) = -\frac{x^2}{\pi^2}(\sum_{n=1}^{\infty}\frac{1}{n^2})$$

Now, Euler links the summation expression for t_2 with the actual square term uncovered using the Taylor Series, $-\frac{x^2}{3!}$ (Dunham, 1999). By doing so, he successfully reaches the conclusion to the Basel Problem:

$$-\frac{x^2}{3!} = -\frac{x^2}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)$$

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3!} = \frac{\pi^2}{6}.\ Q.E.D.$$

Connection to Blind Investigation

What made Euler's approach the most different from mine was how the proof did not begin with the Basel Problem itself. While my investigation attempted to work backwards by manipulating the series and manually solving the summation, Euler discovered this property as a byproduct from an exploration in sinusoids. The unsuccessfulness of my blind investigation resulted from limiting myself to the confines of properties of a series rather than coordinating the problem with concepts related to π such as sinusoidal functions and circular geometry.

Visual Models

Light Beam Model

In searching for a method to visually represent the Basel Problem, a related concept I found was the inverse square law which deals with three-dimensional particle emission such as light, soundwaves, and so forth. According to the law, the intensity of a perceived emission is the reciprocal fraction of the square distance between the source and the observer (Nave, 2000):

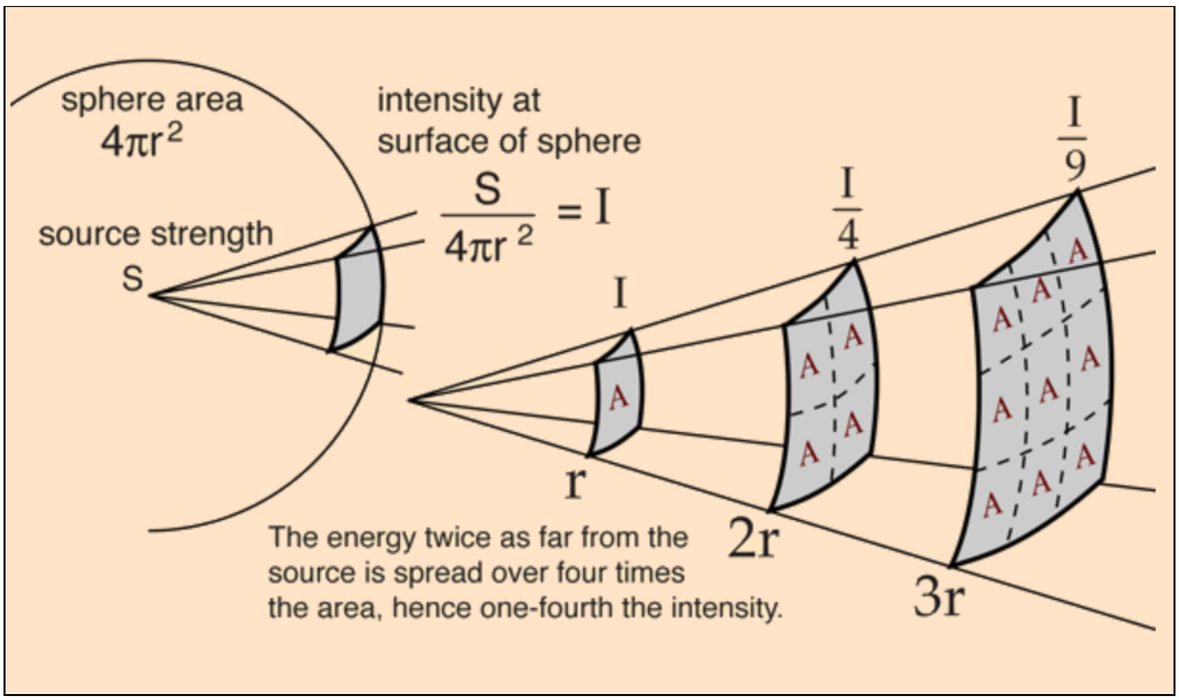

Figure 5. Diagram of Inverse Square Law (Nave, 2000).

