

# Lecture 7: ANCOVA, short introduction to Linear Algebra

BIO144 Data Analysis in Biology

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- ▶ ANCOVA
- ▶ Introduction to linear Algebra

Note: ANCOVA = ANalysi of COVAriance (Kovarianzanalyse)

# Course material covered today

- ▶ "Getting Started with R" chapter 6.3
- ▶ "Lineare regression" chapters 3.A (p. 43-45) and 3.4, 3.5 (p. 39-42)

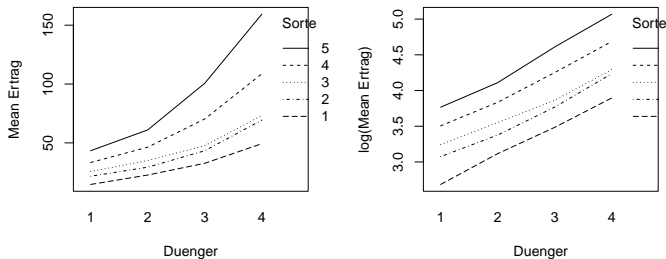
# Recap of ANOVA

- ▶ ANOVA is a method to test if the means of **two or more groups are different**.
- ▶ Post-hoc tests and contrasts, including correction for  $p$ -values, to understand the differences between the groups.
- ▶ Two-way ANOVA for factorial designs, interactions.
- ▶ ANOVA is a special case of linear regression with categorical covariates.

## Recap of two-way ANOVA example

Remember: Influence of four levels of fertilizer (DUENGER) on the yield (ERTRAG) on 5 species (SORTE) of crops was investigated. For each DUENGER  $\times$  ERTRAG combination, 3 repeats were taken.

Interaction plot with ERTRAG and  $\log(\text{ERTRAG})$  as response:



Remember: We used  $\log(\text{ERTRAG})$ , because the residual plots were otherwise not ok.

```
r.duenger2 <- lm(log(ERTRAG) ~ DUENGER*SORTE,d.duenger)
anova(r.duenger2)
```

```
## Analysis of Variance Table
##
## Response: log(ERTRAG)
##           Df Sum Sq Mean Sq  F value Pr(>F)
## DUENGER      3 11.6917   3.8972  854.0505 <2e-16 ***
## SORTE        4  8.5202   2.1300  466.7851 <2e-16 ***
## DUENGER:SORTE 12  0.0929   0.0077   1.6958 0.1045
## Residuals    40  0.1825   0.0046
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

## Questions:

- ▶ Number of parameters?
- ▶ Degrees of freedom (60 data points)?
- ▶ Interpretation?

summary(r.duenger2)

```
##
## Call:
## lm(formula = log(ERTRAG) ~ DUENGER * SORTE, data = d.duenger)
##
## Residuals:
```

	Min	1Q	Median	3Q	Max
	-0.120968	-0.045595	0.008984	0.049072	0.102175

```
##
## Coefficients:
```

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	2.68505	0.03900	68.846	< 2e-16 ***
DUENGER2	0.43165	0.05516	7.826	1.36e-09 ***
DUENGER3	0.79997	0.05516	14.504	< 2e-16 ***
DUENGER4	1.21152	0.05516	21.966	< 2e-16 ***
SORTE2	0.38979	0.05516	7.067	1.51e-08 ***
SORTE3	0.55799	0.05516	10.117	1.38e-12 ***
SORTE4	0.82018	0.05516	14.870	< 2e-16 ***
SORTE5	1.08169	0.05516	19.612	< 2e-16 ***
DUENGER2:SORTE2	-0.12949	0.07800	-1.660	0.105
DUENGER3:SORTE2	-0.10613	0.07800	-1.361	0.181
DUENGER4:SORTE2	-0.04924	0.07800	-0.631	0.531
DUENGER2:SORTE3	-0.12180	0.07800	-1.562	0.126
DUENGER3:SORTE3	-0.18034	0.07800	-2.312	0.026 *
DUENGER4:SORTE3	-0.16061	0.07800	-2.059	0.046 *
DUENGER2:SORTE4	-0.10138	0.07800	-1.300	0.201
DUENGER3:SORTE4	-0.05311	0.07800	-0.681	0.500
DUENGER4:SORTE4	-0.02954	0.07800	-0.379	0.707
DUENGER2:SORTE5	-0.08779	0.07800	-1.125	0.267
DUENGER3:SORTE5	0.04370	0.07800	0.560	0.578
DUENGER4:SORTE5	0.09014	0.07800	1.156	0.255

```
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Analysis of Covariance (ANCOVA)

An ANCOVA is an analysis of variance (ANOVA), **including** also at least one **continuous covariate**.