Assume that there is a viewer *V* that absorbs the light passed through it from light sources denoted as L_n, where *n* is the distance in arbitrary units from the viewer to the respective light. Assume that all of these light sources are in a straight line:

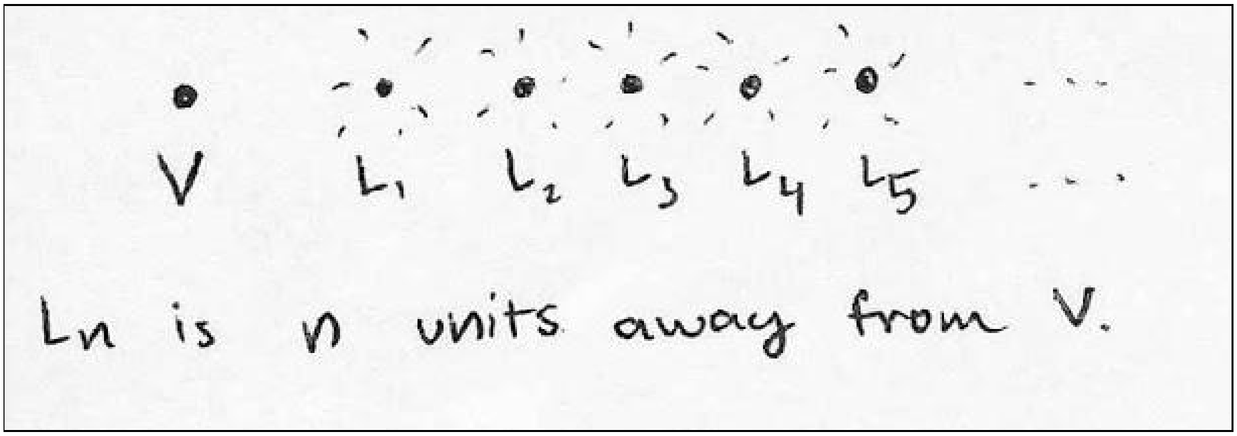

Figure 6. Light beam Model A - Separate L_n *sources.*

Next, assume that the base intensity *I* of the center of each light source is the same and that light can pass through other sources. The cumulative intensity of light received by *V* would be the sum of the Basel series:

Let *T* represent total light received from *L*.

$$T = T(I) = I + \frac{I}{2^2} + \frac{I}{3^2} + \frac{I}{4^2} + \ldots$$

If the absolute distance of a single unit (the distance between adjacent light sources) *d* approaches zero, the infinite set of points will transform into a continuous line of light—a 'light beam.' However, the expression for *T* remains unchanged because the ratio between the base intensity and the total received by *V* is dependent on the unit, not the distance. Therefore, the amount of light received at one end of a beam of light is $\pi^2/6$ times the theoretical intensity of the point lights that compose the beam:

$$\lim_{d \to 0} T(I) = I + \frac{I}{2^2} + \frac{I}{3^2} + \frac{I}{4^2} + \ldots = I(\frac{\pi^2}{6})$$

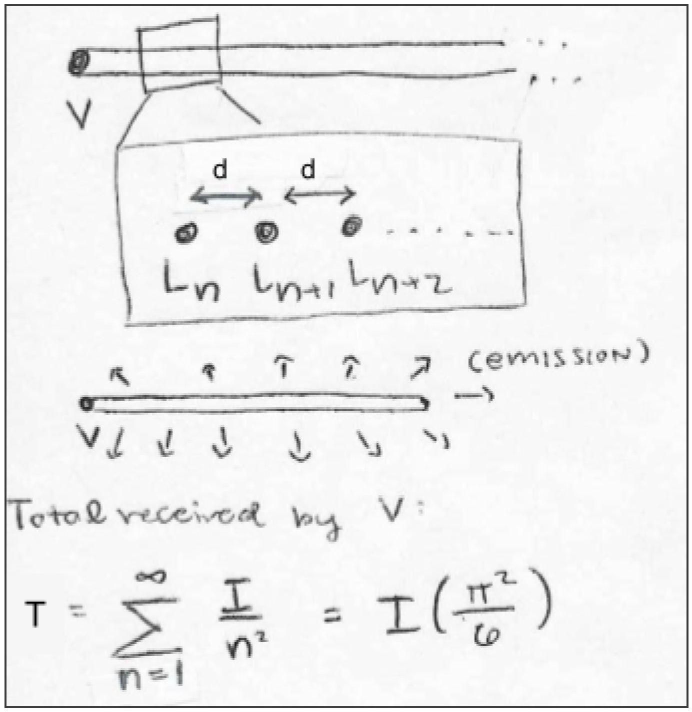

Figure 7. Light beam Model B - Continuous L_n *sources.*

The light beam model can be used to estimate the apparent brightness of a finite line segment of light sources given that each 'point' source is theoretically zero distance away from neighboring points. Therefore, the $\pi^2/6$ ratio could be applied in the real world to measure light intensities of appliances such as LED columns or strip lights.