ANCOVA unifies several concepts that we approached in this course so far:

- ▶ Linear regression
- ▶ Categorical covariates
- ▶ Interactions (of continuous and categorical covariates)
- ▶ Analysis of Variance (ANOVA)

As such, it is a **special case of the linear regression model**.



Given a categorical covariate  $x_i$  and a continuous covariate  $z_i$ . Then the ANCOVA equation (without interactions) is given as

$$y_i = \beta_0 + \beta_1 x_i^{(1)} + \dots + \beta_k x_i^{(k)} + \beta_z z_i + \epsilon_i ,$$

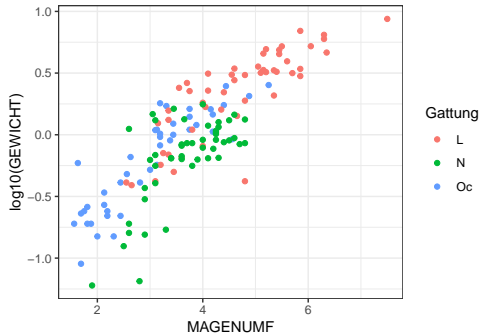
where  $x_i^{(k)}$  is the  $k$ th dummy variable ( $x_i^{(k)}=1$  if  $i$ th observation belongs to category  $k$ , 0 otherwise).

**Note 1:** It is straightforward to add an interaction of  $x_i$  with  $z_i$ .

**Note 2:** Again, for identifiability reason, we typically set  $\beta_1 = 0$ .

## Once more: the earthworms

“Magenumfang” was used to predict “Gewicht” of the worm, including as covariate also the worm species.



**Categorical** and **continuous** covariates were used to predict a continuous outcome → ANCOVA.

```
r.lm <- lm(log(GEWICHT) ~ MAGENUMF + Gattung,d.wurm)
summary(r.lm)$coef
```

##	Estimate	Std. Error	t value	Pr(> t )
## (Intercept)	-2.5355459	0.22147279	-11.4485663	8.617670e-22
## MAGENUMF	0.7118725	0.04528843	15.7186392	1.232126e-32
## GattungN	-0.5151344	0.11009219	-4.6791186	6.760621e-06
## GattungOc	-0.0907298	0.12791000	-0.7093254	4.793107e-01

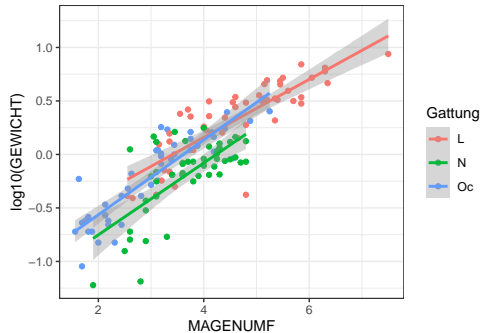
**Important:** The  $p$ -values for the entries GattungN and GattungOc are not very meaningful (why?).

To understand if “Gattung” has an effect, **we need to carry out an  $F$ -test** → ANOVA table:

```
anova(r.lm)
```

```
## Analysis of Variance Table
##
## Response: log(GEWICHT)
##           Df Sum Sq Mean Sq F value    Pr(>F)
## MAGENUMF   1 104.866  104.866  409.69 < 2.2e-16 ***
## Gattung    2   7.177   3.589   14.02 2.842e-06 ***
## Residuals 139  35.579   0.256
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

We can also include an **interaction term** between MAGENUMF and Gattung to allow for different slopes:



→ We again need the **F-test** to check whether the respective interaction term is needed:

```
r.lm2<- lm(log(GEWICHT) ~  MAGENUMF * Gattung,d.wurm)
anova(r.lm2)
```

```
## Analysis of Variance Table
##
## Response: log(GEWICHT)
##
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
MAGENUMF	1	104.866	104.866	414.4743	< 2.2e-16 ***
Gattung	2	7.177	3.589	14.1835	2.521e-06 ***
MAGENUMF:Gattung	2	0.917	0.458	1.8112	0.1673
Residuals	137	34.662	0.253		

```
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

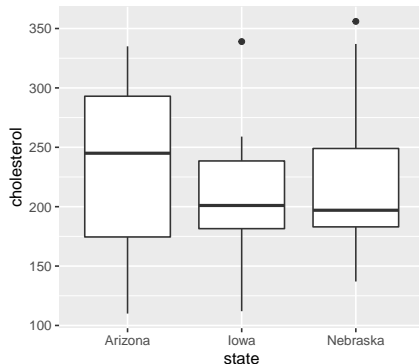
→  $p = 0.167$ , thus interaction is probably not relevant.

## A new example: cholesterol levels

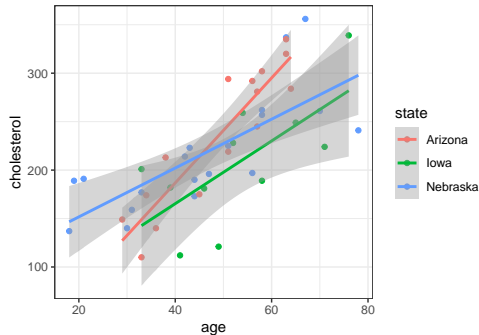
**Example:** Cholesterol levels [mg/ml] for 45 women from three US states (Iowa, Nebraska, Arizona), were measured.

**Question:** Do these levels differ between the states?

Age (years) may be a relevant covariable.



The scatter plot gives an idea about the model that might be useful here:



→ We include state, age and the interaction of the two.



Doing the analysis:

```
r.lm <- lm(cholesterol ~ age*state,data=d.chol)
anova(r.lm)
```

```
## Analysis of Variance Table
```

```
##
```

```
## Response: cholesterol
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
age	1	96524	96524	61.8961	1.424e-09 ***
state	2	11474	5737	3.6789	0.03438 *
age:state	2	12665	6332	4.0606	0.02501 *
Residuals	39	60819	1559		

```
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interpretation?

Compare the results from the previous slide to the estimated coefficients:

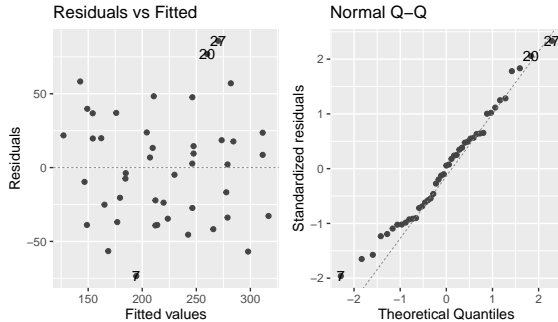
```
r.lm <- lm(cholesterol ~ age*state,data=d.chol)
summary(r.lm)$coef
```

##	Estimate	Std. Error	t value	Pr(> t )
## (Intercept)	-29.895169	43.7353712	-0.6835467	4.983027e-01
## age	5.416908	0.8679635	6.2409400	2.396876e-07
## stateIowa	65.706383	66.7677031	0.9841043	3.311303e-01
## stateNebraska	131.192935	50.8573164	2.5796276	1.377434e-02
## age:stateIowa	-2.178763	1.2672928	-1.7192264	9.350204e-02
## age:stateNebraska	-2.896470	1.0166558	-2.8490174	6.967607e-03

**Note:** The  $p$ -values for the age coefficient is not the same as in the ANOVA table.

**Reason:** `anova()` tests the models against one another in the **order** specified.

As always, some model checking is necessary:



→ This seems ok.

# An introduction to linear Algebra

Who has some knowledge of linear Algebra?

## Overview

- ▶ The basics about
  - ▶ vectors
  - ▶ matrices
  - ▶ matrix algebra
  - ▶ matrix multiplication
- ▶ Why is linear Algebra useful?
- ▶ What does it have to do with data analysis and statistics?
- ▶ Regression equations in matrix notation.

# Motivation

Why are vectors, matrices and their algebraic rules useful?

- **Example 1:** The observations for a covariate  $x$  or the response  $y$  for all individuals  $1 \leq i \leq n$  can be stored in a vector (vectors and matrices are always given in **bold** letters):

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}.$$

- **Example 2:** Covariance matrices for multiple variables. Say we have  $x^{(1)}$  and  $x^{(2)}$ . The **covariance matrix** is then given as

$$\begin{pmatrix} \text{Var}(x^{(1)}) & \text{Cov}(x^{(1)}, x^{(2)}) \\ \text{Cov}(x^{(1)}, x^{(2)}) & \text{Var}(x^{(2)}) \end{pmatrix}.$$

- **Example 3:** The **data** (e.g. of some regression model) can be stored in a **matrix**:

$$\tilde{X} = \begin{pmatrix} 1 & x_1^{(1)} & x_1^{(2)} \\ 1 & x_2^{(1)} & x_2^{(2)} \\ \dots & \dots & \dots \\ 1 & x_n^{(1)} & x_n^{(2)} \end{pmatrix} .$$

This is the so-called **design matrix** with a vector of 1's in the first column.

- **Example 4:** A linear regression model can be written compactly using **matrix multiplication**:

$$y = \tilde{X} \cdot \tilde{\beta} + e ,$$

with  $\tilde{\beta}$  the vector of regression coefficients and  $e$  the vector of errors

Why do we discuss this topic in our course?

- ▶ Useful for **compact notation**.
- ▶ Enables you to **understand many statistical texts** (books, research articles) that remain inaccessible otherwise.
- ▶ Useful for **efficient coding**, e.g. in R, which helps to increase speed and to reduce error rates.
- ▶ More advanced concepts often rely on linear algebra, e.g. **principal component analysis** (PCA) or **random effects** models.
- ▶ Is part of a **general education** (Allgemeinbildung) ;-)

# Matrices

An  $n \times m$  **Matrix** is given as

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix},$$

with rows  $i = 1, \dots, n$  and columns  $j = 1, \dots, m$ .

**Quadratic matrix:**  $n = m$ . Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 6 & 1 & 9 \end{pmatrix}$$



**Symmetric matrix:**  $a_{ij} = a_{ji}$ . Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} .$$

**The diagonal of a quadratic matrix** is given by  $(a_{11}, a_{22}, \dots, a_{nn})$ . Example: the diagonal of the above matrix is given as

$$(a_{11}, a_{22}, a_{33}) = (1, 3, 5) .$$

**Diagonal matrix:** A matrix that has entries  $\neq 0$  **only on the diagonal**. Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} .$$

**Transposing a matrix:** Given a matrix  $A$ . Exchange the rows by the columns and vice versa. This leads to the **transposed matrix**  $A^T$ :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix}.$$

Examples (note also the change of dimensions):

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$$

- ▶ Transposing a matrix **twice** leads to the original matrix:

$$(A^{\top})^{\top} = A .$$

- ▶ When a matrix is **symmetric**, then

$$A^{\top} = A .$$

This is true in particular for diagonal matrices.

# Vectors

A vector is nothing else than  $n$  numbers written in a column:

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

**Transposing** a vector leads to a *row vector*:

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}^{\top} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}$$

**Note:** By definition (by default), a vector is always a column vector.

## Addition and subtraction

- ▶ Adding and subtracting matrices and vectors is only possible when the objects have the **same dimension**.
- ▶ Examples: Elementwise addition (or subtraction)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 \\ 10 & 10 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$