Contact Beam Model

The light beam model can represent any form of three-dimensional emission in addition to light; another representation of the model could be using sound waves. Assume that there is a viewer *V* and an infinite set of objects S_n in a straight line that are *n* units away from *V*. Also, assume that each object produces a sound wave of equivalent intensity *I* when making contact with a flat surface. If all the objects make contact with the surface simultaneously, the viewer will receive sound with $\pi^2/6$ times the volume of the base intensity:

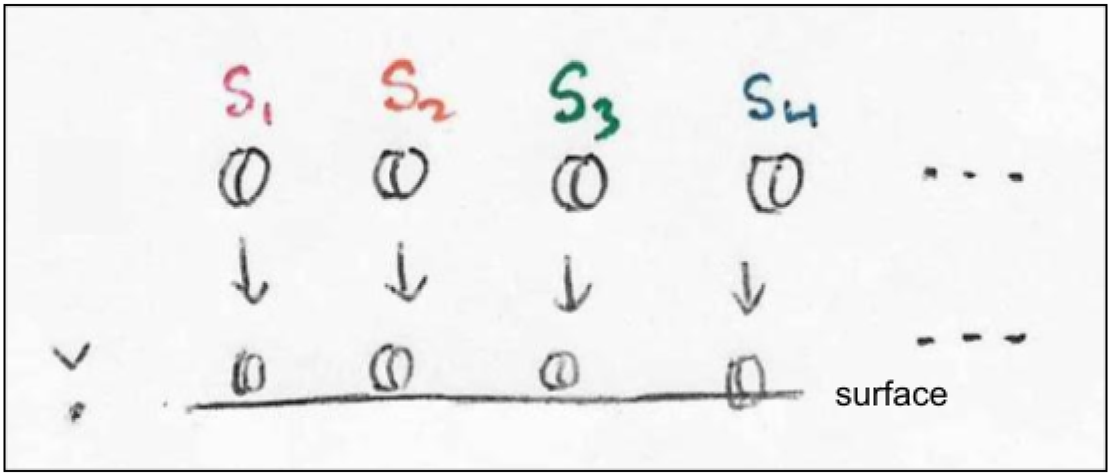

Figure 8. Contact Beam Model A - Objects making contact on a flat surface.

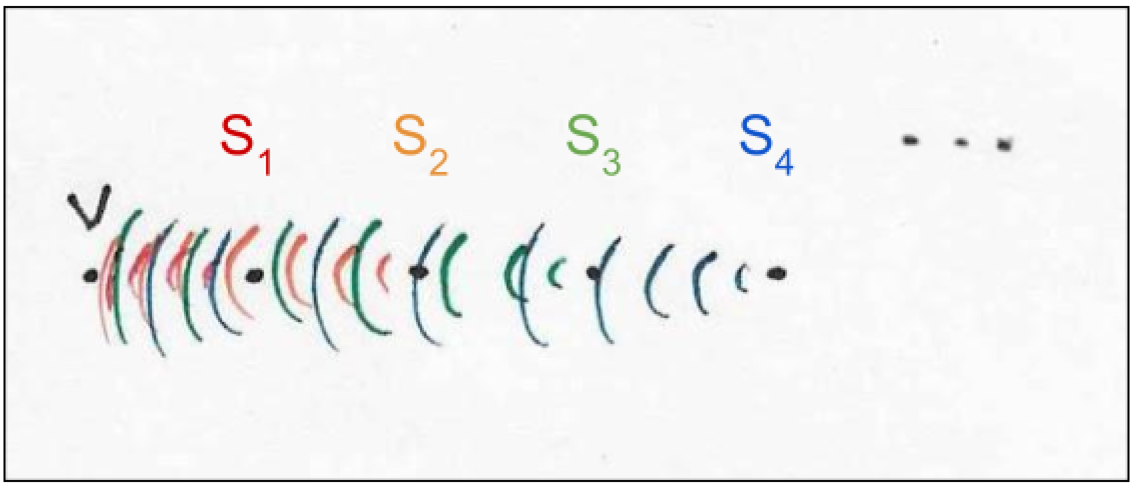

Figure 9. Contact Beam Model B - Production of sound waves upon contact.