- ▶ But this addition is **not defined**:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 2 & 5 \\ 1 & 4 \end{pmatrix} =$$

## Multiplication by a scalar

Multiplication with a “number” (scalar) is simple: Multiply each element in a vector or a matrix.

Examples:

$$3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

$$-2 \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \\ 4 \end{pmatrix}$$

# Matrix multiplication

The multiplication of two matrices  $A$  and  $B$  is **defined if**

number of columns in  $A$  = number of rows in  $B$ .

It is easiest to explain matrix multiplication with an example:

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 + 1 \cdot 4 & 2 \cdot 1 + 1 \cdot (-2) \\ (-1) \cdot 3 + 0 \cdot 4 & (-1) \cdot 1 + 0 \cdot (-2) \\ 3 \cdot 3 + 1 \cdot 4 & 3 \cdot 1 + 1 \cdot (-2) \end{pmatrix} \\ = \begin{pmatrix} 10 & 0 \\ -3 & -1 \\ 13 & 1 \end{pmatrix}$$

► Matrix multiplication app

# Matrix multiplication rules I

Attention: Matrix multiplication does **not** follow the same rules as scalar multiplication!!!

- ▶ It can happen that  $A \cdot B$  can be calculated, but  $B \cdot A$  is not defined (see example on previous slide).
- ▶ In general:  $A \cdot B \neq B \cdot A$ , even if both are defined.
- ▶ It can happen that  $A \cdot B = 0$  (0 matrix), although both  $A \neq 0$  and  $B \neq 0$ .
- ▶ The **Assoziativgesetz** holds:  $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ .
- ▶ The **Distributivgesetz** holds:

$$\begin{aligned} A \cdot (B + C) &= A \cdot B + A \cdot C \\ (A + B) \cdot C &= A \cdot C + B \cdot C \end{aligned}$$



## Matrix multiplication rules II

- ▶ Transposing inverts the order:  $(A \cdot B)^{\top} = B^{\top} \cdot A^{\top}$ .
- ▶ The product  $A \cdot A^{\top}$  is **always symmetric**.
- ▶ All these rules also hold for **vectors**, which can be interpreted as  $n \times 1$  matrices:

$$a \cdot b^{\top} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_m \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_m \\ \vdots & \vdots & & \vdots \\ a_n b_1 & a_n b_2 & \dots & a_n b_m \end{pmatrix}$$

If  $a$  and  $b$  have the **same length**:

$$a^{\top} \cdot b = \sum_i a_i b_i$$

## Short exercise

Given vectors  $a$  and  $b$  and matrix  $C$ :

$$a = \begin{pmatrix} 1 \\ -2 \\ 3 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

Calculate, if defined

- ▶  $a^T \cdot b$
- ▶  $a \cdot b^T$
- ▶  $C \cdot a$
- ▶  $C \cdot b$

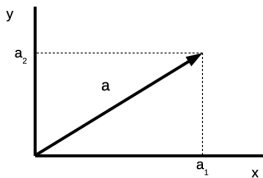
# The length of a vector

The **length of a vector**  $a^T = (a_1, a_2, \dots, a_n)$  is defined as  $\|a\|$  with

$$\|a\|^2 = a^T \cdot a = \sum_i a_i^2 .$$

This is basically the **Pythagoras** idea in 2, 3, ...  $n$  dimensions.

In 2 dimensions:  $\|a\| = \sqrt{a_1^2 + a_2^2}$ :



## Identity matrix (Einheitsmatrix)

The identity matrix (of dimension  $m$ ) is probably the simplest matrix that exists. It has 1's on the diagonal and 0's everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Multiplication with the identity matrix leaves a  $m \times n$  matrix  $A$  unchanged:

$$A \cdot I = A .$$

# Inverse matrix

Given a quadratic matrix  $A$  that fulfills

$$B \cdot A = I ,$$

then  $B$  is called the **inverse** of  $A$  (and vice versa). One then writes

$$B = A^{-1} .$$

Note:

- ▶ In that case it also holds that  $A \cdot B = I$ .
- ▶ Therefore:  $A = B^{-1} \Leftrightarrow B = A^{-1}$

- ▶ The inverse of  $A$  may **not exist**. If it exists,  $A$  is **regular**, otherwise **singular**.
- ▶  $(A^{-1})^{-1} = A$ .
- ▶ The inverse of a matrix product is given as

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1} .$$

- ▶ It is

$$(A^{\top})^{-1} = (A^{-1})^{\top} .$$

Therefore one may also write  $A^{-\top}$ .

## Linear regression in matrix notation

Linear regression with  $n$  data points can be understood as an **equation system with  $n$  equations**.

Remember example 4 from slide 21: We said that a linear regression model can be written compactly using **matrix multiplication**:

$$y = \tilde{X} \cdot \tilde{\beta} + e .$$

Task: Verify this now, using a model with two variables  $x^{(1)}$  and  $x^{(2)}$  and

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 1 & x_1^{(1)} & x_1^{(2)} \\ 1 & x_2^{(1)} & x_2^{(2)} \\ \dots & \dots & \dots \\ 1 & x_n^{(1)} & x_n^{(2)} \end{pmatrix}, \quad \tilde{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}, \quad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} .$$

It can be shown (see Stahel 3.4f,g) that the least-squares estimates  $\hat{\beta}$  can be calculated as

$$\hat{\beta} = (\tilde{X}^{\top} \tilde{X})^{-1} \cdot \tilde{X}^{\top} \cdot y$$

Does this look complicated?

Let's test this in R ....



## Doing linear algebra in R

Let us look at model  $y = \tilde{X} \cdot \tilde{\beta} + e$  with coefficients

$\beta_0 = 10, \beta_1 = 5, \beta_2 = -2$  and variables

$i$	$x_i^{(1)}$	$x_i^{(2)}$
1	0	4
2	1	1
3	2	0
4	3	1
5	4	4

Thus the model is given as

$$y_i = 10 + 5x_i^{(1)} - 2x_i^{(2)} + \epsilon_i, \text{ for } 1 \leq i \leq n.$$

Let us start by generating the “true” response, calculated as  $\tilde{X}\tilde{\beta}$

```
x1 <- c(0,1,2,3,4)
x2 <- c(4,1,0,1,4)
Xtilde <- matrix(c(rep(1,5),x1,x2),ncol=3)
Xtilde
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    4
## [2,]    1    1    1
## [3,]    1    2    0
## [4,]    1    3    1
## [5,]    1    4    4
```

```
t.beta <- c(10,5,-2)
t.y <- Xtilde%*%t.beta
t.y
```

```
##      [,1]
## [1,]    2
## [2,]   13
## [3,]   20
```

Next, we generate the vector containing the  $\epsilon_i \sim N(0, \sigma^2)$  with  $\sigma^2 = 1$ :

```
t.e <- rnorm(5,0,1)
t.e
```

```
## [1]  0.7606833 -0.3257157  0.6830309  0.9070262  0.9342162
```

which we add to the “true”  $y = \tilde{X}\tilde{\beta}$  values, to obtain the “observed” values:

```
t.Y <- t.y + t.e
t.Y
```

```
##           [,1]
## [1,]  2.760683
## [2,] 12.674284
## [3,] 20.683031
## [4,] 23.907026
## [5,] 22.934216
```

It is now possible to fit the model with `lm`:

```
r.lm <- lm(t.Y ~ x1 + x2)
summary(r.lm)$coef
```

##	Estimate	Std. Error	t value	Pr(> t )
## (Intercept)	10.069826	0.5556231	18.12348	0.003030672
## x1	5.157981	0.1866953	27.62780	0.001307540
## x2	-1.896970	0.1577864	-12.02239	0.006847617

Alternatively, we can use formula

$$\hat{\beta} = (\tilde{X}^\top \tilde{X})^{-1} \tilde{X}^\top y$$

to find the parameter estimates:

```
solve(t(Xtilde) %*% Xtilde) %*% t(Xtilde) %*% t.Y
```

```
##           [,1]
## [1,] 10.069826
## [2,]  5.157981
## [3,] -1.896970
```

- ▶ `solve()` calculates the **inverse** (here the inverse of  $\tilde{X}^\top \tilde{X}$ ).
- ▶ `t()` gives the **transposed** (here of  $\tilde{X}^\top$ ).

**Task:** Do this calculation by yourself and verify for each step that the dimensions of the matrices and the vector are indeed fitting, so that this expression is defined.

## Appendix

# Some R commands for matrix algebra

Reading vectors and a matrices into R:

```
a <- c(1,2,3)
a
```

```
## [1] 1 2 3
```

```
A <- matrix(c(1,2,3,4,5,6),byrow=T,nrow=2)
B <- matrix(c(6,5,4,3,2,1),byrow=T,nrow=2)
A
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    3
## [2,]    4    5    6
```

```
B
```

```
##      [,1] [,2] [,3]
## [1,]    6    5    4
## [2,]    3    2    1
```

## Adding and subtracting:

A + B

```
##      [,1] [,2] [,3]
## [1,]    7    7    7
## [2,]    7    7    7
```

A - B

```
##      [,1] [,2] [,3]
## [1,]   -5   -3   -1
## [2,]    1    3    5
```

However, be careful, R sometimes does unreasonable things:

A + a

```
##      [,1] [,2] [,3]
## [1,]    2    5    5
## [2,]    6    6    9
```



# Matrix multiplication:

```
C <- A %*% t(B)
C
```

```
##      [,1] [,2]
## [1,]   28  10
## [2,]   73  28
```

```
A%*%a
```

```
##      [,1]
## [1,]   14
## [2,]   32
```

# Matrix inversion (possible for quadratic matrices only):

```
solve(C)
```

```
##      [,1]      [,2]
## [1,] 0.5185185 -0.1851852
## [2,] -1.3518519  0.5185185
```

```
C %*% solve(C)
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

Why does `solve(A)` or `solve(B)` not work?