Similar to the light beam, theoretically reducing the absolute distance of a unit to zero will result in the 'contact beam,' where the viewer is at the endpoint of the beam yet still receives the same $\pi^2/6$ ratio of volume:

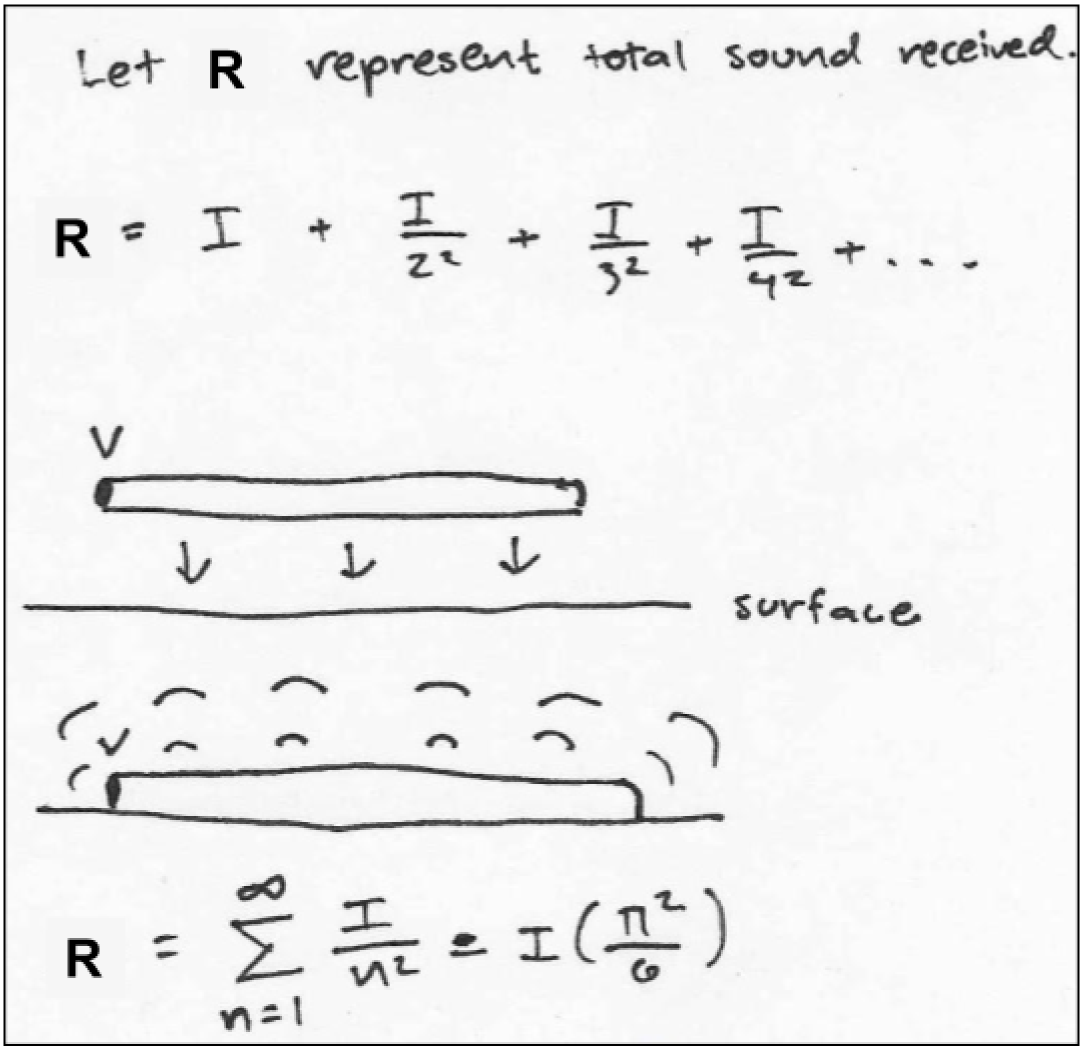

Figure 10. Contact Beam Model C - Continuous beam producing sound.

Connection to Sanderson's Proof

After completing the visual models, I watched a video on a complementary proof by Grant Sanderson which not only utilizes the inverse square law but also incorporates geometry. He begins with similar assumptions about his 'lighthouse' diagram as my light beam model—that light sources have the same intensity and the viewer senses all the light that makes contact with them. Then, he draws a circle such that the viewer and the lighthouse are on opposite ends of a circle's diameter with circumference 2 (Sanderson, 2018). Doing so, he is able to calculate the diameter *d* and hence determine the light intensity *I* received by the viewer *V*. For consistency, I used the same variables from my own models of the inverse square law:

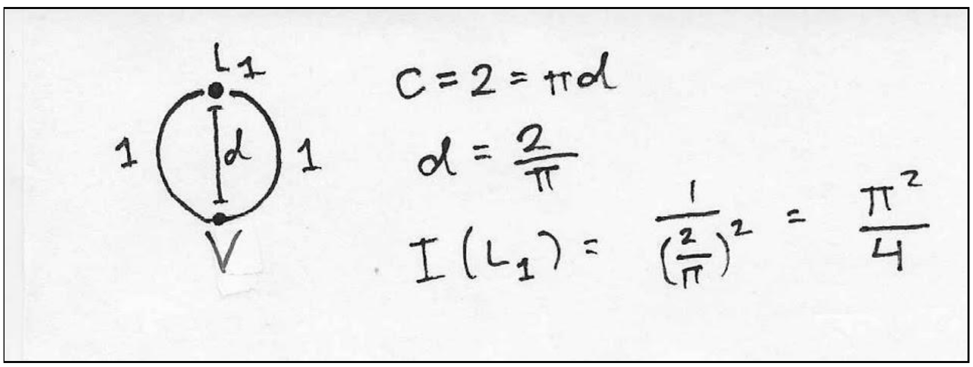

Figure 11. Basic Model - Intensity of light along 2/π diameter (Sanderson, 2018).

Sanderson then introduces two important theorems to expand upon the basic model. First is the Inverse Pythagorean Theorem which states that, of a right triangle, the sum of the reciprocal squares of the triangle's legs is equal to the reciprocal square of the altitude line (Sanderson, 2018). The reciprocal square terms match the expressions for the intensities of light at their respective locations (Sanderson, 2018):

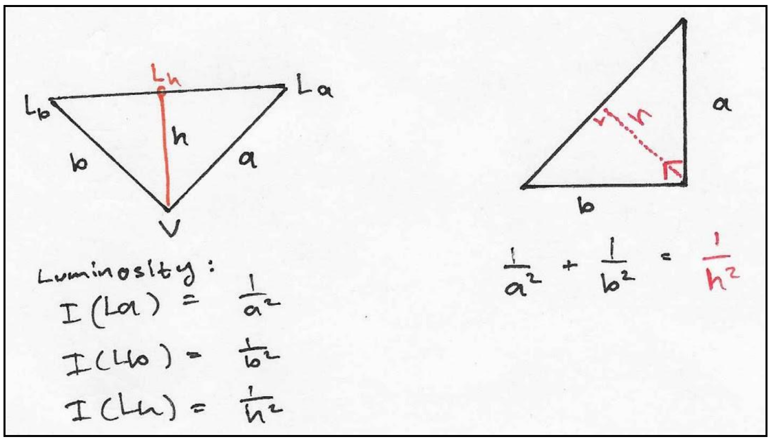

Figure 12. Inverse Pythagorean Theorem and matching light diagram (Sanderson, 2018).

Hence, the following corollary holds true based on the Inverse Pythagorean Theorem: Let L_a and L_b be emission sources at the endpoints of a right triangle. Let L_h be a source at the intersection of the altitude and hypotenuse of the same right triangle. Let I(L) represent the intensity of light received by the origin of the right angle:

$$I(L_a) + I(L_b) = I(L_h)$$

Thales' Theorem states that "the inscribed angle that subtends the diameter of a circle is always a right angle" (Sarig, 2019). Using this theorem, Sanderson can expand upon the model by creating a new circle twice the circumference of the original and passing through point *V*. Drawing a tangent line for the smaller circle through point L_1 creates a diameter for the larger circle because point L_1 is situated at the center of the new circle, halfway to the opposite end of point V (Sanderson, 2018). By connecting point V with the endpoints of the new diameter, a right triangle is formed due to Thales' Theorem; the Inverse Pythagorean Theorem can be used because the line segment between V and L_1 is the altitude. Therefore, the following model with new lighthouses at the endpoints maintains the exact same amount of light as the basic model:

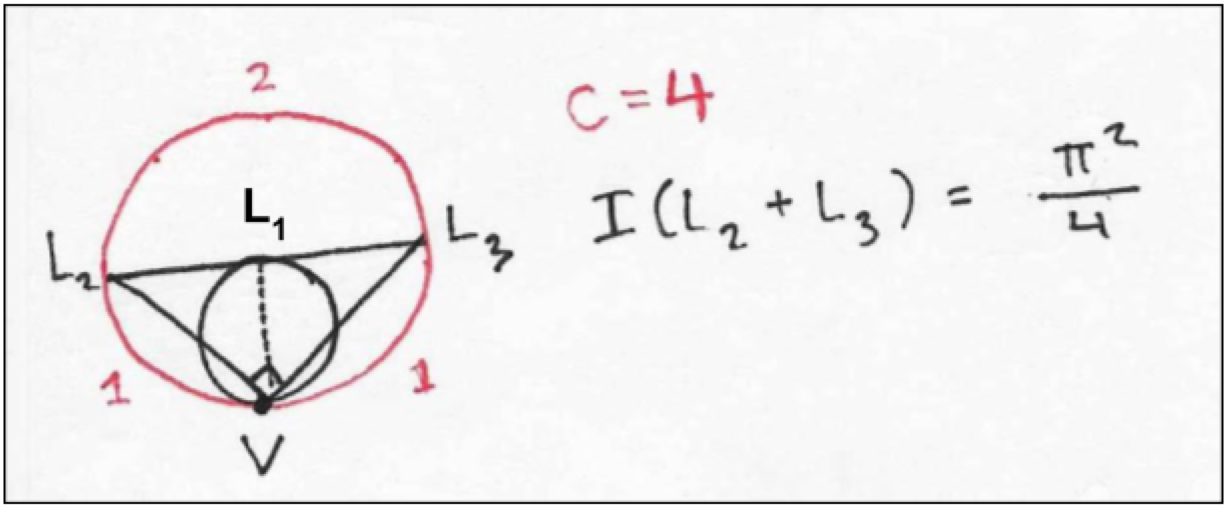

Figure 13. First iteration of Sanderson's circular model.

Repeating this process results in the same light intensity of $\pi^2/4$ received by the viewer. From this iteration onward, the diameters are drawn by connecting the center of the circle with each light source of the previous iteration. Doing so will always create a diameter because any line that passes through the center of a circle divides the circle exactly in half. Below is a diagram showing the completed second iteration. The right triangles are once again formed by connecting the new locations for the lighthouses to the viewer and the new altitude is formed by connecting the viewer to the old lighthouse along the triangle's hypotenuse (Sanderson, 2018). Note that the arcs formed between the diameters are always evenly divided because all the angles formed at the intersection of the diameters are equivalent.

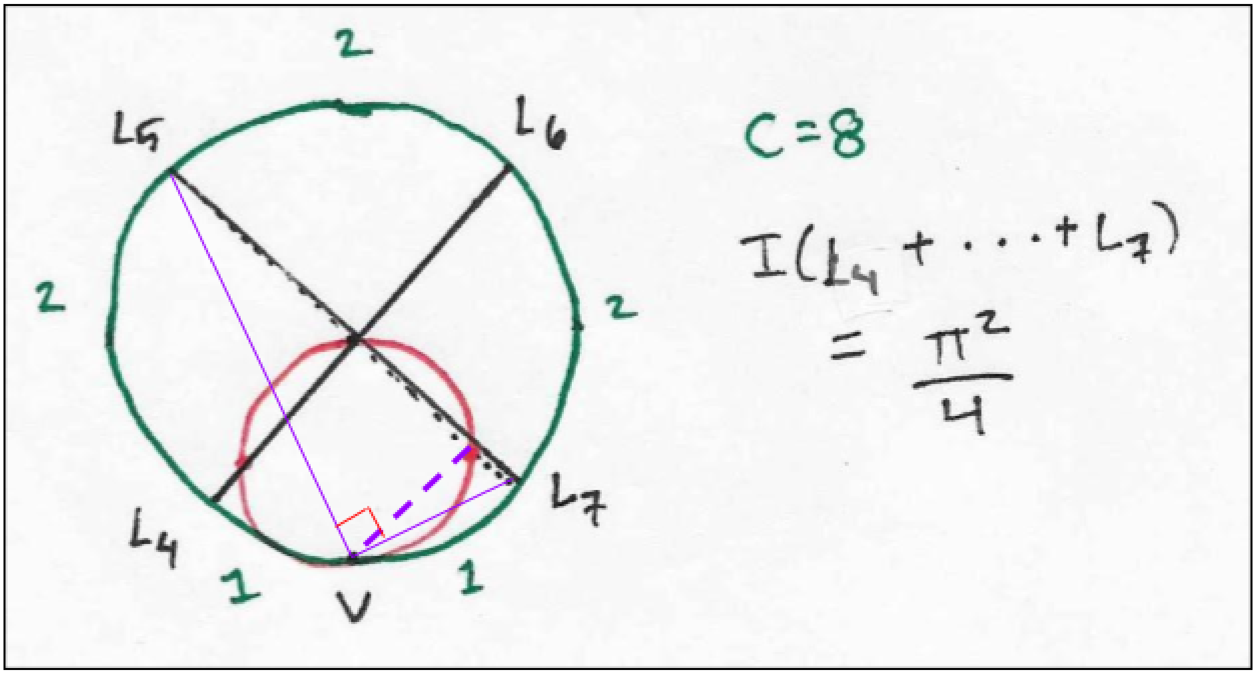

Figure 14. Second iteration of Sanderson's circular model.

After an infinite number of iterations, the circle flattens out into a straight line (Sanderson, 2018). However, due to Thales' Theorem as well as the Inverse Pythagorean Theorem, the amount of light received from the infinite set of light sources still equals $\pi^2/4$. The positions of the light sources can be represented along a number line as the circumference C goes to infinity:

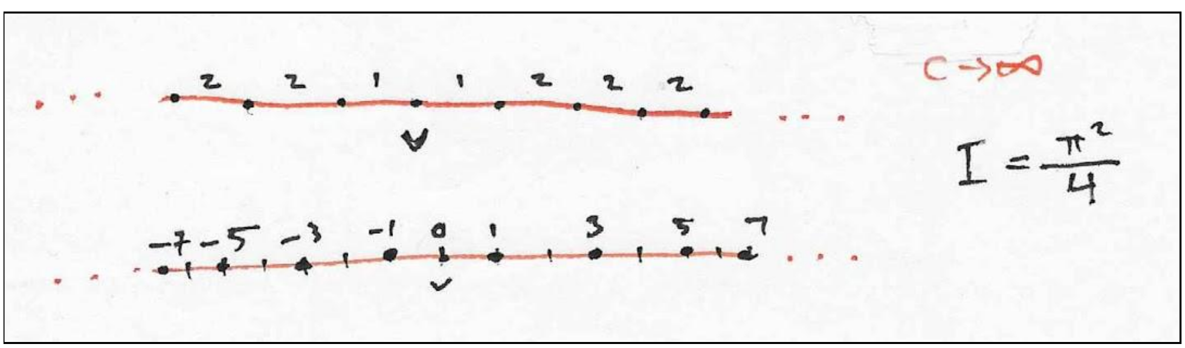

Figure 15. Line of infinite light sources measured by relative distance (above) and absolute location along a number line (below).

Because the light sources are equidistant along evenly divided arcs and because point *V* intersects one of these arcs, each light source is a unique odd number integer distance away from the viewer (Sanderson, 2018). In fact, the light sources cover every odd integer between positive and negative infinity. Hence, the cumulative light intensity received can be written as the sum of the reciprocal squares of all positive and negative odd integers:

$$I = (\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots) + (\frac{1}{(-1)^2} + \frac{1}{(-3)^2} + \frac{1}{(-5)^2} + \frac{1}{(-7)^2} + \ldots)$$

From this point onward, I was able to fill in the rest of the proof using the findings from my blind investigation (see *Grouping Method*):

Squaring any negative integer produces the same value as squaring the positive of that same integer:

$$I = (\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots) + (\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots)$$

Both series are equivalent and can be added:

$$I = 2(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots) = \frac{\pi^2}{4}$$

Dividing by two results in the following:

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$

I have reached the same conclusion about the sum of reciprocal squares of positive odd integers here as I did while investigating the Grouping Method. Recall that I referred to this series with the variable k_{odd}:

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8} = k_{odd}$$

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots = k_{even}$$

The sum of the odd and even positive integer series results in the complete Basel Problem series:

$$k = k_{odd} + k_{even} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots\right)$$

$$= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = k$$

Recall from findings in the Grouping Method that k_{odd} is three-fourth of the value of k:

$$k_{odd} = \frac{\pi^2}{8} = \frac{3}{4}k$$

$$\frac{\pi^2}{6} = k$$

$$k = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.\ Q.E.D.$$

Sanderson's proof further validates my light beam model by tying together the geometric properties of the inverse square law and the abstract concepts in series manipulation to justify the fact that the sum of reciprocal square positive integers is equivalent to $\pi^2/6$.

Conclusion

Overall Conclusions

Because the Basel Problem has strong connections to several branches of mathematics, the techniques used to reach the final answer to the series relies on complex interplay between various mathematical concepts. Euler's proof required a combination of sinusoidal functions, limits, calculus, and algebraic manipulation to piece together and prove the Basel Problem in abstract. I also experimented with mathematical interdependence when findings from my blind investigation merged with the proof for my visual models. Sanderson's proof helped me scaffold my 'unsuccessful' Grouping Method by filling in the geometric properties of an infinite series involving the inverse square law. Similarly, my blind investigation helped complete the algebraic component required to show the relation between the Basel Problem and Sanderson's 'lighthouse' model.

Experimental Evaluation

While the blind investigation on its own failed to answer the research question, it provided necessary insight into understanding the algebraic and geometric properties of the Basel Problem. The Simple Integration Method revealed to me the impossibility of manually calculating the sum of reciprocal squares—that there lacked a plausible integration method to simplify the series independently. The Grouping Method pointed me in a better direction and ultimately became necessary in the process of triangulating my findings with creating visual models. Lastly, comparing my work to Euler's original proof allowed me to understand the multifaceted nature of the Basel Problem, working from several nearly unrelated ends to arrive at the common statement.

Limitations and Extensions

A limitation of this investigation would be my math background. The methods I used in the blind investigation and the analysis I conducted on other proofs mostly do not use math beyond geometry and introductory-level calculus. Researchers with a more advanced background in mathematics might have taken different and possibly more effective approaches into dissecting the Basel Problem. Another limitation of this investigation was a lack of access to resources during the COVID-19 pandemic. All sources used for this investigation were found through online databases; the research I conducted may have benefitted from a greater variety of sources as well as potential hands-on experimenting.

A possible extension to my research would be to expand upon visual models. All the representations I used were derived from properties of the inverse square law. Other researchers could extrapolate my work to incorporate models using other properties in physics and mathematics to demonstrate a different application of the Basel Problem. Another possible extension would be to connect the Basel Problem itself to other theorems and formulae for π. These other formulae related to π are likely to incorporate fields of math that are not within the Basel Problem, allowing for a wider variety of mathematical analysis of π.

Takeaway

In the broader picture, Euler's discovery that the sum of reciprocal whole-number squares equated to $\pi^2/6$ did not come from working directly on untangling the series itself. Looking at his original proof, it was instead a discovery among investigating properties of reciprocal series in general. Instead of singling out the Basel Problem and working backwards, the problem is more efficiently solved by integrating together the algebra, geometry, and calculus that composes it. It may seem to take out part of the mysterious appeal in the formula, but it makes up for it by demonstrating a fundamentally significant connection between abstract and applied mathematics.

Bibliography

Daners, D. (2012). A Short Elementary Proof of $\Sigma\ 1/k^2 = \pi^2/6$. Mathematics Magazine, 85(5), 361–364. https://doi.org/10.4169/math.mag.85.5.361

Desmos Graphing Calculator. (2015). *Desmos Graphing Calculator*. https://desmos.com/calculator.

Dunham, W. (1999). Euler and Infinite Series. In Euler: the master of us all (22nd ed., pp. 46–57). essay, Mathematical Association of America.

Euler, L. (1740). De summis serierum reciprocarum [On the sums of series of reciprocals]. St. Petersburg Academy, 7, 1–8.

Hawking, S. (2005). God created the integers: The mathematical breakthroughs that changed history (pp. 386-399). Philadelphia, Pa: Running Press.

Ivan, M. (2008). A simple solution to Basel Problem. General Mathematics, 16(4), 111–113.

Kalman, D. (1993). Six Ways to Sum a Series. The College Mathematics Journal, 24(5), 402–421. https://doi.org/10.1080/07468342.1993.11973562

Martin, C. (2010). Methods for Evaluating Infinite Series. UCSB Math, 1–4.

Monks, K. M. (2020). Euler's Calculation of the Sum of the Reciprocals of Squares. TRIUMPHS, 9, 1–13.

Moreno, S. G. (2015). A One-Sentence and Truly Elementary Proof of the Basel Problem.

Nave, R. (2000, August 2). Inverse Square Law, General. http://hyperphysics.phy-astr.gsu.edu/hbase/Forces/isq.html. Accessed 23 June 2020.

Sanderson, G. (2018, March 2). Why is pi here? And why is it squared? A geometric answer to the Basel problem. YouTube. https://www.youtube.com/watch?v=d-o3eB9sfls. Accessed 16 July 2020.

Sarig, I. (2019, September 30). Thales' Theorem. https://geometryhelp.net/thales-theorem/. Accessed 3 January 2021.